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A

TREATISE

ON

ALGEBRA,

CONTAINING

THE LATEST IMPROVEMENTS.

ADAPTED TO THE USE OF SCHOOLS AND COLLEGES.

BY

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THIRD EDITION.

HARPER & BROTHERS, PUBLISHERS,

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1849.

PREFACE.

In the preparation of the following work no pains have been spared to obtain from the best sources, such as the later treatises in highest repute, memoirs of scientific bodies, and mathematical journals in English, French, and German, the materials for a book suited to the present state of mathematical science and the wants of teachers and students.

The work contains much that has never before appeared in an English dress, and almost every part will be found to present some new feature. No attempt, however, has been made at originality, unless for the benefit of the student, and in the belief that the existing expositions or processes were inferior. The object has simply been, by any and all means, to make the best book, without aiming so much at individual reputation as at the author's own convenience and that of others, devoted, like himself, to the noble task of guiding the youthful votaries of science.

The French treatises furnish excellent models of the theory of Algebra, the German of ingenuity and brevity of notation and exposition, the English of practical adaptation and variety of illustration and example; and from these, after a careful comparison of many authors in each language, demonstrations have been selected and introduced verbatim when they seemed incapable of improvement; but whenever the slightest alteration or amalgamation, or the entire remodeling of them, could give additional clearness or elegance, the limæ labor has not been spared.

The work will be found to contain all that is important in the higher parts of Algebra, upon which usually separate treatises are thought necessary, as well as the elementary expositions suited to beginners. Every variety of symbol and of example has been introduced.

On page XI. those articles of this volume are indicated which constitute a minimum course of Algebra requisite for the prosecution of the higher branches of mathematics. A more extended course, such as would ordinarily be advisable, is also pointed out. The rest may very well be reserved for reference, as the student's own discovery of his wants, in the advanced stages of mathematical pursuit, shall call it in requisition.

The author desires to acknowledge the effective assistance which he has received, in revising the work and superintending it through the press, from Mr. J. J. Elmendorf, to whom it is indebted for many valuable suggestions.

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A MORE ENLARGED COURSE.

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It may be useful to point out in this connection a course of mathematical study. 1°. Algebra; 2°. Geometry:† these two may be pursued simultaneously; 3°. Plane Trigonometry, with its applications to Surveying and Navigation; Spherical Trigonometry, with its applications to Practical and Nautical Astronomy and Geodesy;‡ 4°. Descriptive Geometry;§ 5°. Analytical Geometry;|| 6°. Theoretic Astronomy;¶ 7°. Differential and Integral Calculus and Calculus of Variations;** 8°. Mechanics;†† 9°. Optics;‡‡ 10°. Physical Astronomy.§§

* The articles are numbered throughout the book at the beginnings of paragraphs.

† Should the present work meet with public favor, it will be followed in the course of a few months by a treatise on Geometry.

‡ The author has already published a work embracing these subjects, a new and greatly improved edition of which will appear in the course of the next year.

§ This branch, though it may be omitted without destroying the connection between those which precede and follow it, is of the highest advantage to the general student, and invaluable to the engineer. It may be best taken up in the excellent treatise by Professor Davies. In the French, Monge, the founder of the science, has written extensively upon the subject; there is also a treatise by that best of French writers of elementary works, Lefebure de Fourcy. Professor Davies has published a fine volume on the application of descriptive geometry to shadows and perspective.

|| On this subject there are numerous writers, Davies, Pierce, and Young, whose work is republished here, the author of a treatise in the Library of Useful Knowledge; and in the French, among the best, Biot, of whom there is an English translation by Professor Smith, of Virginia, and Lefebure de Fourcy, whose work is most generally preferred.

¶ The authors recommended are Norton, Gummery; and in the French, Biot, of whom there is a translation in part, known as the Cambridge Astronomy.

** This is one of the portions of mathematical science on which the author proposes to put forth a treatise at no distant day. We have at present on the calculus, Church and Davies, in America; Young, O'Brien, and Walton, in England; Lacroix, Duhamel, and Moigno, who may be mentioned among the numerous writers in France.

†† Courtenay's Boucharlat; in French, Francœur and Poisson.

‡‡ Bache, Brewster, Bartlett, and Biot. This branch may be pursued to some extent immediately after Geometry.

§§ The authors are Lagrange and Laplace, of whose Mecanique Celeste we have the translation and notes of Bowditch, but for readers of the French, the Système du Monde of Pontécouland is to be preferred.

As Greek letters are frequently used in the following treatise, for the convenience of those unaccustomed to a Greek alphabet, one is here inserted. The names of the letters are given in the last column

Α	α	a	Ἄλφα	Alpha
Β	β, ϐ	b	Βῆτα	Beta
Γ	γ	g	Γάμμα	Gamma
Δ	δ	d	Δέλτα	Delta
Ε	ε	e short	Ἒ ψῖλόν	Epsīlon
Ζ	ζ	z	Ζῆτα	Zeta
Η	η	e long	Ἦτα	Eta
Θ	ϑ θ	th	Θῆτα	Theta
Ι	ι	i	Ἰῶτα	Iōta
Κ	κ	k	Κάππα	Kappa
Λ	λ	l	Λάμϐδα	Lambda
Μ	μ	m	Μῦ	Mu
Ν	ν	n	Νῦ	Nu
Ξ	ξ	x	Ξῖ	Xi
Ο	ο	o short	Ὂ μικρόν	Omĭcron
Π	π	p	Πῖ	Pi
Ρ	ρ	r	Ῥῶ	Rho
Σ	σ, ς	s	Σίγμα	Sigma
Τ	τ	t	Ταῦ	Tau
Υ	υ	u	Ὓ ψῖλόν	Upsīlon
Φ	φ	ph	Φῖ	Phi
Χ	χ	ch	Χῖ	Chi
Ψ	ψ	ps	Ψῖ	Psi
Ω	ω	o long	Ὦ μέγα	Omĕga

INTRODUCTION.

In every question of numbers there are certain conditions which the required numbers in connection with the given ones must fulfill, which conditions are indicated by the question itself.

The *solution* has for its object to determine such required quantities as will verify these conditions. It is necessary, therefore, to endeavor first to seize the different relations by which all the quantities, known and unknown, are connected together, and to find afterward, by means of these relations, what operations ought to be performed upon the given quantities to obtain those which are required. Such is the object proposed in that part of mathematics known by the name of Algebra.

To show how the use of letters and signs arises, let the following simple problem be proposed.

To divide 890 *dollars between three persons in such a manner that the second shall have* 115 *more than the first, and the third* 180 *more than the second.*

Now let us see by what deductions the values of the unknown numbers may be derived.

Since the share of the second is 115 more than that of the first, and the share of the third 180 more than that of the second, it will be 180 added to 115, or 295 more than that of the first.

Then the sum of the three parts will be formed of 3 times the first part, increased by 115, and also by 295, or, what is the same thing, of 3 times the first part increased by 410.

But this is equal to the sum to be divided, viz., 890.

Then 3 times the first part, increased by 410, is equal to 890.

Then 3 times the first part is equal to 890 diminished by 410, or 480

Then the first part will equal the third of 480, or 160 dollars.

The first person, therefore, has 160 dollars; the second, who must have 115 more, will have 275; and the third, who was to have 180 more than the second, 455 dollars. These three sums united make 890 dollars, which confirms the correctness of the result.

This example exhibits the kind of reasonings necessary to be employed in the solution of problems in numbers; and it will be perceived that, to express these reasonings, it is necessary to repeat frequently a number of words, designating the quantities, both known and unknown, as the *first part, the number to be divided,* &c., and other words expressing the relations of these, as *increased by, diminished by,* &c.

To obviate the inconvenience of the periphrases, by means of which the quantities which enter into the question are distinguished, it is customary to represent these quantities by letters. Ordinarily, the given quantities are represented by the first letters of the alphabet, $a, b, c \ldots$, and the required or unknown by the last, $x, y, z \ldots$

The relations are expressed by signs. Thus, *increased by* is written $+$; *diminished by* is written $-$; *multiplied by* is written $\times$; or, a multiplied by b, simply thus, ab; a divided by b, thus, $\frac{a}{b}$; a equal to b, thus, $a=b$.

The reasoning of the above example may, with the aid of such abridgments, if x denote the first share, be written briefly thus:

$$\begin{array}{r} x \\ x+115 \\ x+115+180 \\ \hline 3x+410=890 \\ 3x=890-410 \\ 3x=480 \\ x=\frac{480}{3}=160 \end{array}$$

If the numbers had been different in the above problem, the method of proceeding would have been precisely the same.

Thus, if 1250 had been the sum to be divided, 170 the excess of the second part over the first, and 220 the excess of the third over the second, the reasoning would have had the same form, as seen below.

$$\begin{array}{l} x \\ x+170 \\ x+170+220 \\ \hline 3x+560=1250 \\ 3x=1250-560 \\ 3x=690 \\ x=\frac{690}{3}=230 \end{array} \qquad \begin{array}{lr} \text{share of the 1st,} & 230 \\ & 170 \\ \cline{2-2} \text{share of the 2d,} & 400 \\ & 220 \\ \cline{2-2} \text{share of the 3d,} & 620 \end{array}$$

Proof.

$$\begin{array}{r} 230 \\ 400 \\ 620 \\ \hline 1250 \end{array}$$

All these individual cases of the same kind may be generalized, thus: Let a represent the number to be divided; b the excess of the second over the first share; c that of the third over the second. The reasoning will then stand as follows:

$$\begin{array}{l} x \\ x+b \\ x+b+c \\ \hline 3x+2b+c=a \\ 3x=a-2b-c \\ x=\dfrac{a-2b-c}{3} \end{array}$$

The last expression, $x=\dfrac{a-2b-c}{3}$, shows what operations ought to be performed upon the given numbers to produce the required, and may be interpreted into the following rule.

Subtract double the excess of the second share over the first, together with the excess of the third over the second, from the number to be divided, and divide the remainder by 3. The result will be the first share required.

Applying this rule to the first case above, we have

$$\begin{array}{r} 115\times2=230 \quad 890 \\ 180 \quad\quad\;\; \\ \hline 410 \\ 3)\overline{480} \\ 160 \text{ Ans.} \end{array} \qquad \text{and to the 2d,} \quad \begin{array}{r} 170 \\ 2 \\ \hline 340 \\ 220 \quad 1250 \\ 560 \\ 3)\overline{690} \\ 230 \text{ Ans.} \end{array}$$

The expression $x=\dfrac{a-2b-c}{3}$, from which the rule to be applied is derived, is called a general formula, or simply a formula from which, instead of from the rule, the answers in the particular cases may be obtained by substitution ; thus,

in the 1st case,

$$x=\frac{890-230-180}{3}=\frac{480}{3}=160,$$

in the 2d case,

$$x=\frac{1250-2\times170-220}{3}=\frac{690}{3}=230.$$

The nature and utility of algebra being thus briefly indicated, we proceed to give in detail, first, the methods of representing quantities, and all possible relations and combinations of them, and afterward the use of these methods in the solution of questions.

ALGEBRA.

DEFINITIONS AND NOTATION.

1. Algebra is a species of short-hand writing which, by the aid of certain symbols, serves to abridge and generalize propositions relating to numbers.*

A *Proposition* is any thing propounded as true. If it express the properties or relations of quantity, it is a mathematical proposition. If it be self-evident, it is called an *axiom*. If it require demonstration, it is called a *theorem;* and if it propose something to be done, or that some required or unknown quantity be found, it is called a *problem*.

Symbols may be divided into symbols of quantity, and symbols of relation commonly called signs.

2. The principal symbols employed in algebra are the following:

I. The letters of the alphabet, a, b, c, &c., which are employed to denote the numbers which are the object of our reasonings.

When the Roman letters are exhausted, or when a marked distinction is desirable between the different classes of quantities employed, the Greek letters are also used as representatives of quantity. If different quantities of the same general nature are used together, it is a common custom to represent them by the same letter, distinguishing them from one another by accents, or small numbers written below; thus, a, a', a'', a''', a^{iv}, are representatives of different quantities, and are read a, a prime, a second, &c.; and a_1, a_2, a_3, &c., may be read a one subscript, a two subscript, and so on.

A similar effect is produced by using large and small letters; thus, the diameter of a small circle being represented by d, that of a larger may be by D.

It is customary, in some cases, to represent quantities by symbols, which indicate distinctly the nature of the quantities represented. Thus, the six trigonometrical quantities, which are known by the names of sine, tangent, secant, cosine, cotangent, cosecant, are represented by the symbols sin, tan, sec, cos, cot, cosec; and the astronomical quantities, the longitude of the sun, the longitude of the moon, and the longitude of a node, are represented by the symbols ☉, ☽, and ☋.

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* In the operations of Arithmetic, with the exception of those which relate to compound numbers, quantities are considered as composed of units, but the *kind* of unit is not noticed, only the number. In Algebra, neither the kind nor number of units of which a quantity is composed is regarded, and often the quantity is not considered as composed of units at all. The idea of number may, however, always be introduced, and it is best to keep it in mind in the beginning of Algebra. As in Arithmetic the rules of addition, multiplication, proportion, &c., are the same, whatever be the kind of units which the numbers employed represent, so in Algebra these rules are the same, whatever be either the kind or number of units in the quantities employed (upon which the operations are performed). In every part of Algebra, processes analogous to those prescribed by the rules of Arithmetic are in use. Hence, and because of its character of generalization, it was called by Newton General Arithmetic. Algebra, however, presents many relations of quantity of which Arithmetic takes no cognizance.

These are the symbols of quantity.

The following are symbols of relations :

II. The sign $+$, which is named *plus*, and is employed to denote the addition of two or more numbers.

Thus, $12+30$ signifies 12 *plus* 30, or, 12 *augmented by* 30. In like manner, $a+b$ signifies *a plus b*, or, the number designated by *a augmented by* the number designated by *b*.

III. The sign $-$, which is named *minus*, and is employed to denote the subtraction of one number from another.

Thus, $54-23$ signifies 54 *minus* 23, or, 54 *diminished by* 23. In like manner, $a-b$ signifies *a minus b*, or, the number designated by *a diminished by* the number designated by *b*.

The sign $\sim$ is sometimes employed to denote *the difference* of two numbers, when it is not known which is the greater. Thus, $a \sim b$ signifies *the difference of a* and *b*, when it is not known whether the number designated by a be less or greater than the number designated by b.

IV. The sign $\times$, which may be read *into*, is employed to denote the multiplication of two or more numbers.

Thus, 72×26 is read 72 *into* 26, or, 72 *multiplied by* 26. In like manner, $a\times b$ signifies *a into b*, or, *a multiplied by b*; and $a\times b\times c$ signifies the continued product of the numbers designated by a, b, c; and so on for any number of factors.

The process of multiplication is also frequently indicated by placing a point between the successive factors; thus, $a \, . \, b \, . \, c \, . \, d$ signifies the same thing as $a\times b\times c\times d$.

In general, however, when numbers are represented by letters, their multiplication is indicated by writing the letters in succession, without the interposition of any sign. Thus, ab signifies the same thing as $a \, . \, b$, or $a\times b$; and $abcd$ is equivalent to $a \, . \, b \, . \, c \, . \, d$, or $a\times b\times c\times d$.

Factors expressed by letters are called literal factors, and those expressed by numbers numerical factors.

It must be remarked, that the notation $a \, . \, b$, or ab, can be employed only when the numbers are designated by letters; if, for example, we wished to represent the product of the numbers 5 and 6 in this manner, 5 . 6 would be confounded with an integer followed by a decimal fraction, and 56 would signify the number *fifty-six*, according to the common system of notation.

For the sake of brevity, however, the multiplication of numbers is sometimes expressed by placing a point between them in cases where no ambiguity can arise from the use of this symbol. Thus, 1 . 2 . 3 . 4, may represent the continued product of the numbers 1, 2, 3, 4; and $\frac{2}{3} \cdot \frac{7}{9} \cdot \frac{6}{11}$ may represent the product of $\frac{2}{3}$, $\frac{7}{9}$, and $\frac{6}{11}$.

V. The sign $\div$, which is named *by*, and when placed between two numbers is employed to denote that the former is to be divided by the latter.

Thus, $24\div 6$ signifies 24 *by* 6, or, 24 *divided by* 6. In like manner, $a\div b$ signifies *a by b*, or, *a divided by b*.

Two dots without the horizontal line between are also the sign of division. This form of the sign is used in proportions, where either of the two quantities between which it is placed may be regarded as the dividend, and the other the divisor. It is analogous, in this respect, to the sign $\sim$ in subtraction.

In general, however, the division of two numbers is indicated by writing the dividend above the divisor, and drawing a line between them. Thus, $24 \div 6$ and $a \div b$ are usually written $\frac{24}{6}$ and $\frac{a}{b}$.

Every fraction, then, expresses the quotient of its numerator, divided by its denominator. Thus, $\frac{2}{3}$ of a unit may be regarded as composed of two parts: the one, the third of one unit, and the other, the third of another unit; or both together, the third of 2 units, or the quotient of 2 divided by 3. This reasoning may be generalized.

VI. The sign $=$, called the sign of equality, and read *is equal to*, when placed between two numbers denotes that they are equal to each other.

Thus, $56+6=62$ signifies that the sum of 56 and 6 *is equal to* 62. In like manner, $a=b$ signifies that *a is equal to b*, and $a+b=c-d$ signifies that *a plus b is equal to c minus d*, or that the sum of the numbers designated by *a* and *b is equal to* the difference of the numbers designated by *c* and *d*.

VII. The sign $<$, which is read *is unequal to*, and when placed between two numbers denotes that one of them is greater than the other, the opening of the sign being turned toward the greater number.

Thus, $a>b$ signifies that *a is greater than b*, and $a<b$ signifies that *a is less than b*.

VIII. The *coefficient* is a sign which is employed to denote that a number designated by a letter, or some combination of letters, is added to itself a certain number of times.

Thus, instead of writing $a+a+a+a+a$, which represents 5 *a*'s added together, we write $5a$. In like manner, $10ab$ will signify the same thing as $ab+ab+ab+ab+ab+ab+ab+ab+ab+ab$, or *ten times* the product of *a* and *b*.

The numbers 5 and 10 here are coefficients.

The *coefficient*, then, is a number, written to the left of another number represented by one or more letters, and denotes the number of times that the given letter, or combination of letters, is to be repeated.

Or the coefficient is the numerical factor written before one or more literal factors.

When no coefficient is expressed, the coefficient 1 is always understood; thus, $1a$ and a signify the same thing.

In a more enlarged sense, one literal factor may be regarded as the coefficient of another, especially when the former is one of the first, and the latter one of the last letters of the alphabet. Thus, in the expression ax, a may be called the coefficient of x. So, also, in the expression of $abxy$, ab may be regarded as the coefficient of xy.

IX. The *exponent*, or *index*, is a sign which is employed to denote that a number designated by a letter is multiplied by itself a certain number of times.

Thus, instead of writing $a \times a \times a \times a \times a$, or $aaaaa$, which represents five *a*'s multiplied together, we write a^5, where 5 is called the *exponent* or *index* of *a*. Similarly, $b \times b \times b \times b \times b \times b \times b \times b \times b \times b$, or $b \cdot b \cdot b \cdot b \cdot b \cdot b \cdot b \cdot b \cdot b \cdot b$, or $bbbbbbbbbb$; or the continued product of 10 *b*'s is written more briefly b^{10}, where 10 is the *exponent* or *index* of *b*.

The *exponent* or *index* of a number is, therefore, a number written a little above a letter to the right, and denotes the number of times which the number designated by the letter enters as a factor into a product. When no exponent is expressed, the exponent 1 is always understood; thus, a^1 and a signify the same thing.

The products thus formed by the successive multiplication of the same number by itself, are in general called the *powers* of that number. Thus, a is *the first power* of a; $a \times a = aa = a^2$ is *the second power* of a, or *the square* of a; $aaa = a^3$ is *the third power*, or *cube* of a; $aaaaa = a^5$ is *the fifth power* of a, and $aaaa \ldots$ to n factors $= a^n$, is the nth power of a, or the power of a designated by the number n.

X. The *square root* of any expression is that quantity which, when multiplied by itself, will produce the proposed expression, and is generally denoted by the symbol $\sqrt{\ }$, which is called the *radical sign*. Thus, the square root of 9 is $\sqrt{9} = 3$, and $\sqrt{a^2} = a$, is the square root of a^2; for in the former case $3 \times 3 = 9$, and in the latter $a \times a = a^2$.

XI. The *cube root* of any expression is that quantity which, when multiplied twice by itself, will produce the proposed expression. The *fourth*, or *biquadrate root* of any expression is that quantity which, when multiplied three times by itself, produces the given expression; and the nth root of any expression is that quantity which, multiplied $(n-1)$ times by itself, produces the proposed expression. Thus, the *cube root* of 8 is 2; for $2 \times 2 \times 2 = 8$, the *fourth root* of a^4 is a; for $a \cdot a \cdot a \cdot a = a^4$, and the nth root of x^n is x; for $x \times x \times x \ldots$ to n factors $= x \cdot x \cdot x \cdot x \ldots$ to n factors $= x^n$.

The roots of expressions are frequently designated by fractional or decimal exponents, the figure in the numerator of the fractional exponent denoting the power to which the expression is to be raised or involved, and the figure in the denominator denoting the root to be extracted or evolved. Thus, the symbol of operation for the *square root* of a is either $\sqrt{a}$ or $a^{\frac{1}{2}}$; for the *cube root* it is $\sqrt[3]{a}$, or $a^{\frac{1}{3}}$; for the fourth root, $\sqrt[4]{a}$, or $a^{\frac{1}{4}}$; and $\sqrt[n]{a}$, or $a^{\frac{1}{n}}$, denotes the nth root of a. Also, $\sqrt[6]{a^5}$, or $a^{\frac{5}{6}}$, denotes the *sixth root* of the *fifth power* of a; and $a^{\frac{m}{n}}$, or $\sqrt[n]{a^m}$, signifies the nth *root* of the mth *power* of a.*

XII. A *rational quantity* is that which has no radical sign or fractional exponent annexed to it, as $3mn$, or $5x^2y^2$.

XIII. An *irrational quantity* is a root which can not be exactly extracted, and is expressed by means of the radical sign $\sqrt{\ }$, or a fractional exponent, as $\sqrt{2}$ $\sqrt[3]{a^2}$, or $x^{\frac{1}{2}}y^{\frac{3}{2}}$.

XIV. The *reciprocal* of any quantity is unity divided by that quantity; thus, the reciprocals of a^2, x^3, y^5, $z^{\frac{1}{3}}$, are respectively $\frac{1}{a^2}$, $\frac{1}{x^3}$, $\frac{1}{y^3}$, $\frac{1}{z^{\frac{1}{3}}}$; but the following notation is generally used, as being more commodious: thus, the fractions $\frac{1}{a^2}$, $\frac{1}{x^3}$, $\frac{1}{y^5}$, $\frac{1}{z^{\frac{1}{3}}}$, are expressed by a^{-2}, x^{-3}, y^{-5}, $z^{-\frac{1}{3}}$.*

It will follow from the above, and from the rule for division of fractions, that the reciprocal of a fraction is the fraction inverted. Thus, the reciprocal of $\frac{a}{b}$ is $\frac{1}{\frac{a}{b}} = \frac{b}{a}$.

* The subject of fractional and negative exponents will be fully investigated farther in advance.

XV. The following characters are used to connect several quantities together, viz.:

vinculum, or *bar* ———

parentheses ( )

braces, or *brackets* $\{\ \}$ or $[\ ]$

Thus, $\overline{m+n}\,.\,x$, or $(m+n)x$ signifies that the quantity denoted by $m+n$ is to be multiplied by x, and $\left\{\frac{m}{n}+\frac{p}{q}\right\}\cdot\left\{\frac{m}{n}-\frac{p}{q}\right\}$ signifies that $\frac{m}{n}+\frac{p}{q}$ is to be multiplied by $\frac{m}{n}-\frac{p}{q}$. The vinculum or bar is sometimes placed vertically; thus,

$$\left.\begin{array}{l}+a\\+b\\+c\end{array}\right|x$$

signifies that the sum of a, b, and c is multiplied by x.

XVI. The signs, $\therefore$ *therefore* or *consequently*, and $\because$ *because*, are used to avoid the frequent repetition of these words.

XVII. Every number written in algebraic language, that is, by aid of algebraic symbols, is called an *algebraic quantity*, or, an *algebraic expression.*

Thus, $3a$ is the algebraic expression for three times the number a; $5a^2$ is the algebraic expression for five times the square of the number a; $7a^5b^3$ is the algebraic expression for seven times the fifth power of a multiplied by the cube of b.

$3a^2-6b^3c^4$ is the algebraic expression for the difference between three times the square of a and six times the cube of b multiplied by the fourth power of c.

$2a-3b^2c^3+4d^4e^5f^6$ is the algebraic expression for twice a, diminished by three times the square of b multiplied by the cube of c and augmented by four times the fourth power of d multiplied by the product of the fifth power of e and the sixth power of f.

XVIII. An algebraic quantity, which is not combined with any other by the sign of addition or subtraction, is called a *monomial*, or *monome*, or, a *quantity of one term*, or simply, a *term*. Thus, $3a^2$, $4b^2$, $6c$, are *monomials*. The degree of a term is the number of its literal factors, and is found by adding together the exponents of all the letters contained in the term. Thus, $5a^3b^2c$ is of the sixth degree.

An algebraic expression, which is composed of several terms, separated from each other by the signs $+$ or $-$, is called generally a *polynomial*,* or *polynome*. Thus, $3a^2+4b^2-6c+d$ is a polynomial. A polynomial is said to be homogeneous when all its terms are of the same degree.

A polynomial, consisting of two terms only, is usually called a *binomial*; when consisting of three terms, a *trinomial*. Thus, $a+b$, $3b^2c-xz$, are binomials, and $a+b-c$, $3m^2n^5-6p^3r+9d$, are trinomials.

XIX. Of the different terms which compose a polynomial, some are preceded by the sign $+$, others by the sign $-$. The former are called *additive*, or *positive* terms, the latter, *subtractive*, or *negative* terms.

The first term of a polynomial is not, in general, preceded by any sign; in that case the sign $+$ is always understood.

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* A polynomial is also called a compound quantity. Polynomials, to save the trouble of writing them repeatedly, are often represented by a single large letter. Thus, if we have two polynomials, $x^4-4x^3y+4xy^3-y^4$ and $x^3-3xy^2+3x^2y-y^3$, we may represent the first by A and the second by B, and afterward, in referring to them, may call them the polynomials A and B.

Terms composed of the same letters, affected with the same exponents, are called *similar terms*.

Thus, $7ab$ and $3ab$ are similar terms, so are $6a^2c$ and $7a^2c$; also, $10ab^3c^4d$ and $2ab^3c^4d$; for they are composed of the same letters, and these letters in each are affected with the same exponents. On the other hand, $8ab^3c$ and $3a^2b^3c$ are not similar terms, for, although composed of the same letters, these letters are not each affected with the same exponent in each term.

XX. The *numerical value* of an algebraic expression is the number which results from giving particular values to the letters which compose the expression, and performing the arithmetical operations indicated by the algebraic symbols. This numerical value will, of course, depend upon the particular values assigned to the letters. Thus, the numerical value of $2a^3$ is 54 when we make $a=3$, for the cube of 3 is 27, and twice 27 is 54. The numerical value of the same expression will be 250 if we make $a=5$; for the cube of 5 is 125, and twice 125 is 250.

The numerical value of a polynomial undergoes no change, however we may transpose the order of the terms, provided we preserve the proper sign of each. Thus, the polynomials $4a^3-3a^2b+5ac^2$, $4a^3+5ac^2-3a^2b$, $5ac^2-3a^2b+4a^3$, have all the same numerical value. This follows manifestly from the nature of arithmetical addition and subtraction, for it is evident that if the same amounts be added or taken away, it is immaterial in what order.

Examples of the numerical values of algebraic expressions:

Let $a=4$, $b=3$, $c=2$; then will

(1) $a+b-c=4+3-2=7-2=5$

(2) $a^2+ab+b^2=4^2+4\times3+3^2=16+12+9=37$

(3) $ac-ab+bc=4\times2-4\times3+3\times2=8-12+6=2$

(4) $\dfrac{a^2+b^2-c^2}{ab-ac+bc}=\dfrac{4^2+3^2-2^2}{4\times3-4\times2+3\times2}=\dfrac{16+9-4}{12-8+6}=\dfrac{21}{10}$

(5) $\sqrt{(a+b)c}-\sqrt[3]{(a-b)c^3}=\sqrt{(4+3)\times2}-\sqrt[3]{(4-3)\times2^3}=\sqrt{14}-\sqrt[3]{8}$
$= 3{\cdot}7416574 - 2 = 1{\cdot}7416574$

(6) $\dfrac{a+b}{a-c}+\dfrac{a-c}{b+c}-\dfrac{a-b}{a+b}=\dfrac{7}{2}+\dfrac{2}{5}-\dfrac{1}{7}=\dfrac{263}{70}$

XXI. *Entire quantities* are those which are rational and contain no denominator; such are 47, $2a^2b$, $3a^2-bc$.

XXII. An algebraic expression containing a quantity is called a *function* of that quantity. For example, the expression $3x^2-\sqrt{x}$ is a function of x; the expression $a(x+y)+\frac{b}{2c}(x+y)$ is a function of $x+y$. An entire function of a quantity is one in which this quantity does not enter into a denominator.

A rational function is one in which the quantity does not appear under a radical.

To express, in a general way, a function of x, we write $F(x)$. Where many different functions of x are to be represented, we vary the form of this initial: thus, $F(x)$, $f(x)$, $\phi(x)$, $F'(x)$, &c., which denote, in a general way, different algebraic expressions containing x.

To express functions of the same form of different quantities, we use the same initial before these quantities; thus, $F(x)$, $F(y)$.

To express a function like $x^2+2xy+y^2$ of two quantities, we write $F(x, y)$; of three quantities, $F(x, y, z)$, and so on.

What follows to equations may be called the algebraic *calculus*.

REDUCTION OF TERMS.

3. Reduction of similar terms is the collecting of several similar terms into one.

The rule may be divided into two cases:

(1) When the similar quantities have the same signs.

(2) When the similar quantities have different signs.

CASE I.

When the similar quantities have the same signs.

Add the coefficients; affix the letter or letters of the similar terms, and prefix the common sign + or —.*

Thus, $a+2a+3a+4a+5a=(1+2+3+4+5)a=15a$,

$(-a)+(-2a)+(-3a)+(-4a)=-(1+2+3+4)a=-10a$.

It is convenient to write the similar terms to be reduced under, instead of after one another, they being read in the same order in either way.

EXAMPLES.

(1)	(2)	(3)	(4)	(5)
$3a$	abc	$9axy$	$-5bx$	$\sqrt{a+x}$
$7a$	$2abc$	$3axy$	$-2bx$	$2\sqrt{a+x}$
$2a$	$7abc$	$7axy$	$-bx$	$5\sqrt{a+x}$
a	$3abc$	$5axy$	$-bx$	$\sqrt{a+x}$
$6a$	abc	axy	$-4bx$	$7\sqrt{a+x}$
$8a$	$5abc$	$5axy$	$-10bx$	$4\sqrt{a+x}$
$27a$	$19abc$			

CASE II.

When the similar quantities have different signs.

Collect into one sum the coefficients affected with the sign +, and also those affected with the sign —; to the difference of these sums affix the common literal quantity, and prefix the sign + or —, according as the sum of the + or — coefficients is the greater.†

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* The truth of this rule is evident; for suppose the two terms $3a$ and $5a$ are to be reduced to one, then by the definition of a coefficient we have

$$5a=a+a+a+a+a$$
$$3a=a+a+a$$

Hence $5a+3a=a+a+a+a+a+a+a+a=8a$.

Similarly, $-5a=(-a)+(-a)+(-a)+(-a)+(-a)$

$-3a=(-a)+(-a)+(-a)$.

Hence $-5a+(-3a)=(-a)+(-a)+(-a)+(-a)+(-a)+(-a)+(-a)+(-a)$

$=8(-a)=-8a$.

† The truth of this will be obvious; for to reduce $5a$ and $-3a$, we have

$$5a=a+a+a+a+a$$
$$-3a=(-a)+(-a)+(-a).$$

Thus, $a-2a+3a-4a+5a=(1+3+5)a-(2+4)a=9a-6a=3a.$
And, $3x+4y-2x+3y=(3-2)x+(4+3)y=x+7y.$
Reduce the terms of the polynomials,

(6) $c+2d-2c-3d+3c+4d-4c-5d+c+d$

(7) $3a-2b+5a-6c+3b-9c+a-b+121c$

(8) $k-\frac{a}{4}-\frac{m}{5}-\frac{k}{20}-\frac{m}{7}-\frac{a}{13}-\frac{k}{8}-\frac{m}{9}$

(9) $3a-\frac{1}{4}b+6a-3\frac{2}{5}b+10\frac{1}{2}a-22\frac{5}{9}b-\frac{3}{5}a$

(10) $5xy-4\sqrt[3]{pqr}+4xy-10a^5b^3+7\sqrt[3]{pqr}-9xy+3a^5b^3.$

ADDITION.

Addition is the collecting of several polynomials into one.

RULE.

Write the polynomials one after another, and reduce similar terms.*

EXAMPLES.

(1)
$$\begin{array}{r} 3a^2+\ b^2 \\ 2a^2+3b^2 \\ 6a^2+5b^2 \\ a^2+7b^2 \\ a^2+6b^2 \\ \hline 13a^2+22b^2 \end{array}$$

(2)
$$\begin{array}{r} 2x^2-\ xy \\ 4x^2-7xy \\ 3x^2-4xy \\ x^2-\ xy \\ 8x^2-7xy \\ \hline \end{array}$$

(3)
$$\begin{array}{r} 20\ (a^2-b^2)^{\frac{1}{2}}-15\sqrt{x^2-y^2} \\ \sqrt{a^2-b^2}\ -\ 7\sqrt{x^2-y^2} \\ 12\sqrt{a^2-b^2}\ -\ \sqrt{x^2-y^2} \\ 4\ (a^2-b^2)^{\frac{1}{2}}-\ 3\ (x^2-y^2)^{\frac{1}{2}} \\ 2\ (a^2-b^2)^{\frac{1}{2}}-\ 5\ (x^2-y^2)^{\frac{1}{2}} \\ \hline \end{array}$$

(4)
$$\begin{array}{r} a+\ b \\ -2a+3b \\ 3a-4b \\ -5a+6b \\ 7a-\ b \\ \hline 4a+5b \end{array}$$

(5)
$$\begin{array}{r} xy-\ ab \\ 2xy+3ab \\ -5xy+7ab \\ -\ xy-3ab \\ 8xy-9ab \\ \hline \end{array}$$

(6)
$$\begin{array}{r} \sqrt{x^2+y^2}\ -\ m^2+\ n^2-2mn \\ -2\sqrt{x^2+y^2}\ +\ 3m^2-3\ n^2+5mn \\ -5\sqrt{x^2+y^2}\ -\ 4m^2+5\ n^2-7mn \\ 2\ (x^2+y^2)^{\frac{1}{2}}+12m^2-2\frac{1}{2}n^2+\ mn \\ 8\ (x^2+y^2)^{\frac{1}{2}}-\ 8m^2-\ \frac{1}{2}n^2-6mn \\ \hline \end{array}$$

In example (4), let $a=5$ and $b=3$, then
$$\begin{array}{rl} a+\ b= & 8 \\ -2a+3b= & -1 \\ 3a-4b= & 3 \\ -5a+6b= & -7 \\ 7a-\ b= & 32 \\ \hline 4a+5b= & 35\dagger \end{array}$$

Hence $5a+(-3a)=a+a+a+a+a+(-a)+(-a)+(-a).$
$=a+a=2a.$
Similarly, $2a+(-5a)=a+a+(-a)+(-a)+(-a)+(-a)+(-a)$
$=+(-a)+(-a)+(-a)$
$=3(-a)=-3a.$

* For if certain quantities are to be added and subtracted, it is immaterial in what portions, or what order.

† Similar substitutions may be tried in some of the following examples. Let the learner substitute any other numbers for a and b, and he will find that the sum of the polynomials will be truly expressed by the result $4a+5b$, the correctness of which does not depend on the values of a and b. This illustrates the general principle stated in the note of Art. I.

(7)

$$\begin{array}{l}
5ax^{\frac{1}{2}} - \sqrt[3]{x+y} + (a-b) \\
-7a\sqrt{x}+2(x+y)^{\frac{1}{3}} - 3(a-b) \\
12a\sqrt{x}-3\sqrt[3]{x+y} + 12(a-b) \\
-3a\sqrt{x}-4\sqrt[3]{x+y} - (a-b) \\
\underline{-ax^{\frac{1}{2}} + (x+y)^{\frac{1}{3}} - 3(a-b)}
\end{array}$$

(8)

$$\begin{array}{l}
2\sqrt{xy+xz+yz} + \sqrt[4]{ax+by} \\
-5\sqrt{xy+xz+yz} - 3(ax+by)^{\frac{1}{4}} \\
12(xy+xz+yz)^{\frac{1}{2}} + 5(ax+by)^{\frac{1}{4}} \\
-3\sqrt{xy+xz+yz} - 2\sqrt[4]{ax+by} \\
\underline{(xy+xz+yz)^{\frac{1}{2}} + (ax+by)^{\frac{1}{4}}}
\end{array}$$

(9)

$$\begin{array}{l}
a+b+c+d+e-f \\
a+b+c+d-e+f \\
a+b+c-d+e+f \\
a+b-c+d+e+f \\
a-b+c+d+e+f \\
\underline{-a+b+c+d+e+f}
\end{array}$$

(10)

$$\begin{array}{l}
4(a+b)\sqrt{x^2-y^2} - 2(a-b)\sqrt{x^2+y^2} \\
-3(a+b)\sqrt{x^2-y^2} + (a-b)\sqrt{x^2+y^2} \\
-(a+b)(x^2-y^2)^{\frac{1}{2}} + 3(a-b)(x^2+y^2)^{\frac{1}{2}} \\
6(a+b)(x^2-y^2)^{\frac{1}{2}} - (a-b)(x^2+y^2)^{\frac{1}{2}} \\
10(a+b)\sqrt{x^2-y^2} - 5(a-b)(x^2+y^2)^{\frac{1}{2}} \\
\underline{-2(a+b)(x^2-y^2)^{\frac{1}{2}} + 4(a-b)\sqrt{x^2+y^2}}
\end{array}$$

4. Dissimilar quantities can only be collected by writing them in succession, and prefixing to each its respective sign. Thus, $9xy$, $-5cd$, and $3ab$ are dissimilar quantities, and their sum is $9xy+3ab-5cd$. In like manner, $2ab$, $3ab^2$, $4ab^3$ are dissimilar quantities, and their sum is $2ab+3ab^2+4ab^3$; which, however, admits of another form of expression, as will be explained in the rule of Division. When several polynomials, containing both similar and dissimilar quantities, are to be collected into one polynomial, the process of addition will be much facilitated by writing all the similar terms under each other in vertical columns.

This, however, is not absolutely necessary. The similar terms may be collected together as they stand.

EXAMPLES.

(1) Add together $ax+2by+cz$; $\sqrt{x}+\sqrt{y}+\sqrt{z}$; $3y^{\frac{1}{2}}-2x^{\frac{1}{2}}+3z^{\frac{1}{2}}$; $4cz-3ax-2by$; $2ax-4\sqrt{y}-2z^{\frac{1}{2}}$.

$$\begin{array}{l}
ax+2by+cz+\sqrt{x}+\sqrt{y}+\sqrt{z} \\
-3ax-2by+4cz-2x^{\frac{1}{2}}+3y^{\frac{1}{2}}+3z^{\frac{1}{2}} \\
\underline{2ax \qquad\qquad -4\sqrt{y}-2z^{\frac{1}{2}}} \\
\qquad 5cz-\sqrt{x}+2\sqrt{z} = \text{sum required.}
\end{array}$$

(2) Add together,

$4a^2b+3c^3d-9m^2n$; $4m^2n+ab^2+5c^3d+7a^2b$; $6m^2n-5c^3d+4mn^2-8ab^2$; $7mn^2+6c^3d-5m^2n-6a^2b$; $7c^3d-10ab^2-8m^2n-10d^4$; and $12a^2b-6ab^2+2c^3d+mn$.

Arranging the similar terms in vertical columns, we have

$$\begin{array}{rrrrrrr}
4a^2b+ & 3c^3d- & 9m^2n & & & & \\
7a^2b+ & 5c^3d+ & 4m^2n+ & ab^2 & & & \\
 & -5c^3d+ & 6m^2n- & 8ab^2+ & 4mn^2 & & \\
-6a^2b+ & 6c^3d- & 5m^2n & & +7mn^2 & & \\
 & +7c^3d- & 8m^2n- & 10ab^2 & & -10d^4 & \\
12a^2b+ & 2c^3d & & -6ab^2 & & & +mn \\
\hline
17a^2b+ & 18c^3d- & 12m^2n- & 23ab^2+ & 11mn^2- & 10d^4+ & mn = \text{sum.}
\end{array}$$

(3) Add $11bc+4ad-8ac+5cd$; $8ac+7bc-2ad+4mn$; $2cd-3ab+5ac+an$; and $9an-2bc-2ad+5cd$ together.

(4) Add together, without arranging the similar terms in vertical columns,

$$\begin{array}{l} 2ab^2+3ac^2-\ 8cx^2+\ 9b^2x-\ 8hy^2-10ky \\ 5a^3\ -4ab^2-\ 7bx^2-\ \ b^2x-\ 4ky^2-15hy \\ 5ky-\ hy^2+11x\ +14b^3\ -22ac^2-10x^2 \\ 19ac^2-8b^2x+\ 9x^2\ +\ 6hy+\ 2ky^2+\ 2ab^2 \\ \hline 5a^3\ -8cx^2-\ \ x^2\ +11x\ -\ 9hy^2+14b^3-2ky^2-5ky-9hy-7bx^2. \\ \hline \end{array}$$

(5) Add together $a^3-b^3+3a^2b-5ab^2$; $3a^3-4a^2b+3b^3-3ab^2$; $a^3+b^3+3a^2b$; $2a^3-4b^3-5ab^2$; $6a^2b+10ab^2$; and $-6a^3-7a^2b+4ab^2+2b^3$.

(6) Add $\sqrt{x^2+y^2}-\sqrt{x^2-y^2}-5xy$; $-3(x^2-y^2)^{\frac{1}{2}}+8xy-2(x^2+y^2)^{\frac{1}{2}}$, $2\sqrt{x^2+y^2}-3xy-5\sqrt{x^2-y^2}$; $7xy+10\sqrt{x^2-y^2}-12\sqrt{x^2+y^2}$; and $xy+\sqrt{x^2-y^2}+\sqrt{x^2+y^2}$ together.

(7) Add $\frac{4a}{y}-\frac{3m^3}{c}+\frac{6\sqrt[3]{p}}{z}-\frac{3(q+r)}{s}$ and $\frac{9a}{y}+\frac{8m^3}{c}-\frac{12\sqrt[3]{p}}{z}+\frac{4(q+r)}{s}$ together.

(8) Add together $4A-6\frac{A}{a}+7\frac{B}{C}$ and $7\frac{A}{a}-2A+3\frac{B}{C}$.

(9) Add together $3\cos a-4\sin b+6\tan c$, $2\cos a+2\sin b+7\tan c$, and $\cos a+3\sin b-2\tan c$.

(10) Add together 3.29 ⊙ −2.45 ☽ +1.84 ☋, 4.56 ⊙ +0.59 ☽ +6.41 ☋, and 2.22 ⊙ +3.11 ☽ −4.21 ☋.

ANSWERS.

(3) $16bc+5ac+12cd+4mn-3ab+10an$.

(5) $a^3+a^2b+ab^2+b^3$.

(6) $2\sqrt{x^2-y^2}-10\sqrt{x^2+y^2}+8xy$.

(7) $\frac{13a}{y}+\frac{5m^3}{c}-\frac{6\sqrt[3]{p}}{z}+\frac{(q+r)}{s}$.

(8) $2A+\frac{A}{a}+10\frac{B}{C}$.

(9) $6\cos a+\sin b+11\tan c$.

(10) 10.07 ⊙ +1.25 ☽ +4.04 ☋.

5. When the coefficients are *literal* instead of *numerical*, that is, denoted by letters instead of numbers, their sum may be found by the rules for the addition of similar and dissimilar terms; and the sum thus found being enclosed in a parenthesis, and prefixed to the common literal quantity, will express the sum required.

EXAMPLES.

(1)

$$\begin{array}{l} ax+by+cz \\ bx+cy+az \\ cx+ay+bz \\ \hline \left.\begin{array}{l}(a+b+c)x \\ +(b+c+a)y \\ +(c+a+b)z\end{array}\right\} = \text{sum.} \\ \hline \end{array}$$

(2)

$$\begin{array}{l} 3ax+\ (a+b)\ (x+y)+2mnz^2 \\ -ax+2(a+b)\ (x+y)-5mnz^2 \\ 4mnz^2+5(a+b)\ (x+y)+10ax \\ 2pqz^2+\ (p+q)\ (x+y)+2px \\ \hline \left.\begin{array}{l}(12a+2p)x+\{8(a+b)+p+q\}(x+y) \\ +(mn+2pq)z^2\end{array}\right\} = \text{sum.} \\ \hline \end{array}$$

(3)	(4)
$(a-b)\sqrt{x}+(m-n)\sqrt{y}+\sqrt{2}$	$(m+n)y^2-(a-b)x^2+axy$
$(a+c)x^{\frac{1}{2}}-(m-n)y^{\frac{1}{2}}+2\sqrt{2}$	$(n-p)y^2-(2a+b)x^2-bxy$
$(b-c)\sqrt{x}+3(m-n)\sqrt{y}-3\sqrt{2}$	$(p-2n)y^2-(c-3a)x^2+cxy$
$(c-a)\sqrt{x}-5(m-n)\sqrt{y}-6\sqrt{2}$	$(q-m)y^2-(c+2d)x^2-dxy$

(5) Add ax^2+by+c to dx^2+hy+k.

(6) Add together x^2+xy+y^2; $ax^2-axy+ay^2$; and $-by^2+bxy+bx^2$.

(7) Add $\frac{1}{2}(x+y)$ and $\frac{1}{2}(x-y)$. Also, $\frac{x^2+xy+y^2}{2}$ and $\frac{x^2-xy+y^2}{2}$.

(8) What is the sum of $(a+b)x+(c-d)y-x\sqrt{2}$; $(a-b)x+(3c+2d)y+5x\sqrt{2}$; $2bx+3dy-2x\sqrt{2}$; and $-3bx-dy-4x\sqrt{2}$?

(9) Add $ax+by+cz$; $a'x-b'y+c'z$; and $a''x+b''y-c''z$.

(10) Add together $ax+by+cz$; $a_1x+b_1y-c_1z$; and $a_2x-b_2y+c_2z$.

ANSWERS.

(3) $(a+c)\sqrt{x}-2(m-n)\sqrt{y}-6\sqrt{2}$.

(4) $qy^2-(2c+2d)x^2+(a-b+c-d)xy$.

(5) $(a+d)x^2+(b+h)y+c+k$.

(6) $(1+a+b)x^2+(1-a+b)xy+(1+a-b)y^2$.

(7) First part, x. Second part, x^2+y^2.

(8) $(2a-b)x+(4c+3d)y-2x\sqrt{2}$.

(9) $(a+a'+a'')x+(b-b'+b'')y+(c+c'-c'')z$.

(10) $\left.\begin{array}{r}a\\+a_1\\+a_2\end{array}\right|x+\left.\begin{array}{r}b\\+b_1\\-b_2\end{array}\right|y+\left.\begin{array}{r}c\\-c_1\\+c_2\end{array}\right|z$

SUBTRACTION.

RULE.

6. PLACE the quantity to be subtracted under that from which it is to be taken; change the signs of all the terms in the lower line from + to —, and from — to +, or else conceive them to be changed, and then proceed as directed in Addition.*

* The sign —, prefixed to a monomial, serves to intimate that this monomial ought to enter subtractively into any combination of which it forms a part. If, for example, it be required to add the subtractive quantity $(-d)$ to c, the sum $c+(-d)$ is $c-d$.

If the difference between two quantities, as m and s, be required, m and s being both additive, the expression of the difference is $m-s$. If the difference be required between m, an additive, and $(-s)$, a subtractive quantity, let the difference $=d$; that is, let

$$m-(-s)=d.$$

Adding $(-s)$ to both these equals, there results

$$m-(-s)+(-s)=d+(-s).$$

But $\quad m-(-s)+(-s)=m$, and $d+(-s)=d-s$.

Therefore, $\quad m=d-s$.

Now $\quad m-(-s)=d$, and $m=d-s$.

Hence $m-(-s)$ is greater than m by the additive quantity s, or is equal to $m+s$

The above is the demonstration for isolated terms.

For polynomials we have the following:

It is evident, that if all the terms of the quantity to be subtracted are affected with the

EXAMPLES.

(1)

$$\begin{array}{ll} \text{From} & 4a+3b-2c+8d \\ \text{Take} & a+2b+c+5d \\ \text{Rem.} & 3a+b-3c+3d \end{array}$$

(2)

$$\begin{array}{ll} \text{From} & 12xy+3y^2-17x^2+3\sqrt{2} \\ \text{Take} & -5xy+7y^2-19x^2+2\sqrt{2} \\ \text{Rem.} & 17xy-4y^2+2x^2+\sqrt{2} \end{array}$$

(3)

$$\begin{array}{l} 32a+3b \\ 5a+17b \end{array}$$

(4)

$$\begin{array}{l} 28ax^3-16a^2x^2+25a^3x-13a^4 \\ 18ax^3+20a^2x^2-24a^3x-7a^4 \end{array}$$

(5)

$$\begin{array}{l} 2(a+b)+3(a-x) \\ (a+b)-3(a-x) \end{array}$$

(6)

$$\begin{array}{l} 6aby-3yx+4zx \\ -2aby+6zx+2yx \end{array}$$

(7)

$$\begin{array}{l} \sqrt{x^2-y^2}+4(x+y)-3\sqrt{a+x} \\ 3(x+y)-2(x^2-y^2)^{\frac{1}{2}}+3(a+x)^{\frac{1}{2}} \end{array}$$

(8)

$$\begin{array}{l} x^2+2xy+y^2 \\ x^2-2xy+y^2 \end{array}$$

(9)

$$\begin{array}{l} x^2-2xy+y^2+(x^2-y^2)+2(xy-y^2) \\ x^2+2xy-y^2+(x^2+y^2)-2(xy-y^2) \end{array}$$

(10)

$$\begin{array}{l} 2a^2+ax+x^2-12a^2x+20ax^2-4x^3+6a^2x^2-10ax^3 \\ a^2-3ax+2x^2-16a^2x+12ax^2-12ax^3-4x^3+2a^2x^2 \end{array}$$

(11)

$$\begin{array}{l} 4y^2-4yx+x^2-2a(x+y)+6\sqrt{a^2-x^2}-8\sqrt[3]{b^2-y^2} \\ 4x^2-4xy+y^2-4a(x+y)-10\sqrt[3]{b^2-y^2}+4\sqrt{a^2-x^2} \end{array}$$

7. In order to *indicate* the subtraction of a polynomial, without actually performing the operation, we have simply to inclose the polynomial to be subtracted within *brackets* or *parentheses*, and prefix the sign —. Thus, $2a^3$

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sign +, we must take away, in succession, all the parts or terms of the quantity to be subtracted; and this is indicated by affecting all its terms with the sign —. But if some of the terms of the subtrahend are affected with the sign —, as, for instance, if $c-d$ is to be subtracted from $a+b$; then, if c be subtracted, we shall have subtracted too much by d; hence the remainder $a+b-c$ is too small by d; and therefore, to make up the defect, the quantity d must be added, which gives $a+b-c+d$; by inspecting which we perceive that the signs of the subtrahend have been changed.

This reasoning may be generalized by supposing c to represent the sum of the additive terms, and d to represent the sum of the subtractive terms of the lower line, or quantity to be subtracted.

Another mode of proving the rule for the signs in subtraction is the following:

By subtraction we solve the problem, "Given one of two quantities, and their algebraical sum, to find the other."

Let A be any algebraical quantity, simple or compound, from which it is proposed to subtract another simple or compound quantity, B. The quantity A may be conceived to be the algebraical sum of B, and some other quantity which it is proposed to discover. Call it x. As A was obtained by annexing to x the polynomial expressed by B, with its proper signs, the effect of this process will be destroyed by annexing to A the polynomial represented by B, *with its signs changed.*

$-3a^2b+4ab^2-(a^3+b^3+ab^2)$ signifies that the quantity $a^3+b^3+ab^2$ is to be subtracted from $2a^3-3a^2b+4ab^2$. When the operation is actually performed, we have by the rule,

$$2a^3-3a^2b+4ab^2-(a^3+b^3+ab^2)=2a^3-3a^2b+4ab^2-a^3-b^3-ab^2$$
$$=a^3-3a^2b+3ab^2-b^3.$$

When, therefore, brackets are removed which have the sign — before them, the signs of all the terms within the brackets must be changed.

8. According to this principle, we may make polynomials undergo several transformations, which are of great utility in various algebraic calculations. Thus,

$$a^3-3a^2b+3ab^2-b^3=a^3-(3a^2b-3ab^2+b^3)$$
$$=a^3-b^3-(3a^2b-3ab^2)$$
$$=a^3+3ab^2-(3a^2b+b^3)$$
$$=-(-a^3+3a^2b-3ab^2+b^3)$$

And $$x^2-2xy+y^2=x^2-(2xy-y^2)=y^2-(2xy-x^2).$$

EXAMPLES OF QUANTITIES WITH LITERAL COEFFICIENTS.

(1)

From	$ax^2+byx+cy^2$
Take	$dx^2-hxy+ky^2$
Rem.	$(a-d)x^2+(b+h)xy+(c-k)y^2$.

(2)

From	$(a+b)\sqrt{x^2+y^2}+(a+c)(a+x)^3$
Take	$(a-b)\sqrt{x^2+y^2}+\quad c\,(a+x)^3$
Rem.	$2b\sqrt{x^2+y^2}+a(a+x)^3$.

(3) From $m^2n^2x^2-2mnpqx+p^2q^2$ take $p^2q^2x^2-2pqmnx+m^2n^2$.

(4) From $a(x+y)-bxy+c(x-y)$ take $4(x+y)+(a+b)xy-7(x-y)$.

(5) From $(a+b)(x+y)-(c-d)(x-y)+h^2$ take $(a-b)(x+y)+(c+d)(x-y)+k^2$.

(6) From $(2a-5b)\sqrt{x+y}+(a-b)xy-cz^2$ take $3bxy-(5+c)z^2-(3a-b)(x+y)^{\frac{1}{2}}$.

(7) From $2x-y+(y-2x)-(x-2y)$ take $y-2x-(2y-x)+(x+2y)$

(8) To what is $a+b+c-(a-b)-(b-c)-(-b)$ equal?

(9) From Ax^3+Bx^2+Cx+D take $A_1x^3+B_1x^2+C_1x+D_1$.

ANSWERS.

(3) $(m^2n^2-p^2q^2)x^2+p^2q^2-m^2n^2$, or $(m^2n^2-p^2q^2)x^2-(m^2n^2-p^2q^2)$, or $(m^2n^2-p^2q^2)(x^2-1)$.

(4) $(a-4)(x+y)-(a+2b)xy+(c+7)(x-y)$.

(5) $2b(x+y)-2c(x-y)+h^2-k^2$.

(6) $(5a-6b)\sqrt{x+y}+(a-4b)xy+5z^2$.

(7) $y+x$.

(8) $2b+2c$.

(9) $(A-A_1)x^3+(B-B_1)x^2+(C-C_1)x+D-D_1$.

MULTIPLICATION.

9. Multiplication is usually divided into three cases:

(1) When both multiplicand and multiplier are simple quantities.

(2) When the multiplicand is a compound, and the multiplier a simple quantity.

(3) When both multiplicand and multiplier are compound quantities.

CASE I.

10. *When both multiplicand and multiplier are simple quantities, or monomials.* To the product of the coefficients affix that of the letters.*

Thus, to multiply $5x$ by $4y$, we have

$$5\times4=20;\quad x\times y=xy;$$
$$\therefore 5x\times4y=20\times xy=20xy=\text{ product.}$$

11. *Powers* of the same quantity are multiplied by simply adding their indices; for since, by the definition of a power,

$$a^5=aaaaa;\quad a^7=aaaaaaa,$$
$$\therefore a^5\times a^7=aaaaa\times aaaaaaa=aaaaaaaaaaaa=a^{12}=a^{5+7}.$$

Also, $a^m=aaa\ldots.$ to m factors; $a^n=aaa\ldots.$ to n factors;

$$\therefore a^m\times a^n=aaa\ldots.\text{ to } m \text{ factors }\times aaa\ldots.\text{ to } n \text{ factors};$$
$$=aaaaaa\ldots..\text{ to } (m+n) \text{ factors};$$
$$=a^{m+n}.$$

It is proved, in the same manner, that $a^m\times a^n\times a^h\times a^k=a^{m+n+h+k}$.

---

* I. The rule is derived in the following manner: We begin by assuming that when several letters are written one after another without any sign, their continued multiplication is understood, and that the operation proceeds from left to right. Then $abcd$ will signify a multiplied by b, that product by c, and that again by d. We shall now prove that in whatever order these letters or simple factors are arranged, their continued product will always be the same;† and, moreover, that they may be grouped into partial products at pleasure, provided all the letters be employed each time. Thus the above product may be written $badc$ (the multiplication here, as before, going on by each factor successively from left to right), and the result will be the same as before; or it may be written $a\times b\times cd$, understanding the products separated by the sign $\times$ as being previously formed and then multiplied together.

The demonstration depends upon three propositions, which we shall first establish:

(1) . . . $a\times b=b\times a$

$$a\left\{\overbrace{\begin{array}{cccccc}1&1&1&1&1&1\\1&1&1&1&1&1\\1&1&1&1&1&1\\1&1&1&1&1&1\\1&1&1&1&1&1\end{array}}^{b}\right.$$

For in the adjoining table of units let b denote the number of units in each horizontal row, and a the number of rows, then b multiplied by a, or repeated a times, will give the number of units in the table. But a, which is the number of horizontal rows, is also the number of units in each column; and b is the number of columns; then a multiplied by b, or repeated b times, will produce the number of units in the table again; whence b multiplied by a is equal to a multiplied by b.

$$c\left\{\overbrace{\begin{array}{ccccc}a&a&a&a&a\\a&a&a&a&a\\a&a&a&a&a\\a&a&a&a&a\end{array}}^{b}\right.$$

In a similar manner, from the adjoining table, it may be proved that

$$a\,.\,b\,.\,c=a\,.\,c\,.\,b \qquad (2)$$

Also that $a\,.\,b\,.\,c=a\,.\,(bc)$ (3)

II. By (1) $abcd=bacd=$ by (2) $bcad=$ by (2) $bcda$. Thus, we perceive that the factor a has been made to occupy successively every place from the first to the last. The same might now be done with the factor b, and so with all the others. Therefore a product is the same, whatever be the order of its factors.

III. Again. Take $a\times b\times c\times d\times e$. It may be written by (3) $a\times bc\times d\times e$ or by (3) $a\times bcd\times e$, or, instead, by (3) $ab\times cd\times e$. From which it appears that the factors of a product may be grouped into partial products at pleasure, and then afterward multiplied together or conversely.

IV. Let us now suppose that the product $3a^3b^2$ is to be multiplied by the product $5a^2b^4$. Instead of multiplying by the whole product $5a^2b^4$, multiply by its factors separately, and we have $5a^2b^4 3a^3b^2$. Since the order may be changed at pleasure, bring the numerical factors together, and the different powers of the same letters; thus, $5\times3a^2a^3b^4b^2$. Grouping the different powers of the same letters into partial products, as well as the numerical factors, the result is $15a^5b^7$, which has evidently been obtained by multiplying the coefficients and adding the exponents of like letters.

† Such a relation as that of a product to its factors is called a *symmetrical relation.*

RULE OF SIGNS IN MULTIPLICATION.

The product of quantities with like signs is affected with the sign $+$; the product of quantities with unlike signs is affected with the sign $-$;

or

$+$ multiplied by $+$ and $-$ multiplied by $-$ give $+$;
$+$ multiplied by $-$ and $-$ multiplied by $+$ give $-$;

or

like signs produce $+$ and unlike signs $-$.

The continued product of an even number of negative factors is positive; of an uneven number, negative.*

EXAMPLES.

(1) $4a^2b^2cd \times 3abc^2d^2 = 12a^3b^3c^3d^3$.
(2) $12\sqrt{ay} \times 4bx = 48bx\sqrt{ay}$.
(3) $5\frac{1}{2}x^2y^3z^4 \times 6xy^4z^3 = 33x^3y^7z^7$.
(4) $13a^2b^3x^3y \times -5abxy^3 = -65a^3b^4x^4y^4$.
(5) $-5x^my^n \times -4x^ny^m = +20x^{m+n}y^{m+n}$.
(6) $-20a^pb^q \times 5a^mb^nc^r = -100a^{m+p}b^{n+q}c^r$.

CASE II.

12. *When the multiplicand is a compound, and the multiplier a simple quantity.*

Multiply each term of the multiplicand by the multiplier, beginning at the left hand; and these partial products, being connected by their respective signs, will give the complete product.†

EXAMPLES.

(1)

Multiply a^2+ab+b^2
By $4a$
Product, $4a^3+4a^2b+4ab^2$.

(2)

Multiply $a^2-2ab+b^2$
By $3xy$
Product, $3a^2xy-6abxy+3b^2xy$.

(3) Multiply $5mn+3m^2-2n^2$ by $12abn$.
(4) Multiply $3ax-5by+7xy$ by $-7abxy$.
(5) Multiply $-15a^2b+3ab^2-12b^3$ by $-5ab$.
(6) Multiply ax^3-bx^2+cx-d by $-x^5$.
(7) Multiply $\sqrt{a+b}+\sqrt{x^2-y^2}-3xy$ by $-2\sqrt{x}$.
(8) Multiply $a^mx^n+b^my^n-c^ny^m-d^nx^m$ by x^my^n.

* Let m, m' be two monomial quantities whose product is required. If m, m' are both additive quantities, the product mm' is an additive quantity. This is the case of arithmetic. If the multiplicand m is an additive quantity, and the multiplier m' a subtractive quantity, the expression $m\times(-m')$ indicates that the multiplicand m is to be subtracted as many times as there are units in m', or that m' repetitions of the quantity m are to be subtracted, which is expressed by $-mm'$.

If m is subtractive and m' additive, $-m$ taken once is $-m$; taken twice is $-2m$; taken m' times is $-m'm$.

If m and m' are both subtractive, the quantity $-m$ is to be subtracted m' times. Now $-m$ subtracted once is $+m$, twice is $+2m$; and m' times is $+m'm$.

† 1st. Suppose the signs to be all plus. The whole multiplicand being to be taken as many times as is denoted by the multiplier, each of its parts or terms must be taken so many times. 2d. For the case where some of the signs are negative, see the demonstration in the next note.

CASE III.

13. *When both multiplicand and multiplier are compound quantities.*

Multiply each term of the multiplicand, in succession, by each term of the multiplier, and the sum of these partial products will give the complete product.*

EXAMPLES.

(1)

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \quad +ab+b^2 \\ \hline a^2+2ab+b^2 \\ \hline \end{array}$$

(2)

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline a^2-b^2 \\ \hline \end{array}$$

(3)†

$$\begin{array}{l} a-b \\ a-b \\ \hline a^2-ab \\ \quad -ab+b^2 \\ \hline a^2-2ab+b^2 \\ \hline \end{array}$$

(4)

$$\begin{array}{l} ab+cd \\ ab-cd \\ \hline a^2b^2+abcd \\ \quad -abcd-c^2d^2 \\ \hline a^2b^2-c^2d^2 \\ \hline \end{array}$$

(5)

$$\begin{array}{l} a^2+2ab+b^2 \\ a^2-b^2 \\ \hline a^4+2a^3b+a^2b^2 \\ \quad -a^2b^2-2ab^3-b^4 \\ \hline a^4+2a^3b-2ab^3-b^4 \\ \hline \end{array}$$

(6) Multiply $4a^3-5a^2b-8ab^2+2b^3$ by $2a^2-3ab-4b^2$.

$$\begin{array}{l} 4a^3-5a^2b-8ab^2+2b^3 \\ 2a^2-3ab-4b^2 \\ \hline 8a^5-10a^4b-16a^3b^2+4a^2b^3 \\ \quad -12a^4b+15a^3b^2+24a^2b^3-6ab^4 \\ \quad\quad -16a^3b^2+20a^2b^3+32ab^4-8b^5 \\ \hline 8a^5-22a^4b-17a^3b^2+48a^2b^3+26ab^4-8b^5 = \text{product} \\ \hline \end{array}$$

---

* 1st. Suppose all the terms of the multiplier to be affected with the sign +. The multiplicand, being to be taken as many times additively as is denoted by the multiplier, must be taken as many times as is denoted by each term of the multiplier separately, and the separate results added together. 2d. When there are both additive and subtractive terms in the multiplier and multiplicand. The rule for the signs may be thus demonstrated. Let $a-b$ be multiplied by $c-d$. First multiplying a by c, the product is ac; but b should have been subtracted from a before the multiplication; b units have, therefore, been taken c times in the a, which ought not to have been so taken; hence b, taken c times, must be subtracted, and there results $ac-bc$ as the product of $a-b$ by c. But the multiplier was $c-d$ instead of c; therefore the multiplicand has been taken times too often; d times the multiplicand, which will be of the same form as c times the multiplicand, viz., $ad-bd$, must be subtracted, and the rule for subtraction is to change the signs of the quantity to be subtracted. The result is, therefore, $ac-bc-ad+bd$; comparing which with the given quantities we perceive that like signs have produced + and unlike —. To render the demonstration still more general, a may represent the assemblage of the additive terms of the multiplicand, and b that of the subtractive; c and d the same for the multiplier.

$$\begin{array}{l} a-b \\ c-d \\ \hline ac-bc \\ \quad ad-bd \\ \hline ac-bc-ad+bd. \end{array}$$

† The results in examples (1), (2), and (3) show, 1. That the square of the sum of two numbers or quantities is equal to the square of the first of the two quantities plus twice the product of the first and second, plus the square of the second. 2. That the product of the sum and difference is equal to the difference of the squares; and, 3. That the square of the difference is equal to the sum of the squares minus twice the product.

(7) Multiply $a'b-ab'$ by $h'k-hk'$.

$$\begin{array}{l} a'b-ab' \\ h'k-hk' \\ \hline a'bh'k-ab'h'k \\ \qquad -a'bhk'+ab'hk' \\ \hline a'bh'k-ab'h'k-a'bhk'+ab'hk' \end{array} = \text{product.}$$

(8) Multiply $x^m+x^{m-1}y+x^{m-2}y^2+x^{m-3}y^3+$ &c., by $x+y$.

$$\begin{array}{l} x^m+x^{m-1}y+x^{m-2}y^2+x^{m-3}y^3+\cdots\cdots \\ x\ +y \\ \hline x^{m+1}+\ x^my+\ x^{m-1}y^2+\ x^{m-2}y^3+\cdots\cdots \\ \qquad +\ x^my+\ x^{m-1}y^2+\ x^{m-2}y^3+\cdots\cdots \\ \hline x^{m+1}+2x^my+2x^{m-1}y^2+2x^{m-2}y^3+\cdots\cdots \end{array}$$

(9) Multiply x^2+y^2 by x^2-y^2.
(10) Multiply $x^2+2xy+y^2$ by $x-y$.
(11) Multiply $5a^4-2a^3b+4a^2b^2$ by $a^3-4a^2b+2b^3$.
(12) Multiply $x^4+2x^3+3x^2+2x+1$ by x^2-2x+1.
(13) Multiply $\frac{5}{2}x^2+3ax-\frac{7}{3}a^2$ by $2x^2-ax-\frac{1}{2}a^2$.
(14) Multiply $a^2+2ab+b^2$ by $a^2-2ab+b^2$.
(15) Multiply x^2+xy+y^2 by x^2-xy+y^2.
(16) Multiply $x^2+y^2+z^2-xy-xz-yz$ by $x+y+z$.
(17) Multiply together $x-a$, $x-b$, and $x-c$.
(18) Multiply together $g+h$, $g+h$, $g-h$, and $g-h$.
(19) Multiply together $p+q$, $p+2q$, $p+3q$, and $p+4q$.
(20) Multiply together $z-3$, $z-5$, $z-7$, and $z-9$.
(21) $(a^m-a^n+a^2)\times(a^m-a)$.
(22) $(5a^5x^3-4b^4y^5)\times(5a^5x^3+4b^4y^5)$ as ex. 2.

ANSWERS.

(9) x^4-y^4.
(10) $x^3+x^2y-xy^2-y^3$.
(11) $5a^7-22a^6b+12a^5b^2-6a^4b^3-4a^3b^4+8a^2b^5$.
(12) x^6-2x^3+1.
(13) $5x^4+\frac{7}{2}ax^3-\frac{107}{12}a^2x^2+\frac{5}{6}a^3x+\frac{7}{6}a^4$.
(14) $a^4-2a^2b^2+b^4$.
(15) $x^4+x^2y^2+y^4$.
(16) $x^3+y^3+z^3-3xyz$.
(17) $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$.
(18) $g^4-2g^2h^2+h^4$.
(19) $p^4+10p^3q+35p^2q^2+50pq^3+24q^4$.
(20) $z^4-24z^3+206z^2-744z+945$.
(21) $a^{2m}-a^{m+n}+a^{m+2}-a^{m+1}+a^{n+1}-a^3$.
(22) $25a^{10}x^6-16b^8y^{10}$.

When the multiplicand and multiplier are each homogeneous, the product will be also; and the degree of each term of the product will be equal to the sum of the degrees of a term in the multiplier, and a term in the multiplicand.

This serves conveniently to verify the accuracy of the operation. It is applicable in the above examples to all except the 12th, 20th, 21st, and 22d.

In multiplying one polynomial by another, there are always two terms of the total product which are not produced by the reduction of similar terms in the partial products. These two terms are the term affected with the highest exponent of any letter, and the term affected with the lowest exponent. If the terms of the multiplicand, multiplier, and product be arranged in the order of the powers of some letter,* as is usual, and as may be seen in the above examples, then the two terms in question of the product will be the first and last, the one being produced by the multiplication of the first of the multiplicand by the first of the multiplier, and the other by the multiplication of the last of the multiplicand by the last of the multiplier. The first of the multiplicand by the second of the multiplier usually produces a term similar to that which is produced from the multiplication of the second of the multiplicand by the first of the multiplier. The same is the case with the first and third of each, the first and fourth, the second and fourth, the third and fourth, and so on.

When a polynomial, arranged according to the powers of some letter, contains many terms in which this letter has the same exponent, these terms, after suppressing from them the letter of arrangement, may be placed in a parenthesis, or in a vertical column with a vinculum placed vertically on the right, and the letter of arrangement, with its proper exponent, following after. The polynomial in the parenthesis, or vertical column, is to be regarded as the coefficient of the power of the letter which follows, and is to be operated with exactly as we do with a numerical coefficient; *i. e.*, multiply the coefficient of the letter of arrangement in the multiplicand by the coefficient of the same letter in the multiplier, and afterward add the exponents of this letter.

EXAMPLE.

Multiplicand	$2b$ -1	a^2	$-4b^2$ $+2b$ -1	a	$+8b^3$ $-4b^2$		
Multiplier	$2b$ $+1$	a	$-4b^2$ $+1$				
Product of the multiplicand by $\begin{array}{r\|l} 2b & a \\ +1 & \end{array}$	$4b^2$ $-2b$ $+2b$ -1	a^3	$-8b^3$ $+4b^2$ $-2b$ $-4b^2$ $+2b$ -1	a^2	$+16b^4$ $-8b^3$ $+8b^3$ $-4b^2$	a	
Product of the multiplicand by $-4b^2$ $+1$			$-8b^3$ $+4b^2$ $+2b$ -1		$+16b^4$ $-8b^3$ $+4b^2$ $-4b^2$ $+2b$ -1		$-32b^5$ $+16b^4$ $+8b^3$ $-4b^2$
Total product simplified	$4b^2$ -1	a^3	$-16b^3$ $+4b^2$ $+2b$ -2	a^2	$+32b^4$ $-8b^3$ $-4b^2$ $+2b$ -1	a	$-32b^5$ $+16b^4$ $+8b^3$ $-4b^2$

* The letter chosen for this purpose is called the letter of arrangement.

Or thus,

$$\begin{array}{l}
(2b-1)a^2-(4b^2-2b+1)a+8b^3-4b^2 \\
(2b+1)a-(4b^2-1) \\
\hline
(4b^2-1)a^3-(8b^3+1)a^2+(16b^4-4b^2)a \\
\qquad -(8b^3-4b^2-2b+1)a^2+(16b^4-8b^3+2b-1)a-(32b^5-16b^4-8b^3+4b^2) \\
\hline
(4b^2-1)a^3-(16b^3-4b^2-2b+2)a^2+(32b^4-8b^3-4b^2+2b-1)a-(32b^5-16b^4-8b^3+4b^2)
\end{array}$$

1st Partial Multiplication.

$$\begin{array}{l}
2b-1 \\
2b+1 \\
\hline
4b^2-2b \\
\quad +2b-1 \\
\hline
4b^2 \qquad -1
\end{array}$$

2d Partial Multiplication.

$$\begin{array}{l}
4b^2-2b+1 \\
2b+1 \\
\hline
8b^3-4b^2+2b \\
\quad +4b^2-2b+1 \\
\hline
8b^3 \qquad +1
\end{array}$$

3d Partial Multiplication.

$$\begin{array}{l}
8b^3-4b^2 \\
2b+1 \\
\hline
16b^4-8b^3 \\
\quad +8b^3-4b^2 \\
\hline
16b^4 \qquad -4b^2
\end{array}$$

4th Partial Multiplication.

$$\begin{array}{l}
2b-1 \\
4b^2-1 \\
\hline
8b^3-4b^2 \\
\qquad -2b+1 \\
\hline
8b^3-4b^2-2b+1
\end{array}$$

5th Partial Multiplication.

$$\begin{array}{l}
4b^2-2b+1 \\
4b^2-1 \\
\hline
16b^4-8b^3+4b^2 \\
\qquad -4b^2+2b-1 \\
\hline
16b^4-8b^3 \qquad +2b-1
\end{array}$$

6th Partial Multiplication.

$$\begin{array}{l}
8b^3-4b^2 \\
4b^2-1 \\
\hline
32b^5-16b^4 \\
\qquad -8b^3+4b^2 \\
\hline
32b^5-16b^4-8b^3+4b^2
\end{array}$$

MULTIPLICATION BY DETACHED COEFFICIENTS.

14. In many cases the powers of the quantity or quantities in the multiplication of polynomials may be omitted, and the operation performed by the coefficients alone; for the same powers occupy the same vertical columns, when the polynomials are arranged according to the successive powers of the letters; and these successive powers, generally increasing or decreasing by a common difference, are readily supplied in the final product.

EXAMPLES.

(1) Multiply $x^3+x^2y+xy^2+y^3$ by $x-y$.

Coefficients of multiplicand $1+1+1+1$

———————— multiplier $1-1$

$$\begin{array}{l}
1+1+1+1 \\
\quad -1-1-1-1 \\
\hline
1+0+0+0-1
\end{array}$$

Since $x^3 \times x = x^4$, the highest power of x is 4, and decreases successively by unity, while that of y increases by unity; hence the product is

$$x^4 + 0.x^3y + 0.x^2y^2 + 0.xy^3 - y^4 = x^4 - y^4 = \text{product.}$$

(2) Multiply $3a^2 + 4ax - 5x^2$ by $2a^2 - 6ax + 4x^2$.

$$\begin{array}{l} 3+\ 4-\ 5 \\ \underline{2-\ 6+\ 4} \\ 6+\ 8-10 \\ \quad -18-24+30 \\ \underline{\qquad\quad +12+16-20} \\ 6-10-22+46-20 \end{array}$$

$\therefore$ Product $= 6a^4 - 10a^3x - 22a^2x^2 + 46ax^3 - 20x^4$.

(3) Multiply $2a^3 - 3ab^2 + 5b^3$ by $2a^2 - 5b^2$.

Here the coefficients of a^2 in the multiplicand, and a in the multiplier, are each zero; hence

$$\begin{array}{l} 2+0-\ 3+\ 5 \\ \underline{2+0-\ 5} \\ 4+0-\ 6+10 \\ \underline{\qquad -10-\ 0+15-25} \\ 4+0-16+10+15-25 \end{array}$$

Hence $4a^5 - 16a^3b^2 + 10a^2b^3 + 15ab^4 - 25b^5 =$ product.

The coefficient of a^4 being zero in the product, causes that term to disappear.

(4) Multiply $x^3 - 3x^2 + 3x - 1$ by $x^2 - 2x + 1$.

(5) Multiply $y^2 - ya + \frac{1}{4}a^2$ by $y^2 + ya - \frac{1}{4}a^2$.

(6) Multiply $ax - bx^2 + cx^3$ by $1 - x + x^2 - x^3 + x^4$.

(7) $(x^3 - ax^2 + bx - c) \times (x^2 - dx + e)$.

ANSWERS.

(4) $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$.

(5) $y^4 - a^2y^2 + \frac{1}{2}a^3y - \frac{1}{16}a^4$.

(6) $\begin{array}{r|r|r|r|r|l} ax-a & x^2+a & x^3-a & x^4+a & x^5-b & x^6+cx^7 \\ -b & b & -b & b & -c & \\ & c & -c & c & & \end{array}$

Or, $ax - (a+b)x^2 + (a+b+c)x^3 - (a+b+c)x^4 + (a+b+c)x^5 - (b+c)x^6 + cx^7$.

(7) $x^5 - (a+d)x^4 + (b+ad+e)x^3 - (c+bd+ae)x^2 + (cd+eb)x - ce$

DIVISION.

15. The object of algebraic division is to discover one of the factors of a given product, the other factor being given; and as multiplication is divided into three cases, so, in like manner, is division.

(1) When both dividend and divisor are monomials.

(2) When the dividend is a polynomial, and the divisor a monomial.

(3) When both dividend and divisor are polynomials.

CASE I.

16. *When both dividend and divisor are monomials.*

Write the divisor under the dividend, in the form of a fraction; cancel like quantities in both divisor and dividend, and suppress the greatest factor common to the two coefficients.

17. Powers of the same quantity are divided by subtracting the exponent of the divisor from that of the dividend, and writing the remainder as the exponent of the quotient.*

Thus, $a^7=aaaaaaa$; $a^4=aaaa$

$$\therefore \frac{a^7}{a^4}=\frac{aaaaaaa}{aaaa}=aaa=a^3=a^{7-4}.$$

Generally, $a^m=aaaa\ldots\ldots$ to m factors; $a^n=aaa\ldots\ldots$ to n factors;
$b^p=bbbb\ldots\ldots$ to p factors; $b^q=bbb\ldots\ldots$ to q factors;

$$\therefore \frac{a^mb^p}{a^n b^q}=\frac{aaa\ldots \text{to } m \text{ factors} \times bbb\ldots \text{to } p \text{ factors};}{aaa\ldots \text{to } n \text{ factors} \times bbb\ldots \text{to } q \text{ factors};}$$
$$=aaa\ldots \text{to } (m-n) \text{ factors} \times bbb\ldots \text{to } (p-q) \text{ factors},$$
$$=a^{m-n}b^{p-q}.$$

When a quantity has the same exponent in the dividend and divisor, we have

$$\frac{a^m}{a^m}=a^{m-m}=a^0; \text{ but } \frac{a^m}{a^m}=1.$$
$$\therefore a^0=1.$$

Hence *every quantity whose exponent is* 0 *is equal to* 1.

$$\frac{a^3}{a^5}=\frac{aaa}{aaaaa}=\frac{1}{aa}=\frac{1}{a^2}.$$

But we may subtract 5, the greater exponent, from 3, the less, and affect the difference with the sign —; hence

$$\frac{a^3}{a^5}=a^{3-5}=a^{-2}; \text{ but } \frac{a^3}{a^5}=\frac{1}{a^2};$$
$$\therefore \frac{1}{a^2}=a^{-2}.$$

---

* The rule for division follows from its object, which is, having one of the factors of a product given to find the other. As in multiplication we join together the factors of a product without any sign, and without regard to order, in division we suppress from the product, *i. e.*, the dividend, one of the factors, *i. e.*, the divisor, to obtain the other, which is the quotient. *Note.*—The quotient must contain those factors of the dividend which are not in the divisor. Note, also, that dividing one of the factors of a product divides the whole product. Thus, dividing a^5bc by a^3, we divide the single factor a^5, and get a^2bc; so to divide 16×12 by 8, we divide 16 alone, and get 2×12 for the quotient.

When there are factors in the divisor which are not in the dividend, the quotient may be expressed in the form of a fraction, as has been previously shown (2, V.). Suppressing the common factors in this case amounts to dividing both numerator and denominator by the same quantity. That such a division does not alter the value of the fraction, will be obvious from the following considerations:

1. If the numerator of a fraction be increased any number of times, the fraction itself will be increased as many times; and if the denominator be diminished any number of times, the fraction must still be increased as many times.

2. If the denominator of a fraction be increased any number of times, or the numerator diminished the same number of times, the fraction itself will, in either case, be diminished the same number of times.

3. If the numerator of a fraction be increased any number of times, the fraction is increased the same number of times; and if the denominator be increased as many times, the fraction is again diminished the same number of times, and must therefore have its original value. Hence both terms of a fraction may be multiplied by the same number, and, by similar considerations, it will appear, may be divided by the same number without changing the value of the fraction.

Corollary.—Rule. To multiply a fraction by a whole number, multiply the numerator of the fraction, or divide its denominator by the whole number. To divide a fraction, divide its numerator, or multiply its denominator.

Similarly, $\frac{1}{a+x}=(a+x)^{-1}$; $\frac{1}{(x+y)^2}=(x+y)^{-2}$,

And $\frac{1}{(x^2+y^2)^3(x^2-y^2)^{\frac{1}{2}}}=(x^2+y^2)^{-3}(x^2-y^2)^{-\frac{1}{2}}$, and so on.

So, also, $\frac{a^5}{a^3}=\frac{1}{a^{3-5}}=\frac{1}{a^{-2}}$;

But $\frac{a^5}{a^3}=a^2$;

$\therefore \frac{1}{a^{-2}}=a^2$.

From this it appears that a factor may be transferred from the denominator to the numerator, and vice versa, by changing the sign of its exponent.

EXAMPLES.

(1) Write a^2b^3c with the factors all in the denominator.

(2) Write $\frac{a^3bc^5}{d^4f^6}$ with the factors all in one line, and also all in the denominator.

For more of the theory of *negative exponents*, see a subsequent article.

18. In multiplication, the product of two terms, having the same sign, is affected with the sign $+$; and the product of two terms, having different signs, is affected with the sign $-$; hence we may conclude,

(1) That if the term of the dividend have the sign $+$, and that of the divisor the sign $+$, the resulting term of the quotient must have the sign $+$; because $+\times+$ gives $+$.

(2) That if the term of the dividend have the sign $+$, and that of the divisor the sign $-$, the resulting term of the quotient must have the sign $-$; because $-\times-$ gives $+$.

(3) That if the term of the dividend have the sign $-$, and that of the divisor the sign $+$, the resulting term of the quotient must have the sign $-$; because $+\times-$ gives $-$.

(4) That if the term of the dividend have the sign $-$, and that of the divisor the sign $-$, the resulting term of the quotient must have the sign $+$.

RULE OF SIGNS IN DIVISION.

$+$ divided by $+$, and $-$ divided by $-$, give $+$,
$-$ divided by $+$, and $+$ divided by $-$, give $-$;
or,
like signs give $+$, and unlike $-$, the same as in multiplication.

$$\frac{+ab}{+a}=+b;\quad \frac{-ab}{-a}=+b;\quad \frac{-ab}{+a}=-b;\quad \frac{+ab}{-a}=-b.$$

EXAMPLES.

(1) Divide $48a^3b^3c^2d$ by $12ab^2c$.

$$\frac{48a^3b^3c^2d}{12ab^2c}=\frac{48aaabbbccd}{12abbc}=4aabcd=4a^2bcd.$$

(2) $\frac{150a^5b^8cd^3}{30a^3b^5d^2}=5a^{5-3}b^{8-5}cd^{3-2}=5a^2b^3cd.$

(3) $\frac{-16a^2b^2c^2}{-\ 4abc}=4a^{2-1}b^{2-1}c^{2-1}=4abc.$

(4) $\frac{15a^{2m}x^{3n}y^{4n}}{3a^mx^{2n}y^{5n}}=5a^{2m-m}x^{3n-2n}y^{4n-5n}=5a^mx^ny^{-n}$.

(5) $\frac{-48a^mb^n}{6a^pb^q}=-8a^{m-p}b^{n-q}$.

(6) $\frac{-63a^3b^4c^5d^6x^2yz}{-7a^2bcd^3x^4y^5z^6}=+9ab^3c^4d^3x^{-2}y^{-4}z^{-5}$.

(7) $a^mb^nc^r\div a^nb^nc=a^{m-n}c^{r-1}$.

(8) $a^{3m}b^{n+1}c^{r-2}\div a^mb^nc=a^{2m}bc^{r-3}$.

(9) $5a^p\div 3a^{p+r}bc^{-1}=\frac{5}{3}a^{-r}b^{-1}c$.

(10) $a^{m-n}\div a^{p-q}=a^{m-n-p+q}$.

(11) $ab\div -ab=-1$.

(12) $-abc\div abc=-1$.

(13) $-b^m\div -b^m=1$.

(14) $96a^3b^4c^5d\div 84ab^4c^7d^5=\frac{8}{7}\frac{a^2}{c^2d^4}$.

(15) $x^{-5}y^{-n}z^{-q-3}\div x^{-7}y^{-m}z^{-p}=x^2y^{m-n}z^{p-q-3}$.

CASE II.

19. *When the dividend is a polynomial, and the divisor a monomial* Divide each of the terms of the dividend separately by the divisor.*

EXAMPLES.

(1) Divide $6a^2x^4y^6-12a^3x^3y^6+15a^4x^5y^3$ by $3a^2x^2y^2$.

$$\frac{6a^2x^4y^6-12a^3x^3y^6+15a^4x^5y^3}{3a^2x^2y^2}=2x^2y^4-4axy^4+5a^2x^3y.$$

(2) Divide $15a^2bc-20acy^2+5cd^2$ by $-5abc$. Ans. $-3a+4\frac{y^2}{b}-\frac{d^2}{ab}$.

(3) Divide $x^{n+1}-x^{n+2}+x^{n+3}-x^{n+4}$ by x^n. Ans. $x-x^2+x^3-x^4$.

(4) Divide $5(a+b)^3-10(a+b)^2+15(a+b)$ by $-5(a+b)$.
Ans. $-(a+b)^2+2(a+b)-3$.

(5) Divide $12a^4y^6-16a^5y^5+20a^6y^4-28a^7y^3$ by $-4a^4y^3$.
Ans. $-3y^3+4ay^2-5a^2y+7a^3$.

CASE III.

20. *When both dividend and divisor are polynomials.*

1. Arrange the dividend and divisor according to the powers of the same letter in both.

2. Divide the first term of the dividend by the first term of the divisor, and the result will be the first term in the quotient, by which multiply all the terms in the divisor, and subtract the product from the dividend.

3. Then to the remainder annex as many of the remaining terms of the dividend as are necessary, and find the next term in the quotient as before.

(1) Divide $a^4-4a^3x+6a^2x^2-4ax^3+x^4$ by $a^2-2ax+x^2$.

$$
\begin{array}{rl}
a^2-2ax+x^2) & a^4-4a^3x+6a^2x^2-4ax^3+x^4\ (a^2-2ax+x^2 \\
 & \underline{a^4-2a^3x+\ a^2x^2} \\
 & \quad -2a^3x+5a^2x^2-4ax^3 \\
 & \quad \underline{-2a^3x+4a^2x^2-2ax^3} \\
 & \qquad\qquad a^2x^2-2ax^3+x^4 \\
 & \qquad\qquad \underline{a^2x^2-2ax^3+x^4}
\end{array}
$$

* This rule follows from that for multiplication, which requires each term of the multiplicand to be repeated as many times as is expressed by the multiplier.

Arranging the terms according to the descending powers of x, we have

$$\begin{array}{r|l}
x^2-2ax+a^2) & x^4-4ax^3+6a^2x^2-4a^3x+a^4\ (x^2-2ax+a^2 \\
 & \underline{x^4-2ax^3+\ a^2x^2} \\
 & \quad -2ax^3+5a^2x^2-4a^3x \\
 & \quad \underline{-2ax^3+4a^2x^2-2a^3x} \\
 & \qquad\qquad a^2x^2-2a^3x+a^4 \\
 & \qquad\qquad \underline{a^2x^2-2a^3x+a^4.}^{*}
\end{array}$$

(2) Divide $x^4+x^2y^2+y^4$ by x^2+xy+y^2.

$$\begin{array}{r|l}
x^2+xy+y^2) & x^4+x^2y^2+y^4\ (x^2-xy+y^2 \\
 & \underline{x^4+x^3y\ +x^2y^2} \\
 & \quad -x^3y\ +y^4 \\
 & \quad \underline{-x^3y\ -x^2y^2-xy^3} \\
 & \qquad\qquad x^2y^2+xy^3+y^4 \\
 & \qquad\qquad \underline{x^2y^2+xy^3+y^4.}
\end{array}$$

---

* It has been shown (13) that when the dividend (which is the product of the divisor and quotient) is arranged as directed in the rule, its first term is produced without reduction by the multiplication of the first term of the divisor by the first of the quotient. Hence the rule above for finding the latter. This first term of the quotient being found, and the divisor being taken away from the dividend as many times as is expressed by this term, the remainder must contain the divisor as many times as is expressed by the second and remaining terms of the quotient. Hence the remainder may be regarded as a new dividend, and the object being to find how many times it contains the divisor, it must be arranged in the same manner as was the given dividend, and the first step will be the same as before. Similar reasoning will apply to the rest of the process.

Note.—The arrangement of the terms is for convenience. The term having the highest or lowest exponent of some letter might be *selected* from the dividend and remainders *without* any arrangement. The operation must always, however, begin with this term, as a reference to the last example will show; for if we attempt to commence with the term $6a^2x^2$, the third of the dividend, for instance, we perceive that this is produced by reduction from the term a^2x^2 in the second line, the term $4a^2x^2$ in the fourth line, and the term a^2x^2 in the sixth. The first of these is produced by the multiplication of the first of the quotient by the last of the divisor, the second by the multiplication of the second of the quotient by the second of the divisor, and the third by the last of the quotient and first of the divisor. It is not till the first and second terms of the quotient have been found by the rule above given, that any portion of the term $6a^2x^2$ presents itself to be divided, or that we can know what part of it is to be used as a dividend.

In the same manner, it may be shown that it would be impossible to begin with the second term of the dividend $4ax^3$ until the first term of the quotient has been found, which, multiplied by the second of the divisor, produces $-2ax^3$, a part of $-4ax^3$, and the subtraction leaves the other part $-2ax^3$, which now we know is the product of the first of the divisor by the second of the quotient, which latter we may then find.

The first of the divisor multiplied by the second of the quotient, and the second of the divisor by the first of the quotient, usually produce the same power of the letter of arrangement, and reduce together; the first and third of each, together with the two second terms of each, usually produce the same power, and so on. It is only the first of the divisor and first of the quotient, or last of the divisor and last of the quotient, which always produce a term that does not reduce with any other term.

N.B.—The arrangement may begin with the lowest as well as the highest power of any letter, and go on increasing instead of decreasing. When either of these arrangements is observed, if the first term of the divisor in any part of the operation is not contained exactly in the first term of the remainder, the division is impossible. By varying the arrangement, therefore, or simply considering which terms would come first, using different letters of arrangement, we may often determine beforehand by inspection whether the division is possible or not.

Another form of the work which has the convenience of placing the quotient, which is the multiplier, under the divisor, which is the multiplicand, is the following.

$$\begin{array}{l|l} \text{Dividend, } x^4+x^2y^2+\quad y^4 & x^2+xy+y^2, \text{ divisor.} \\ \underline{x^4+x^3y\ +x^2y^2} & x^2-xy+y^2, \text{ quotient.} \\ \quad -x^3y\ +y^4 & \\ \quad \underline{-x^3y\ -x^2y^2-xy^3} & \\ \qquad\qquad x^2y^2+xy^3+y^4 & \\ \qquad\qquad \underline{x^2y^2+xy^3+y^4} & \end{array}$$

(3) Divide $a^5-a^3b^2+2a^2b^3-ab^4+b^5$ by a^2-ab+b^2.

$$\begin{array}{l} a^2-ab+b^2)\ a^5-a^3b^2+2a^2b^3-ab^4+b^5\ (a^3+a^2b-ab^2+\dfrac{b^5}{a^2-ab+b^2} \\ \qquad\qquad \underline{a^5-a^4b+\ a^3b^2} \\ \qquad\qquad\quad a^4b\ -2a^3b^2+2a^2b^3 \\ \qquad\qquad\quad \underline{a^4b\ -\ a^3b^2+\ a^2b^3} \\ \qquad\qquad\qquad -\ a^3b^2+\ a^2b^3-ab^4 \\ \qquad\qquad\qquad \underline{-\ a^3b^2+\ a^2b^3-ab^4} \\ \qquad\qquad\qquad\quad *\quad *\quad *\ +b^5. \end{array}$$

Arranging the terms according to powers of b, we get

$$\begin{array}{l} b^2-ab+a^2)\ b^5-ab^4+2a^2b^3-a^3b^2+a^5\ (b^3+a^2b+\dfrac{-a^4b+a^5}{b^2-ab+a^2} \\ \qquad\qquad \underline{b^5-ab^4+\ a^2b^3} \\ \qquad\qquad\qquad a^2b^3-a^3b^2+a^5 \\ \qquad\qquad\qquad \underline{a^2b^3-a^3b^2+a^4b} \\ \qquad\qquad\qquad\qquad -a^4b+a^5. \end{array}$$

The results we have obtained in these two arrangements are apparently different; but their equivalence will be established as follows:

$$\begin{array}{lll} (1)\ (a^2-ab+b^2)\ (a^3+a^2b-ab^2) & = & a^5-a^3b^2+2a^2b^3-a\,b^4 \\ \quad \text{Add remainder} & = & \underline{\qquad\qquad\qquad\qquad +b^5} \\ \qquad \text{Proof} \ \ldots\ldots & & a^5-a^3b^2+2a^2b^3-a\,b^4+b^5 \\ (2)\ (b^2-ab+a^2)\ (b^3+a^2b) & = & b^5-a\,b^4+2a^2b^3-a^3b^2+a^4b \\ \quad \text{Add remainder} & = & \underline{\qquad\qquad\qquad\qquad -a^4b+a^5} \\ \qquad \text{Proof} \ \ldots\ldots & & b^5-a\,b^4+2a^2b^3-a^3b^2+a^5. \end{array}$$

The moment we arrive at a term of the quotient in which the exponent of the letter of arrangement is less than the difference of the exponents of this letter in the last terms of the divisor and dividend, we may be sure that the division will not terminate. If the divisor and dividend be arranged in the reverse order, that is, beginning with the lowest power of a letter, then the division will not terminate when the exponent of this letter in the term of the quotient is *greater* than the difference of its exponents in the last terms of the divisor and dividend.

Thus in the following example.

$$\begin{array}{l|l} x^9+x^7-ax^5+ax^4 & x^4+x^3+a \\ \underline{x^9+x^8+ax^5} & x^5-x^4 \\ \quad -x^8+\ x^7-2ax^5+\ ax^4 & \\ \quad \underline{-x^8-\ x^7-\ ax^4} & \\ \qquad\qquad 2x^7-2ax^5+2ax^4. & \end{array}$$

The last term of the quotient must be x^4, in order that, multiplied by a, the last of the divisor, it may produce the last of the dividend. If, therefore, the division is not completed when this term containing x^4 is obtained, it will not be.

EXAMPLES FOR PRACTICE.

(1) Divide $a^2-2ab+b^2$ by $a-b$.
(2) Divide $a^2+4ax+4x^2$ by $a+2x$.
(3) Divide $12x^4-192$ by $3x-6$.
(4) Divide $6x^6-6y^6$ by $2x^2-2y^2$.
(5) Divide $a^6-3a^4b^2+3a^2b^4-b^6$ by $a^3-3a^2b+3ab^2-b^3$.
(6) Divide $x^3+5x^2y+5xy^2+y^3$ by $x^2+4xy+y^2$.
(7) Divide x^5-y^5 by $x-y$.
(8) Divide a^4-b^4 by $a^3+a^2b+ab^2+b^3$.
(9) Divide $x^3-9x^2+27x-27$ by $x-3$.
(10) Divide x^4+y^4 by $x+y$.
(11) Divide $48x^3-76ax^2-64a^2x+105a^3$ by $2x-3a$.
(12) Divide $\frac{1}{2}x^3+x^2+\frac{3}{8}x+\frac{3}{4}$ by $\frac{1}{2}x+1$.
(13) Divide $52m^5-93m^4p-70m^3p^2+48m^2p^3-27mp^4$ by $13m^3-7m^2p+3mp^2$.
(14) Divide $33a^3b^3-77a^2b^4+121a^2b^5$ by $3a^2b-7ab^2+11ab^3$.
(15) Divide $(6p^4-12pq^3-6p^3q+12q^4)$ by $(p-q)$.
(16) Divide $(100a^5-440a^4k+235a^3k^2-30a^2k^3)$ by $(5a^3-2a^2k)$.
(17) Divide $(g^4-4g^3h+6g^2h^2-4gh^3+h^4)$ by $(h^2-2hg+g^2)$.
(18) Divide $(37a^2m^2-26a^3m+3a^4-14am^3)$ by $(3a^2-5am+2m^2)$.
(19) Divide (a^6-b^6) by $(a-b)$ and (a^6+b^6) by $(a+b)$.
(20) Divide (a^7-b^7) by $(a-b)$ and (a^7+b^7) by $(a+b)$.
(21) $(\frac{1}{3}-6z^2+27z^4)\div(\frac{1}{3}+2z+3z^2)=1-6z+9z^2$.

ANSWERS.

(1) $a-b$.
(2) $a+2x$.
(3) $4x^3+8x^2+16x+32$.
(4) $3x^4+3x^2y^2+3y^4$.
(5) $a^3+3a^2b+3ab^2+b^3$.
(6) $x+y$.
(7) $x^4+x^3y+x^2y^2+xy^3+y^4$.
(8) $a-b$.
(9) x^2-6x+9.
(10) $x^3-x^2y+xy^2-y^3+\dfrac{2y^4}{x+y}$.
(11) $24x^2-2ax-35a^2$.
(12) $x^2+\frac{3}{4}$.
(13) $4m^2-5mp-9p^2$.
(14) $11ab^2$.
(15) $6p^3-12q^3$.
(16) $20a^2-80ak+15k^2$.
(17) $g^2-2gh+h^2$.
(18) a^2-7am.
(19) $\begin{cases} a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5, \text{ and} \\ a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5+\dfrac{2b^6}{a+b}. \end{cases}$

$$(20)\ \begin{cases} a^6+a^5b+a^4b^2+a^3b^3+a^2b^4+ab^5+b^6, \text{ and} \\ a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6-\dfrac{2b^7}{a+b}. \end{cases}$$

EXAMPLES WITH LITERAL EXPONENTS.

(1) Divide $2a^{3n}-6a^{2n}b^n+6a^nb^{2n}-2b^{3n}$ by a^n-b^n.

$$\begin{array}{r|l} a^n-b^n) & 2a^{3n}-6a^{2n}b^n+6a^nb^{2n}-2b^{3n}\ (2a^{2n}-4a^nb^n+2b^{2n} \\ & \underline{2a^{3n}-2a^{2n}b^n} \\ & \quad -4a^{2n}b^n+6a^nb^{2n} \\ & \quad \underline{-4a^{2n}b^n+4a^nb^{2n}} \\ & \qquad\qquad 2a^nb^{2n}-2b^{3n} \\ & \qquad\qquad \underline{2a^nb^{2n}-2b^{3n}.} \end{array}$$

(2) Divide $x^{m+1}+x^my+xy^m+y^{m+1}$ by x^m+y^m.

(3) Divide a^n-x^n by $a-x$.

(4) Divide $x^{4n}+x^{2n}y^{2n}+y^{4n}$ by $x^{2n}+x^ny^n+y^{2n}$.

(5) Divide $a^{m+n}b^n-4a^{m+n-1}b^{2n}-27a^{m+n-2}b^{3n}+42a^{m+n-3}b^{4n}$ by $a^nb^n-7a^{n-1}b^{2n}$.

(6) Divide $a^{3m-2n}b^{2p}c-a^{2m+n-1}b^{1-p}c^n+a^{-n}b^{-1}c^m+a^{3m-n}b^{3p+2}c^n-a^{2m+2n-1}b^3c^{2n-1}+b^{p+1}c^{m+n-1}$ by $a^{-n}b^{-p-1}+bc^{n-1}$.

ANSWERS.

(2) $x+y$.

(3) $a^{n-1}+a^{n-2}x+a^{n-3}x^2+\dfrac{a^{n-3}x^3-x^n}{a-x}$.

(4) $x^{2n}-x^ny^n+y^{2n}$.

(5) $a^m+3a^{m-1}b^n-6a^{m-2}b^{2n}$.

(6) $a^{3m-n}b^{3p+1}c-a^{2m+2n-1}b^2c^n+b^pc^m$.

EXAMPLES WITH LITERAL COEFFICIENTS.*

(1) Divide $ax^5+ax^4+bx^4+ax^3+bx^3+cx^3+ax^2+bx^2+cx^2+bx+cx+c$ by ax^2+bx+c.

Arrange the terms of the dividend in the following manner, in order to keep the operation within the breadth of the page.

$$ax^2+bx+c)\ ax^5+\left.\begin{array}{r}a\\b\\ \end{array}\right|x^4+\left.\begin{array}{r}a\\b\\c\end{array}\right|x^3+\left.\begin{array}{r}a\\b\\c\end{array}\right|x^2+\left.\begin{array}{r}b\\c\\ \end{array}\right|x+c\ (x^3+x^2+x+1.$$

$$\underline{ax^5+b\,x^4+c\,x^3}$$

$$a\,x^4+\left.\begin{array}{r}a\\b\\ \end{array}\right|x^3+\left.\begin{array}{r}a\\b\\c\end{array}\right|x^2$$

$$\underline{a\,x^4+b\,x^3+c\,x^2}$$

$$a\,x^3+\left.\begin{array}{r}a\\b\end{array}\right|x^2+\left.\begin{array}{r}b\\c\end{array}\right|x$$

$$\underline{a\,x^3+b\,x^2+c\,x}$$

$$a\,x^2+b\,x+c$$

$$\underline{a\,x^2+b\,x+c.}$$

* The literal multipliers of each power of the letter of arrangement are to be collected together, and regarded as a polynomial coefficient of that power, which is to be treated exactly in the process of division as a numerical coefficient would be, observing only the four ground rules applicable to polynomials instead of numbers.

(2)*

Divid.	$4b^2$ $-\ c^2$	a^3-16b^3 $+\ 4b^2c$ $+\ 2bc^2$ $-\ 2c^3$	a^2+32b^4 $-\ 8b^3c$ $-\ 4b^2c^3$ $+\ 2bc^3$ $-\ c^4$	$a-32b^5$ $+16b^4c$ $+\ 8b^3c^2$ $-\ 4b^2c^3$
Product to subtract.		$+\ 8b^3$ $-\ 4b^2c$ $-\ 2bc^2$ $+\ c^3$		
1st rem. or 2d divid.		$-\ 8b^3$ $-\ c^3$	a^2+32b^4 $-\ 8b^3c$ $-\ 4b^2c^2$ $+\ 2bc^3$ $-\ c^4$	$a-32b^5$ $+16b^4c$ $+\ 8b^3c^2$ $-\ 4b^2c^3$
Product to subtract.			$-16b^4$ $+\ 8b^3c$ $-\ 4b^2c^2$ $+\ 4b^2c^2$ $-\ 2bc^3$ $+\ c^4$	
2d rem. or 3d divid.			$+16b^4$ $-\ 4b^2c^2$	$a-32b^5$ $+16b^4c$ $+\ 8b^3c^2$ $-\ 4b^2c^3$
Product to subtract.				$+32b^5$ $-16b^4c$ $-\ 8b^3c^2$ $+\ 4b^2c^3$
3d rem.				o.

Divis.	$2b$ $+\ c$	$a\ -4b^2$ $+\ c^2$	
Quot.	$2b$ $-\ c$	a^2-4b^2 $+2bc$ $-\ c^2$	$a+8b^3$ $-4b^2c$

1st Partial Division.

$$\begin{array}{r|l} 4b^2-\ c^2 & 2b+c \\ -2bc & 2b\ -c \\ \hline -2bc-c^2 & \\ +c^2 & \\ \hline \text{o.} & \end{array}$$

2d Partial Division.

$$\begin{array}{r|l} -8b^3-\ c^3 & 2b+c \\ +4b^2c & -4b^2+2bc-c^2 \\ \hline +4b^2c-\ c^3 & \\ -2bc^2 & \\ \hline -2bc^2-\ c^3 & \\ +\ c^3 & \\ \hline \text{o.} & \end{array}$$

3d Partial Division.

$$\begin{array}{r|l} 16b^4-4b^2c^2 & 2b+c \\ -8b^3c & 8b^3-4b^2c \\ \hline -8b^3c-4b^2c^2 & \\ +4b^2c^2 & \\ \hline \text{o.} & \end{array}$$

(3) Divide x^3+ax^2+bx+c by $x-r$.

$$\begin{array}{l} x-r)\ x^3+ax^2+bx+c\ (x^2+(r+a)x+(r^2+ar+b) \\ \qquad\quad x^3-rx^2 \\ \hline \qquad\qquad (r+a)x^2+bx \\ \qquad\qquad (r+a)x^2-(r^2+ar)x \\ \hline \qquad\qquad\qquad\qquad (r^2+ar+b)x+c \\ \qquad\qquad\qquad\qquad (r^2+ar+b)x-(r^3+ar^2+br) \\ \hline \qquad\qquad\qquad\qquad\qquad\qquad r^3+ar^2+br+c, \text{ remainder} \end{array}$$

In the preceding and similar examples, the remainder differs only from the dividend in having r instead of x. That this is always the case when the divisor is x minus some quantity, will be shown hereafter. (Art. 238, Pr. I.)

(4) Divide x^3-ax^2+bx-c by $x-r$.

(5) Divide $x^3-(a+b+c)x^2+(ab+bc+ca)x-abc$ by $x-a$.

(6) Divide $x^3-(a+2)x^2+(2a+b)x-2b$ by $x-2$.

(7) Divide $11a^2b-19abc+10a^3-15a^2c+3ab^2+15bc^2-5b^2c$ by $5a^2+3ab-5bc$.

(8) Divide $x^3-(a+b+d)x^2+(ad+bd+c)x-cd$ by $x^2-(a+b)x+c$.

(9) Divide $x^m+px^{m-1}+qx^{m-2}+rx^{m-3}+$, &c.... $+tx+u$ by $x-a$.

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* N.B. The signs of the products to subtract are actually changed in this example before they are written; a method which is sometimes practised. Their first terms need not be written, since they are cancelled by the first terms of the corresponding dividends.

(10) Divide

$$\begin{array}{r|l|l|l|l} a^4 & x^4+a^5 & x^3-a^5b & x^2+a^4b^3 & x-a^2b^6 \\ -a^3b & -a^4b & -2a^4b^2 & +2a^3b^4 & \\ +a^2b^2 & +a^3b^2 & -ab^5 & & \\ -ab^3 & +a^2b^3 & & & \end{array} \text{ by } \begin{array}{r|l} a^2 & x^2+a^3x-a^2b^2. \\ -ab & \end{array}$$

When there are negative exponents of the letter of arrangement, they come after the term containing x^0, *i. e.*, the term in which x does not appear, those which have the greatest absolute value being placed last.

(11) Divide $-x^3-x^2+10x+\frac{5}{3}-\frac{13}{2}x^{-1}-\frac{23}{6}x^{-2}+3x^{-3}$ by $x^2-2x-2+\frac{1}{3}x^{-1}+\frac{3}{2}x^{-2}$.

ANSWERS.

(4) $x^2+(r-a)x+(r^2-ar+b)$, and remainder is r^3-ar^2+br-c.

(5) $x^2-(b+c)x+bc$.

(6) x^2-ax+b.

(7) $2a+b-3c$.

(8) $x-d$.

(9) $$\begin{array}{r|l|l|l} x^{m-1}+a & x^{m-2}+a^2 & x^{m-3}+a^3 & x^{m-4}+, \text{ \&c.} \ldots\ldots + a^{m-1} \\ +p & \quad +ap & \quad +a^2p & \qquad\qquad\qquad +a^{m-2}p \\ & \quad +q & \quad +aq & \qquad\qquad\qquad +a^{m-3}q \\ & & \quad +r & \qquad\qquad\qquad +a^{m-4}r \\ & & & \qquad\qquad\qquad \ldots\ldots\ldots \\ & & & \qquad\qquad\qquad \ldots\ldots\ldots \\ & & & \qquad\qquad\qquad +t. \end{array}$$

(10) $$\begin{array}{r|l|l} a^2 & x^2-a^2b & x+b^4. \\ +b^2 & -ab^2 & \end{array}$$

(11) $-x-3+2x^{-1}$.

21. In those cases in which the division does not terminate, and the quotient may be continued to an unlimited number of terms, the quotient is termed *an infinite series*, and then the successive terms of the quotient are generally regulated by a law which, in most cases, is readily discoverable.

EXAMPLES.

(1) Divide 1 by $1-x$.

$$\begin{array}{l} 1-x)\ 1 \qquad (1+x+x^2+x^3+x^4+x^5+\ldots\ldots \\ \qquad\ \underline{1-x} \\ \qquad\quad +x \\ \qquad\quad \underline{+x-x^2} \\ \qquad\qquad\quad +x^2 \\ \qquad\qquad\quad \underline{+x^2-x^3} \\ \qquad\qquad\qquad\quad \underline{+x^3} \end{array}$$

The quotient in this case is called an infinite series, and the law of formation of this series is, that any term in the quotient is the product of the immediately preceding term by x.

(2) Divide 1 by $1+x$. Ans. $1-x+x^2-x^3+x^4-\ldots.$

(3) Divide $1+x$ by $1-x$. Ans. $1+2x+2x^2+2x^3+2x^4+\ldots.$

(4) Divide 1 by $x+1$. Ans. $x^{-1}-x^{-2}+x^{-3}-x^{-4}+x^{-5}-\ldots$

(5) Divide $x-a$ by $x-b$.

Ans. $1-(a-b)x^{-1}-(a-b)bx^{-2}-(a-b)b^2x^{-3}-.$

(6) Divide 1 by $1-2x+x^2$. Ans. $1+2x+3x^2+4x^3+5x^4+\ldots.$

22. When a polynomial is the product of two or more factors, it is often requisite to resolve it into the factors of which it is composed, and merely to indicate the multiplication. This can frequently be done by inspection, and by the aid of the following formulas:

$$(x+a)(x+b)=x^2+(a+b)x+ab \ldots\ldots (1)$$
$$(x+a)(x-b)=x^2+(a-b)x-ab \ldots\ldots (2)$$
$$(x-a)(x+b)=x^2-(a-b)x-ab \ldots\ldots (3)$$
$$(x-a)(x-b)=x^2-(a+b)x+ab \ldots\ldots (4)$$
$$(a+b)(a-b)=a^2-b^2 \ldots\ldots (5)$$
$$(n+1)(n+1)=n^2+2n+1 \ldots\ldots (6)$$
$$(n-1)(n-1)=n^2-2n+1 \ldots\ldots (7)$$

EXAMPLES.

(1) Resolve $ax^2+bx^2-cx^2$ into its component factors.

Here $ax^2+bx^2-cx^2=x^2(a+b-c)$.

(2) Transform the expression n^3+2n^2+n into factors.

Here $n^3+2n^2+n=n(n^2+2n+1)$
$=n(n+1)(n+1)$ by (6)
$=n(n+1)^2$.

(3) Decompose the expression x^2-x-72 into two factors.

By inspecting formula (3), we have $-1=-9+8$, and $-72=-9\times 8$; hence $x^2-x-72=(x-9)(x+8)$.

(4) Decompose $5a^2bc+10ab^2c+15abc^2$ into two factors.

(5) Transform $3m^4n^6-6m^3n^5p+3m^2n^4p^2$ into factors.

(6) Transform $3b^3c-3bc^3$ into factors.

(7) Decompose $x^2+8x+15$ into two factors.

(8) Decompose x^3-2x^2-15x into three factors.

(9) Decompose x^2-x-30 into factors.

(10) Transform $a^2-b^2+2bc-c^2$ into two factors.

(11) Transform a^2x-x^3 into factors.

ANSWERS.

(4) $5abc(a+2b+3c)$.
(5) $3m^2n^4(mn-p)^2$.
(6) $3bc(b+c)(b-c)$.
(7) $(x+3)(x+5)$.
(8) $x(x+3)(x-5)$.
(9) $(x+5)(x-6)$.
(10) $(a+b-c)(a-b+c)$.
(11) $x(a+x)(a-x)$.

23. By the usual process of division we might obtain the quotient of a^n-b^n divided by $a-b$, when any particular number is substituted for n; but we shall here prove generally that a^n-b^n is always exactly divisible by $a-b$, and exhibit the quotient.

It is required to divide a^n-b^n by $a-b$.

$$a-b)\ a^n-b^n \quad (a^{n-1}+\frac{b(a^{n-1}-b^{n-1})}{a-b}$$
$$\underline{a^n-a^{n-1}b}$$
$$\text{Rem.}\quad a^{n-1}b-b^n;$$

Rem. under another form, $b(a^{n-1}-b^{n-1})$.

Hence, $$\frac{a^n-b^n}{a-b}=a^{n-1}+\frac{b(a^{n-1}-b^{n-1})}{a-b} \ldots\ldots\ldots (1)$$

Now it appears from this result, that a^n-b^n will be exactly divisible by $a-b$, if $a^{n-1}-b^{n-1}$ be divisible by $a-b$; that is, if the difference of the same powers of two quantities is divisible by their difference, then the difference of the powers of the next higher degree is also divisible by that difference.

But a^2-b^2 is exactly divisible by $a-b$, and we have

$$\frac{a^2-b^2}{a-b}=a+b \quad \ldots\ldots\ldots\ldots (2)$$

And since a^2-b^2 is divisible by $a-b$, it appears, from what has been just proved, that a^3-b^3 must be exactly divisible by $a-b$; and since a^3-b^3 is divisible, a^4-b^4 must be divisible, and so on *ad infinitum.*

Hence, generally, a^n-b^n will always be exactly divisible by $a-b$, and give the quotient

$$\frac{a^n-b^n}{a-b}=a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots\ldots a^2b^{n-3}+ab^{n-2}+b^{n-1} \quad \ldots\ldots (5)$$

In a similar manner, we find, when n is an *odd* number,

$$\frac{a^n+b^n}{a+b}=a^{n-1}-a^{n-2}b+a^{n-3}b^2-\ldots\ldots+a^2b^{n-3}-ab^{n-2}+b^{n-1} \quad \ldots\ldots (6)$$

And when n is an *even* number

$$\frac{a^n-b^n}{a+b}=a^{n-1}-a^{n-2}b+a^{n-3}b^2-\ldots\ldots-a^2b^{n-3}+ab^{n-2}-b^{n-1} \quad \ldots\ldots (7)$$

By substituting particular numbers for n, in the formulas (5), (6), (7), we may deduce various algebraical formulas, several of which will be found in the following deductions from the rules of multiplication and division.

USEFUL ALGEBRAIC FORMULAS.

(1) $a^2-b^2=(a+b)(a-b)$.
(2) $a^4-b^4=(a^2+b^2)(a^2-b^2)=(a^2+b^2)(a+b)(a-b)$.
(3) $a^3-b^3=(a^2+ab+b^2)(a-b)$.
(4) $a^3+b^3=(a^2-ab+b^2)(a+b)$.
(5) $a^6-b^6=(a^3+b^3)(a^3-b^3)=(a^3+b^3)(a^2+ab+b^2)(a-b)$.
(6) $a^6-b^6=(a^3+b^3)(a^3-b^3)=(a^3-b^3)(a^2-ab+b^2)(a+b)$.
(7) $a^6-b^6=(a^3+b^3)(a^3-b^3)=(a^2-b^2)(a^4+a^2b^2+b^4)$.
(8) $a^6-b^6=(a+b)(a-b)(a^2+ab+b^2)(a^2-ab+b^2)$.
(9) $(a^2-b^2)\div(a-b)=a+b$.
(10) $(a^3-b^3)\div(a-b)=a^2+ab+b^2$.
(11) $(a^3+b^3)\div(a+b)=a^2-ab+b^2$.
(12) $(a^4-b^4)\div(a+b)=a^3-a^2b+ab^2-b^3$.
(13) $(a^5-b^5)\div(a-b)=a^4+a^3b+a^2b^2+ab^3+b^4$.
(14) $(a^5+b^5)\div(a+b)=a^4-a^3b+a^2b^2-ab^3+b^4$.
(15) $(a^6-b^6)\div(a^2-b^2)=a^4+a^2b^2+b^4$.

DIVISION BY DETACHED COEFFICIENTS.

24. Arrange the terms of the divisor and dividend according to the successive powers of the letter, or letters, common to both; write down simply the coefficients with their respective signs, supplying the coefficients of the absent terms with zeros, and proceed as usual. Divide the highest power of the omitted letters in the dividend by that of the omitted letters in the divisor, and the result will be the literal part of the first term in the quotient. The

literal parts of the successive terms follow the same law of increase or decrease as those in the dividend. The coefficients prefixed to the literal parts will give the complete quotient, omitting those terms whose coefficients are zero.

EXAMPLES.

(1) Divide $6a^4-96$ by $3a-6$.

$$\begin{array}{l}
3-6)\ 6+0+0+0-96\ (2+4+8+16 \\
\quad 6-12 \\
\qquad 12 \\
\qquad 12-24 \\
\qquad\quad 24 \\
\qquad\quad 24-48 \\
\qquad\qquad 48-96 \\
\qquad\qquad 48-96
\end{array}$$

But $a^4\div a=a^3$, and the literal parts of the successive terms, are, therefore a^3, a^2, a^1, a^0, or a^3, a^2, a, 1; hence, $2a^3+4a^2+8a+16=$ quotient.

(2) Divide $8a^5-4a^4x-2a^3x^2+a^2x^3$ by $4a^2-x^2$.

$$\begin{array}{l}
4+0-1)\ 8-4-2+1\ (2-1 \\
\qquad 8+0-2 \\
\qquad -4+0+1 \\
\qquad -4-0+1
\end{array}$$

Now, $a^5\div a^2=a^3$; hence a^3 and a^2x are the literal parts of the terms in the quotient, for there are only two coefficients in the quotient; therefore

$$2a^3-a^2x= \text{quotient required.}$$

(3) Divide $x^4-3ax^3-8a^2x^2+18a^3x-8a^4$ by $x^2+2ax-2a^2$.

(4) Divide $3y^3+3xy^2-4x^2y-4x^3$ by $x+y$.

(5) Divide $10a^4-27a^3x+34a^2x^2-18ax^3-8x^4$ by $2a^2-3ax+4x^2$.

(6) Divide a^4+5a^3+a+5 by a^3+1.

ANSWERS.

(3) $x^2-5ax+4a^2$.

(4) $-4x^2+3y^2$.

(5) $5a^2-6ax-2x^2$.

(6) $a+5$.

SYNTHETIC DIVISION.

RULE.*

25. (1) Divide the divisor and dividend by the coefficient of the first term in

* The rule here given for *Synthetic Division* is due to the late W. G. Horner, Esq., of Bath, whose researches in science have issued in several elegant and useful processes, especially in the higher branches of algebra, and in the evolution of the roots of equation of all dimensions.

In the common method of division, the several terms in the divisor are multiplied by the first term in the quotient, and the product subtracted from the dividend; but subtraction is performed by changing all the signs of the quantities to be subtracted, and then *adding* the several terms in the lower line to the similar terms in the higher. If, therefore, the signs of the terms in the divisor were changed, we should have to *add* the product of the divisor and quotient instead of subtracting it. By this process, then, the second dividend would be *identically* the same as by the usual method. We may omit altogether the products of the first term in the divisor by the successive terms in the quotient, because in the usual method the first term in each successive dividend is cancelled by these products. Omitting, therefore, these products, the coefficient of the first term in any dividend the divisor, which will make the leading coefficient of the divisor unity, and the first term of the quotient will be identical with that of the dividend.

(2) Set the coefficients of the dividend in a horizontal line with their proper signs, and those of the divisor, with the signs all changed except that of the first, in a vertical column on the right or left, drawing a line under the whole, underneath which to write the quotient.

(3) Multiply all the terms so changed by the first term in the quotient, and place the products successively under the corresponding terms of the dividend, in a diagonal column.

(4) Add the results in the second column, which will give the second term of the quotient; and multiply the changed terms in the divisor by this, placing the products in a diagonal series, as before.

(5) Add the results in the third column for the next term in the quotient, by which, again, multiply the changed terms in the divisor, placing the products as before.

(6) This process, continued till the last line of products extends as far to the right as the dividend, will give the same series of terms as the usual mode of division.

EXAMPLES.

(1) Divide $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^2-2ax+x^2$.

$$\begin{array}{r|rrrrrr} 1 & 1- & 5+ & 10- & 10+ & 5- & 1 \\ +2 & & +2- & 6+ & 6- & 2 & \\ -1 & & & -1+ & 3- & 3+ & 1 \\ \hline & 1- & 3+ & 3- & 1 & * & * \end{array}$$

Hence $a^3-3a^2x+3ax^2-x^3=$ quotient.

In this example the coefficients of the dividend are written horizontally, and those of the divisor vertically, with all the signs of the latter changed, except the first. Then $+2$ and -1, the changed terms in the divisor, are multiplied by 1, the first term of the quotient, which is written in the horizontal line at the bottom, and is the same as the first term of the dividend; the products $+2$ and -1 are placed diagonally, under -5 and $+10$, the corresponding terms of the dividend. Then, by adding the second column, we have -3 for the second term in the quotient, and the changed terms $+2$ and -1 in the divisor, multiplied by -3, give -6 and $+3$, which are placed diagonally under $+10$ and -10. The sum of the third column is $+3$, the next term in the quotient, which, multiplied into the changed terms of the divisor, gives $+6-3$, for the next diagonal column. The sum of the fourth column is -1, and by this we obtain the last diagonal column $-2+1$. The process here terminates, and the sums of the fifth and sixth columns are zero, which shows that there is no remainder. If the last terms did not reduce to zero by addition, their sum would be the coefficients of the remainder; the quotient is completed by restoring the letters, as in detached coefficients.

Having made the coefficient of the first term in the divisor unity, that co-

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will be the coefficient of the succeeding term in the quotient, the coefficient in the first term of the divisor being unity; for in all cases it can be made unity by dividing both divisor and dividend by the coefficient of the first term in the divisor. The operation, thus simplified, may, however, be farther abridged by omitting the successive additions, except so much only as is necessary to show the first term in each dividend, which, as before remarked, is also the coefficient of the succeeding term in the quotient.

efficient may be omitted entirely, since it is of no use whatever in continuing the operation here described.

(2) Divide $x^6-5x^5+15x^4-24x^3+27x^2-13x+5$ by $x^4-2x^3+4x^2-2x+1$.

$$\begin{array}{r|rrrrrrr}
 & 1 & -5 & +15 & -24 & +27 & -13 & +5 \\
+2 & & +2 & -6 & +10 & & & \\
-4 & & & -4 & +12 & -20 & & \\
+2 & & & & +2 & -6 & +10 & \\
-1 & & & & & -1 & +3 & -5 \\
\hline
 & 1 & -3 & +5 & 0 & 0 & 0 & 0
\end{array}$$

Hence $x^2-3x+5=$ quotient required.

(3) Divide $a^5+2a^4b+3a^3b^2-a^2b^3-2ab^4-3b^5$ by $a^2+2ab+3b^2$.

$$\begin{array}{r|rrrrrr}
 & 1 & +2 & +3 & -1 & -2 & -3 \\
-2 & & -2 & +0 & +0 & +2 & \\
-3 & & & -3 & +0 & +0 & +3 \\
\hline
 & 1 & +0 & +0 & -1 & &
\end{array}$$

Hence $a^3+0\cdot a^2b+0\cdot ab^2-b^3=a^3-b^3=$ quotient.

(4) Divide $1-x$ by $1+x$. Ans. $1-2x+2x^2-2x^3+$, &c.

(5) Divide 1 by $1-x$. Ans. $1+x+x^2+x^3+$, &c.

(6) Divide x^7-y^7 by $x-y$. Ans. $x^6+x^5y+x^4y^2+x^3y^3+x^2y^4+xy^5+y^6$.

(7) Divide $a^6-3a^4x^2+3a^2x^4-x^6$ by $a^3-3a^2x+3ax^2-x^3$.
Ans. $a^3+3a^2x+3ax^2+x^3$.

(8) Divide $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^2-2ax+x^2$.
Ans. $a^3-3a^2x+3ax^2-x^3$.

(9) Divide $4y^6-24y^5+60y^4-80y^3+60y^2-24y+4$ by $2y^3-4y+2$.
Ans. $2y^4-8y^3+12y^2-8y+2$.

THE GREATEST COMMON MEASURE.

26. A *measure* of a quantity is any quantity that is contained in it exactly, or divides it without a remainder; and, on the other hand, a *multiple* of a quantity is any quantity that contains it exactly. Thus, 5 is a measure of 15, and 15 is a multiple of 5; for 5 is contained in 15 exactly 3 times, and 15 contains 5 exactly 3 times, or is produced by *multiplying* 5.

27. A *common measure*, or *common divisor*, of two or more quantities, is a quantity which is contained exactly in each of them.

28. The *greatest common measure*, of two or more quantities, is composed of all the prime* factors, whether numerical, monomial, or polynomial factors, common to each of the quantities; $3x$ is a common measure of $12ax$ and $18bx$, and $6x$ is the greatest common measure of $12ax$ and $18bx$. The greatest common divisor of $2\times 7a(b+c)d$ and $2\times 3am(b+c)$ is composed of the common prime factors $2a(b+c)$; the factors $7d$ of the one and 3 of the other make no part of the common divisor.

29. *To find the greatest common measure of two polynomials.*

Arrange the polynomials according to the powers of some letter, and divide that which contains the highest power of the letter by the other, as in division; then divide the last divisor by the remainder arising from the first division; consider the remainder that arises from this second division as a divisor, and

* A prime number or a prime algebraic quantity is one which is divisible only by itself or unity.

the last divisor as the corresponding dividend, and continue this process of division till the remainder is 0; then the last divisor is the greatest common measure.

Note 1. When the highest power of the leading quantity is the same in both polynomials, it is indifferent which of the polynomials is made the divisor, the only guide being the coefficients of the leading terms of the polynomials.

Note 2. If the two given polynomials have a monomial factor common to all the terms of both, it may be suppressed; but as it forms part of the common measure (28), it must be restored at the end of the process by multiplying it into the common measure which is in consequence obtained.

Note 3. If any divisor contain a factor, which is not a factor also of the dividend, that factor may be rejected, as such factor can form no part of the greatest common measure, which is composed of the common factors alone.

Note 4. If the coefficient of the leading term of any dividend be not divisible by that of the divisor, it may be rendered so by multiplying every term of the dividend by a proper factor, to make it divisible. This new factor thus introduced, not being a common factor, does not affect the common measure.

If it were already a factor of the divisor, it could not be thus used; the remedy, in this case, would be to suppress it in the divisor, according to Note 3.

In order to prove the truth of this rule, we shall premise two lemmas.*

Lemma 1. If a quantity measure another quantity, it will also measure any multiple of that quantity. Thus, if d measures a, it will also measure m times a, or ma; for, let $a=hd$, then $ma=mhd$, and, therefore, d measures ma, the quotient being mh.

Lemma 2. If a quantity measure two other quantities, it will also measure both their sum and difference, or any multiples of them. For, let $a=hd$, and $b=kd$, then d measures both a and b; hence $a \pm b=hd \pm kd=d(h \pm k)$, and, therefore, d measures both $a+b$ and $a-b$, the quotient being $h+k$ in the former case, and $h-k$ in the latter: and by lemma 1, d measures any multiples of $a+b$ and $a-b$.

Now, let a and b be two polynomials, or the terms of a fraction, and let

a divided by b leave a remainder c

b . . . . . . c . . . . d

c . . . . . . d leave no remainder, as is shown in the marginal scheme. Then we have, by the nature of division, these six equalities, viz.:

$$
\begin{array}{ll}
a-mb=c \quad \ldots\ (1) & a=mb+c \quad \ldots\ (4) \\
b-nc=d \quad \ldots\ (2) & b=nc+d \quad \ldots\ (5) \\
c-pd=0 \quad \ldots\ (3) & c=pd \quad \ldots\ (6)
\end{array}
$$

$$
\begin{array}{l}
b)\ a\ (m \\
\underline{m b} \\
\quad c)\ b\ (n \\
\quad \underline{n c} \\
\qquad d)\ c\ (p \\
\qquad \underline{p d}
\end{array}
$$

where the equalities marked (4), (5), (6) are not deduced from those marked (1), (2), (3), but from the consideration that the dividend is always equal to the product of the divisor and quotient, increased by the remainder.

Now, by (6) it is obvious that d measures c, since $c=pd$; hence (Lemma 1) d measures nc, and it likewise measures itself; therefore (Lemma 2) d measures $nc+d$, which by (5) is equal to b; hence, again, d, measuring b and c, measures $mb+c$ by the Lemmas 1 and 2.

* A *lemma* is a preparatory proposition, to aid in the demonstration of the main proposition which follows it.

$\therefore d$ measures a, which is equal to $mb+c$ by (4).

Hence d measures both the polynomials a and b, and is consequently a common measure of these polynomials; but d is also the greatest common measure of a and b; for if d' is a greater common measure of a and b than d is, it is obvious that by (1) d' measures $a-mb$, or c; and d' measuring both b and c, it measures $b-nc$, or d by (2); hence d' measures d, which is absurd, since no quantity measures a quantity less than itself; therefore d is the greatest common measure. Q. E. D.*

30. If the greatest common measure of three quantities be required, find the greatest common measure of two of them, and then that of this measure and the remaining quantity will be the greatest common measure of all three.†

31. If the two polynomials be the terms of a fraction, as $\frac{a}{b}$, and d their greatest common measure, then we may put $a=da'$, and $b=db'$; hence $\frac{a}{b}=\frac{da'}{db'}=\frac{a'}{b'}$, and, since a', b' contain no common factor (28), by dividing both numerator and denominator of a fraction by their greatest common measure, the resulting fraction will be simplified to its utmost extent, and thus the proposed fraction will be reduced to its lowest terms.

---

* These letters stand for the Latin words quod erat demonstrandum, signifying which was to be demonstrated. Another mode of demonstrating the same is as follows: Let A and B represent the two given quanties, D their greatest common divisor, Q the quotient of A by B, and R the remainder. We shall first prove that the greatest common divisor of A and B is the same as the greatest common divisor of B and R. Represent the latter by D′.

$$A=BQ+R, \therefore \frac{A}{D}=\frac{BQ}{D}+\frac{R}{D}, \text{ and } \frac{A}{D'}=\frac{BQ}{D'}+\frac{R}{D'}.$$

A and B being divisible by D, R must be, because a whole number can not be equal to a whole number plus a fraction. Again, B and R being divisible by D′, A must be, for the sum of two whole numbers can not equal a fraction. Finally, D, a common divisor of B and R, can not be greater than their greatest common divisor D′; and D′, a *c.d.* of A and B, can not be greater than their *g.c.d.* D; *i. e.*, D can not be greater than D′, and D′ can not be greater than D.

Or thus: since

$$A=BQ+R,$$

the greatest common divisor D of A and B, must divide R. Represent the three quotients by A′, B′, and R′; then

$$A'=B'Q+R'.$$

B′ and R′ have no farther common factor, for if they had, it must by this equality divide A; then A′ and B′ would have still a common factor, and D, the greatest common divisor of A and B, would not contain all the common factors of these quantities, which is contrary to the definition. Since B′ and R′, which are the quotients of B and R by D, can have no farther common factor, it follows that the greatest common divisor of B and R is equal to D; then it is the same as that of the quantities A and B.

In pursuing the rule for finding the *g.c.d.*, we arrive at a remainder which exactly divides the preceding divisor, and which is, therefore, the *g.c.d.* of itself and this preceding divisor; also by the above demonstration of that divisor and its dividend, and so on up to the given quantities.

† For suppose we have the three quantities A, B, C; let D be the greatest common divisor of A and B, and D′ that of D and C. According to the definition, D is the product of the common factors of A and B, and D′ is that of the common factors of D and C; then D′ is the product of the common factors of the three quantities A, B, C; therefore D′ is their greatest common divisor.

EXAMPLES.

(1) What is the greatest common measure of $4x^2y^3z^4$ and $8x^4y^3z^2$?

Here 4 is the greatest common measure of 4 and 8, and $x^2y^3z^2$ is that of the literal parts; hence $4x^2y^3z^2$ is the greatest common measure required.

(2) Find the greatest common measure of $\dfrac{x^3+y^3}{x^2-y^2}$.

$$\begin{array}{l}
x^2-y^2)\ x^3+y^3\ (x \\
\qquad\quad \underline{x^3-xy^2} \\
\qquad\qquad\quad xy^2+y^3=y^2(x+y)\text{; rejecting the factor } y^2 \\
\qquad\qquad\quad x+y)\ \ x^2-y^2\ (x-y \\
\qquad\qquad\qquad\qquad \underline{x^2+xy} \\
\qquad\qquad\qquad\qquad\ -xy-y^2 \\
\qquad\qquad\qquad\qquad\ \underline{-xy-y^2.}
\end{array}$$

Hence $x+y$ is the greatest common measure sought, and

$$\frac{x^3+y^3}{x^2-y^2}=\frac{(x^3+y^3)\div(x+y)}{(x^2-y^2)\div(x+y)}=\frac{x^2-xy+y^2}{x-y}=\text{reduced fraction.}$$

(3) Required the greatest common measure of the two polynomials

$$6a^3-6a^2y+2ay^2-2y^3 \quad \ldots\ (a)$$
$$12a^2-15ay+3y^2 \quad \ldots\ (b).$$

Here
$$6a^3-6a^2y+2ay^2-2y^3=2(3a^3-3a^2y+ay^2-y^3)$$
$$12a^2-15ay+3y^2=3(4a^2-5ay+y^2);$$

And therefore, by suppressing the factors 2 and 3, which have no common measure, and which, not being common factors of the two given quantities, do not affect the common divisor, we have to find the greatest common measure of

$$3a^3-3a^2y+ay^2-y^3 \text{ and } 4a^2-5ay+y^2.$$

$$\begin{array}{l}
4a^2-5ay+y^2)\ 3a^3-3a^2y+ay^2-y^3 \\
\qquad\qquad\qquad\quad \underline{4} \\
\qquad\qquad\qquad\quad 12a^3-12a^2y+4ay^2-4y^3\ (3a \\
\qquad\qquad\qquad\quad \underline{12a^3-15a^2y+3ay^2} \\
\qquad\qquad\qquad\qquad\qquad 3a^2y+ay^2-4y^3 \\
\qquad\qquad\qquad\qquad\qquad \underline{4} \\
\qquad\qquad\qquad\qquad\qquad 12a^2y+4ay^2-16y^3\ (3y \\
\qquad\qquad\qquad\qquad\qquad \underline{12a^2y-15ay^2+3y^3} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 19ay^2-19y^3=19y^2\ (a-y)
\end{array}$$

Suppressing $19y^2$, by note 3, rule,

$$\begin{array}{l}
a-y)\ 4a^2-5ay+y^2\ (4a-y \\
\qquad\ \underline{4a^2-4ay} \\
\qquad\quad -ay+y^2 \\
\qquad\quad \underline{-ay+y^2.}
\end{array}$$

Hence $a-y$ is the greatest common measure of the polynomials a and b.

The factor 4 is introduced into the dividend in this example to render it divisible by the divisor. This can be done, because 4 is not a factor of every term of the divisor, and therefore not a factor of the divisor. The quantities employed, after introducing or suppressing factors, are different from the given, but as they have the same greatest common divisor, and as the object is to find this, the circumstance is immaterial.

(4) Required the greatest common measure of the terms of the fraction

$$\frac{a^6-a^2x^4}{a^6+a^5x-a^4x^2-a^3x^3}.$$

Here a^2 is a simple factor of the numerator, and a^3 is a factor of the denominator; hence a^2 is the greatest common measure of these simple factors, which must be reserved to be introduced into the greatest common measure of the other factors of the terms of the proposed fraction; viz.:

$$a^4-x^4 \text{ and } a^3+a^2x-ax^2-x^3.$$

$$\begin{array}{l} a^3+a^2x-ax^2-x^3)\ a^4-x^4 \qquad (a-x \\ \qquad a^4+a^3x-a^2x^2-ax^3 \\ \qquad -a^3x+a^2x^2+ax^3-x^4 \\ \qquad -a^3x-a^2x^2+ax^3+x^4 \\ \qquad\qquad 2a^2x^2-2x^4=2x^2(a^2-x^2);\ \text{rejecting } 2x^2 \\ \qquad\qquad a^2-x^2)\ a^3+a^2x-ax^2-x^3\ (a+x \\ \qquad\qquad\qquad a^3-ax^2 \\ \qquad\qquad\qquad\quad a^2x-x^3 \\ \qquad\qquad\qquad\quad a^2x-x^3 \end{array}$$

Therefore, restoring a^2, the greatest common measure, is $a^2(a^2-x^2)$.

$$\therefore \frac{a^6-a^2x^4}{a^6+a^5x-a^4x^2-a^3x^3}=\frac{(a^6-a^2x^4)\div a^2(a^2-x^2)}{(a^6+a^5x-a^4x^2-a^3x^3)\div a^2(a^2-x^2)}=\frac{a^2+x^2}{a^2+ax}.$$

ADDITIONAL EXAMPLES.

(1) Find the greatest common measure of $2a^2x^2$, $4x^2y^2$, and $6x^3y$.

(2) Find the greatest common measure of the two polynomials $a^3-a^2b+3ab^2-3b^3$, and $a^2-5ab+4b^2$.

(3) What is the greatest common measure of x^3-xy^2 and $x^2+2xy+y^2$?

(4) Find the greatest common measure of x^8-y^8 and $x^{13}-y^{13}$.

(5) Find the greatest common measure of the polynomials

$$(b-c)x^2-b(2b-c)x+b^3 \quad \ldots\ldots \quad (a)$$
$$(b+c)x^3-b(2b+c)x^2+b^3x \quad \ldots\ldots \quad (b).$$

(6) Find the greatest common measure of the polynomials

$$x^4-8x^3+21x^2-20x+4 \quad \ldots\ldots \quad (a)$$
$$2x^3-12x^2+21x-10 \quad \ldots\ldots \quad (b).$$

(7) $y^3-5y^2z-4yz^2+2z^3$ and $7y^2z+10yz^2+3z^3$.

(8) Also of $(x^4+a^2x^2+a^4)$ and $(x^4+ax^3-a^3x-a^4)$.

(9) Also of $(7a^2-23ab+6b^2)$ and $(5a^3-18ba^2+11ab^2-6b^3)$.

(10) Also of $(5a^5+10a^4b+5a^3b^2)$ and $(a^3b+2a^2b^2+2ab^3+b^4)$.

(11) Also of $(6a^5+15a^4b-4a^3c^2-10a^2bc^2)$ and $(9a^3b-27a^2bc-6abc^2+18bc^3)$.

(12) Also of $(a^{\alpha+\gamma}+a^{\gamma}b^{\beta}+a^{\alpha}b^{\delta}+b^{\beta+\delta})$ and $(a^{\alpha}m+a^{\alpha}n+b^{\beta}m+b^{\beta}n)$.

(13) Find the *g. c. d.* of the three quantities $a^3+3a^2b+3ab^2+b^3$, $a^2+2ab+b^2$, and a^2-b^2.

ANSWERS.

(1) $2x^2$.	(5) $x-b$.	(8) x^2+ax+a^2.	(11) $3a^2-2c^2$.
(2) $a-b$.	(6) $x-2$.	(9) $a-3b$.	(12) $a^{\alpha}+b^{\beta}$.
(3) $x+y$.	(7) $y+z$.	(10) $a+b$.	(13) $a+b$.
(4) $x-y$.			

A quantity is said to be independent of a letter when it does not contain this letter, and, therefore, does not depend upon it for its value.

Note.—In seeking for a common divisor, we find ourselves often working with polynomials different from the given, but always with such as have the same common measure with the given polynomials.

PROPOSITION.—A divisor of a polynomial, which is independent of the letter of arrangement of that polynomial, must divide separately each of the multipliers of the different powers of that letter.

DEMONSTRATION.—Let $Ax^m+Bx^{m-1}+Cx^{m-2}$, &c., be the polynomial, and D the divisor. The quotient must contain every power of the letter of arrangement that the dividend does, since the quotient, multiplied by the divisor, must produce the dividend, and the letter of arrangement is not contained in the divisor. The quotient must, therefore, be of the form $A'x^m+B'x^{m-1}+C'x^{m-2}$, &c., multiplying which by the divisor gives $DA'x^m+DB'x^{m-1}+DC'x^{m-2}$, &c., the original dividend, the multiplier of each power of x in which is evidently divisible by D. Q. E. D.

N.B.—A′ is a different quantity from A, B′ from B, &c.

EXAMPLES.

(1) Find a common divisor, independent of the letter a, of the two quantities
$b\,a^2-ca^2+b^2a-c^2a+b^2-2bc+c^2$ and
$b^3a^3-3b^2ca^3+3bc^2a^3-c^3a^3+b^4a^2-c^4a^2+b^3a-c^3a+b^3-3b^2c+3bc^2-c^3$.
Collecting together in the first of these two quantities the multipliers of a^2 and a; and observing that $b^2-2bc+c^2$ is the square of $b-c$, we have

$$(b-c)a^2+(b^2-c^2)a+(b-c)^2,$$

and from the second by a similar process,

$$(b-c)^3a^3+(b^4-c^4)a^2+(b^3-c^3)a+(b-c)^3.$$

The multipliers of the different powers of a in the two quantities are, therefore, $b-c$, b^2-c^2, $(b-c)^2$, $(b-c)^3$, b^4-c^4, and b^3-c^3. The only number which will divide them all is their common divisor $b-c$, which is, therefore, the answer required.

(2) Find the greatest common divisor of

$$(b-c)a^2-2b\,(b-c)a+b^2(b-c) \text{ and}$$
$$(b^2-c^2)a^2-\ b^2(b^2-c^2).$$

Here the common divisor, independent of a, is $b-c$; suppressing which, we have left the two quantities

$$a^2-2ba+b^2 \text{ and}$$
$$(b+c)(a^2-b^2).$$

Suppressing the factor $(b+c)$ not common to both, we shall find the common divisor of these last two quantities to be $a-b$, and the greatest common divisor of the two original quantities is

$$(b-c)\ (a-b) \text{ or } ab-ac-b^2+bc.$$

The success of the process for finding a greatest common divisor depends upon the fact that the quantities being arranged according to the powers of some letter, each division leads to a remainder of a degree inferior to the divisor. When the polynomials contain many terms of the same degree, a precaution is necessary, without which this reduction would not always obtain, which consists in uniting all these terms under a single multiplier

Let there be the two polynomials:

$$A=x^3+yx^2+x^2-y^2x+2yx-y^3+y^2$$
$$B=yx^2+x^2+y^2x+yx+x+y.$$

I write them thus:

$$A=x^3+(y+1)x^2-(y^2-2y)x-y^3+y^2$$
$$B=\quad(y+1)x^2+(y^2+y+1)x+y.$$

The first term, x^3, not being divisible by $(y+1)x^2$, on account of the factor $y+1$, I know (Prop. above), that if a quantity is arranged like the preceding, every divisor of this quantity, independent of x, must divide separately the multiplier of each power of x. From this it follows that $y+1$ has no common factor with B, because, if it had, this factor would be found in y^2+y+1 and in y; but it is evident that y has no factor common with $y+1$.

We can then multiply A by $y+1$ without affecting the common divisor sought; and as it would be necessary to multiply again by $y+1$, we multiply at once by $(y+1)^2$, or y^2+2y+1.* In this manner we arrive at the remainder

$$R=(-y^4-y^3+y^2)x-y^5-y^4+y^3.$$

Before passing to the second division, it is necessary to suppress in R the factors common to the multipliers of the powers of x. But the two parts of R are evidently divisible by $-y^4-y^3+y^2$, and after this simplification there remains $x+y$. We can then take $x+y$ for a divisor, and as the division is effected exactly, it follows that the common divisor sought is $x+y$.

The process is not always so easy. To develop the general method to be pursued in such cases, let us consider the polynomials A and B, which contain two letters, x and y. Take first the greatest monomial common divisor of the terms of A; let a be this divisor, and A′ the quotient of A by a: we shall have $A=aA'$. Arrange A according to the powers of x, taking care to collect all the terms containing a same power of this letter, and let us suppose, for example, that we have

$$A'=Lx^2+Mx+N.$$

All the factors of A′, independent of x, must be factors of the quantities L, M, N, which multiply the different powers of x. These quantities containing only the single letter y, it will be easy to find their greatest common divisor; let us name this divisor a', and the quotient of A′ by a', A″; we shall have $A'=a'A''$, and, consequently,

$$A=aa'A''.$$

a will be the product of the monomial factors of A, a' the product of the polynomial factors which do not contain x, and A″ the product of the factors which contain x.

Let us effect the same decomposition of the polynomial B, and let

$$B=\beta\beta'B''.$$

Then I determine the greatest common divisor of the monomials a and β, as well as that of the polynomials a' and β', which contain only the letter y; and if I can also find that of the polynomials A″ and B″, which contain y and x, I shall have three quantities, the product of which will be the greatest common divisor of A and B.

But I say that we can find the greatest common divisor of the quantities A″ and B″, in subjecting them to the same calculus as the preceding examples. It is clear, indeed, that, these quantities having no longer either monomial factors or polynomial independent of x, it will be proper to multiply the partial dividends of the first division by the polynomial which is placed before the highest power of x in the divisor, and that we shall thus arrive at a remainder of a degree less in x than the divisor. It will be easy to take from this remainder all the monomial factors which it contains, as well as the polynomial factors independent of x, and then proceed to a second division, taking for a divisor this remainder simplified. We operate as in the first; then we pass to the third, and continuing always in this manner, we are sure of arriving finally at a remainder zero, or independent of x.

In the first case the quantities A″ and B″ have, for greatest common divisor, the divisor of the last division.

We have thus seen that the finding of a common divisor, when the polynomials contain two letters, depends upon finding it when they contain one; so the case where they contain three depends upon that where they contain two, and so on, whatever be the number of letters.

There is, therefore, no case in which the common divisor can not be found by the above rules.

THE LEAST COMMON MULTIPLE.

32. We have already defined a *multiple* of a quantity to be any quantity that contains it exactly; and a *common multiple* of two or more quantities is a quantity that contains each of them exactly.

---

* Let N be the dividend, D the divisor, c the coefficient of the first term of the divisor. Multiplying by the square of this coefficient, the dividend becomes c^2N. The first term of the quotient will contain the first power of c. Multiplying the whole divisor by this term of the quotient, every term of the product will contain the first power of c, and the whole product may be represented by cP. Subtracting this from the dividend, the remainder is c^2N-cP, every term of which contains c, and, therefore, its first term is ready for division without multiplying again by c.

The *least common multiple*, of two or more quantities, is, therefore, the least quantity that contains each of them exactly.

N. B. The least common multiple must, from its nature, contain all the distinct factors in either of the quantities.

33. *To find the least common multiple of two quantities.*

(1) Divide the product of the two proposed quantities by their greatest common measure, and the quotient is the least common multiple of these quantities; or divide one of the quantities by their greatest common measure, and multiply the quotient by the other.

Let a and b be two quantities, d their greatest common measure, and m their least common multiple; then let

$$a=hd, \text{ and } b=kd;$$

and since d is the greatest common measure, h and k can have no common factor, and hence their least common multiple is hk; therefore, hkd is the least common multiple of hd and kd; hence,

$$m=hkd=\frac{hkd^2}{d}=\frac{hd\times kd}{d}=\frac{a\times b}{d}=\frac{ab}{d} \qquad \text{Q. E. D.}$$

(2) Also, the least common multiple is composed of the highest powers of all the factors which enter into the given quantities.*

EXAMPLES.

(1) Find the least common multiple of $2a^2x$ and $8a^3x^3$.

Here $m=\frac{ab}{d}=\frac{2a^2x\times 8a^3x^3}{2a^2x}=8a^3x^3=$ least common multiple;

or, by (2), the two quantities being $2a^2x$ and $2^3a^3x^3$, $2^3a^3x^3$ is the *l. c. m.*; because 2^3 is the highest power of 2, a^3 the highest power of a, and x^3 the highest power of x, in either of the given quantities.

(2) Find the least common multiple of $4x^2(x^2-y^2)$ and $12x^3(x^3-y^3)$.

Here $d=4x^2(x-y)$, and, therefore, we have

$$m=\frac{ab}{d}=\frac{4x^2(x^2-y^2)\times 12x^3(x^3-y^3)}{4x^2(x-y)}=12x^3(x+y)\,(x^3-y^3);$$

or, $\qquad m=12x^7+12x^6y-12x^4y^3-12x^3y^4;$

or, the two given quantities being $2^2x^2(x+y)(x-y)$ and $2^2.3x^3(x-y)(x^2+xy+y^2)$, the *l. c. m.* is $2^2.3x^3(x+y)(x-y)(x^2+xy+y^2)$.

(3) Find the least common multiple of $x^2+2xy+y^2$ and x^3-xy^2.

Here $d=x+y$, and, therefore, we get

$$m=\frac{a}{d}.b=\frac{x^2+2xy+y^2}{x+y}.(x^3-xy^2)$$
$$=(x+y)\,(x^3-xy^2)$$
$$=x(x+y)\,(x^2-y^2)=\text{ least common multiple}$$

or, the two given quantities being $(x+y)^2$ and $x(x+y)\,(x-y)$, the *l. c. m.* is $x(x+y)^2(x-y)$.

(4) What is the least common multiple of $x^4-5x^3+9x^2-7x+2$, and x^4-6x^2+8x-3?

By the process for finding the greatest common measure, we find

$$d=x^3-3x^2+3x-1$$

$$\therefore m=\frac{x^4-5x^3+9x^2-7x+2}{x^3-3x^2+3x-1}(x^4-6x^2+8x-3)$$
$$=(x-2)\,(x^4-6x^2+8x-3)$$
$$=x^5-2x^4-6x^3+20x^2-19x+6, \text{ the least common multiple.}$$

* The ordinary arithmetic method depends on this principle.

(5) Find the least common multiple of $a^2-2ab+b^2$ and a^4-b^4.
(6) Find the least common multiple of a^2-b^2 and a^3+b^3.
(7) Find the least common multiple of x^2-y^2 and x^3-y^3.
(8) Find the least common multiple of y^2-8y+7 and y^2+7y-8.

ANSWERS.

(5) $(a-b)(a^4-b^4)$.
(6) $(a-b)(a^3+b^3)$.
(7) $(x+y)(x^3-y^3)$.
(8) $y^3-57y+56$.

34. *Every common multiple of two quantities,* a *and* b, *is a multiple of* m, *their least common multiple.*

For let m' be a common multiple of a and b, then, because m' is greater than m, if we suppose that m' is not a multiple of m, we have, as in the annexed scheme,

$$m'=hm+k \quad . \; . \; . \; (1)$$
$$m'-hm=k \quad . \; . \; . \; (2)$$

$m)\; m'\; (h$
$\underline{hm}$
$k=$ remainder.

Now the remainder k is always less than m the divisor; hence, since a and b measure m and m', it is evident that a and b measure $m'-hm$, or, by (2), k; therefore, k is a common multiple of a and b, and it has been proved to be less than m, the least common multiple, which is absurd; hence the supposition that m' is not a multiple of m is false, or m' is a multiple of m.

35. *To find the least common multiple of three or more quantities*

Let a, b, c, d, &c., be the proposed quantities;

find m, the least common multiple of a and b;
find m', ... c and m;
find m'', ... d and m'; &c.

The last, say m'', is evidently a multiple of a, b, c, d, &c.

Then, since every multiple of a and b is a multiple of m, their least common multiple (34), the quantity sought, x, is a multiple of m; but x also is by hypothesis a multiple of c; therefore, x is a multiple of both c and m, and, therefore, it is (34) a multiple of m'; but x is a multiple of d and m', and, therefore, of m''; hence x can not be less than m'', and, therefore, m'' is the least common multiple.

EXAMPLES.

(1) Find the least common multiple of $2a^2$, $4a^3b^2$, and $6ab^3$.

Here, taking $2a^2$ and $4a^3b^2$, we find $d=2a^2$, and, therefore,

$$m=\frac{ab}{d}=\frac{2a^2\times 4a^3b^2}{2a^2}=4a^3b^2.$$

Again, taking m, or $4a^3b^2$, and $6ab^3$, we find $d=2ab^2$; hence

$$m'=\frac{cm}{d}=\frac{6ab^3\times 4a^3b^2}{2ab^2}=12a^3b^3=\text{answer required.}$$

Or, the three given quantities being $2a^2$, $2^2a^3b^2$, and $2 . 3ab^3$, the *l. c. m.*, by 33 . (2), is $2^2 . 3a^3b^3$.

(2) Find the least common multiple of $a-x$, a^2-x^2, and a^3-x^3.

Taking $a-x$ and a^2-x^2, we have $d=a-x$; and hence

$$m=\frac{ab}{d}=\frac{a-x}{a-x}\times(a^2-x^2)=a^2-x^2.$$

Again, taking a^2-x^2 and a^3-x^3, we find $d=a-x$; hence

$$m'=\frac{cm}{d}=\frac{(a^3-x^3)(a^2-x^2)}{a-x}=(a+x)(a^3-x^3)=\text{answer sought.}$$

Or, the three given quantities being $(a-x)$, $(a-x)(a+x)$, and $(a-x)(a^2+ax+x^2)$, the least common multiple is $(a-x)(a+x)(a^2+ax+x^2)$.

(3) Find the least common multiple of $15a^2b^2$, $12ab^3$, and $6a^3b$.

(4) Find the least common multiple of $6a^2x^2(a-x)$, $8x^3(a^2-x^2)$, and $12(a-x)^2$.

(5) Find the least common multiple of $x^3-x^2y-xy^2+y^3$, $x^3-x^2y+xy^2-y^3$, and x^4-y^4.

(6) Find the least common multiple of $(a+b)^2$, (a^2-b^2), $(a-b)^2$, and $a^3+3a^2b+3ab^2+b^3$.

(7) Find the least common multiple of 45, 50, and 75, or $3^2.5$, 2.5^2, and 3.5^2.

ANSWERS.

(3) $60a^3b^3$.
(4) $24a^2x^3(a-x)(a^2-x^2)$.
(5) $x^5-xy^4-x^4y+y^5$.
(6) $(a+b)(a^2-b^2)^2$.
(7) $3^2.2.5^2=450$.

OF ALGEBRAIC FRACTIONS.

36. Algebraic fractions differ in no respect from arithmetical fractions, and all the rules for the latter apply equally to the former. We shall, therefore, merely repeat the rules, adding a few examples of the application of each. It may be proper to remind the reader that all operations with regard to fractions are founded upon the three following principles:

1. *In order to multiply a fraction by any number, we must multiply the numerator, or divide the denominator of the fraction by that number.*

2. *In order to divide a fraction by any number, we must divide the numerator, or multiply the denominator of the fraction by that number.*

3. *The value of a fraction is not changed, if we multiply or divide both the numerator and denominator by the same number.*—See (17, Note).

REDUCTION OF FRACTIONS.

I. *To reduce a fraction to its lowest terms.*

37. Rule.—*Divide both numerator and denominator by their greatest common measure, and the result will be the fraction in its lowest terms.*

When the numerator and denominator are, one or both of them, monomials, their greatest common factor is immediately detected by inspection; thus

$$\frac{a^2bc}{5a^2b^2}=\frac{a^2b\times c}{a^2b\times 5b}=\frac{c}{5b} \text{ in its lowest terms.}$$

So, also,

$$\frac{ax^2}{ax+x^2}=\frac{x\times ax}{x(a+x)}=\frac{ax}{a+x} \text{ in its lowest terms.}$$

If, however, both numerator and denominator are polynomials, we must have recourse to the method of finding the greatest common measure of two algebraic quantities, developed in a former article. Thus, let it be required to reduce the following fraction to its lowest terms:

$$\frac{6a^3-6a^2y+2ay^2-2y^3}{12a^2-15ay+3y^2}.$$

The greatest common measure of the two terms of this fraction was found at page 37 to be $a-y$; therefore, dividing both numerator and denominator by this quantity, we obtain as our result the fraction in its lowest terms; or,

$$\frac{6a^2+2y^2}{12a-3y}.$$

In like manner, taking the fraction $\frac{4a^4-4a^2b^2+4ab^3-b^4}{6a^4+4a^3b-9a^2b^2-3ab^3+2b^4}$, the greatest common measure of the two terms is found to be $2a^2+2ab-b^2$; and, dividing both numerator and denominator by this quantity, the reduced fraction is

$$\frac{2a^2-2ab+b^2}{3a^2-ab-2b^2}.$$

EXAMPLES FOR PRACTICE.

(1) Reduce $\frac{2x^3-16x-6}{3x^3-24x-9}$ to its lowest terms.

(2) Reduce $\frac{48x^3+36x^2-15}{24x^3-21x^2+18x-6}$ to its lowest terms.

(3) Reduce $\frac{20x^4+x^2-1}{25x^4+5x^3-x-1}$ to its lowest terms.

(4) Reduce $\frac{3m^2n-m^2p-6mn^2+2mnp}{12mn-15n^2-4mp+5np}$ to its lowest terms.

Ans. $\frac{m^2-2mn}{4m-5n}$.

(5) Reduce $\frac{4a^3cx-4a^3dx+24a^2bcx-24a^2bdx+36ab^2cx-36ab^2dx}{7abcx^3-7abdx^3+7ac^2x^3-7acdx^3-21b^2dx^3+21b^2cx^3+21bc^2x^3-21bcdx^3}$

to its lowest terms. Ans. $\frac{4a(a+3b)}{7x^2(b+c)}$.

38. It frequently happens, however, that when the polynomials which form the numerator and denominator of a fraction which can be decomposed are not very complicated, we are enabled by a little practice to detect the common factor and effect the reduction without performing the operation of finding the greatest common measure, which is generally a tedious process. The results to which we called the attention of the reader at the end of algebraic division (see page 30) will be found particularly useful in simplifications of this nature.

Thus, for example:

(6) $\frac{3x^2y+3xy^2}{3x^2+6xy+3y^2}=\frac{3xy(x+y)}{3(x+y)^2}=\frac{3xy(x+y)}{3(x+y)(x+y)}=\frac{xy}{x+y}$.

(7) $\frac{a^2-b^2}{a^2-2ab+b^2}=\frac{(a-b)(a+b)}{(a-b)^2}=\frac{a+b}{a-b}$.

(8) $\frac{5a^3+10a^2b+5ab^2}{8a^3+8a^2b}=\frac{5a(a^2+2ab+b^2)}{8a^2(a+b)}=\frac{5a(a+b)^2}{8a^2(a+b)}=\frac{5(a+b)}{8a}$.

(9) $\frac{a^3-x^3}{a^2-2ax+x^2}=\frac{(a^2+ax+x^2)(a-x)}{(a-x)^2}=\frac{a^2+ax+x^2}{a-x}$.

(10) $\frac{ac+bd+ad+bc}{af+2bx+2ax+bf}=\frac{(a+b)c+(a+b)d}{(a+b)f+2x(a+b)}=\frac{(c+d)(a+b)}{(f+2x)(a+b)}=\frac{c+d}{f+2x}$.

(11) $\dfrac{6ac+10bc+9ad+15bd}{6c^2+9cd-2c-3d}=\dfrac{3a(2c+3d)+5b(2c+3d)}{3c(2c+3d)-(2c+3d)}=\dfrac{(3a+5b)(2c+3d)}{(3c-1)(2c+3d)}$
$=\dfrac{3a+5b}{3c-1}.$

(12) $\dfrac{ax^m-bx^{m+1}}{a^2bx-b^3x^3}=\dfrac{x^m(a-bx)}{bx(a^2-b^2x^2)}=\dfrac{x^m(a-bx)}{bx(a+bx)(a-bx)}=\dfrac{x^{m-1}}{b(a+bx)}.$

(13) $\dfrac{a^4-b^4}{a^3+ab^2}=\dfrac{a^2-b^2}{a}.$

(14) $\dfrac{2xy+3y^2+2x^2+3xy}{8cx+12cy-10dx-15dy}=\dfrac{y+x}{4c-5d}.$

II. *To reduce a mixed quantity to an improper fraction.*

39. Rule.—*Multiply the integral part by the denominator of the fraction, and to the product add the numerator with its proper sign; then the result placed over the denominator will give the improper fraction required.* Thus,

(1) $\dfrac{a}{b}+1=\dfrac{a+b}{b}.$

(2) $1+\dfrac{a^2-x^2}{a^2+x^2}=\dfrac{a^2+x^2+a^2-x^2}{a^2+x^2}=\dfrac{2a^2}{a^2+x^2}.$

(3) $ab+cd+\dfrac{abc-c^2d-2cd^2}{c+2d}=\dfrac{abc+c^2d+2abd+2cd^2+abc-c^2d-2cd^2}{c+2d}$

$$=\frac{2abc+2abd}{c+2d}$$

$$=\frac{2ab(c+d)}{c+2d}$$

(4) $1+\dfrac{b^2+c^2-a^2}{2bc}=\dfrac{2bc+b^2+c^2-a^2}{2bc}=\dfrac{(b+c)^2-a^2}{2bc}.$

40. It is to be remarked that when a fraction has the sign —, it signifies that the whole fraction is to be subtracted; the negative sign, therefore, as will be shown hereafter, applies to the numerator alone; when the numerator is a polynomial, the negative sign extends to all its terms; the bar which separates the numerator from the denominator is to be regarded as a vinculum, and if it have the negative sign before it, when removed, all the signs of the numerator must be changed.

(5) $1-\dfrac{b}{a}=\dfrac{a-b}{a}.$

(6) $c-\dfrac{ef}{d}=\dfrac{cd-ef}{d}.$

(7) $1-\dfrac{a^2-2ab+b^2}{a^2+b^2}=\dfrac{a^2+b^2-(a^2-2ab+b^2)}{a^2+b^2}=\dfrac{2ab}{a^2+b^2}.$

(8) $1-\dfrac{b^2+c^2-a^2}{2bc}=\dfrac{2bc-(b^2+c^2-a^2)}{2bc}$

$$=\frac{a^2-(b^2-2bc+c^2)}{2bc}$$

$$=\frac{a^2-(b-c)^2}{2bc}.$$

(9) $$x^2+2xy+y^2-\frac{x^3-3x^2y+3xy^2-y^3}{x+y}$$
$$=\frac{x^3+3x^2y+3xy^2+y^3-(x^3-3x^2y+3xy^2-y^3)}{x+y}$$
$$=\frac{6x^2y+2y^3}{x+y}$$
$$=\frac{2y(3x^2+y^2)}{x+y}.$$

(10) $$mn-pq-\frac{2mn^2-2pqn}{m+n}=\frac{m^2n-mpq+mn^2-npq-(2mn^2-2pqn)}{m+n}$$
$$=\frac{m^2n-mpq-mn^2+pqn}{m+n}$$
$$=\frac{mn(m-n)-pq(m-n)}{m+n}$$
$$=\frac{(mn-pq)(m-n)}{m+n}.*$$

III. *To reverse this process, or to reduce an improper fraction to a mixed quantity.*

Rule.—*Divide the numerator by the denominator; the quotient obtained as far as practicable, will be the entire part, and the remainder, set over the denominator, will be the fractional part. Then the two, joined together with the proper sign, will form the mixed quantity required.* Thus,

(11) $$\frac{ay+b}{y}=a+\frac{b}{y}.$$

(12) $$\frac{a^2+x^2}{a-x}=a+x+\frac{2x^2}{a-x}.$$

(13) $$\frac{20x^3-10x+4}{5x}=4x^2-2+\frac{4}{5x}.$$

(14) $$\frac{p^2+2pq+q^2-r-s}{p+q}=p+q-\frac{r+s}{p+q}.$$

(15) $$\frac{m^2(m^4-n^4)+3m^3-3mn^2}{m^2(m^2-n^2)}=m^2+n^2+\frac{3}{m}.$$

IV. *To reduce fractions to others equivalent, and having a common denominator.*

41. Rule.—*Multiply each of the numerators, separately, into all the denominators, except its own, for the new numerators, and all the denominators together for a common denominator.*†

Thus, reduce $\frac{a}{b}$ and $\frac{c}{d}$ to equivalent fractions having a common denominator

$a\times d$ is the new numerator of the first,
$c\times b$ is the new numerator of the second,
$b\times d$ is the common denominator;

Therefore, the fractions required are $\frac{ad}{bd}$ and $\frac{bc}{bd}$.

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* The rationale of the above examples is given in the note on the next page.

† The numerator and denominator of each fraction will thus be multiplied by the same number, viz., the product of the other denominators, and, consequently, its value will be unchanged.

Reduce $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}, \frac{k}{l}, \frac{m}{n}$, to a common denominator.

$\frac{adfhln}{bdfhln}, \frac{cbfhln}{bdfhln}, \frac{ebdhln}{bdfhln}, \frac{gbdfln}{bdfhln}, \frac{kbdfhn}{bdfhln}, \frac{mbdfhl}{bdfhln}$, are the fractions required.

Reduce $\frac{1+x}{1-x}, \frac{1+x^2}{1-x^2}, \frac{1+x^3}{1-x^3}$, to a common denominator.

$\frac{(1+x)(1-x^2)(1-x^3)}{(1-x)(1-x^2)(1-x^3)}, \frac{(1+x^2)(1-x)(1-x^3)}{(1-x)(1-x^2)(1-x^3)}, \frac{(1+x^3)(1-x)(1-x^2)}{(1-x)(1-x^2)(1-x^3)}$, are the fractions required.

ADDITION OF FRACTIONS.

42. Rule.—*Reduce the fractions to a common denominator, add the numerators together, and subscribe the common denominator.* Thus,

(1) $\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}$.

(2) $$\frac{a}{b}+\frac{m}{n}+\frac{p}{q}+\frac{x}{y}=\frac{anqy}{bnqy}+\frac{mbqy}{bnqy}+\frac{pbny}{bnqy}+\frac{xbnq}{bnqy}$$
$$=\frac{anqy+mbqy+pbny+xbnq}{bnqy}.$$

(3) $$\frac{a}{bx}+\frac{c}{dx^2}+\frac{e}{fx^3}=\frac{adfx^5}{bdfx^6}+\frac{cbfx^4}{bdfx^6}+\frac{ebdx^3}{bdfx^6}$$
$$=\frac{adfx^5+bcfx^4+bdex^3}{bdfx^6}.$$

(4) $$\frac{1+x^2}{1-x^2}+\frac{1-x^2}{1+x^2}=\frac{(1+x^2)^2}{(1-x^2)(1+x^2)}+\frac{(1-x^2)^2}{(1-x^2)(1+x^2)}$$
$$=\frac{(1+x^2)^2+(1-x^2)^2}{(1-x^2)(1+x^2)}$$
$$=\frac{2(1+x^4)}{1-x^4}.$$

(5) $$\frac{1}{1+x}+\frac{1}{1-x}=\frac{1-x}{(1+x)(1-x)}+\frac{1+x}{(1+x)(1-x)}$$
$$=\frac{1-x+1+x}{(1+x)(1-x)}$$
$$=\frac{2}{1-x^2}.$$

SUBTRACTION OF FRACTIONS.

43. Rule.—*Reduce the fractions to a common denominator, subtract the numerator or the sum of the numerators of the fractions to be subtracted, from the numerator or the sum of the numerators of the others, and subscribe the common denominator.**

(1) $\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd}$.

(2) $$\frac{a}{b}+\frac{m}{n}-\left(\frac{p}{q}+\frac{x}{y}\right)=\frac{anqy}{bnqy}+\frac{mbqy}{bnqy}-\frac{pbny}{bnqy}-\frac{xbnq}{bnqy}$$
$$=\frac{anqy+mbqy-pbny-xbnq}{bnqy}.$$

* The rules for addition and subtraction of fractions follow from the general principle that quantities to be added or subtracted must be of the same denomination.

(3) $$\frac{a}{bx}+\frac{c}{dx^2}-\frac{e}{fx^3}-\frac{g}{hx^4}=\frac{adfhx^9}{bdfhx^{10}}+\frac{bcfhx^8}{bdfhx^{10}}-\frac{bedhx^7}{bdfhx^{10}}-\frac{bdfgx^6}{bdfhx^{10}}$$
$$=\frac{adfhx^3+bcfhx^2-bedhx-bdfg}{bdfhx^4}.$$

(4) $$\frac{a+b}{a-b}-\frac{a-b}{a+b}=\frac{(a+b)^2-(a-b)^2}{(a+b)(a-b)}$$
$$=\frac{4ab}{a^2-b^2}.$$

(5) $$\frac{1+x^2}{1-x^2}-\frac{1-x^2}{1+x^2}=\frac{(1+x^2)^2}{(1-x^2)(1+x^2)}-\frac{(1-x^2)^2}{(1-x^2)(1+x^2)}$$
$$=\frac{(1+x^2)^2-(1-x^2)^2}{(1-x^2)(1+x^2)}$$
$$=\frac{4x^2}{1-x^4}.$$

(6) $$\frac{1}{a^{m-n}}-\frac{1}{a^m}=\frac{a^n-1}{a^m}$$

(7) $$\frac{a^2}{l^{m-r-1}}-\frac{b^3d}{l^{m+1}}=\frac{a^2l^{r+2}-b^3d}{l^{m+1}}.$$

44. When the denominators of the fractions which it is required to reduce have a common multiple less than their continued product, the result will frequently be much simplified by finding this least common multiple, and then reducing the fractions to their least common denominator by multiplying the numerator and denominator of each fraction by the quotient of the least common multiple, divided by the denominator of that fraction.

Thus, if we are required to reduce the following fractions:

$$\frac{a-3x}{4}+\frac{3a-5x}{5}+\frac{3a-5x}{20}.$$

The least common multiple of 4 and 5 is 20, the denominator of the third fraction; therefore the fractions, when reduced to their least common denominator, are

$$\frac{5a-15x}{20}+\frac{12a-20x}{20}+\frac{3a-5x}{20}=\frac{5a-15x+12a-20x+3a-5x}{20}$$
$$=\frac{20a-40x}{20}$$
$$=a-2x.$$

So, also, in

$$x+\frac{27-9x}{4}-\frac{5x+2}{6}-\frac{61}{12}+\frac{2x+5}{3}+\frac{29+4x}{12}-\frac{5-37x}{12},$$

the least common multiple of 3, 4, 6 is 12, which will be the least common denominator, and the above fractions become

$$\frac{12x}{12}+\frac{81-27x}{12}-\frac{10x+4}{12}-\frac{61}{12}+\frac{8x+20}{12}+\frac{29+4x}{12}-\frac{5-37x}{12}.$$

Or,

$$\frac{12x+81-27x-10x-4-61+8x+20+29+4x-5+37x}{12}=\frac{24x+60}{12}$$
$$=2x+5.$$

MULTIPLICATION OF FRACTIONS.

45. Rule.—*Multiply all the numerators together for a new numerator, and all the denominators together for a new denominator.* Thus,*

(1) $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$.

(2) $\frac{a}{b}\times\frac{m}{n}\times\frac{p}{q}\times\frac{x}{y}=\frac{ampx}{bnqy}$.

(3) $\frac{a+b}{c+d}\times\frac{e-f}{g-h}\times\frac{k+l}{m-n}\times\frac{p-q}{r+s}=\frac{(a+b)(e-f)(k+l)(p-q)}{(c+d)(g-h)(m-n)(r+s)}$.

(4) $\frac{a}{bx}\times\frac{b}{cx^2}\times\frac{c}{dx^3}\times\frac{d}{ex^4}\times\frac{e}{fx^5}=\frac{abcde}{bcdefx^{15}}=\frac{a}{fx^{15}}$.

DIVISION OF FRACTIONS.

46. Rule.—*Invert the divisor and proceed as in Multiplication.*†

(1) $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$. Proof, $\frac{c}{d}\times\frac{ad}{bc}=\frac{acd}{bcd}=\frac{a}{b}$.

(2) $\frac{a+b}{c+d}\div\frac{e-f}{g-h}=\frac{a+b}{c+d}\times\frac{g-h}{e-f}=\frac{(a+b)(g-h)}{(c+d)(e-f)}$.

(3) $\frac{1+x^2}{1-x^2}\div\frac{1-x^2}{1+x^2}=\frac{1+x^2}{1-x^2}\times\frac{1+x^2}{1-x^2}=\frac{(1+x^2)^2}{(1-x^2)^2}$.

(4) $$\frac{x^4-b^4}{x^2-2bx+b^2}\div\frac{x^2+bx}{x-b}=\frac{x^4-b^4}{x^2-2bx+b^2}\times\frac{x-b}{x^2+bx}$$
$$=\frac{(x^4-b^4)(x-b)}{(x^2-2bx+b^2)(x^2+bx)}$$
$$=\frac{(x^2-b^2)(x^2+b^2)(x-b)}{(x-b)^2\,.\,x\,.\,(x+b)}$$
$$=\frac{(x+b)(x-b)(x^2+b^2)(x-b)}{x(x-b)(x-b)(x+b)}$$
$$=\frac{x^2+b^2}{x}.$$

47. *Miscellaneous Examples in the operations performed in Algebraic Fractions.*

(1) $\frac{3a}{4b}+\frac{5f}{8e}-\frac{x}{7y}=\frac{42aey+35bfy-8bex}{56bey}$

(2) $\frac{2a}{3bc}+\frac{5df}{8b^2c}-\frac{deg}{6b^2c^2}=\frac{16abc+15cdf-4deg}{24b^2c^2}$.

(3) $e-f-\frac{g^3}{2ef}+\frac{f^m}{3eg}=\frac{6efg(e-f)-3g^4+2f^{m+1}}{6efg}$.

(4) $\frac{a}{x^n}-\frac{c}{x^{n-1}}+\frac{d}{x^{n-r-s}}=\frac{a-cx+dx^{r+s}}{x^n}$.

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* To multiply a quantity by the fraction $\frac{2}{3}$, for instance, is to take it as many times as is expressed by this multiplier, that is, two thirds of a time, or to take two thirds of it, which is done by dividing it by 3, and multiplying by 2. If the multiplicand be a fraction, this is done, as has been before shown (17, Note), by multiplying its numerator by 2, and its denominator by 3, which accords with the rule above given.

† This rule depends upon the principle that the divisor, multiplied by the quotient, must produce the dividend.

(5) $c+2ab-3ac-\frac{b^2c-5ab^2c+a^3}{b^2-bc}=\frac{2ab^3-bc^2+3abc^2-a^3}{b^2-bc}$.

(6) $\frac{a+b}{2}+\frac{a-b}{2}=a$.

(7) $\frac{a+b}{2}-\frac{a-b}{2}=b$.

(8) $\frac{13a-5b}{4}-\frac{7a-2b}{6}-\frac{3a}{5}=\frac{89a-55b}{60}$.

(9) $\frac{3a-4b}{7}-\frac{2a-b-c}{3}+\frac{15a-4c}{12}=\frac{85a-20b}{84}$.

(10) $\frac{a}{b}+\frac{a-3b}{cd}+\frac{a^2-b^2-ab}{bcd}=\frac{acd-4b^2+a^2}{bcd}$.

(11) $\frac{a^3}{(a+b)^3}-\frac{ab}{(a+b)^2}+\frac{b}{a+b}=\frac{a^3+ab^2+b^3}{(a+b)^3}$.

(12) $\frac{3}{4(1-x)^2}+\frac{3}{8(1-x)}+\frac{1}{8(1+x)}-\frac{1-x}{4(1+x^2)}=\frac{1+x+x^2}{1-x-x^4+x^5}$.

(13) $\frac{a^2+b^2}{a^2-b^2}\times\frac{a-b}{a+b}=\frac{a^2+b^2}{a^2+2ab+b^2}$.

(14) $\frac{x^2-9x+20}{x^2-6x}\times\frac{x^2-13x+42}{x^2-5x}=\frac{x^2-11x+28}{x^2}$.*

(15) $\frac{x^2+3x+2}{x^2+2x+1}\times\frac{x^2+5x+4}{x^2+7x+12}=\frac{x+2}{x+3}$.*

(16) $\frac{\frac{a}{b}+\frac{c}{d}}{\frac{e}{f}+\frac{g}{h}}=\frac{(ad+bc)fh}{(eh+fg)bd}$.

(17) $\frac{\frac{a}{a+b}+\frac{b}{a-b}}{\frac{a}{a-b}-\frac{b}{a+b}}=1$.

(18) $\frac{\frac{a+x}{a-x}+\frac{a-x}{a+x}}{\frac{a+x}{a-x}-\frac{a-x}{a+x}}=\frac{a^2+x^2}{2ax}$.

(19) $\frac{1+\frac{n-1}{n+1}}{1-\frac{n-1}{n+1}}=n$.

(20) $\frac{a^3-a^2x+ax^2-x^3}{a^5-a^4x+a^3x^2-a^2x^3+ax^4-x^5}=\frac{a^4-x^4}{a^6-x^6}$

$$=\frac{a^2+x^2}{a^4+a^2x^2+x^4}.$$

* These examples admit of the application of the formulas at the top of page 30

ON THE FORMATION OF POWERS, AND THE EXTRACTION OF ROOTS OF ALGEBRAIC QUANTITIES.

48. We begin by considering the case of monomials, and, in order to simplify the subject as much as possible, we shall first treat of the formation of the square and the extraction of the square root only, and then proceed to generalize our reasonings in such a manner as to embrace powers and roots of any degree whatsoever.

Definition.—The *square root* of any expression is that quantity which, when multiplied by itself, will produce the proposed expression. Thus, the square root of a^2 is a, because a, when multiplied by itself, produces a^2; the square root of $(a+b)^2$ is $a+b$, because $a+b$, when multiplied by itself, produces $(a+b)^2$; in like manner, 8 is the square root of 64, 12 of 144, and so on. The process of finding the square root of any quantity is called the *extraction of the square root.*

The extraction of the square root is *indicated* by prefixing the symbol $\sqrt{\ }$ to the quantity whose root is required. Thus, $\sqrt{a^4}$ signifies that the square root of a^4 is to be extracted; $\sqrt{a^2+2ab+b^2}$, or $\sqrt{(a^2+2ab+b^2)}$, signifies that the square root of $a^2+2ab+b^2$ is to be extracted, &c.

In order to discover the method which we must pursue to extract the square root of a monomial, let us consider in what manner we form its square. According to the rule for the multiplication of monomials,

$$(5a^2b^3c)^2=5a^2b^3c\times 5a^2b^3c=25a^4b^6c^2.$$

So,

$$(9ab^2c^3d^4)^2=9ab^2c^3d^4\times 9ab^2c^3d^4=81a^2b^4c^6d^8.$$

And,

$$(\mathrm{A}x^my^nz^h\text{---})^2=\mathrm{A}x^my^nz^h\text{---}\times \mathrm{A}x^my^nz^h\text{---}=\mathrm{A}^2x^{2m}y^{2n}z^{2h}\text{---};$$

i. e., we add the exponent of each letter of the given monomial to itself.

49. Hence it appears that, in order to square a monomial, we must *square its coefficient, and multiply the exponents of each of the different letters by* 2. Therefore, in order to derive the square root of a monomial from its square, we must,

I. *Extract the square root of its coefficient according to the rules of Arithmetic.*

II. *Divide each of the exponents by* 2.

Thus, we shall have

$$\sqrt{64a^6b^4}=8a^3b^2.$$

This is manifestly the true result, for

$$(8a^3b^2)^2=8a^3b^2\times 8a^3b^2=64a^6b^4.$$

Similarly,

$$\sqrt{625a^2b^8c^6}=25ab^4c^3.$$

Here, also,

$$(25ab^4c^3)^2=25ab^4c^3\times 25ab^4c^3=625a^2b^8c^6.$$

Again,

$$\sqrt{25a^6p^{-18}c^4d^{-32}}=5a^3p^{-9}c^2d^{-16}=\frac{5a^3c^2}{p^9d^{16}}.$$

Also,

$$\sqrt{81a^{2m}x^{4m}y^{6n}z^{2p-2}}=9a^mx^{2m}y^{3n}z^{p-1}.$$

Also,

$$\sqrt{16c^m d^{n+p-q} g}=4c^{\frac{m}{2}} d^{\frac{n+p-q}{2}} g^{\frac{1}{2}}.$$

If the given quantity be a fraction, extract the square root of its numerator and denominator separately. This rule follows from that for multiplication of fractions. Thus,

$$\sqrt{\frac{49a^4b^6}{16c^2d^4}}=\frac{7a^2b^3}{4cd^2}.$$

Also,

$$\sqrt{\frac{36a^{2m}b^{8n}}{64a^{2p}c^4}}=\frac{6a^m b^{4n}}{8a^p c^2}.$$

Also,

$$\sqrt{\frac{a^6b^{10}c^2}{(a+x)^2h^{10}y^4}}=\frac{a^3b^5c}{(a+x)h^5y^2}.$$

Also,

$$\sqrt{\left(\frac{a^4x^4y^4}{m^4}\times\frac{1}{a^6n^4f^8}\right)}=\frac{x^2y^2}{am^2n^2f^4}.$$

50. It appears, from the preceding rule, that *a monomial can not be the square of another monomial unless its coefficient be a square number, and the exponents of the different letters all even numbers.* Thus, $98ab^4$ is not a perfect square, for 98 is not a square number, and the exponent of a is not an even number. In this case we introduce the quantity into our calculations affected with the sign $\sqrt{\ }$, and it is written under the form $\sqrt{98ab^4}$. Expressions of this nature are called *Surds*, or *Radicals of the Second Degree.**

51. Such expressions can frequently be simplified by the application of the following principle: *The square root of the product of two or more factors is equal to the product of the square roots of these factors.* Or, in algebraic language,

$$\sqrt{abcd\text{-----}}=\sqrt{a}\times\sqrt{b}\times\sqrt{c}\times\sqrt{d}\text{-----}.$$

In order to demonstrate this principle, let us remark that, according to our definition of the square root of any expression, we have

$$(\sqrt{abcd\text{----------}})^2=abcd\text{-----}.$$

Again,

$$(\sqrt{a}\times\sqrt{b}\times\sqrt{c}\times\sqrt{d}\text{---})^2=(\sqrt{a})^2\times(\sqrt{b})^2\times(\sqrt{c})^2\times(\sqrt{d})^2\text{---}\dagger$$
$$=abcd\text{-----}.$$

Hence, since the squares of the quantities $\sqrt{abcd\text{-----}}$, and $\sqrt{a}\,.\,\sqrt{b}\,.\,\sqrt{c}\,.\,\sqrt{d}\text{---}$ are equal, the quantities themselves must be equal.

This being established, the expression given above, $\sqrt{98ab^4}$, may be put under the form $\sqrt{49b^4\times 2a}=\sqrt{49b^4}\times\sqrt{2a}$, but $\sqrt{49b^4}$ is by (Art. 49) $=7b^2$; hence

$$\sqrt{98b^4a}=\sqrt{49b^4}\times\sqrt{2a}=7b^2\sqrt{2a}.$$

Similarly,

$$\sqrt{45a^2b^3c^2d}=\sqrt{9a^2b^2c^2\times 5bd}=\sqrt{9a^2b^2c^2}\times\sqrt{5bd}$$
$$=3abc\sqrt{5bd}.$$

---

* From the Latin *surdus.* They are sometimes called incommensurable, having no common measure with unity. They are also called irrational, because their ratio with unity can not be expressed in numbers. Fractions have both a common measure and ratio with unity. Thus the fraction $\frac{5}{8}$ has $\frac{1}{8}$ for a common measure with unity, and its ratio with unity is $\frac{5}{8}$.

† This follows from (10, III., note).

So, also,

$$\sqrt{864a^2b^5c^{11}} = \sqrt{144a^2b^4c^{10} \times 6bc} = \sqrt{144a^2b^4c^{10}} \times \sqrt{6bc}$$
$$= 12ab^2c^5\sqrt{6bc}.$$

Also,

$$\sqrt{24a^2b} = 2a\sqrt{6b}.$$

Also,

$$\sqrt{54a^3b^3c} = 3ab\sqrt{6abc}.$$

Also,

$$2\sqrt{8a^{2m+1}b} = 4a^m\sqrt{2ab}.$$

Also,

$$3\sqrt{75p^9q^4} = 15p^4q^2\sqrt{3p}.$$

Also,

$$9\sqrt{\frac{48x^{2p+3}y^5}{12a^2b}} = \frac{36x^{p+1}y^2}{2a}\sqrt{\frac{3xy}{3b}} = \frac{18x^{p+1}y^2}{a}\sqrt{\frac{xy}{b}}.$$

In general, therefore, in order to simplify a monomial radical of the second degree, *separate those factors which are perfect squares, extract their root* (Art. 49), *place the product of all these roots before the radical sign, and place all those factors which are not perfect squares under the radical sign.*

In the expressions, $7b^2\sqrt{2a}$, $3abc\sqrt{5bd}$, $12ab^2c^5\sqrt{6bc}$, &c., the quantities $7b^2$, $3abc$, $12ab^2c^5$, are called the *coefficients of the radical.*

52. We have not hitherto considered the sign with which the radical may be affected. But since, as will be seen hereafter, in the solution of problems we are led to consider monomials affected with the sign —, as well as the sign +, it is necessary that we should know how to treat such quantities. Now the square of a monomial being the product of the monomial by itself, it necessarily follows that, *whatever may be the sign of a monomial, its square must be affected with the sign +.* Thus, the square of $+5a^2b^3$, or of $-5a^2b^3$, is $+25a^4b^6$.

Hence we conclude that, *if a monomial be positive, its square root may be either positive or negative.* Thus, $\sqrt{9a^4} = +3a^2$, or $-3a^2$, for either of these quantities, when multiplied by itself, produces $9a^4$; we therefore always affect the square root of a quantity with the double sign $\pm$, which is called *plus or minus.* Thus, $\sqrt{9a^4} = \pm 3a^2$, $\sqrt{144a^2b^4c^6} = \pm 12ab^2c^3$.*

53. If the monomial be affected with a negative sign, the extraction of its square root is impossible, since we have just seen that the square of every quantity, whether positive or negative, is essentially positive. Thus, $\sqrt{-9}$,

* The double sign may be omitted, being always understood before $\sqrt{}$. An important proposition, not usually noticed, should be demonstrated here; it is, that the quantity A has no other square root than the two, $+\sqrt{A}$ and $-\sqrt{A}$. To prove this, let us observe that the different square roots of A are the values of x in the equation $x^2 = A$, or what is the same,

$$x^2 - A = 0.$$

Instead of $x^2 - A$, we may write $x^2 - (\sqrt{A})^2$; then, decomposing this difference into two factors, we have

$$x^2 - A = (x - \sqrt{A})(x + \sqrt{A}).$$

Under this form we perceive that every value of x which is not either $+\sqrt{A}$ or $-\sqrt{A}$, will fail to render either of these two factors zero; then it will not render the product $x^2 - A$ zero; therefore the quantity A has no other square root than $\pm\sqrt{A}$.

The square root of a quantity has, therefore, two values, which are equal with contrary signs, and it has no other values.

$\sqrt{-4a^2}$, $\sqrt{-5}$, are algebraic symbols which represent operations which it is impossible to execute. Quantities of this nature are called *imaginary* or *impossible quantities*, and are symbols of absurdity which we frequently meet with in resolving quadratic equations.

By an extension of our principles, however, we perform the same operations upon quantities of this nature as upon ordinary surds. Thus, by (Art 51),

$$\sqrt{-9} = \sqrt{9\times -1} = \sqrt{9}\,.\sqrt{-1} = 3\sqrt{-1}.$$
$$\sqrt{-4a^2} = \sqrt{4a^2\times -1} = \sqrt{4a^2}\sqrt{-1} = 2a\sqrt{-1}$$
$$\sqrt{-8a^2b} = \sqrt{2\times 4a^2\times b\times -1} = \sqrt{4a^2}\times\sqrt{2b}\times\sqrt{-1} = 2a\sqrt{2b}\sqrt{-1}$$

54. Let us now proceed to consider the formation of powers and extraction of roots of any degree in monomial algebraic quantities.

Definition.—The *cube root* of any expression is that quantity which, multiplied twice by itself, or taken three times as a factor, will produce the proposed expression. The *fourth*, or *biquadrate*, *root* of any expression is that quantity which, multiplied three times by itself, or taken four times as a factor, will produce the proposed expression; and in general, the n^{th} root of any expression is that quantity which, multiplied $(n-1)$ times by itself, or taken n times as a factor, will produce the proposed expression. Thus, the cube root of a^3b^3 is ab, because ab, multiplied by itself twice, or taken three times as a factor, produces a^3b^3; for the same reason, $(a+b)$ is the 6^{th} root of $(a+b)^6$, 2 is the seventh root of 128, and so on.

55. Let it be required to form the fifth power of $2a^3b^2$.

$$(2a^3b^2)^5 = 2a^3b^2\times 2a^3b^2\times 2a^3b^2\times 2a^3b^2\times 2a^3b^2$$
$$= 32a^{15}b^{10}.$$

Where we perceive, 1°. That the coefficient has been raised to the fifth power; 2°. That the exponent of each of the letters has been multiplied by 5

In like manner,

$$(8a^2b^3c)^3 = 8a^2b^3c\times 8a^2b^3c\times 8a^2b^3c$$
$$= 8^3a^{2+2+2}b^{3+3+3}c^{1+1+1}$$
$$= 512a^6b^9c^3.$$

So, also,

$$(2ab^2c^3d^4)^n = 2ab^2c^3d^4\times 2ab^2c^3d^4\times \text{-----} \text{ to } n \text{ factors}$$
$$= 2^na^nb^{2n}c^{3n}d^{4n}.$$

Hence we deduce the following general

RULE TO RAISE A MONOMIAL TO ANY POWER.

*Raise the numerical coefficient to the given power, and multiply the exponents of each of the letters by the index of the power required.**

And hence, reciprocally, we obtain a

RULE TO EXTRACT THE ROOT, OF ANY DEGREE, OF A MONOMIAL.

1°. *Extract the root of the numerical coefficient according to the rules of arithmetic.*

2°. *Divide the exponent of each letter by the index of the required root.*

Thus,

$$\sqrt[3]{64a^9b^3c^6} = 4a^3bc^2$$
$$\sqrt[4]{16a^8b^{12}c^{16}d^4} = 2a^2b^3c^4d$$

---

* When a quantity is positive, all its powers are positive; but if it is negative, all its even powers will be positive, and its uneven negative.

EXAMPLES.

(1) $(2abc)^5=32a^5b^5c^5$.

(2) $(3a^2m^3n^4)^3=27a^6m^9n^{12}$.

(3) $(x^my^nz^p)^8=x^{8m}y^{8n}z^{8p}$.

(4) $\left(\frac{x^{m+1}y^{n-3}}{z^{n-p+1}}\right)^7=\frac{x^{7m+7}y^{7n-21}}{z^{7n-7p+7}}$.

(5) $(m^{0,432}\times p^{4,234}\times q^{3,789}\times r^{0,04})^{0.13}=m^{0.05616}p^{0.55042}q^{0.49257}r^{0.0052}$.

(6) $\left(\frac{a^k}{b^l}\right)^m=\frac{a^{km}}{b^{lm}}$.

(7) $(x^{\frac{p}{q}})^n=x^{\frac{pn}{q}}$.

(8) $(y^{\frac{p}{n}})^{\frac{q}{m}}=y^{\frac{pq'}{nm}}$.

(9) $\left(\frac{a^{m+1}b^{n-2}c^{p+r}}{d^{\alpha}c^{\gamma}f^{\beta}}\right)^{\delta}=\frac{a^{m\delta+\delta}b^{n\delta-2\delta}c^{p\delta+r\delta}}{d^{\alpha\delta}c^{\gamma\delta}f^{\beta\delta}}$.

(10) $\left(\frac{A^4b^{kp}c^{mn}}{D^5f^{\varepsilon}g^{\eta}}\right)^{\lambda}=\frac{A^{4\lambda}b^{kp\lambda}c^{mn\lambda}}{D^{5\lambda}f^{\varepsilon\lambda}g^{\eta\lambda}}$.

(11) $\sqrt[5]{\frac{32m^5n^{10}}{p^{15}q^{20}}}=\frac{2mn^2}{p^3q^4}$.

(12) $\sqrt[7]{\frac{B^{14}k^{7p}z^{21q}}{a^{35}\beta^{42}}}=\frac{B^2k^pz^{3q}}{a^5\beta^6}$.

(13) $\sqrt[8]{\frac{\odot^{32}\,\text{☽}^{16}\,\text{☋}^{24}}{256}}=\frac{\odot^{4}\,\text{☽}^{2}\,\text{☋}^{3}}{2}$.

(14) $\sqrt{\frac{\sin^2\phi\cos^2\phi}{\tan^4\psi\sec^6\psi}}=\frac{\sin\phi\cos\phi}{\tan^2\psi\sec^3\psi}$.

(15) $\sqrt[5]{\left\{\frac{3^{10}a^{-5}(a+b)^{15}(x+y)^{-10}(b+c-x)^{20}}{32(p+q-r)^{100}}\right\}}=\frac{3^2a^{-1}(a+b)^3(x+y)^{-2}(b+c-x)^4}{2(p+q-r)^{20}}$.

When the root to be extracted is of an uneven degree, its sign should be that of the given quantity; when of an even degree, it should be $\pm$. (See last note.)

56. By the rule for extracting a root, we perceive that, in order that a monomial may be a perfect power of that degree whose root is required, its coefficient must be a perfect power of that degree, and the exponent of each letter must be divisible by the index of the root.

When the monomial whose root is required is not a perfect power of the required degree, we can only *indicate* the operation by placing the radical sign $\sqrt{}$ before the quantity, and writing within it the index of the root. Thus, if it be required to extract the cube root of $4a^2b^5$, the operation will be indicated by writing the expression,

$$\sqrt[3]{4a^2b^5}.$$

Expressions of this nature are called *surds*, or, *irrational quantities*, or *radicals of the second*, *third*, or n^{th} *degree*, according to the index of the root required.

57. We can frequently simplify these quantities by the application of the following principle, which is merely an extension of that already proved in (Art. 51).

The n*th* *root of the product of any number of factors is equal to the product of the* n*th* *roots of the different factors.* Or, in algebraic language,

$$\sqrt[n]{abcd------}=\sqrt[n]{a}\times\sqrt[n]{b}\times\sqrt[n]{c}\times\sqrt[n]{d}\times----.$$

Raise each of these expressions to the power of the degree n, then

$$(\sqrt[n]{abcd-----})^n=abcd-----.$$

And,

$$(\sqrt[n]{a}\times\sqrt[n]{b}\times\sqrt[n]{c}\times\sqrt[n]{d}---)^n=(\sqrt[n]{a})^n\times(\sqrt[n]{b})^n\times(\sqrt[n]{c})^n\times(\sqrt[n]{d})^n----$$
$$=abcd----.$$

Hence, since the n^{th} powers of the quantities $\sqrt[n]{abcd}$, and $\sqrt[n]{a}.\sqrt[n]{b}.\sqrt[n]{c}.\sqrt[n]{d}----$ are equal, the quantities themselves must be equal. Q. E. D.

This being established, let us take the expression $\sqrt[3]{54a^4b^3c^2}$, whose root can not be exactly extracted, since 54 is not a perfect cube, and the exponents of a and c are not exactly divisible by 3.

We have,

(1) $\sqrt[3]{54a^4b^3c^2}=\sqrt[3]{27\times2\times a^3\times a\times b^3\times c^2}$
$=\sqrt[3]{27}\times\sqrt[3]{a^3}\times\sqrt[3]{b^3}\times\sqrt[3]{2ac^2}$

by the principle just proved,

$=3ab\sqrt[3]{2ac^2}$.

So, also,

(2) $\sqrt[4]{48a^5b^8c^6}=\sqrt[4]{16\times3\times a^4\times a\times b^8\times c^4\times c^2}$
$=\sqrt[4]{16}\times\sqrt[4]{a^4}\times\sqrt[4]{b^8}\times\sqrt[4]{c^4}\times\sqrt[4]{3}\times\sqrt[4]{a}\times\sqrt[4]{c^2}$
$=2ab^2c\sqrt[4]{3ac^2}$.

(3) $\sqrt[6]{192a^7bc^{12}}=\sqrt[6]{64\times3\times a^6\times a\times b\times c^{12}}$
$=\sqrt[6]{64}\times\sqrt[6]{a^6}\times\sqrt[6]{c^{12}}\times\sqrt[6]{3}\times\sqrt[6]{a}\times\sqrt[6]{b}$
$=2ac^2\sqrt[6]{3ab}$.

(4) $\sqrt[3]{192}=4\sqrt[3]{3}$.*

(5) $5\sqrt[3]{56a^4b^5}=10ab\sqrt[3]{7ab^2}$.

(6) $\sqrt[5]{x^{10}y^{-5}z^{5m+1}}=x^2y^{-1}z^m\sqrt[5]{z}$.

(7) $\sqrt[8]{\frac{\alpha^9\beta^{10}\gamma^{11}}{\delta^8\varepsilon^{16}}}=\frac{\alpha\beta\gamma}{\delta\varepsilon^2}\sqrt[8]{\alpha\beta^2\gamma^3}$.

(8) $\sqrt[7]{\frac{A^{2a}B^{7b}C^{14c}}{m}}=B^bC^{2c}\sqrt[7]{\frac{A^{2a}}{m}}$.

In the above expressions, the quantities $3ab$, $2ab^2c$, $2ac^2$, &c., placed before the radical sign, are called the *coefficients of the radical.*

58. There is another principle which can frequently be employed with advantage in treating these quantities; this is,

The m*th* *power of the* n*th* *power of any quantity is equal to the* mn*th* *power of that quantity.* Or, in algebraic language,

$$\{a^n\}^m=a^{mn}.$$

---

* A good way of separating a number into factors, some of which are perfect powers, is to try perfect powers upon it as divisors, beginning with powers of the lowest numbers. Thus, in the 4th example, 8, the cube of 2, will divide 192, and the quotient is 24; again, 8 will divide 24, and the original number, 192, may be put under the form $8\times8\times3=64\times3$, and the cube root will be $2\times2\times\sqrt[3]{3}$, or $4\sqrt[3]{3}$. The cube root of 1080 may be found by first dividing by 2^3, and that quotient by 3^3, or 27. The result is $\sqrt[3]{2^3\times3^3\times5}=2\times3\sqrt[3]{5}=6\sqrt[3]{5}$

For we have,

$$\{a^3\}^4=a^3\times a^3\times a^3\times a^3$$
$$=a^{3+3+3+3}=a^{12}.$$

And, in general,

$$\{a^n\}^m=a^n\times a^n\times a^n\times a^n\text{---to } m \text{ factors};$$
$$=a^{n+n+n+n\text{---to } m \text{ terms}};$$
$$=a^{mn}.$$

And, reciprocally,

The mn*th* *root of any quantity is equal to the* m*th* *root of the* n*th* *root of that quantity.* Or, in algebraic language,

$$\sqrt[mn]{a}=\sqrt[m]{\sqrt[n]{a}}.$$

For, let

$$\sqrt[m]{\sqrt[n]{a}}=p;$$

Raise the two quantities to the power m,

$$\sqrt[n]{a}=p^m;$$

Again, raise both to the power n,

$$a=p^{mn};$$

Extract the mn^{th} root,

$$\sqrt[mn]{a}=p;$$

But, by supposition,

$$\sqrt[m]{\sqrt[n]{a}}=p,$$

$$\therefore \sqrt[mn]{a}=\sqrt[m]{\sqrt[n]{a}}.$$

Hence, as often as the index of the root is a number composed of two or more factors, we may obtain the root required by extracting, in succession, the roots whose indices are the factors of that number. Thus,

(1) $\sqrt[6]{4a^2}=\sqrt[3\times 2]{4a^2}$,

$=\sqrt[3]{\sqrt[2]{4a^2}}$ by the above principle,

$=\sqrt[3]{2a}.$

(2) $\sqrt[4]{36a^2b^2}=\sqrt{\sqrt{36a^2b^2}}$

$=\sqrt{6ab}.$

(3) $\sqrt[8]{256}=\sqrt[4]{\sqrt[2]{256}}=\sqrt[4]{16}=2.$

(4) $\sqrt[10]{32a^5b^5}=\sqrt{2ab}.$

(5) $\sqrt[12]{16a^4x^2y^{2m}z^{4n-4}}=\sqrt[6]{4a^2xy^mz^{2n-2}}.$

(6) In general,

$$\sqrt[mn]{a^n}=\sqrt[m]{\sqrt[n]{a^n}}$$
$$=\sqrt[m]{a}.$$

That is to say, *When the index of the radical is multiplied by a certain number* n, *and the quantity under the radical sign is an exact* n*th* *power, we can, without changing the value of the radical, divide its index by* n, *and extract the* n*th* *root of the quantity under the sign.*

Thus,

$$\sqrt[8]{25a^4b^2c^6} = \sqrt[4]{5a^2bc^3},$$
$$\sqrt[6]{27m^{15}n^9p^6} = \sqrt[2]{3m^5n^3p^2},$$
$$\sqrt[9]{27a^3x^{3m}y^{3(p-q)}} = \sqrt[3]{3ax^my^{p-q}},$$
$$\sqrt[10]{q^5p^{-5}r^{5n}} = \sqrt{qp^{-1}r^n}.$$

59. This last proposition is the converse of another not less important, which consists in this, *that we may multiply the index of a radical by any number, provided we raise the quantity under the sign to the power whose degree is marked by that number*, or, in algebraic language,

$$\sqrt[n]{a} = \sqrt[mn]{a^m}.$$

For, if the last rule be applied to the second of these quantities, it will produce the first.

60. By aid of this last principle, we can always reduce two or more radicals of different degrees to others which shall have the same index. Let it be required, for example, to reduce the two radicals $\sqrt[3]{2a}$ and $\sqrt[5]{3bc}$ to others which shall be equivalent, and have the same index. If we multiply 3, the index of the first, by 5, the index of the second, and, at the same time, raise $2a$ to the 5th power; if, in like manner, we multiply 5, the index of the second, by 3, the index of the first, and, at the same time, raise $3bc$ to the 3d power, we shall not change the value of the two radicals, which will thus become

$$\sqrt[3]{2a} = {}^{5\times}\sqrt[3]{(2a)^5} = \sqrt[15]{32a^5}$$
$$\sqrt[5]{3bc} = {}^{3\times}\sqrt[5]{(3bc)^3} = \sqrt[15]{27b^3c^3}.$$

We shall thus have the following general

RULE.

In order to reduce two or more radicals to others which shall be equivalent and have the same index, multiply the index of each radical by the product of the indices of all the others, and raise the quantity under the sign to the power whose degree is marked by that product.

Thus, let it be required to reduce $\sqrt{2a}$, $\sqrt[3]{3b^2c^3}$, $\sqrt[5]{4d^4e^5f^6}$ to the same index,

$$\sqrt[2]{2a} = {}^{3\times5\times}\sqrt[2]{(2a)^{3\times5}} = \sqrt[30]{2^{15}a^{15}}$$
$$\sqrt[3]{3b^2c^3} = {}^{2\times5\times}\sqrt[3]{(3b^2c^3)^{2\times5}} = \sqrt[30]{3^{10}b^{20}c^{30}}$$
$$\sqrt[5]{4d^4e^5f^6} = {}^{2\times3\times}\sqrt[5]{(4d^4e^5f^6)^{2\times3}} = \sqrt[30]{4^6d^{24}e^{30}f^{36}}.$$

The above rule, which bears a great analogy to that given for the reduction of fractions to a common denominator, is susceptible of the same modifications.

RULE.

*To reduce radicals to their least common index, find the least common multiple of all the indices, divide it by the index of each radical, and raise the quantity under the radical to the power expressed by the quotient.**

This rule, applied to the radicals $\sqrt[4]{a}$, $\sqrt[6]{5b}$, $\sqrt[8]{2c}$, gives

$$\sqrt[4]{a} = \sqrt[24]{a^6},\ \sqrt[6]{5b} = \sqrt[24]{625b^4},\ \sqrt[8]{3c} = \sqrt[24]{27c^3}.$$

EXAMPLES.

(1) Reduce $\sqrt[3]{a^m}$, $\sqrt[4]{b^n}$, and $\sqrt[6]{c^p}$ to the same index.

(2) Reduce $\sqrt[m]{a}$, $\sqrt[n]{b}$, and $\sqrt[p]{c}$ to the same index.

(3) Reduce $\sqrt[\alpha]{a^5}$, $\sqrt[\beta]{b^4}$, $\sqrt[\gamma]{c^3}$, and $\sqrt[\delta]{d^2}$ to the same index.

* This is, in effect, multiplying the index of each radical, and the exponents under that radical, by the quotient.

(4) Reduce $\sqrt[p]{\frac{C}{a}}$, $\sqrt[q]{\frac{D}{b}}$, and $\sqrt[r]{\frac{m}{n}}$ to the same index.

(5) Reduce $\sqrt{\frac{1}{a-b}}$, $\sqrt[3]{\frac{1}{(x+y)^2}}$, and $\sqrt[6]{\frac{1}{z^5}}$ to the same index.

ANSWERS.

(1) $\sqrt[12]{a^{4m}}$, $\sqrt[12]{b^{3n}}$, and $\sqrt[12]{c^{2p}}$.

(2) $\sqrt[mnp]{a^{np}}$, $\sqrt[mnp]{b^{mp}}$, and $\sqrt[mnp]{c^{mn}}$.

(3) $\sqrt[\alpha\beta\gamma\delta]{a^5\beta\gamma\delta}$, $\sqrt[\alpha\beta\gamma\delta]{b^4\alpha\gamma\delta}$, $\sqrt[\alpha\beta\gamma\delta]{c^3\alpha\beta\delta}$, and $\sqrt[\alpha\beta\gamma\delta]{d^2\alpha\beta\gamma}$.

(4) $\sqrt[rqp]{\frac{C^{qr}}{a^{qr}}}$, $\sqrt[rqp]{\frac{D^{pr}}{b^{pr}}}$, and $\sqrt[rqp]{\frac{m^{pq}}{n^{pq}}}$.

(5) $\sqrt[6]{\frac{1}{(a-b)^3}}$, $\sqrt[6]{\frac{1}{(x+y)^4}}$, and $\sqrt[6]{\frac{1}{z^5}}$.

61. Let us now proceed to execute upon radicals the fundamental operations of arithmetic.

ADDITION AND SUBTRACTION OF RADICALS.

DEFINITION.—Radicals are said to be *similar* when they have the same index, and when, also, the quantity under the radical sign is the same in each; thus, $3\sqrt{a}$, $12ac\sqrt{a}$, $15b\sqrt{a}$, are similar radicals, as are, also, $4a^2b\sqrt[4]{mn^2p^3}$, $5l\sqrt[4]{mn^2p^3}$, $25d\sqrt[4]{mn^2p^3}$, &c.

This being premised, in order to add or subtract two similar radicals we have the following

RULE.

Add or subtract their coefficients, and place the sum or difference as a coefficient before the common radical. For example,

(1) $3\sqrt[3]{b}+2\sqrt[3]{b}=5\sqrt[3]{b}$.

(2) $3\sqrt[3]{b}-2\sqrt[3]{b}=\sqrt[3]{b}$.

(3) $3pq\sqrt[5]{mn}+4l\sqrt[5]{mn}=(3pq+4l)\sqrt[5]{mn}$.*

(4) $9cd\sqrt{a}-4cd\sqrt{a}=5cd\sqrt{a}$.

If the radicals are not similar, we can only *indicate* the addition or subtraction by interposing the signs $+$ or $-$.

It frequently happens that two radicals, which do not at first appear similar, may become so by simplification; thus,

(5) $\sqrt{48ab^2}+b\sqrt{75a}=\sqrt{3\times16\times a\times b^2}+b\sqrt{3\times25\times a}$
$=4b\sqrt{3a}+5b\sqrt{3a}$
$=9b\sqrt{3a}$.

(6) $2\sqrt{45}-3\sqrt{5}=2\sqrt{5\times9}-3\sqrt{5}$
$=3\sqrt{5}$.

(7) $\sqrt[3]{8a^3b+16a^4}-\sqrt[3]{b^4+2ab^3}=\sqrt[3]{8a^3(b+2a)}-\sqrt[3]{b^3(b+2a)}$
$=(2a-b)\sqrt[3]{2a+b}$.

* When two products, consisting each of several factors, have any common factors, the other factors may be regarded as the coefficients of these, since they show how many times the common factors are repeated, and the addition may be performed by adding the coefficients, and annexing the common factors to the sum; thus, $abcd+mncd=(ab+mn)cd$, and $5ab\sqrt{x}+4cb\sqrt{x}=(5a+4c)b\sqrt{x}$, on the same principle as $8a+4a=12a$.

(8) $3\sqrt[6]{4a^2}+2\sqrt[3]{2a}=3\sqrt[3]{2a}+2\sqrt[3]{2a}$
$=5\sqrt[3]{2a}.$

(9) $\sqrt{8}+\sqrt{50}-\sqrt{18}=4\sqrt{2}.$

(10) $b\sqrt[m]{ab^m}+c\sqrt[m]{ac^m}-d\sqrt[m]{ad^m}=(b^2+c^2-d^2)\sqrt[m]{a}.$

(11) $2\sqrt{\frac{5}{3}}+\sqrt{60}-\sqrt{15}+\sqrt{\frac{3}{5}}=\frac{23}{15}\sqrt{15}.$*

(12) $4a\sqrt[3]{a^3b^4}+b\sqrt[3]{8a^6b}-\sqrt[3]{125a^6b^4}=a^2b\sqrt[3]{b}.$

(13) $\sqrt{(3a^2c+6abc+3b^2c)}=(a+b)\sqrt{3c}.$

(14) $\sqrt{45c^3}-\sqrt{80c^3}+\sqrt{5a^2c}=(a-c)\sqrt{5c}.$

MULTIPLICATION AND DIVISION OF RADICALS.

62. In the first place, with regard to radicals which have the same index, let it be required to multiply or divide $\sqrt[n]{a}$ by $\sqrt[n]{b}$, then we shall have

$$\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}, \text{ and } \sqrt[n]{a}\div\sqrt[n]{b}=\sqrt[n]{\frac{a}{b}}.$$

For, if we raise $\sqrt[n]{a}\times\sqrt[n]{b}$, and $\sqrt[n]{ab}$, each to the n^{th} power, we obtain the same result, ab; hence these two expressions are equal. The same principle is demonstrated in (57).

In like manner, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\sqrt[n]{\frac{a}{b}}$, when raised to the n^{th} power, give $\frac{a}{b}$; hence the two expressions are equal. We shall thus have the following

RULE.

In order to multiply or divide two radicals which have the same index, multiply or divide the quantities under the sign by each other, and affect the result with the common radical sign. If there be any coefficients, we commence by multiplying or dividing them separately. The latter part of this rule depends upon the principles set forth and alluded to in 17, note; the coefficients, or rational parts, and the radical parts being regarded as factors composing a product.

(1) $2a\sqrt[3]{\frac{(a^2+b^2)}{c}}\times-3a\sqrt[3]{\frac{(a^2+b^2)^2}{d}}=-6a^2\sqrt[3]{\frac{(a^2+b^2)^3}{cd}}$
$=-\frac{6a^2(a^2+b^2)}{\sqrt[3]{cd}}$

(2) $3a\sqrt[4]{8a^2}\times2b\sqrt[4]{4a^2c}=6ab\sqrt[4]{32a^4c}$
$=12a^2b\sqrt[4]{2c}.$

(3) $2a\sqrt{bc}\times3b\sqrt{abc}\times a\sqrt{2a}=6a^2b\sqrt{2a^2b^2c^2}$
$=6a^3b^2c\sqrt{2}.$

(4) $\frac{5a\sqrt{b}}{2b\sqrt{c}}=\frac{5a}{2b}\sqrt{\frac{b}{c}}.$

(5) $\frac{25a^2b\sqrt{m^3n}}{5ab^2\sqrt{mn^2}}=\frac{25a^2b}{5ab^2}\sqrt{\frac{m^3n}{mn^2}}$
$=\frac{5a}{b}\sqrt{\frac{m^2}{n}}$
$=\frac{5am}{b}\sqrt{\frac{1}{n}}.$

---

* The numerator and denominator of each of the two fractions in this example are multiplied by its denominator. The denominator becomes thus a perfect square, and may be set outside the radical sign.

(6) $\dfrac{\sqrt[3]{a^2b^2+b^4}}{\sqrt[3]{\dfrac{a^2-b^2}{8b}}}=\sqrt[3]{\dfrac{8b(a^2b^2+b^4)}{a^2-b^2}}$

$=2b\sqrt[3]{\dfrac{a^2+b^2}{a^2-b^2}}.$

(7) $(a+b\sqrt{-1})\times(a-b\sqrt{-1})=a^2+b^2.$

(8) $\sqrt[m]{a}\times\sqrt[m]{b}\times\sqrt[m]{c}=\sqrt[m]{abc}.$

(9) $a\sqrt[n]{x}\times b\sqrt[n]{y}\times c\sqrt[n]{z}=abc\sqrt[n]{xyz}.$

(10) $4\times2\sqrt[6]{3}\times\sqrt[6]{72}=8\sqrt{6}.$

(11) $c\sqrt{a}\times d\sqrt{a}=acd.$

(12) $5\sqrt{8}\times3\sqrt{5}=30\sqrt{10}.$

(13) $\sqrt[3]{18}\times5\sqrt[3]{4}=10\sqrt[3]{9}.$

(14) $\frac{1}{4}\sqrt{6}\times\frac{2}{15}\sqrt{9}=\frac{1}{10}\sqrt{6}.$

(15) $2\sqrt[\alpha]{a^\gamma}\times3\sqrt[\alpha]{a^\beta}\times4\sqrt[\alpha]{b^\delta}=24\sqrt[\alpha]{a^{\gamma+\beta}b^\delta}.$

(16) $\sqrt[pm]{\dfrac{1}{(x-y)^2}}\times\sqrt[pm]{(x+y)^2}=\sqrt[pm]{\dfrac{(x+y)^2}{(x-y)^2}}.$

(17) $(\sqrt{-15}+\sqrt{-12}-\sqrt{-21})\div\sqrt{-3}=2+\sqrt{5}-\sqrt{7}.$

If the radicals have not the same index, we must reduce them to others having the same index, and then operate upon them as above; thus,

(1) $3a\sqrt[6]{b}\times5b\sqrt[8]{2c}=3a\sqrt[24]{b^4}\times5b\sqrt[24]{8c^3}$

$=15ab\sqrt[24]{8b^4c^3}.$

(2) $\sqrt{5abc^3}\times\sqrt[3]{2a^2bc^2}=\sqrt[6]{125a^3b^3c^9}\times\sqrt[6]{4a^4b^2c^4}$

$=\sqrt[6]{500a^7b^5c^{13}}$

$=ac^2\sqrt[6]{500ab^5c}.$

(3) $m\sqrt[4]{a}\times\sqrt[3]{b}\times n\sqrt[5]{c}=mn\sqrt[60]{a^{15}b^{20}c^{12}}.$

(4) $\dfrac{a}{b}\sqrt{\dfrac{c}{d}}\times\dfrac{x}{y}\sqrt[3]{\dfrac{z}{u}}=\dfrac{ax}{by}\sqrt[6]{\dfrac{c^3z^2}{d^3u^2}}.$

(5) $x\sqrt[4]{m^\delta}\times y\sqrt[12]{m^\mu}=xy\sqrt[12]{m^{3\delta+\mu}}.$

(6) $\sqrt[8]{\dfrac{1}{a^2-b^2}}\times\sqrt[4]{a^4-b^4}\times\sqrt[12]{\dfrac{1}{(a^4-b^4)^2}}=\sqrt[24]{\dfrac{(a^2+b^2)^2}{a^2-b^2}}.$

(7) $\sqrt[m]{a}\times\sqrt[n]{b}\times\sqrt[p]{c}=\sqrt[mnp]{a^{np}b^{mp}c^{mn}}.$

(8) $A\sqrt[\alpha]{a^m}\times B\sqrt[\beta]{b^n}\times C\sqrt[\gamma]{c^r}=ABC\sqrt[\alpha\beta\gamma]{a^{m\beta\gamma}b^{n\alpha\gamma}c^{r\alpha\beta}}.$

(9) $\sqrt{\dfrac{a^mb}{c^2d}}\div\sqrt[3n]{\dfrac{a^{m-1}c^3}{d^5}}=\sqrt[6n]{\dfrac{a^{m(3n-2)+2}b^{3n}d^{10-3n}}{c^{6n+6}}}.$

(10) $c\sqrt{a^2-x^2}\div\sqrt[3]{a+x}=c\sqrt[6]{(a-x)^2(a^2-x^2)}.$

(11) $\sqrt{a^2-z^2}\div(a-z)=\sqrt{\dfrac{a+z}{a-z}}.$

(12) $A_1\sqrt[m]{\dfrac{x^\alpha}{y^\rho}}\div A_2\sqrt[n]{\dfrac{x^\nu}{y^\delta}}=\dfrac{A_1}{A_2}\sqrt[mn]{\dfrac{x^{\alpha n-\nu m}}{y^{\rho n-\delta m}}}.$

FORMATION OF POWERS AND EXTRACTION OF ROOTS OF RADICALS.

63. Let it be required to raise $\sqrt[m]{a}$ to the nth power; then,

$(\sqrt[m]{a})^n=\sqrt[m]{a}\times\sqrt[m]{a}\times\sqrt[m]{a}$ - - - to n factors,

$=\sqrt[m]{a^n}$, according to the rule for multiplication just established. Hence we have the following

RULE.

In order to raise a radical quantity to any given power, raise the quantity under the sign to that power, and place over the result the radical sign with its original index. If there be any coefficient, we must raise the coefficient separately to the required power. Thus,

$$(1)\quad (\sqrt[4]{4a^3})^2 = \sqrt[4]{16a^6}$$
$$= 2a\sqrt[4]{a^2}.$$
$$(2)\quad (3\sqrt[3]{2a})^5 = 3^5\sqrt[3]{32a^5}$$
$$= 243\sqrt[3]{32a^5}$$
$$= 486a\sqrt[3]{4a^2}.$$

When the index of the radical is a multiple of the exponent of the power which we wish to form, the operation may be simplified.

Let it be required, for example, to square $\sqrt[4]{2a}$; we have seen (Art. 58) that $\sqrt[4]{2a} = \sqrt{\sqrt{2a}}$; but in order to square this quantity, it is sufficient to suppress the first radical sign; hence, $(\sqrt[4]{2a})^2 = \sqrt{2a}$. Again, let it be required to raise $\sqrt[10]{abc}$ to the 5th power; now, $\sqrt[10]{abc} = \sqrt[5]{\sqrt{abc}}$; but in order to raise this quantity to the 5th power, it is sufficient to suppress the first radical sign; hence, $(\sqrt[10]{abc})^5 = \sqrt{abc}$, and, in general,

$$(\sqrt[mn]{a})^m = \left(\sqrt[m]{\sqrt[n]{a}}\right)^m = \sqrt[n]{a};$$

that is to say,

*If the index of the radical be divisible by the index of the required power, we may divide the index of the radical by the index of the power, and leave the quantity under the sign unchanged.**

64. With regard to the extraction of roots, either by virtue of the principle established in (Art. 59), or by reversing the last rule, we shall manifestly have the following

RULE.

In order to extract any root of a radical quantity, multiply the index of the radical by the index of the root required, and leave the quantity under the sign unchanged. If there be a coefficient, we must extract its root separately. Thus,

$$(1)\quad \sqrt[3]{\sqrt[4]{3c}} = \sqrt[12]{3c}.$$
$$(2)\quad \sqrt[2]{\sqrt[3]{5a}} = \sqrt[6]{5a}.$$
$$(3)\quad \sqrt[3]{8c^3\sqrt[5]{a^2b}} = 2c\sqrt[15]{a^2b}.$$

If the quantity under the sign be a perfect power of the same degree as the root required, we may simplify. Thus,

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* It may be well to note here that the even power of a radical of the second degree is rational, and the uneven power irrational, the latter being formed by the multiplication of the proposed radical by a rational quantity.

$$(4)\ \sqrt[3]{\sqrt[4]{8a^3}}=\sqrt[4]{\sqrt[3]{8a^3}}^{*}$$
$$=\sqrt[4]{2a}.$$

$$(5)\ \sqrt[2]{\sqrt[9]{9a^2}}=\sqrt[9]{\sqrt[2]{9a^2}}$$
$$=\sqrt[9]{3a};$$

that is, *we may extract the root of the quantity under the radical sign.*

MISCELLANEOUS EXAMPLES.

(1) $\sqrt{24}+\sqrt{54}-\sqrt{6}=4\sqrt{6}.$

(2) $\sqrt{12}+2\sqrt{27}+3\sqrt{75}+9\sqrt{48}=59\sqrt{3}.$

(3) $\sqrt[3]{81}-2\sqrt[3]{24}+\sqrt{28}+2\sqrt{63}=8\sqrt{7}-\sqrt[3]{3}.$

(4) $\sqrt{45c^3}-\sqrt{80c^3}+\sqrt{5a^2c}=(a-c)\sqrt{5c}.$

(5) $\sqrt{18a^5b^3}+\sqrt{50a^3b^3}=(3a^2b+5ab)\sqrt{2ab}.$

(6) $\sqrt[4]{2^{14}a^{13}b^5c}-\sqrt[4]{4\times 5^4a^5b^9c^5}+\sqrt[4]{4\times 6^4ab^5c}$
$=(8a^3b-5ab^2c+6b)\sqrt[4]{4abc}.$

(7) $\sqrt[3]{\frac{27a^5x}{2b}}-\sqrt[3]{\frac{a^2x}{2b}}=(3a-1)\sqrt[3]{\frac{a^2x}{2b}}.$

(8) $\sqrt[3]{54a^{m+6}b^3}-\sqrt[3]{16a^{m-3}b^6}+\sqrt[3]{2a^{4m+9}}+\sqrt[3]{2c^3a^m}$
$=(3a^2b-\frac{2b^2}{a}+a^{m+3}+c)\sqrt[3]{2a^m}.$

(9) $\frac{\sqrt[6]{3\times 2^3c^3f^4}}{\sqrt[6]{d^4g}}+\frac{\sqrt[6]{2^3g^{11}}}{\sqrt[6]{3^5c^3d^4f^2}}=\left\{\frac{f}{d}+\frac{g^2}{3cd}\right\}\sqrt[6]{\frac{3\times 2^3c^3d^2}{f^2g}}.$

(10) $x\sqrt[3]{\left(\frac{8a^4}{27b^3}+\frac{16a^3}{27b^2}\right)}=\frac{2ax}{3b}\sqrt[3]{a+2b}.$

(11) $\sqrt{4a^2y+8aby+4b^2y}=2(a+b)\sqrt{y}.$

(12) $\sqrt{4a^5b^2-20a^3b^3+25ab^4}=(2a^2-5b)\sqrt{ab^2}.$

(13) $\frac{\sqrt{a^2x-2ax^2+x^3}}{\sqrt{a^2+2ax+x^2}}=\frac{a-x}{a+x}\sqrt{x}.$

(14) $\frac{a-b}{a+b}\cdot\frac{\sqrt{ac}}{\sqrt{a^2-2ab+b^2}}=\frac{\sqrt{ac}}{a+b}.$

(15) $\frac{a+b}{a-b}\cdot\sqrt{\frac{a-b}{a+b}}=\sqrt{\frac{a+b}{a-b}}.$

(16) $\sqrt[3]{2}\times\sqrt[6]{\frac{1}{3}}\times\sqrt[8]{3}=\sqrt[24]{\frac{256}{3}}.$

(17) $\sqrt[5]{4}\times\sqrt[10]{3}\times\sqrt[15]{6}=\sqrt[30]{3981312}$

(18) $a\sqrt[m]{x}\times b\sqrt[n]{y}\times c\sqrt[p]{z}=abc\sqrt[mnp]{x^{np}y^{mp}z^{mn}}.$

(19) $\sqrt[12]{\frac{a}{bc}}\times\sqrt[8]{\frac{a^m}{b}}=\sqrt[24]{\frac{a^{3m+2}}{b^5c^2}}.$

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* It is manifest that, in general, $\sqrt[m]{\sqrt[n]{a}}=\sqrt[n]{\sqrt[m]{a}}$; for, by (Art. 58), each of these expressions is $=\sqrt[mn]{a}$.

(20) $\frac{ac}{b^3d^3}\sqrt[3]{\frac{bcd}{e}}\times\frac{\sqrt[6]{b^{10}d^7e}}{\sqrt[6]{a^2c^5}}=\frac{1}{bd}\sqrt[6]{\frac{a^4c^3}{d^3e}}$.

65. Let us now inquire with what sign a monomial root is to be affected.

We have seen (Art. 52) that, whatever may be the sign of a monomial, its square is always positive; and it is evident that, in like manner, every *even* power must be positive, whatever may be the sign of the original monomial, and that every *uneven power* will be affected with the same sign as the original monomial.

Thus, $-a$, when raised to different powers in succession will give

$$-a,\ +a^2,\ -a^3,\ +a^4,\ -a^5,\ +a^6,\ -a^7,\ \&c.$$

And $+a$, in like manner, will give

$$+a,\ +a^2,\ +a^3,\ +a^4,\ +a^5,\ +a^6,\ +a^7,\ \&c.$$

In fact, every even power $2n$ may be considered as the square of the n^{th} power or $a^{2n}=(a^n)^2$, and must, therefore, be positive; and, in like manner, every power of an uneven degree $(2n+1)$ may be considered as the product of the $2n^{\text{th}}$ power by the original monomial, and must, therefore, have the same sign with the monomial.

Hence it appears,

I. *That every root of an uneven degree of a monomial quantity must be affected with the same sign as the quantity itself.* Thus,

$$\sqrt[3]{+8a^3}=2a\,;\ \sqrt[3]{-8a^3}=-2a\,;\ \sqrt[5]{-32a^{10}b^5}=-2a^2b.$$

II. *That every root of an even degree of a positive monomial may be affected with the sign* $+$, *or the sign* $-$, *indifferently.* Thus,

$$\sqrt[4]{81a^4b^{12}}=\pm 3ab^3\,;\ \sqrt[6]{64a^{18}}=\pm 2a^3.$$

III. *That every root of an even degree of a negative monomial is an impossible root;* for no quantity can be found which, when raised to an even power, can give a negative result. Thus, $\sqrt[4]{-a}$, $\sqrt[6]{-c}$, ... are symbols of operations which can not be performed, and are called *impossible*, or *imaginary*, *quantities*, as $\sqrt{-a}$, $\sqrt{-b}$, in (Art. 53).

66. The different rules which have been established for the calculation of radicals are exact so long as we treat of absolute numbers; but are subject to some modifications when we consider expressions or symbols which are purely algebraical, such as the *imaginary expressions* just mentioned.

Let it be required, for example, to determine the product of $\sqrt{-a}$ by $\sqrt{-a}$; by the rule given in (Art. 62),

$$\sqrt{-a}\times\sqrt{-a}=\sqrt{-a\times -a}$$
$$=\sqrt{+a^2}.$$

But $\sqrt{+a^2}=\pm a$, so that there is apparently a doubt as to the sign with which a ought to be affected in order to answer the question. However, the true result is $-a$; because, in general, in order to square $\sqrt{m}$, it is sufficient to suppress the radical sign; but $\sqrt{-a}\times\sqrt{-a}$ is the same thing as $(\sqrt{-a})^2$, and, consequently, is equal to $-a$.

Next, let it be required to determine the product of $\sqrt{-a}$ by $\sqrt{-b}$; by the rule (Art. 62)

$$\sqrt{-a}\times\sqrt{-b}=\sqrt{-a\times -b}$$
$$=\sqrt{+ab}$$
$$=\pm\sqrt{ab}.$$

The true result, however, is $-\sqrt{ab}$, so long as we suppose the radicals $\sqrt{-a}$, $\sqrt{-b}$ to be each preceded by the sign $+$; for we have, according to (Art. 53),

$$\sqrt{-a} = \sqrt{a}\cdot\sqrt{-1}$$
$$\sqrt{-b} = \sqrt{b}\cdot\sqrt{-1}$$

Hence,

$$\sqrt{-a}\times\sqrt{-b} = \sqrt{ab}(\sqrt{-1})^2$$
$$= \sqrt{ab}\times -1$$
$$= -\sqrt{ab}.$$

According to this principle, we shall find for the different powers of $\sqrt{-1}$ the following results:

$$\sqrt{-1} = \sqrt{-1}$$
$$(\sqrt{-1})^2 = -1$$
$$\sqrt{-1})^3 = (\sqrt{-1})^2\cdot\sqrt{-1}$$
$$= -\sqrt{-1}$$
$$(\sqrt{-1})^4 = (\sqrt{-1})^2\times(\sqrt{-1})^2$$
$$= -1\times -1$$
$$= +1.$$

Since the four following powers will be found by multiplying $+1$ by the first, the second, the third, and the fourth, we shall again find for the four new powers $+\sqrt{-1}$, -1, $-\sqrt{-1}$, $+1$; so that all the powers of $\sqrt{-1}$ will form a repeating cycle of four terms, being successively, $\sqrt{-1}$, -1, $-\sqrt{-1}$, $+1$.*

Finally, let it be required to determine the product of $\sqrt[4]{-a}$ by $\sqrt[4]{-b}$, which, according to the rule, would be $\sqrt[4]{+ab}$. To determine the true result, we must observe that

$$\sqrt[4]{-a} = \sqrt[4]{a}\cdot\sqrt[4]{-1}$$
$$\sqrt[4]{-b} = \sqrt[4]{b}\cdot\sqrt[4]{-1}.$$

And ∴.

$$\sqrt[4]{-a}\times\sqrt[4]{-b} = \sqrt[4]{ab}\cdot(\sqrt[4]{-1})^2.$$

But,

$$(\sqrt[4]{-1})^2 = \left(\sqrt{\sqrt{-1}}\right)^2$$
$$= \sqrt{-1}.$$

Hence,

$$\sqrt[4]{-a}\times\sqrt[4]{-b} = \sqrt[4]{ab}\cdot\sqrt{-1}.$$

The above principles will enable the student to operate upon these quantities without embarrassment.

THEORY OF FRACTIONAL AND NEGATIVE EXPONENTS.

67. This is the proper place to explain a species of notation which is found extremely useful in algebraic calculations.

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* This may be expressed in its most general form thus, if n be any whole number:

$$(a\sqrt{-1})^{4n} = a^{4n}\times +1 = a^{4n}$$
$$(a\sqrt{-1})^{4n+1} = a^{4n+1}\times +\sqrt{-1} = a^{4n+1}\cdot\sqrt{-1}$$
$$(a\sqrt{-1})^{4n+2} = a^{4n+2}\times -1 = -a^{4n+2}$$
$$(a\sqrt{-1})^{4n+3} = a^{4n+3}\times -\sqrt{-1} = -a^{4n+3}\cdot\sqrt{-1}.$$

The first in the note corresponds to the last in the text, the second in the note to the first in the text, and the third in the note to the second in the text.

I. Let it be required to extract the n^{th} root of a quantity such as a^m. We have seen by (Art. 55) that, if m is a multiple of n, we must divide m, the index of the power, by n, the index of the root required. But if m is not divisible by n, in which case the extraction of the root is algebraically impossible, we may agree to *indicate* that operation by indicating the division of the exponents. We shall thus have

$$\sqrt[n]{a^m}=a^{\frac{m}{n}},$$

the expression $a^{\frac{m}{n}}$ being understood to signify the n^{th} root of a^m, by a convention founded upon the rule for the extraction of roots of monomial quantities. According to this convention or definition, we shall have

$$\sqrt[3]{a^2}=a^{\frac{2}{3}};\ \sqrt[4]{a^7}=a^{\frac{7}{4}}.$$

It may be observed that the denominator of the fractional exponent is the index of the radical, and the numerator the exponent of the quantity under the radical.

II. Let it be required to divide a^m by a^n. According to the rule in (Art. 17), we must subtract the index of the divisor from the index of the dividend; so that

$$\frac{a^m}{a^n}=a^{m-n};$$

it is to be remarked, however, that here it is supposed that $m>n$. But if $m<n$, in which case the division is algebraically impossible, we may agree to indicate the division by the aid of a negative index equal to the excess of n over m. Let p be the absolute difference of m and n, so that $n=m+p$; we shall then have

$$\begin{aligned}\frac{a^m}{a^n}&=\frac{a^m}{a^{m+p}}\\&=a^{m-(m+p)}\\&=a^{-p}.\end{aligned}$$

But $\frac{a^m}{a^{m+p}}$ may also be put under the form $\frac{1}{a^p}$, by suppressing the factor a^m common to both terms of the fraction; we shall then have

$$a^{-p}=\frac{1}{a^p}.$$

The expression a^{-p} is then the symbol of a division which can not be executed; and the true value of the expression is unity divided by the same letter a affected with the exponent p, taken positively. According to this convention, we shall have

$$a^{-3}=\frac{1}{a^3};\ a^{-7}=\frac{1}{a^7},\ \&c.$$

Again, by supposing the exponent of the numerator to be larger by p than the exponent of the denominator, it may be proved in a similar manner that

$$a^p=\frac{1}{a^{-p}}.$$

From these expressions it appears that a factor may be transferred from the denominator to the numerator of a fraction, or *vice versa*, by changing the sign of its exponent.

EXAMPLES.

Write $\frac{a^2b^2}{c^2d^3}$ in one line. Ans. $a^2b^2c^{-2}d^{-3}$.

Write $\frac{3a^mc^n}{d^pe^q}$ in one line. Ans. $3a^mc^nd^{-p}e^{-q}$.

Write $\frac{2g^m}{3h^{p-q}k^{-5}}$ in one line. Ans. $2\times3^{-1}g^mh^{q-p}k^5$.

Write $\frac{a^5b^4}{c^3d^2}$ all in the lower line. Ans. $\frac{1}{a^{-5}b^{-4}c^3d^2}$.

Write $\frac{A^{\alpha}B^{\beta}C}{M^{\gamma}}$ all in the lower line. Ans. $\frac{1}{A^{-\alpha}B^{-\beta}C^{-1}M^{\gamma}}$.

Write $\frac{A^5C^{-4}}{B^{-6}D^3}$ with all positive exponents. Ans. $\frac{A^5B^6}{C^4D^3}$.

Write $\frac{a^{\alpha}b^{-\beta}}{c^{\gamma}d^{-\delta}}$ with all positive exponents. Ans. $\frac{a^{\alpha}d^{\delta}}{c^{\gamma}b^{\beta}}$.

III. By combining the last two conventions, we arrive at a third notation, which is the *negative and fractional exponent.*

Let it be required to extract the n^{th} root of $\frac{1}{a^m}$.

In the first place, $\frac{1}{a^m}=a^{-m}$; hence $\sqrt[n]{\frac{1}{a^m}}=\sqrt[n]{a^{-m}}=a^{-\frac{m}{n}}$, substituting the fractional exponent for the ordinary sign of the radical.

As in words, a^m is usually enunciated *a to the power m*, *m* being a positive integer; so by analogy, $a^{\frac{m}{n}}$, a^{-m}, $a^{-\frac{m}{n}}$ are usually enunciated, *a to the power m by n*, *a to the power minus m*, and *a to the power minus m by n.*

All that has been hitherto said, with regard to fractional and negative exponents must be considered as a mere matter of definition; in short, that by a *convention* among algebraists $a^{\frac{m}{n}}$ is understood to mean the same thing as $\sqrt[n]{a^m}$, a^{-m} to be the same as $\frac{1}{a^m}$, and $a^{-\frac{m}{n}}$ as $\sqrt[n]{\frac{1}{a^m}}$. We shall now proceed *to prove* that the rules already established for the multiplication, division, formation of powers, and extraction of roots of quantities affected with positive integral exponents, are applicable without any modification, when the exponents are fractional or negative. We shall examine the different cases in succession.

68. MULTIPLICATION. Let it be required to multiply $a^{\frac{3}{5}}$ by $a^{\frac{2}{3}}$; then it is asserted that it will be sufficient to add the two exponents, and that

$$a^{\frac{3}{5}}\times a^{\frac{2}{3}}=a^{\frac{3}{5}+\frac{2}{3}}$$
$$=a^{\frac{19}{15}}.$$

For, by our definition,

$$a^{\frac{3}{5}}=\sqrt[5]{a^3}.$$

And,

$$a^{\frac{2}{3}}=\sqrt[3]{a^2};$$
$$\therefore a^{\frac{3}{5}}\times a^{\frac{2}{3}}=\sqrt[5]{a^3}\times\sqrt[3]{a^2}$$
$$=\sqrt[15]{a^{19}}$$
$$=a^{\frac{19}{15}} \text{ by definition in (Art. 67, I.).}$$

Again, let it be required to multiply $a^{-\frac{3}{4}}$ by $a^{\frac{5}{6}}$; then it is asserted that

$$a^{-\frac{3}{4}} \times a^{\frac{5}{6}} = a^{-\frac{3}{4}+\frac{5}{6}}$$
$$= a^{-\frac{9}{12}+\frac{10}{12}}$$
$$= a^{\frac{1}{12}}.$$

For,

$$a^{-\frac{3}{4}} = \sqrt[4]{\frac{1}{a^3}}, \text{ and } a^{\frac{5}{6}} = \sqrt[6]{a^5}.$$

$$\therefore a^{-\frac{3}{4}} \times a^{\frac{5}{6}} = \sqrt[4]{\frac{1}{a^3}} \times \sqrt[6]{a^5}$$
$$= \sqrt[12]{\frac{1}{a^9}} \times \sqrt[12]{a^{10}}$$
$$= \sqrt[12]{\frac{a^{10}}{a^9}}$$
$$= \sqrt[12]{a}$$
$$= a^{\frac{1}{12}} \text{ by definition in (Art. 67, I.)}$$

Generally, let it be required to multiply $a^{-\frac{m}{n}}$ by $a^{\frac{p}{q}}$; then

$$a^{-\frac{m}{n}} \times a^{\frac{p}{q}} = a^{-\frac{m}{n}+\frac{p}{q}}$$
$$= a^{\frac{np-mq}{nq}}.$$

For,

$$a^{-\frac{m}{n}} = \sqrt[n]{\frac{1}{a^m}} \text{ and } a^{\frac{p}{q}} = \sqrt[q]{a^p}$$

$$a^{-\frac{m}{n}} \times a^{\frac{p}{q}} = \sqrt[n]{\frac{1}{a^m}} \times \sqrt[q]{a^p}$$
$$= \sqrt[nq]{a^{np-mq}}$$
$$= a^{\frac{np-mq}{nq}} \text{ by definition.}$$

69. Hence we have the following general

RULE FOR EXPONENTS IN MULTIPLICATION.

In order to multiply quantities expressed by the same letter, add the exponents of that letter, whatever may be the nature of the exponents.

This is the same rule as was established in (Art. 11) for quantities affected with integral and positive exponents. According to this rule, we shall find

$$a^{\frac{3}{4}} b^{-\frac{3}{2}} c^{-1} \times a^2 b^{\frac{2}{3}} c^{\frac{3}{5}} = a^{\frac{11}{4}} b^{-\frac{5}{6}} c^{-\frac{2}{5}}$$
$$3a^{-2} b^{\frac{2}{3}} \times 2a^{-\frac{4}{5}} b^{\frac{1}{2}} c^2 = 6a^{-\frac{14}{5}} b^{\frac{7}{6}} e^2.$$

70. Division. Let it be required to divide $a^{\frac{3}{2}}$ by $a^{\frac{1}{4}}$; then it is asserted that it will be sufficient to subtract the index of the divisor from the index of the dividend, and that we shall thus have

$$\frac{a^{\frac{3}{2}}}{a^{\frac{1}{4}}} = a^{\frac{3}{2}-\frac{1}{4}}$$
$$= a^{\frac{5}{4}}.$$

For,

$$a^{\frac{3}{2}}=\sqrt[2]{a^3}, \text{ and } a^{\frac{1}{4}}=\sqrt[4]{a},$$

$$\therefore \frac{a^{\frac{3}{2}}}{a^{\frac{1}{4}}}=\frac{\sqrt[2]{a^3}}{\sqrt[4]{a}}$$

$$=\sqrt[4]{\frac{a^6}{a}} \text{ by (Art. 62)}$$

$$=\sqrt[4]{a^5}$$

$$=a^{\frac{5}{4}} \text{ by definition.}$$

In like manner, we can prove that

$$\frac{a^{\frac{5}{4}}}{a^{-\frac{3}{8}}}=a^{\frac{5}{4}-(-\frac{3}{8})}$$

$$=a^{\frac{13}{8}}.$$

Generally, let it be required to divide $a^{\frac{m}{n}}$ by $a^{\frac{p}{q}}$. Then,

$$a^{\frac{m}{n}}\div a^{\frac{p}{q}}=a^{\frac{m}{n}-\frac{p}{q}}$$

$$=a^{\frac{mq-np}{nq}}.$$

For,

$$a^{\frac{m}{n}}=\sqrt[n]{a^m}, \text{ and } a^{\frac{p}{q}}=\sqrt[q]{a^p},$$

$$\therefore a^{\frac{m}{n}}\div a^{\frac{p}{q}}=\frac{\sqrt[n]{a^m}}{\sqrt[q]{a^p}}$$

$$=\sqrt[nq]{\frac{a^{mq}}{a^{np}}}$$

$$=\sqrt[nq]{a^{mq-np}}$$

$$=a^{\frac{mq-np}{nq}} \text{ by definition.}$$

71. Hence we have the following general

RULE FOR EXPONENTS IN DIVISION.

In order to divide quantities expressed by the same letter, subtract the exponent of the divisor from the exponent of the dividend, whatever may be the nature of the exponents.

This is the same rule as that established in (Art. 17) for quantities affected with integral and positive exponents. According to this rule, we have

$$a^{\frac{2}{3}}\div a^{-\frac{3}{4}}=a^{\frac{2}{3}-(-\frac{3}{4})}$$

$$=a^{\frac{17}{12}}.$$

$$a^{\frac{3}{4}}\div a^{\frac{4}{5}}=a^{-\frac{1}{20}}.$$

$$a^{\frac{2}{5}}b^{\frac{3}{4}}\div a^{-\frac{1}{2}}b^{\frac{7}{8}}=a^{\frac{9}{10}}b^{-\frac{1}{8}}.$$

72. Formation of powers.—In order to raise a monomial to any power, the rule given in the case of positive and integral exponents was, to multiply the index of the quantity by the index of the power sought. We have now to prove that this holds good, whatever may be the nature of the exponent.

Let it be required to raise $a^{\frac{5}{7}}$ to the 4th power.
Then,

$$\left(a^{\frac{5}{7}}\right)^4 = a^{\frac{5}{7}\times 4}$$
$$= a^{\frac{20}{7}}.$$

For,

$$a^{\frac{5}{7}} = \sqrt[7]{a^5}, \text{ and } \left(a^{\frac{5}{7}}\right)^4 = (\sqrt[7]{a^5})^4.$$

But,

$$(\sqrt[7]{a^5})^4 = \sqrt[7]{a^{20}}, \text{ by (Art. 63)}$$
$$= a^{\frac{20}{7}}.$$

Generally, let it be required to raise $a^{\frac{m}{n}}$ to the power p.
Then,

$$\left(a^{\frac{m}{n}}\right)^p = a^{\frac{m}{n}\times p}$$
$$= a^{\frac{mp}{n}}.$$

For,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}, \text{ and } \left(a^{\frac{m}{n}}\right)^p = (\sqrt[n]{a^m})^p.$$

But,

$$(\sqrt[n]{a^m})^p = \sqrt[n]{a^{mp}}$$
$$= a^{\frac{mp}{n}}.$$

The demonstration will manifestly be precisely the same if we suppose one or both of the indices to be negative.

73. Hence we have the following general

RULE FOR RAISING A MONOMIAL TO ANY POWER.

Multiply the exponent of the monomial by the exponent of the power required, whatever may be the nature of the exponents.

This is the same rule as that established in (Art. 55) for quantities affected with positive integral exponents. According to this rule, we have

$$\left(a^{\frac{3}{4}}\right)^5 = a^{\frac{3}{4}\times 5}$$
$$= a^{\frac{15}{4}}$$
$$\left(a^{\frac{2}{3}}\right)^3 = a^{\frac{2}{3}\times 3}$$
$$= a^2$$
$$\left(2a^{-\frac{1}{2}}b^{\frac{3}{4}}\right)^6 = 2^6 a^{-\frac{1}{2}\times 6} b^{\frac{3}{4}\times 6}$$
$$= 64a^{-3}b^{\frac{9}{2}}.$$

74. Extraction of Roots.—In order to extract the n^{th} root of any quantity according to the rule in (Art. 55), we must divide the exponent of each letter by the index n of the root. Let us examine the case of fractional exponents.

Let it be required to extract the cube root of $a^{\frac{5}{3}}$.

Then,

$$\sqrt[3]{a^{\frac{5}{3}}}=a^{\frac{5}{3}\div 3}$$
$$=a^{\frac{5}{9}}.$$

For,

$$a^{\frac{5}{3}}=\sqrt[3]{a^5},\text{ and }\therefore\sqrt[3]{a^{\frac{5}{3}}}=\sqrt[3]{\sqrt[3]{a^5}}.$$

But,

$$\sqrt[3]{\sqrt[3]{a^5}}=\sqrt[9]{a^5},$$
$$=a^{\frac{5}{9}},\text{ by definition.}$$

Generally, let it be required to extract the p^{th} root of $a^{\frac{m}{n}}$.
Then,

$$\sqrt[p]{a^{\frac{m}{n}}}=a^{\frac{m}{np}}.$$

For,

$$a^{\frac{m}{n}}=\sqrt[n]{a^m},\text{ and }\therefore\sqrt[p]{a^{\frac{m}{n}}}=\sqrt[p]{\sqrt[n]{a^m}}.$$

But,

$$\sqrt[p]{\sqrt[n]{a^m}}=\sqrt[np]{a^m},\text{ (by Art. 58),}$$
$$=a^{\frac{m}{np}},\text{ by definition.}$$

75. Hence we have the following

RULE FOR THE EXTRACTION OF ANY ROOT OF AN ALGEBRAIC MONOMIAL.

Divide the exponent of the monomial by the exponent of the root required, whatever may be the nature of the exponents. Thus,

$$\sqrt[5]{a^{\frac{3}{4}}}=a^{\frac{3}{4}\div 5}$$
$$=a^{\frac{3}{20}}$$

$$\sqrt[3]{a^{-\frac{2}{5}}}=a^{-\frac{2}{5}\div 3}$$
$$=a^{-\frac{2}{15}}.$$

$$\sqrt[7]{a^{\frac{3}{5}}b^{-2}}=a^{\frac{3}{5}\div 7}b^{-2\div 7}$$
$$=a^{\frac{3}{35}}b^{-\frac{2}{7}}.$$

76. We shall close this discussion by an operation which includes the demonstration of every possible variety of the two preceding rules.

Let it be required to raise $a^{\frac{m}{n}}$ to the power of $-\frac{r}{s}$; we must prove that

$$\left(a^{\frac{m}{n}}\right)^{-\frac{r}{s}}=a^{\frac{m}{n}\times-\frac{r}{s}}$$
$$=a^{-\frac{mr}{ns}}.$$

If we recur to the origin of this notation, we find that

$$\left(a^{\frac{m}{n}}\right)^{-\frac{r}{s}}=\sqrt[s]{\frac{1}{\left(a^{\frac{m}{n}}\right)^r}}$$

$$=\sqrt[s]{\frac{1}{(\sqrt[n]{a^m})^r}}$$

$$=\sqrt[s]{\frac{1}{\sqrt[n]{a^{mr}}}}$$

$$=\sqrt[s]{\sqrt[n]{\frac{1}{a^{mr}}}}$$

$$=\sqrt[ns]{\frac{1}{a^{mr}}}$$

$$=\sqrt[ns]{a^{-mr}}$$

$$=a^{-\frac{mr}{ns}}, \text{ by definition.}$$

77. The notation above explained can be extended to polynomials, by including them within brackets, in the same manner as was explained in the case of integral exponents.

Thus, $(x+a)^{\frac{1}{2}}$ signifies the same thing as $\sqrt{x+a}$, or *the square root of* $x+a$.

So, $(x+a)^{-\frac{1}{2}}$ is equivalent to $\frac{1}{\sqrt{x+a}}$, or *unity divided by the square root of* $x+a$.

In like manner, $(x+a+b)^{\frac{3}{4}}$ will be the same as $\sqrt[4]{(x+a+b)^3}$, or *the fourth root of the third power of the quantity* $x+a+b$, and $(x+a+b)^{-\frac{3}{4}}$ will be unity divided by the last-mentioned quantity. Since unity is always understood to be the exponent when no other is expressed, $(x+a)^{-1}$ is the same as $\frac{1}{x+a}$, and so on. The same rules which have been established for the treatment of monomials affected with exponents will also manifestly apply to polynomials under the same restrictions.*

EXAMPLES.

(1) $a^{-\frac{3}{4}}\times a^{-\frac{7}{8}}=a^{-\frac{13}{8}}=\frac{1}{a\sqrt[8]{a^5}}$.

(2) $a^{-\frac{3}{4}}b^{-2}\times a^{\frac{5}{6}}b^{\frac{1}{2}}c=a^{\frac{1}{12}}b^{-\frac{3}{2}}c=\frac{c}{b}\sqrt[12]{\frac{a}{b^6}}$.

(3) $\frac{a}{b^{\frac{1}{2}}c^{\frac{3}{4}}}\times\frac{a^{\frac{7}{8}}b}{c^{-\frac{1}{2}}}=a^{\frac{15}{8}}b^{\frac{1}{2}}c^{-\frac{1}{4}}=a^2\sqrt[8]{\frac{b^4}{ac^2}}$.

* The calculus of fractional exponents, says Lacroix, is one of the most remarkable examples of the utility of signs, when they are well chosen. The analogy which exists between fractional and entire exponents renders the rules to be followed in the calculus of the latter applicable to the former, while particular rules are requisite for the calculus of radicals. The farther we advance in algebra, the more we perceive the numerous advantages which have resulted to that science from the notation of exponents, invented by Descartes.

(4) $a^{-\frac{m}{n}} \div a^{-\frac{p}{q}} = a^{\frac{p}{q}-\frac{m}{n}} = a^{\frac{np-mq}{nq}}$

(5) $ca^{\frac{3}{4}} \div da^{\frac{5}{6}} = \frac{c}{d} \cdot a^{-\frac{1}{12}}.$

(6) $a^{\frac{3}{5}}b^{\frac{1}{2}} \div a^{-\frac{7}{5}}b^{-\frac{1}{4}}c = \frac{a^2b^{\frac{3}{4}}}{c}.$

(7) $\frac{a^{-\frac{9}{2}}b^{\frac{2}{3}}}{c^{\frac{1}{6}}d^3} \div \frac{a^{-\frac{29}{4}}d^{\frac{11}{3}}}{b^{\frac{8}{5}}} = \frac{a^{\frac{11}{4}}b^{\frac{34}{15}}}{c^{\frac{1}{6}}d^{\frac{20}{3}}}.$

(8) $\left(a^{\frac{3}{4}}b^{\frac{2}{3}}\right)^{\frac{1}{3}} = a^{\frac{1}{4}}b^{\frac{2}{9}}.$

(9) $\left(a^2b^{-\frac{1}{2}}c^{-\frac{2}{5}}\right)^{-\frac{1}{4}} = a^{-\frac{1}{2}}b^{\frac{1}{8}}c^{\frac{1}{10}}$

(10) $\left\{\frac{c^2d}{(a+b)^{\frac{3}{2}}}\right\}^{-\frac{1}{3}} = \frac{c^{-\frac{2}{3}}d^{-\frac{1}{3}}}{(a+b)^{-\frac{1}{2}}}$

(11) $\left(a^{\frac{5}{2}}+a^2b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}+ab+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}}\right) \times \left(a^{\frac{1}{2}}-b^{\frac{1}{3}}\right) = a^3-b^2.$

(12) $\left(x^{\frac{1}{2}}+x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}\right) \times \left(x^{\frac{1}{4}}-y^{\frac{1}{4}}\right) = x^{\frac{3}{4}}-y^{\frac{3}{4}}.$

(13) $\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right) \times \left(x^{-\frac{1}{2}}+y^{-\frac{1}{2}}\right) = x^{\frac{1}{2}}y^{-\frac{1}{2}}+2+x^{-\frac{1}{2}}y^{\frac{1}{2}}.$

(14) $\frac{a^3-b^3}{a^{\frac{3}{4}}+b^{\frac{3}{4}}} = a^{\frac{9}{4}}-a^{\frac{3}{2}}b^{\frac{3}{4}}+a^{\frac{3}{4}}b^{\frac{3}{2}}-b^{\frac{9}{4}}.$

(15) $\frac{a^{\frac{7}{3}}-a^2b^{-\frac{2}{3}}-a^{\frac{1}{3}}b+b^{\frac{1}{3}}}{a^{\frac{1}{3}}-b^{-\frac{2}{3}}} = a^2-b.$

(16) $\left(a^{\frac{3}{4}}-b^{\frac{3}{4}}\right) : \left(a^{\frac{1}{4}}-b^{\frac{1}{4}}\right) = a^{\frac{1}{2}}+b^{\frac{1}{2}}+(ab)^{\frac{1}{4}}.$

(17) $a^{-5}b^{47}c^{10} \times b^{-50}a^6c^{-8} \times a^5b^2c^2 = a^6b^{-1}c^4.$

(18) $m^{\frac{3}{8}}p^{\frac{5}{7}}q^{\frac{1}{2}}r^4 \times p^{-\frac{17}{15}}q^9r^{-\frac{3}{4}}m^{12} \times p^{32}q^4 = m^{\frac{99}{8}}p^{\frac{3316}{105}}q^{\frac{27}{2}}r^{\frac{13}{4}}.$

(19) $a^{\frac{1}{2}}b^{\frac{3}{4}}c^{-5}d^{\frac{2}{3}} \div a^{\frac{1}{4}}b^2c^{\frac{7}{6}}d^{-8} = \frac{a^{\frac{1}{4}}d^{\frac{26}{3}}}{b^{\frac{5}{4}}c^{\frac{37}{6}}}.$

(20) $\left(Z^{\frac{2}{7}}+6Z^{\frac{1}{7}}a^{\frac{3}{4}}+9a^{\frac{6}{4}}\right) \cdot \sqrt{\frac{4}{5}k^8} \cdot \left(\sqrt[7]{Z}+3\sqrt[4]{a^3}\right) = \left(Z^{\frac{1}{7}}+3a^{\frac{3}{4}}\right)^3 \cdot \frac{2}{5}\sqrt{5k^8}.$

It may be asked here whether the rules for the calculus of exponents apply to incommensurable and imaginary exponents.

With regard to incommensurable exponents, it may be said that they have not absolutely of themselves any signification, and that, in order to give them one, it is necessary to conceive them in imagination, replaced by their approximate commensurable values. A formula, therefore, into which incommensurable exponents enter, should be considered as representing the limit toward which the values deduced from it tend by the substitution of commensurable numbers for the exponents, differing from them by as small a quantity as we choose to assign; in this way we perceive that the proposed expression will represent exactly this same limit, when the same operations shall have been executed upon the incommensurable exponents which it contains, as would be if they were commensurable.

Thus, for example, m and n being incommensurable quantities, we shall always have

$$a^m \times a^n = a^{m+n}.$$

For, if m' and n' represent their approximate commensurable values, we have

$$a^{m'} \times a^{n'} = a^{m'+n'}.$$

The first members of this equality tend toward the same limit as the second. But $a^m \times a^n$ represents the limit of the one, and a^{m+n} that of the other; hence, $a^m \times a^n = a^{m+n}$.

With regard to imaginary exponents, there is necessary here, as every where, a tacit admission that the general relations of real quantities, represented by letters, hold good when these letters are replaced by symbols of quantities which are imaginary.

This subject will be better understood after the student has been over that of extraction of roots by approximation.

78. Having thus discussed the formation of powers, and the extraction of roots in monomial quantities, we shall now direct our attention to polynomials; and, in the first place, let it be required to determine the square of $x+a$; then,

$$(x+a)^2 = (x+a) \times (x+a)$$
$$= x^2 + 2xa + a^2 \text{ by rules of multiplication.}$$

By inspection of this result, it is perceived that the square of a binomial contains the square of each term together with twice the product of the two.

Next, let it be required to form the square of a trinomial $(x+a+b)$. Let us represent, for a moment, the two terms, $x+a$, by the single letter z.

Then,

$$(x+a+b)^2 = (z+b)^2$$
$$= z^2 + 2zb + b^2 \ldots . (1).$$

But,

$$z^2 = (x+a)^2$$
$$= x^2 + 2xa + a^2.$$

And,

$$2zb = 2b(x+a)$$
$$= 2xb + 2ab.$$

Therefore, substituting for z^2 and $2zb$ their values in (1), we find

$$(x+a+b)^2 = x^2 + a^2 + b^2 + 2xa + 2xb + 2ab.$$

Hence it appears that *the square of a trinomial is composed of the sum of the squares of all the terms, together with the sum of twice the products of all the terms multiplied together two and two.*

We shall now prove that this law of formation extends to all polynomials, whatever may be the number of terms. In order to demonstrate this, let us suppose that it is true for a polynomial consisting of n terms, and then endeavor to ascertain whether it will hold good for a polynomial composed of $(n+1)$ terms.

Let $x+a+b+c+ \text{----} +k+l$ be a polynomial consisting of $n+1$ terms, and let us represent the sum of the first n terms by the single letter z; then

$$(x+a+b+c+ \text{---} +k+l) = (z+l),$$

and $\therefore (x+a+b+c+ \text{---} +k+l)^2 = (z+l)^2$

$$= z^2 + 2zl + l^2;$$

or, putting for z its value, $= (x+a+b+c+ \text{---} +k)^2 + 2(x+a+b+c+ \text{---} +k)l + l^2.$

But the first part of this expression, being the square of a polynomial consisting of n terms, is, by hypothesis, composed of the sum of the squares of all the terms, together with twice the sum of the products of all the terms multiplied two and two; the second part of the above expression is equal to twice the sum of the products of all the first n terms of the proposed polynomial, multiplied by the $(n+1)^{th}$ term l; and the third part is the square of the $(n+1)^{th}$ term l.

Hence, if the *law of formation* already enounced holds good for a polynomial composed of n terms, it will hold good for a polynomial composed of $(n+1)$ terms.

But we have seen above that it does hold good for a polynomial composed of *three* terms; therefore it must hold for a polynomial composed of *four* terms, and therefore for a polynomial of *five* terms, and so on in succession. Therefore the law is general, and we have the following

RULE FOR THE FORMATION OF THE SQUARE OF A POLYNOMIAL.

The square of any polynomial is composed of the sum of the squares of all the terms, together with twice the sum of the products of all the terms multiplied together two and two. According to this rule, we shall have,

(1) $(a+b+c+d+e)^2=a^2+b^2+c^2+d^2+e^2+2ab+2ac+2ad+2ae+2bc+2bd+2be+2cd+2ce+2de.$

(2) $(a-b-c+d)^2=a^2+b^2+c^2+d^2-2ab-2ac+2ad+2bc-2bd-2cd.$

If any of the terms of the proposed polynomial be affected with exponents or coefficients, we must square these monomials according to the rules already established.

(3) $(2a-4b^2c^3)^2=4a^2+16b^4c^6-16ab^2c^3.$

(4) $(3a^2-2ab+4b^2)^2=9a^4+4a^2b^2+16b^4-12a^3b+24a^2b^2-16ab^3$
$=9a^4-12a^3b+28a^2b^2-16ab^3+16b^4$, arranging according to powers of a, and reducing.

(5) $(5a^2b-4abc+6bc^2-3a^2c)^2=25a^4b^2+16a^2b^2c^2+36b^2c^4+9a^4c^2-40a^3b^2c+60a^2b^2c^2-30a^4bc-48ab^2c^3+24a^3bc^2-36a^2bc^3.$
$=25a^4b^2-40a^3b^2c+76a^2b^2c^2-48ab^2c^3+36b^2c^4-30a^4bc+24a^3bc^2-36a^2bc^3+9a^4c^2.$

79. Let us now pass on to the extraction of the square root of algebraic quantities.

Let P be the polynomial whose root is required, and let R represent the root which for the moment we suppose to be determined; let us also suppose the two polynomials, P and R, to be arranged according to the powers of some one of the letters which they contain; a, for example.

If we reflect upon the law just given of the formation of the square of a polynomial, it will be seen that the first two terms of the polynomial P, when thus arranged, are formed without reduction, and will enable us at once to determine the first two terms of the root sought; for,

1°. The square of the first term of R must involve a, affected with an exponent greater than any that is to be found in the other terms which compose the square of R; because this exponent is double the highest exponent of a in R, and must be greater than the double of any lower exponent, or than the result produced by adding it to one of the lower exponents, or by adding any two of them together.

2°. Twice the product of the first term of R by the second must contain a, affected with an exponent greater than any to be found in the succeeding terms; for it will be the sum of the highest, and the next to the highest exponent of a in R.

It follows from this, that if P be a perfect square,

I. The first term must be a perfect square; and the square root of this term, when extracted according to the rule for monomials (Art. 49), is the first term of R.

II. The second term must be divisible by twice the first term of R thus found, and the quotient will be the second term of R.

III. In order to obtain the remaining terms of R, *square the two terms of* R *already determined*, and subtract the result from P; we thus obtain a new polynomial, P′, which contains twice the product of the first term of R by the third term, together with a series of other terms. But twice the product of the first term of R by the third must contain a, affected with an exponent greater than any that is to be found in the succeeding terms, and hence this double product must form the first term of P′.*

IV. The first term of P′ must be divisible by twice the first term of R, and the quotient will be the third term of R.

V. In order to obtain the remaining terms of R, square the three terms of the root already determined, and subtract the result from the original polynomial P;† we thus obtain a new polynomial, P″, concerning which we may reason precisely in the same manner as for P′, and continuing to repeat the operation until we find no remainder, we shall arrive at the root required.

The above observations may be collected and imbodied in the following

RULE FOR THE EXTRACTION OF THE SQUARE ROOT OF ALGEBRAIC POLYNOMIALS.

1°. *Arrange the polynomial according to the powers of some one letter.*

2°. *Extract the square root of the first term according to the rule for monomials, and the result will be the first term of the root required.*

3°. *Square the first term of the root thus determined, and subtract it from the original polynomial.*

4°. *Double the first term of the root, and divide by it the first term of the remainder, and annex the result (which will be the second term of the root), with its proper sign, to the divisor.*

5°. *Multiply the whole of this divisor by the second term of the root, and subtract the product from the first remainder.*

6°. *Divide this second remainder by twice the sum of the first two terms of the root already found, and annex the result (which will be the third term of the root), with its proper sign, to the divisor.*

7°. *Multiply the whole of this divisor by the third term of the root, and subtract the product from the second remainder; continue the operation in this manner until the whole root is ascertained.*

The above process will be readily understood by attending to the following examples:

EXAMPLE 1.

Extract the square root of $10x^4-10x^3-12x^5+5x^2+9x^6-2x+1$.

Or, arranging according to the powers of x,

---

* The square of the second term of R usually contains the same exponent of the letter of arrangement, but this is already subtracted from P, and not left in P′.

† In practice, this operation is dispensed with by following the precepts 5°, 7°, in the following rule, which evidently come to the same thing.

$$\begin{array}{r|l|l}
 & 9x^6 - 12x^5 + 10x^4 - 10x^3 + 5x^2 - 2x + 1 & 3x^3 - 2x^2 + x - 1 \\
 & 9x^6 & \\
\hline
6x^3 - 2x^2 & -12x^5 + 10x^4 - 10x^3 + 5x^2 - 2x + 1 & \\
 & -12x^5 + 4x^4 & \\
\hline
6x^3 - 4x^2 + x & 6x^4 - 10x^3 + 5x^2 - 2x + 1 & \\
 & 6x^4 - 4x^3 + x^2 & \\
\hline
6x^3 - 4x^2 + 2x - 1 & -6x^3 + 4x^2 - 2x + 1 & \\
 & -6x^3 + 4x^2 - 2x + 1 & \\
\hline
 & 0. &
\end{array}$$

Having arranged the polynomial according to powers of x, we first extract the square root of $9x^6$, the first term; this gives $3x^3$ for the first term of the root required; this we place on the right hand of the polynomial, as in division; squaring this quantity, and subtracting it from the whole polynomial, we obtain for a first remainder, $-12x^5+10x^4-10x^3+5x^2-2x+1$; we now double $3x^3$, and place it as a divisor on the left of this remainder, and dividing by it $-12x^5$, the first term of the remainder, we obtain the quotient $-2x^2$ (the second term of the root sought), which we annex, with its proper sign, to the double root $6x^3$; multiplying the whole of this quantity, $6x^3-2x^2$, by $-2x^2$ (which produces twice the product of the first term of the root by the second, together with the square of the second), and subtracting the product from the first remainder, we obtain for a second remainder, $6x^4-10x^3+5x^2-2x+1$. Next, doubling $3x^3-2x^2$, the two terms of the root thus found, and dividing $6x^4$, the first term of the new remainder, by $6x^3$, the first term of the double root, we obtain x for a quotient (which is the third term of the root sought), and annex it to the double root $6x^3-4x^2$, multiplying the whole of this quantity $6x^3-4x^2+x$ by x (which produces twice the first by the third, twice the second by the third, and the square of the third), and subtracting the product from the second remainder, we obtain a third remainder, $-6x^3+4x^2-2x+1$; we now double $3x^3-2x^2+x$, the three terms of the root already found, and dividing $-6x^3$, the first term of the new remainder, by $6x^3$, the first term of the double root, we obtain -1 for the quotient (which is the fourth term of the root sought), and annex it to the double root $6x^3-4x^2+2x$; multiplying the whole of this quantity $6x^3-4x^2+2x-1$ by -1, and subtracting it from the third remainder, we find 0 for a new remainder, which shows that the root required is

$$3x^3-2x^2+x-1.$$

EXAMPLE 2.

Extract the square root of

$$
\begin{array}{l}
25a^4b^2-40a^3b^2c+76a^2b^2c^2-48ab^2c^3+36b^2c^4-30a^4bc+24a^3bc^2-36a^2bc^3+9a^4c^2 \,\big|\, 5a^2b-4abc+6bc^2-3a^2c \\
25a^4b^2 \\
10a^2b-4abc \,\big|\, -40a^3b^2c+76a^2b^2c^2 - \cdots \\
\qquad -40a^3b^2c+16a^2b^2c^2 \\
10a^2b-8abc+6bc^2 \,\big|\, 60a^2b^2c^2-48ab^2c^3+36b^2c^4 - \cdots \\
\qquad 60a^2b^2c^2-48ab^2c^3+36b^2c^4 \\
10a^2b-8abc+12bc^2-3a^2c \,\big|\, -30a^4bc+24a^3bc^2-36a^2bc^3+9a^4c^2 \\
\qquad -30a^4bc+24a^3bc^2-36a^2bc^3+9a^4c^2 \\
\qquad * \qquad *
\end{array}
$$

The root required is, therefore, $5a^2b-4abc+6bc^2-3a^2c$.

EXAMPLE 3.

Extract the square root of

$$
\begin{array}{l}
a^{2m}x^{2n+2}+10ca^{2m-2}x^{2n+3}+25c^2a^{2m-4}x^{2n+4}-6a^{m+1}x^{n+1}-30ca^{m-1}x^{n+2}+9a^2 \,\big|\, a^mx^{n+1}+5ca^{m-2}x^{n+2}-3a \\
a^{2m}x^{2n+2} \\
2a^mx^{n+1}+5ca^{m-2}x^{n+2} \,\big|\, 10ca^{2m-2}x^{2n+3}+25c^2a^{2m-4}x^{2n+4} - \cdots \\
\qquad 10ca^{2m-2}x^{2n+3}+25c^2a^{2m-4}x^{2n+4} \\
2a^mx^{n+1}+10ca^{m-2}x^{n+2}-3a \,\big|\, -6a^{m+1}x^{n+1}-30ca^{m-1}x^{n+2}+9a^2 \\
\qquad -6a^{m+1}x^{n+1}-30ca^{m-1}x^{n+2}+9a^2 \\
\qquad * \qquad *
\end{array}
$$

The root is, therefore, $a^mx^{n+1}+5ca^{m-2}x^{n+2}-3a$.

EXAMPLE 4.*

Extract the square root of

$$4x^{\frac{3}{2}}-5x^{\frac{3}{4}}a^{\frac{2}{3}} \Big| 4x^3-20x^{\frac{9}{4}}a^{\frac{2}{3}}+25x^{\frac{3}{2}}a^{\frac{4}{3}}+24x^{\frac{3}{2}}y^{\frac{5}{6}}b^{\frac{1}{2}}-60x^{\frac{3}{4}}y^{\frac{5}{6}}a^{\frac{2}{3}}b^{\frac{1}{2}}+36by^{\frac{5}{3}} \Big| 2x^{\frac{3}{2}}-5x^{\frac{3}{4}}a^{\frac{2}{3}}+6y^{\frac{5}{6}}b^{\frac{1}{2}}$$

$$-20x^{\frac{9}{4}}a^{\frac{2}{3}}+25x^{\frac{3}{2}}a^{\frac{4}{3}}\text{------}$$

$$4x^{\frac{3}{2}}-10x^{\frac{3}{4}}a^{\frac{2}{3}}+6y^{\frac{5}{6}}b^{\frac{1}{2}} \Big| 24x^{\frac{3}{2}}y^{\frac{5}{6}}b^{\frac{1}{2}}-60x^{\frac{3}{4}}y^{\frac{5}{6}}a^{\frac{2}{3}}b^{\frac{1}{2}}+36by^{\frac{5}{3}}$$

$$24x^{\frac{3}{2}}y^{\frac{5}{6}}b^{\frac{1}{2}}-60x^{\frac{3}{4}}y^{\frac{5}{6}}a^{\frac{2}{3}}b^{\frac{1}{2}}+36by^{\frac{5}{3}}$$

$$* \qquad *$$

The root is, therefore, $2x^{\frac{3}{2}}-5x^{\frac{3}{4}}a^{\frac{2}{3}}+6y^{\frac{5}{6}}b^{\frac{1}{2}}$.

(5) Extract the square root of $4x^4+8ax^3+4a^2x^2+16b^2x^2+16ab^2x+16b^4$. Ans. $2x^2+2ax+4b^2$.

(6) Extract the square root of $\frac{4}{9}a^2x^4-\frac{4}{3}abx^3z+\frac{8}{3}a^2bx^2z^2+b^2x^2z^2-4ab^2xz^3+4a^2b^2z^4$. Ans. $\frac{2}{3}ax^2-bxz+2abz^2$.

(7) Extract the square root of $\frac{9a^{2m-2}c^2}{4d^{6p}}-\frac{3a^{m+n-1}b^{2n-1}c}{d^{3p-3}}-\frac{2^8a^{m-1}b^xc}{d^{3p}}+a^{2n}b^{4n-2}d^6+\frac{2^9}{3}a^nb^{x+2n-1}d^3+\frac{2^{16}b^{2x}}{9}$. Ans. $\frac{3a^{m-1}c}{2d^{3p}}-a^nb^{2n-1}d^3-\frac{2^8b^x}{3}$.

(8) Extract the square root of $\frac{a^2x^2+2ab^2x^3+b^4x^4}{a^{2m}+2a^mx^n+x^{2n}}$. Ans. $\frac{ax+b^2x^2}{a^m+x^n}$.

(9) Extract the square root of $x^3+4x^{\frac{3}{2}}y^{\frac{5}{4}}+6x^{\frac{3}{2}}z^{\frac{7}{3}}+4y^{\frac{5}{2}}+12y^{\frac{5}{4}}z^{\frac{7}{3}}+9z^{\frac{14}{3}}$. Ans. $x^{\frac{3}{2}}+2y^{\frac{5}{4}}+3z^{\frac{7}{3}}$.

(10) Extract the square root of $4xa+12x^{\frac{1}{2}}y^{\frac{3}{2}}a^{\frac{1}{2}}b^{\frac{3}{2}}-16x^{\frac{1}{2}}z^{\frac{5}{2}}a^{\frac{1}{2}}c^{\frac{5}{2}}-20x^{\frac{1}{2}}a^{\frac{1}{2}}d^{\frac{7}{2}}+9b^3y^3-24b^{\frac{3}{2}}y^{\frac{3}{2}}c^{\frac{5}{2}}z^{\frac{5}{2}}-30b^{\frac{3}{2}}y^{\frac{3}{2}}d^{\frac{7}{2}}+16c^5z^5+40c^{\frac{5}{2}}z^{\frac{5}{2}}d^{\frac{7}{2}}+25d^7$.

Ans. $2x^{\frac{1}{2}}a^{\frac{1}{2}}+3b^{\frac{3}{2}}y^{\frac{3}{2}}-4c^{\frac{5}{2}}z^{\frac{5}{2}}-5d^{\frac{7}{2}}$.

* This example is abbreviated by omitting to square the first term, and subtract.

80. If the proposed polynomial contain several terms affected with the same power of the principal letter, we must arrange the polynomial in the manner explained in division (Art. 20); and in applying the above process we shall be obliged to perform several *partial extractions of the square roots of the coefficients* of the different powers of the principal letter, before we can arrive at the root required.

Extract the square root of

$$(a^2-2ab+b^2)x^4+2(a-b)(c-d)x^3+\{2(a-b)(f+g)+(c-d)^2\}x^2+2(c-d)(f+g)x+f^2+2fg+g^2.$$

Ans. $(a-b)x^2+(c-d)x+f+g$.

Such examples, however, very rarely occur.

Before quitting this subject, we may make the following remarks:

I. *No binomial can be a perfect square;* for the square of a monomial is a monomial, and the square of the most simple polynomial, that is, a binomial, consists of three distinct terms, which do not admit of being reduced with each other. Thus, such an expression as a^2+b^2 is not a square; it wants the term $\pm 2ab$ to render it the square of $(a\pm b)$.

II. *In order that a trinomial, when arranged according to the powers of some one letter, may be a perfect square, the two extreme terms must be perfect squares,* and the middle term must be equal to twice the product of the square roots of the extreme terms.* When these conditions are fulfilled, we may obtain the square root of a trinomial immediately, by the following

RULE.

Extract the square roots of the extreme terms, and connect the two terms thus found by the sign +, when the second term of the trinomial is positive, and by the sign —, when the second term of the trinomial is negative. Thus, the expression

$$9a^6-48a^4b^2+64a^2b^4$$

is a perfect square; for the two extreme terms are perfect squares, and the middle term is twice the product of the square roots of the extreme terms; hence the square root of the trinomial is

$$\sqrt{9a^6}-\sqrt{64a^2b^4}.$$

Or,

$$3a^3-8ab^2.$$

An expression such as $4a^2+12ab-9b^2$ can not be a perfect square, although $4a^2$ and $9b^2$, considered independently of their signs, are perfect squares, and $12ab=2(2a\,.\,3b)$; for $-9b^2$ is not a square, since no quantity, when multiplied by itself, can have the sign —.

III. In performing the operations required by the general rule, if we find that the first term of one of the remainders is not exactly divisible by twice the first term of the root, we may immediately conclude that the polynomial is not a perfect square; and when we arrive at a term in the root having a power of the letter of arrangement of a degree less than half that of this letter in the last term of the given polynomial, we may be sure that the operation will not terminate. This is on the supposition that the given polynomial is ar-

* In order that any polynomial may be a perfect square, the two extreme terms must be perfect squares, if it be arranged according to the powers of some letter.

ranged according to the decreasing powers of the letter. If it be according to the increasing powers, substitute the word greater for "less" in the above precept.

IV. We may apply to the square roots of polynomials which are not perfect squares the simplifications already employed in the case of monomials (Art. 51). Thus, in the expression

$$\sqrt{a^3b+4a^2b^2+4ab^3}.$$

The quantity under the radical sign is not a perfect square, but it may be put under the form

$$\sqrt{ab(a^2+4ab+4b)^2}.$$

The factor within brackets is manifestly the square of $a+2b$; hence

$$\begin{aligned}\sqrt{a^3b+4a^2b^2+4ab^3} &= \sqrt{ab(a^2+4ab+4b)^2}\\ &= \sqrt{ab(a+2b)^2}\\ &=(a+2b)\sqrt{ab}.\end{aligned}$$

81. Let us next proceed to form the *cube* of $x+a$.

$$\begin{aligned}(x+a)^3 &= (x+a)\times(x+a)\times(x+a)\\ &= x^3+3x^2a+3xa^2+a^3 \text{ by rules of multiplication.}\end{aligned}$$

Let it be required to form the cube of a trinomial $(x+a+b)$; represent the last two terms $a+b$ by the single letter s; then

$$\begin{aligned}(x+a+b)^3 &= (x+s)^3\\ &= x^3+3x^2s+3xs^2+s^3\\ &= x^3+3x^2(a+b)+3x(a+b)^2+(a+b)^3\\ &= x^3+3x^2a+3x^2b+3xa^2+6xab+3xb^2+a^3\\ &\quad +3a^2b+3ab^2+b^3.\end{aligned}$$

This expression is composed of the sum of the cubes of all the terms, together with three times the sum of the squares of each term, multiplied by the simple power of each of the others in succession, together with six times the product of the simple power of all the terms.

By following a process of reasoning analogous to that employed in (Art. 78), we can prove that the above law of formation will hold good for any polynomial of whatever number of terms. We shall thus find

$$\begin{aligned}(a+b+c+d)^3 &= a^3+b^3+c^3+d^3+3a^2b+3a^2c+3a^2d+3b^2a+3b^2c+3b^2d\\ &\quad +3c^2a+3c^2b+3c^2d+3d^2a+3d^2b+3d^2c+6abcd\\ (2a^2-4ab+3b^2)^3 &= 8a^6-64a^3b^3+27b^6-48a^5b+36a^4b^2+96a^4b^2+144a^2b^4\\ &\quad +54a^2b^4-108ab^5-144a^3b^3\\ &= 8a^6-48a^5b+132a^4b^2-208a^3b^3+198a^2b^4-108ab^5+27b^6.\end{aligned}$$

In a similar manner, we can obtain the 4th, 5th, &c., powers of any polynomial.

For more upon this subject, see a subsequent article (105).

82. We shall now explain the process by which we can extract the cube root of any polynomial, a method analogous to that employed for the square root, and which may easily be generalized, so as to be applicable to the extraction of roots of any degree.

Let P be the given polynomial, R its cube root. Let these two polynomials be arranged according to the powers of some one letter, a, for example. It follows, from the law of formation of the cube of a polynomial, that the cube of R contains two terms, which are not susceptible of reduction with any others; these are, the cube of the first term, and three times the square of the first term multiplied by the second term; for it is manifest that these two terms will involve a affected with an exponent higher than any that is to be found in the succeeding terms. Consequently, these two terms must form the first two terms of P. Hence, if we extract the cube root of the first term of P, we shall obtain the first term of R, and then, dividing the second term of P by three times the square of the first term of R thus found, the quotient will be the second term of R. Having thus determined the first two terms of R, cube this binomial, and subtract it from P. The remainder, P′, being arranged, its first term will be three times the product of the square of the first term of R by the third, together with a series of terms involving a, affected with a less exponent than that with which it is affected in this product. Dividing the first term of P′ by three times the square of the first term of R, the quotient will be the third term of R. Forming the cube of the trinomial root thus determined, and subtracting this cube from the original polynomial P, we obtain a new polynomial, P″, which we may treat in the same manner as P′, and continue the operation till the whole root is determined.*

EXAMPLES.

(1) Extract the cube root of $27x^3-135x^2+225x-125$.

(2) $\sqrt[3]{(8x^6+48zx^5+60z^2x^4-80z^3x^3-90z^4x^2+108z^5x-27z^6)}$.

ANSWERS.

(1) $3x-5$. | (2) $2x^2+4zx-3z^2$.

EXTRACTION OF THE SQUARE ROOT OF NUMBERS.

83. Rules are given in Arithmetic for extracting the square and cube roots of any proposed number; we shall now proceed to explain the principles upon which these rules are founded.

The numbers

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 100, 1000,

when squared, become

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 10000, 1000000,

and reciprocally, the numbers in the first line are the square roots of the numbers in the second.

Upon inspecting these two lines we perceive that, among numbers expressed by one or two figures, there are only nine which are the squares of other whole numbers; consequently, the square root of all other numbers consisting of one or two figures must be a whole number plus a fraction.

Thus, the square root of 53, which lies between 49 and 64, is 7 plus a fraction. So, also, the square root of 91 is 9 plus a fraction.

84. It is, however, very remarkable *that the square root of a whole number, which is not a perfect square, can not be expressed by an exact fraction, and is, therefore, incommensurable with unity.*

To prove this, let $\frac{a}{b}$, a fraction in its lowest terms, be, if possible, the square root of some whole number; then the square of $\frac{a}{b}$, or $\frac{a^2}{b^2}$, must be equal to this whole number. But since a and b are, by supposition, prime to each other

* This subject will be resumed a few pages farther on.

(*i. e.*, have no common divisor), a^2 and b^2 are also prime to each other;* therefore $\frac{a^2}{b^2}$ is an irreducible fraction, and can not be equal to a whole number.

85. The difference between the squares of two consecutive whole numbers is greater in proportion as the numbers themselves are greater; the expression for this difference can easily be found.

Let a and $a+1$ be two consecutive whole numbers;

Then,

$$(a+1)^2 \quad =a^2+2a+1.$$

Hence.

$$(a+1)^2-a^2=2a+1;$$

that is to say, *the difference of the squares of two consecutive whole numbers is equal to twice the less of the two numbers plus unity.*

Thus, the difference between the squares of 348 and 347 is equal to $2\times 347+1$, or 695.

---

* This depends upon the principle that, if any prime number, P, will divide the product of two numbers, it must divide one of them, which may be demonstrated as follows:

Let A and B be the two numbers, and let it be supposed that P will not divide A, we are to prove that it must divide B.

Dividing A by P, and denoting the quotient by Q and the remainder by P′, we have $A=PQ+P'$ ∴ multiplying by B, $AB=PQB+P'B$ ∴ dividing by P, $\frac{AB}{P}=QB+\frac{P'B}{P}$.

Since by hypothesis AB is divisible by P, P′B must be, else we should have a whole number, equal to a whole number plus a fraction, which is impossible. Proceed now with P and P′ after the method for finding a common divisor, and let P″, P‴, &c., be the successive remainders, which can none of them be zero, because P is by hypothesis a prime number (*i. e.*, a number divisible only by itself and unity): these remainders must go on diminishing till the last becomes unity, and we shall have the series of equalities,

$$P=P'Q'+P'',\ P'=P''Q''+P''', \text{ \&c.};$$

or, multiplying by B and dividing by P,

$$B=\frac{P'Q'B}{P}+\frac{P''B}{P},\ \frac{P'B}{P}=\frac{P''Q''B}{P}+\frac{P'''B}{P}, \text{ \&c.}$$

The first of these equalities shows that if P′B is divisible by P, P″B must also be divisible; and if both these are divisible, the second equality shows that P‴B is divisible by P, and so on. But the remainders, P″, P‴, &c., diminish till the last becomes unity, and we shall thus have, finally, $1\times B$, or B divisible by P. Q. E. D.

Now, since a^2 is the product of a and a, any prime number which divides a^2 must divide a, or which divides b^2 must divide b, so that any prime number which divides both a^2 and b^2 must divide a and b.

Every number is either prime or composed of prime numbers as factors, and if this number will divide the two terms of a fraction, its prime factors will successively divide them. This follows from (10, I., 2).

As an addition to this note may be demonstrated the following theorem: *A literal quantity can not be decomposed into prime factors in different ways.*

Let ABCD... be a product of prime factors, and suppose that it could be equal to another product, $abcd$..., the factors a, b, c, d... being also prime. The factor a, dividing $abcd$, must divide the equal ABCD...; but if the prime quantity a is different from each of the quantities A, B, C, D, &c., it can not divide any of them. Not dividing either A or B according to the above theorem, it can not divide the product AB. Not dividing either AB or C, it will not divide the product ABC, and so on. The factor a must, therefore, necessarily be equal to one of the factors A, B, C, &c. Suppose a=A. Dividing the two products by A. the remaining products, BCD... and bcd..., are still equal, and applying to them the preceding reasoning, we conclude that b ought to be equal to one of the factors of the product, BCD..., and so on. The two products, ABCD... and $abcd$..., must, therefore, be composed of the same prime factors. Q. E. D

The square of a number will always consist of twice as many digits, or one less than twice as many, as the number itself. Thus, the square of 10 is 100, and the square of any number less than 10 must be less than 100, or contain not more than two figures. The square of 100 is 10000, and the square of all numbers between 10 and 100 must be between 100 and 10000; *i. e.*, consist of 3 or 4 figures. In the same way it may be shown that the square of a number containing three figures must be one containing five or six figures, and so on; *i. e.*, the square of a number consists of twice as many digits as the number itself, or one less than twice as many.

Let us now proceed to investigate a process for the extraction of the square root of any number, beginning with whole numbers.

EXTRACTION OF THE SQUARE ROOT OF WHOLE NUMBERS.

86. If the number proposed consist of one or two figures only, its root may be found immediately by inspecting the squares of the nine first numbers in (Art. 83). Thus, the square root of 25 is 5, the square root of 42 is 6 plus a fraction, or 6 is the approximate square root of 42, and is within one unit of the true value; for 42 lies between 36, which is the square of 6, and 49, which is the square of 7.

Let us consider, then, a number composed of more than two figures, 6084 for example.

Since this number consist of four figures, its root must necessarily consist of two figures, that is to say, of tens and units. Designating the tens in the root sought by a, and the units by b, we have

```
      60′84 | 78
      49    |
148 | 118′4
    | 118′4
    |   0.
```

$$6084=(a+b)^2=a^2+2ab+b^2,$$

which shows that *the square of a number consisting of tens and units is composed of the square of the tens, plus twice the product of the tens by the units, plus the square of the units.*

This being premised, since the square of a certain number of tens must be a certain number of hundreds, or have two ciphers on the right, it follows that the squares of the tens contained in the root must be found in the part 60 (or 60 hundreds), to the left of the last two figures of 6084 (which written at full length is $6000+80+4$), the 84 forming no part of the square of the tens; we, therefore, separate the last two figures from the others by a point. The part 60 is comprised between the two perfect squares 49, and 64, the roots of which are 7 and 8; hence 7 is the figure which expresses the number of tens in the root sought; for 6000 is evidently comprised between 4900 and 6400, which are the squares of 70 and 80, and the root of 6084 must, therefore, be comprised between 70 and 80; hence, the root sought is composed of 7 tens and a certain number of units less than ten.

The figure 7 being thus found, we place it on the right of the given number, in the place of tens, separated by a vertical line as in division; we then subtract 49, which is the square of 7, from 60, which leaves as remainder 11 (which is 11 hundreds), after which we write the remaining figures, 84. Having taken away the square of the tens, the remainder, 1184, contains, as we have seen above, twice the product of the tens multiplied by the units plus the square of the units. But the product of the tens multiplied by the units must be tens, or have one cipher on the right, and, therefore, the last figure 4 can not form any part of the product of the tens by the units; we, therefore, separate it from the others by a point.

If we double the tens, which gives 14, and divide the 118 tens by 14, the quotient 8 is the figure of units in the root sought, or a figure greater than the one required. It may manifestly be greater than the figure sought, for 118 may contain, in addition to twice the product of the tens by the units, other tens arising from the square of the units, which may exceed the denomination units. In order to determine whether 8 expresses the real number of units in the root, it is sufficient to place it on the right of 14, and then multiply the number 148, thus obtained, by 8. In this manner we form, 1°, the square of the units; 2°, twice the product of the units by the tens. This operation being effected, the product is 1184; subtracting this product, the remainder is 0, which shows that 6084 is a perfect square, and 78 the root sought.

It will be seen, in reviewing the above process, that we have successively subtracted from 6084, the square of 7 tens or 70, plus twice the product of 70 by 8, plus the square of 8, that is, the three parts which enter into the composition of the square of 70+8, or 78; and since the result of this subtraction is 0, it follows that 6084 is the square of 78.

The quotient obtained from dividing by double the tens is a trial figure; it will never be too small, but may be too great, and on trial may require to be diminished by one or two units.

Take as a second example the number 841.

```
    8'41 | 29
    4    |
49 |44'1
   |441
   | 0.
```

This number being comprised between 100 and 10000, its root must consist of two figures, that is to say, of tens and units. We can prove, as in the last example, that the root of the greatest square contained in 8, or in that portion of the number to the left of the last two figures, expresses the number of tens in the root required. But the greatest square contained in 8 is 4, whose root is 2, which is, therefore, the figure of the tens. Squaring 2, and subtracting the result from 8, the remainder is 4; bringing down the figures of the second period 41, and annexing them on the right of 4, the result is 441, a number which contains twice the product of the tens by the units, plus the square of the units.

We may farther prove, as in the last case, that if we point off the last figure 1, and divide the preceding figures 44 by twice the tens, or 4, the quotient will be either the figure which expresses the number of units in the root, or a figure greater than the one sought. In this case the quotient is 11, but it is manifest that we can not have a number greater than 9 for the units, for otherwise we must suppose that the figure already found for the tens is incorrect. Let us try 9; place 9 to the right of 4, and then multiply this number 49 by 9; the product is 441, which, when subtracted from the result of the first operation, leaves a remainder 0, proving that 29 is the root required.

Let us take, as a third example, a number which is not a perfect square, such as 1287.

```
    12'87 | 35
     9    |
65 |38'7
   |325
   | 62
```

Applying to this number the process described in the preceding example, we find that the root is 35, with a remainder 62. This shows that 1287 is not a perfect square, but that it is comprised between the square of 35 and that of 36. Thus, when the number is not a perfect square, the above process enables us at least to determine the root of the greatest square contained in the number, or the integral part of the root of the number.

87. Let us pass on to consider the extraction of the square root of a number composed of more than four figures.

Let 56821444 be the number.

```
        56'82'14'44|7538
        49         |
145   | 78'2
      | 725
1503  |  571'4
      |  4509
15068 |   12054'4
      |   120544
      |        0.
```

Since the number is greater than 10000, its root must be greater than 100; that is to say, it must consist of more than two figures.* But, whatever the number may be, we may always consider it as composed of units and of tens, the tens being expressed by one or more figures. (Thus, any number such as 37142 may be resolved into 37140+2, or 3714 tens, plus two units.)

Now the square of the root sought, that is, the proposed number, contains the square of the tens, plus twice the product of the tens by the units, plus the square of the units. But the square of the tens must give at least hundreds; hence the last two figures, 44, can form no part of it, and it is in the portion of the number to the left hand that we must look for that square. But this portion containing more than two figures, its root will consist of units and tens; it will, therefore, be necessary to commence the process for finding the root of this portion by cutting off its two right-hand figures, 14, and the square of the tens of the tens is to be sought in the figures now remaining at the left, 5682. This number being the square of two figures, we again separate 82, and seek for the square of the tens of the tens of the tens in the two remaining figures, 56. The given number is thus separated into periods of two figures each, beginning on the right. We then go on to extract the root of the number 5682, as in the previous examples; this will give the tens of the root of the number 568214. We then double these tens for a divisor, and take the remainder after the last operation, with 14 annexed for a dividend; we divide this dividend, after cutting off the right-hand figure, and the quotient will be the units of the root of 568214. All the figures now found of the root will constitute the tens of the root of the given number, and we find the units by the rule previously given. The detail of the whole operation is as follows:

Extracting the root of 56, we find 7 for the root of 49, the greatest square contained in 56; we place 7 on the right of the proposed number, and squaring it, subtract 49 from 56, which gives a remainder 7, to which we annex the following period, 82. Separating the last figure to the right of 782, and then dividing 78 by 14, which is twice the root already found, we have 5 for a quotient, which we annex to 14; we then multiply the whole number 145 by 5, and subtract the product 725 from 782. We next bring down the period 14, annex it to the second remainder 57, and point off the last figure of this number 5714. Dividing 571 by 150, which is twice the root already found, the quotient is 3, which we place to the right of 150, and multiplying the whole number 1503 by 3, we subtract the product 4509 from 5714.

Finally, we bring down the last period 44, annex it to the third remainder 1205, and point off the last figure of this number 120544. Dividing 12054 by

* We have seen in the last article that it will consist of four figures, half as many as the given number. Had the given number contained but seven figures, the root would still be composed of four.

1506, which is twice the root already found, the quotient is 8, which we place on the right of 1506, and multiplying the whole number 15068 by 8, we subtract the product 120544 from the last result 120544. The remainder is 0; hence 7538 is the root sought.

From what has been said above, it is easy to deduce the rule, ordinarily given in Arithmetic, for the extraction of the square root of a number consisting of any number of figures, and which it is unnecessary here to repeat.

EXTRACTION OF THE SQUARE ROOT BY APPROXIMATION.

88. When a whole number is not the square of another whole number, we have seen (Art. 84) that its root can not be expressed by a whole number and an exact fraction; but although it is impossible to determine the precise value of the fraction which completes the root sought, we can approximate it as nearly as we please.

Suppose that a is a whole number which is not a perfect square, and that we are required to extract the root to within $\frac{1}{n}$, that is, to determine a number which shall differ from the true root of a, by a quantity less than the fraction $\frac{1}{n}$.

To effect this, let us observe that the quantity a may be put under the form $\frac{an^2}{n^2}$; if we designate the integral, or whole number, portion of the root of an^2 by r, this number an^2 will be comprised between r^2 and $(r+1)^2$; hence, $\frac{an^2}{n^2}$ is comprised between $\frac{r^2}{n^2}$ and $\frac{(r+1)^2}{n^2}$, and consequently, the root of a is comprised between the roots of $\frac{r^2}{n^2}$ and $\frac{(r+1)^2}{n^2}$, that is, between $\frac{r}{n}$ and $\frac{r+1}{n}$. Thus, it appears that $\frac{r}{n}$ represents the square root of a within $\frac{1}{n}$ of the true value.

From this we derive the following

RULE.

To extract the square root of a whole number to within a given fraction, multiply the given number by the square of the denominator of the given fraction; extract the integral part of the square root of the product, and divide this integral part by the given denominator.

Let it be required, for example, to find the square root of 59 within $\frac{1}{12}$ of the true value.

Multiply 59 by the square of 12, that is, 144, the product is 8496; the integral part of the root of 8496 is 92. Hence $\frac{92}{12}$ or $7\frac{8}{12}$ is the approximate root of 59, the result differing from the true value by a quantity less than $\frac{1}{12}$.

So, also,

$$\sqrt{11} = 3\tfrac{4}{15} \text{ true to } \tfrac{1}{15},$$
$$\sqrt{223} = 14\tfrac{37}{40} \text{ true to } \tfrac{1}{40}.$$

89. The method of *approximation in decimals*, which is the process most frequently employed, is an immediate consequence of the preceding rule.

In order to obtain the square root of a whole number within $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000} \ldots$ of the true value, we must, according to the above rule, multiply the proposed number by $(10)^2$, $(100)^2$, $(1000)^2$, ... or, which comes to the same thing, place to the right of the number, two, four, six, ... ciphers, then extract the integral part of the root of the product, and divide the result by 10, 100, 1000 .....

Hence, in order to obtain any required number of decimals in the root, we must

Place on the right hand of the proposed number twice as many zeros as we wish to have decimal figures; extract the integral part of the root of this new number, and then mark off in the result the required number of decimal places.

EXAMPLES.

(1) Extract the square root of 3 to six places of decimals.

Ans. 1.732050.

(2) Extract the square root of 5 to six places of decimals.

Ans. 2.236068.

(3) Extract the square root of 12 to six places of decimals.

Ans. 3.464101.

When half, or one more than half, the figures are found, the rest may be found by division.

(4) Extract the square root of 2 to nine places of decimals.

The first five figures of the root found by the ordinary method are 1.4142; with the remainder, 3836. The next divisor is 28284. Dividing 3836 by 28284, according to the ordinary method of division, produces 1356 for a quotient, which, annexed to 1.4142, before found, gives for the root required 1.41421356.*

Extract the square root of 11 to six places of decimals.

Ans. 3.316624.

EXTRACTION OF THE SQUARE ROOT OF FRACTIONS.

We have seen (Art. 62) that $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$; hence, in order to extract the square root of a fraction, it is sufficient to extract the square roots of the numerator and denominator, and then divide the former result by the latter. This method may be employed with advantage when either one or both of the terms of the proposed fraction are perfect squares; but when this is not the case, it will be found inconvenient in practice. If, for example, we take the fraction $\frac{3}{5}$, although $\sqrt{\frac{3}{5}}=\frac{\sqrt{3}}{\sqrt{5}}$ (since each of these expressions, when multiplied by itself, produces the same quantity, $\frac{3}{5}$), we must find an approximate value both for $\sqrt{3}$ and also for $\sqrt{5}$, and, after all, we shall not be able to determine at once the degree of approximation in the result. Under such circumstances the following process may be employed:

Let the proposed fraction be $\frac{a}{b}$, this may be put under the form $\frac{ab}{b^2}$; this being premised, let r represent the integral part of the root of the numerator

* The reason for this rule may be given thus: Let k be the part of the root already found, and z the remaining part. Then $k+z$ will be the whole root, and $(k+z)^2=k^2+2kz+z^2$ the given number; as z is but a small fraction of k, z^2 will be a still smaller fraction, and may be neglected, so that the given number may, without sensible error, be considered equal to k^2+2kz. But k^2 has been taken away, and the remainder, $2kz$, divided by $2k$, gives z.

ab; hence $\frac{ab}{b^2}$, or $\frac{a}{b}$, is comprised between $\frac{r^2}{b^2}$ and $\frac{(r+1)^2}{b^2}$; consequently, the root of $\frac{a}{b}$ is comprised between $\frac{r}{b}$ and $\frac{r+1}{b}$. Thus, it appears that $\frac{r}{b}$ represents the root of $\frac{a}{b}$ within $\frac{1}{b}$ of the true value. Hence, in order to obtain the square root of a fraction,

Make the denominator of the fraction a perfect square, by multiplying both terms of the fraction by the denominator; extract the integral part of the root of the numerator, and divide the result by the denominator.

Let it be required to extract the square root of $\frac{7}{13}$.

This fraction is the same as $\frac{7\times 13}{(13)^2}$, or $\frac{91}{(13)^2}$. But the integral part of the square root of 91 is 9; hence $\frac{9}{13}$ is the root sought, a result within $\frac{1}{13}$ of the true value.

A greater degree of approximation may, perhaps, be required. In this case, returning to the number $\frac{91}{(13)^2}$, extract the root of 91 to any required degree of approximation. Suppose, for example, we wish to find the root of 91 within $\frac{1}{100}$ of the real value, it will become by (Art. 88) $\sqrt{91}=9.53\ldots.$ Hence the root of $\frac{7}{13}$, or $\frac{91}{(13)^2}$, will be $\frac{9.53}{13}$, or a result within $\frac{1}{1300}$ of the true value.

Remark.—It frequently happens that the denominator of the fraction, although not a perfect square, has a perfect square for one of its factors, in which case the above operation may be simplified.

Let the fraction, for example, be $\frac{23}{48}$. 48 is equal to 16×3, or $(4)^2\times 3$; hence, multiplying both terms of the fraction by 3, it becomes $\frac{23\times 3}{(4)^2\times(3)^2}$, or $\frac{69}{(12)^2}$; and the denominator is thus made a perfect square. Extracting the root of 69 to $\frac{1}{10}$, which gives 8.3, we find $\frac{8.3}{12}$, or $\frac{83}{120}$ for the root required, a result within $\frac{1}{120}$ of the true value.

In general, therefore, *whenever the denominator of the fraction involves a factor which is a perfect square, multiply both terms of the fraction by the factor which is not a perfect square.*

Extract the square root of $\frac{5}{6}$ to within $\frac{1}{48}$.

$$\frac{5}{6}=\frac{5\times 6\times 8^2}{6^2\times 8^2}=\frac{1920}{6^2\times 8^2},\ \sqrt{1920}=43\ \therefore\ \sqrt{\frac{5}{6}}=\frac{43}{48}.$$

EXTRACTION OF THE SQUARE ROOT OF DECIMAL FRACTIONS.

90. This process is an immediate consequence of the preceding remark. Required, for example, the square root of 2.36.

This fraction is the same as $\frac{236}{100}$; in this case the denominator is a perfect square; extracting, therefore, the integral part of the root of the numerator, we have $\frac{15}{10}$, a result within $\frac{1}{10}$ of the true value.

Again, let it be required to extract the square root of 3.425.

This fraction is the same as $\frac{3425}{1000}$. But 1000 is not a perfect square; it is, however, equal to 100×10, or $(10)^2 \times 10$; thus, in order to render the denominator a perfect square, it is sufficient to multiply both terms of the fraction by 10, which gives $\frac{34250}{10000}$, or $\frac{34250}{(100)^2}$. Extracting the integral part of the root 34250, we find 185; hence the root required is $\frac{185}{100}$, or 1.85, a result which is within $\frac{1}{100}$ of the true value.

It appears from the above that the number of decimal places must always be made even before the operation commences.

If we wish to have a greater number of decimal places in the root, we must add on the right of 34250 twice as many zeros as we wish to have additional decimal figures.

From what has just been observed, we readily deduce for the extraction of the square root of a decimal fraction the following

RULE.

Annex ciphers till there are twice as many decimal places as are required in the root, and then proceed as in whole numbers; or, beginning at the decimal point, point off both ways the usual periods of two figures each.

EXTRACTION OF THE CUBE ROOT OF NUMBERS.

91. The numbers

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 100, 1000,

when cubed, become

1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1000000, 1000000000;

and, reciprocally, the numbers in the first line are the cube roots of the numbers in the second.

Upon inspecting the two lines, we perceive that, among the numbers expressed by one, two, or three figures, there are only nine which are *perfect cubes*; consequently, the cube root of all the rest must be a whole number plus a fraction.

92. But we can prove, in the same manner as in the case of the square root, that *the cube root of a whole number, which is not the perfect cube of some other whole number, can not be expressed by an exact fraction, and, consequently, its cube root is incommensurable with unity.*

93. The difference between the cubes of two consecutive whole numbers is greater in proportion as the numbers themselves are greater; the expression for this difference can easily be found.

Let

$$a \text{ and } a+1 \text{ be two consecutive whole numbers;}$$

Then,

$$(a+1)^3=a^3+3a^2+3a+1;$$

Hence,

$$(a+1)^3-a^3=3a^2+3a+1;$$

that is to say, *the difference of the cubes of two consecutive whole numbers is equal to three times the square of the less of the two numbers, plus three times the simple power of the number, plus unity.*

Thus, the difference between the cube of 90 and the cube of 89 is equal to $3\times(89)^2+3\times89+1=24031$.

Let us now proceed to investigate a process for the extraction of the cube root of any number.

EXTRACTION OF THE CUBE ROOT.

94. The cube root of a proposed number, consisting of one, two, or three figures only, will be found immediately by inspecting the cubes of the first nine numbers in (Art. 91). Thus, the cube root of 125 is 5, and the cube root of 54 is 3 plus a fraction, for $3\times3\times3=27$, and $4\times4\times4=64$; therefore 3 is the approximate cube root of 54, within one unit of the true value.

For the purpose of investigating a new and simple rule for the extraction of the cube root, it will be necessary to attend to the composition of a complete power of the third degree. Now, since we have

$$(a+b)^3=(a+b)(a+b)(a+b)=a^3+3a^2b+3ab^2+b^3,$$

it is obvious that the cube of a number, consisting of tens and units, will be algebraically indicated by the polynomial

$$a^3+3a^2b+3ab^2+b^3,$$

where a designates the number of tens, and b the number of units in the root sought. The number in the tens' place will evidently be found by extracting the cube root of the monomial a^3, for $\sqrt[3]{a^3}=a$, and removing a^3 from the polynomial $a^3+3a^2b+3ab^2+b^3$, we have the remainder,

$$3a^2b+3ab^2+b^3=(3a^2+3ab+b^2)b;$$

and the difficulty that has been hitherto experienced in the extraction of the cube root entirely consists in the composition of the expression $3a^2+3ab+b^2$, which is obviously the true divisor by which to divide the remainder, after subtracting a^3, or the cube of the tens, for the determination of b, the figure of the root in the place of units. The part $3a^2$ of the expression $3a^2+3ab+b^2$, being independent of b, the yet unknown part of the root, is employed as a *trial* divisor for the determination of b; but since the expression $3a^2+3ab+b^2$ involves the unknown part of the root in its composition, it is obvious that the trial divisor $3a^2$, which does not contain b, will, at the first step of the operation, give no certain indication of the next figure of the root, unless the figure denoted by b be very small in comparison with that denoted by a; for the trial divisor $3a^2$ will be considerably augmented by the addend $3ab+b^2$ when b is a large number, while the augmentation, when b is a small number, will not so materially affect the trial divisor.

When the figure in the tens' place is a small number, as 1 or 2, it is hence obvious that little or no dependence can be placed on the trial divisor; but if a

be great and b small, the trial divisor, $3a^2$, will generally point out the value of b. All this will be evident if we consider that the relative values of a and b materially affect the true divisor, $3a^2+3ab+b^2$. In the successive steps, however, of the cube root this uncertainty diminishes; for, conceiving a to designate a number consisting of tens and hundreds, and b the number o units, then the value of b being small in comparison with a, the amount of the effect of b in the addend $3ab+b^2$ will be very inconsiderable; hence the trial divisor, $3a^2$, will generally indicate the next figure in the root.

To remove, in some measure, the difficulty which has hitherto been experienced in the extraction of the cube root, we shall proceed to point out *two* methods of composing the true divisor, $3a^2+3ab+b^2$, and leave the student to select that which he conceives to possess the greater facility of operation.*

95. *First method of composition of* $3a^2+3ab+b^2$.

$a \times a$	$= a^2$	$a^3+3a^2b+3ab^2+b^3$ ($a+b=$ root sought.
a	$a^2 \times a = \ldots\ldots$	a^3
a	a^2	———
—	——	$3a^2b+3ab^2+b^3$
	$3a^2$	
$(3a+b)\times b=$	$3ab+\ b^2$	
b	———	
b	$(3a^2+3ab+\ b^2)\times b = \ldots\ldots$	$3a^2b+3ab^2+b^3$
	b^2	———
——	———	
$3a+3b$	$3a^2+6ab+3b^2$.	

Distinguishing the three columns from left to right by *first*, *second*, and *third* columns, we write a in the root, and also three times vertically in the first column; then $a\times a$ produces a^2, which write, also, three times vertically in the second column; multiply the second a^2 by a, placing the product, a^3, under a^3 in the third column; then, subtracting a^3 from the proposed quantity, we have the remainder, $3a^2b+3ab^2+b^3$. The sum of the three quantities in the second column gives $3a^2$ for the trial divisor, by which find b, the next figure of the root, and to $3a$, the sum of the last three written quantities in the first column, annex b; then the sum, $3a+b$, is multiplied by b, and the product, $3ab+b^2$, is placed in the second column; then the trial divisor, $3a^2$, and the addend, $3ab+b^2$, being collected, give the true divisor, $3a^2+3ab+b^2$, which multiply by b, and place the product, $3a^2b+3ab^2+b^3$, under the remainder, $3a^2b+3ab^2+b^3$. When there is a remainder after this operation, the process may be continued by writing b twice in the first column, under $3a+b$, and b^2 once in the second column, under the last true divisor; then $3a^2+6ab+3b^2$, the sum of the last written three lines in the second column, will be another trial divisor, with which proceed as above. We have written a^2 in the second column three times in succession, to assimilate the first step in the operation to the other successive steps, but the first trial divisor, $3a^2$, may be written at once, and the symmetry of the disposition of the quantities in the first steps disregarded.†

* These methods may be passed over by the student, as well as that given for the biquadrate root, and the method employed, which is described at (Art. 112), which is applicable to the extraction of the root of the third and fourth, as well as of any other degree.

† Three quantities are added each time; in the method on next page, two.

96. *Second method of composing* $3a^2+3ab+b^2$, *the true divisor.*

a		$a^3+3a^2b+3ab^2+b^3\ (a+b$
a	a^2	a^3
$2a$	$2a^2$	$3a^2b+3ab^2+b^3$
a	$3a^2$	
$3a+b$	$3ab+b^2$	
b	$3a^2+3ab+b^2$	$3a^2b+3ab^2+b^3$
$3a+2b$	$3ab+2b^2$	
b		
$3a+3b$	$3a^2+6ab+3b^2=$ second trial divisor.	

In this method we write a under a in the first column, and the sum $2a$ being multiplied by a, gives $2a^2$ to place under a^2 in the second column, and the sum of $2a^2$ and a^2 is $3a^2$ for the trial divisor. Again, under $2a$ in the first column write a, and the sum of $2a$ and a gives $3a$. Now, having found b by the trial divisor, annex it to $3a$ in the first column, making $3a+b$, which, multiplied by b, and the product placed in the second column, gives, by addition, the true divisor, $3a^2+3ab+b^2$, as before. We shall exhibit the operation of extracting the cube root by both these methods.

EXAMPLES.

(1) What is the cube root of $x^6-9x^5+39x^4-99x^3+156x^2-144x+64$?

By the first method.

x^2	x^4	$x^6-9x^5+39x^4-99x^3+156x^2-144x+64\ (x^2-3x+4$
x^2	x^4	x^6
x^2	x^4	
	$3x^4$	$-9x^5+39x^4-99x^3$
$3x^2-3x$ ..	$-9x^3+9x^2$	
$-3x$	$3x^4-9x^3+9x^2$	$-9x^5+27x^4-27x^3$
	$9x^2$	
$-3x$		$12x^4-72x^3+156x^2-144x+64$
	$3x^4-18x^3+27x^2$	
$3x^2-9x+4$ ...	$12x^2-36x+16$	
	$3x^4-18x^3+39x^2-36x+16$ ...	$12x^4-72x^3+156x^2-144x+64.$

(2) What is the cube root of $x^6+6x^5-40x^3+96x-64$?

By the second method.

x^2		$x^6+6x^5-40x^3+96x-64\ (x^2+2x-4$
x^2	x^4	x^6
$2x^2$	$2x^4$	$6x^5-40x^3$
x^2	$3x^4$	
$3x^2+2x$ ...	$6x^3+4x^2$	
$2x$	$3x^4+6x^3+4x^2$	$6x^5+12x^4+8x^3$
$3x^2+4x$ ...	$6x^3+8x^2$	
$2x$		$-12x^4-48x^3+96x-64$
	$3x^4+12x^3+12x^2$	
$3x^2+6x-4$ ..	$-12x^2-24x+16$	
	$3x^4+12x^3-24x+16$	$-12x^4-48x^3+96x-64.$

(3) What is the cube root of $a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$? Ans. $a+b+c$.

(4) Extract the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

Ans. x^2-2x+1.

97. The same process is employed in the extraction of the cube root of numbers, as in the subsequent examples.

EXAMPLES.

(1) Extract the cube root of 403583419.

		403583419 (739 = root
7	49	
7	49	343
7	49	
		60583
	147	
213	639	
3		
3	15339	46017
	9	
		14566419
	15987	
2199	19791	
	1618491	14566419.

(2) What is the cube root of 115501303?

		115501303 (487 = root
4	16	64
4		51501
8	32	
4	48	
128	1024	
8		
	5824	46592
136	1088	
8		4909303
	6912	
1447.	10129	
	701329	4909303.

98. The local values of the figures in the root determine the arrangement of the figures in the several columns, as is exemplified by working the last example as on next page; by omitting the terminal ciphers, the arrangement is precisely the same as in the preceding example.

		115501303 (400+80+7
400	160000	64000000
400		
		51501303
800	320000	
400		
	480000	
1200		
80		
1280	102400	
80		
	582400	46592000
1360	108800	
80		4909303
	691200	
1440		
7		
1447	10129	
	701329	4909303

99. *Extraction of the fourth root of whole numbers.*

The investigation of a method for extracting the fourth root of any number is similar to that employed for the cube root. Thus, since

$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4,$$

we may conceive a to denote the number of tens, and b the number of units in the root of the number expressed by $a^4+4a^3b+6a^2b^2+4ab^3+b^4$. Then $\sqrt[4]{a^4}=a$, the figure in the tens' place, and the remainder, when a^4 is removed, is

$$4a^3b+6a^2b^2+4ab^3+b^4=(4a^3+6a^2b+4ab^2+b^3)b.$$

The method of composing the divisor $4a^3+6a^2b+4ab^2+b^3$, for the determination of b, the figure in the units' place, may be illustrated as follows:

$a\times a=a^2$				$a^4+4a^3b+6a^2b^2+4ab^3+b^4\ (a+b$
a	$a^2\times a=a^3$			
$2a\times a=2a^2$		$a^3\times a=a^4$		
a	$3a^2\times a=3a^3$			$4a^3b+6a^2b^2+4ab^3+b^4$
$3a\times a=3a^2$	$4a^3$			
a				
$6a^2$				

$$(4a+b)b=4ab+b^2$$

$$(6a^2+4ab+b^2)b=6a^2b+4ab^2+b^3$$

$$(4a^3+6a^2b+4ab^2+b^3)b=4a^3b+6a^2b^2+4ab^3+b^4.$$

100. From this mode of composing the complete divisor we easily derive the following process for the extraction of the fourth root of any number.

EXAMPLE.

What is the fourth root of 1185921?

3×3	$=$	9				1185921 $(33 =$ root
3		9×3	$=$	27		
6×3	$=$	18		27×3	$=$	81
3		27×3	$=$	81		375921
9×3	$=$	27		$108\ .\ .\ .$		
3						
		$54\ .\ .$				
123×3	$=$	369				
		5769×3	$=$	17307		
				125307×3	$=$	375921

In the same manner, the student may readily investigate rules for the extraction of the higher roots of numbers, simply observing to use an additional column for each successive root.

101. *To represent a rational quantity as a surd.*

Let it be required to represent a in the form of a surd of the nth order; then, by (Art. 63), the form will be $\sqrt[n]{a^n}$, or $(a^n)^{\frac{1}{n}}$; for by raising a to the nth power, and then extracting the nth *root* of the nth *power* of a, we must evidently revert to the proposed quantity, a. Hence we have

$$a=\sqrt{a^2}=\sqrt[3]{a^3}=\sqrt[4]{a^4}=\sqrt[15]{a^{15}}=\sqrt[m]{a^m}=\sqrt[n]{a^n}$$

$$a=(a^2)^{\frac{1}{2}}=(a^{\frac{3}{2}})^{\frac{2}{3}}=(a^{\frac{6}{5}})^{\frac{5}{6}}=(a^{\frac{m}{n}})^{\frac{n}{m}}.$$

102. When the given quantity is the product of a rational quantity and a surd, we must represent the rational quantity in the form of the given surd, and then express the product with a single radical sign, or fractional index. Thus, we have

$$a\sqrt{b}=\sqrt{a^2}\times\sqrt{b}=\sqrt{a^2b}$$

$$3a\sqrt{5b}=\sqrt{3a\times 3a}\times\sqrt{5b}=\sqrt{9a^2\times 5b}=\sqrt{45a^2b}$$

$$a\sqrt[3]{xy}=\sqrt[3]{a\times a\times a}\times\sqrt[3]{xy}=\sqrt[3]{a^3}\times\sqrt[3]{xy}=\sqrt[3]{a^3xy}$$

$$12\sqrt{7}=\sqrt{144}\times\sqrt{7}=\sqrt{144\times 7}=\sqrt{1008}$$

$$a(1-a^{-2}x^2)^{\frac{1}{2}}=(a^2)^{\frac{1}{2}}(1-a^{-2}x^2)^{\frac{1}{2}}=(a^2-a^0x^2)^{\frac{1}{2}}=\sqrt{a^2-x^2}.$$

EXAMPLES.

(1) Represent a^2 in the form of a surd, whose index is 5.

(2) Represent $2-\sqrt{3}$ in the form of a quadratic surd.

(3) Transform $6\sqrt{11}$ into the form of a quadratic surd.

(4) Transform $a\sqrt{a-b}$ into the form of a quadratic surd.

(5) Represent as a surd the mixed quantity $(x+y)\sqrt{\dfrac{x-y}{x+y}}$.

(6) Represent as a surd the mixed quantity $(x+4)\sqrt{\dfrac{1}{x+4}}$.

ANSWERS.

(1) $\sqrt[5]{a^{10}}$ or $(a^{10})^{\frac{1}{5}}$.

(2) $\sqrt{7-4\sqrt{3}}$.

(3) $\sqrt{396}$.

(4) $\sqrt{a^3-a^2b}$ or $(a^3-a^2b)^{\frac{1}{2}}$.

(5) $\sqrt{x^2-y^2}$ or $(x^2-y^2)^{\frac{1}{2}}$.

(6) $\sqrt{x+4}$ or $(x+4)^{\frac{1}{2}}$.

103. *To find multipliers which will render binomial surds rational.*

The product of two irrational quantities is, in many instances, a rational quantity, and, therefore, an irrational quantity may frequently be found, which, employed as a factor to multiply some other given irrational quantity, will produce a rational result; thus,

$$\sqrt{a}\times\sqrt{a}=a$$
$$\sqrt[3]{x}\times\sqrt[3]{x^2}=x$$
$$\sqrt[m]{y}\times\sqrt[m]{y^{m-1}}=y.$$

Again, since the product of the sum and difference of two quantities is equal to the difference of their squares, we have, evidently,

$$(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})=a-b$$
$$(x+\sqrt{y})(x-\sqrt{y})=x^2-y$$
$$(\sqrt{x}-y)(\sqrt{x}+y)=x-y^2.$$

Hence it is obvious that, in these and similar equalities, if one of the factors be given, the other factor or multiplier is readily known, and the proposed irrational quantity is thus rendered rational. By a double operation of this kind, multiplying $(\sqrt{n}+\sqrt{p}+\sqrt{q})$ by $(\sqrt{n}+\sqrt{p}-\sqrt{q})$, we have $(\sqrt{n}+\sqrt{p})^2-q$, or $n+p-q+2\sqrt{np}$; and multiplying this by $n+p-q-2\sqrt{np}$, the given expression, $\sqrt{n}+\sqrt{p}+\sqrt{q}$, is rationalized. In the same manner, since

$$(x\pm y)(x^2\mp xy+y^2)=x^3\pm y^3$$
$$\therefore (\sqrt[3]{x}\pm\sqrt[3]{y})(\sqrt[3]{x^2}\mp\sqrt[3]{xy}+\sqrt[3]{y^2})=x\pm y,$$

and the expression $\sqrt[3]{x}\pm\sqrt[3]{y}$ may, therefore, be rationalized by multiplying it by $\sqrt[3]{x^2}\mp\sqrt[3]{xy}+\sqrt[3]{y^2}$; and $\sqrt[3]{x^2}\mp\sqrt[3]{xy}+\sqrt[3]{y^2}$, multiplied by $\sqrt[3]{x}\pm\sqrt[3]{y}$, will produce a rational result.

Again, by division [see Art. 23 (5), (6), (7)],

$$\frac{x^n-y^n}{x-y}=x^{n-1}+x^{n-2}y+x^{n-3}y^2+x^{n-4}y^3+\ldots\ldots+y^{n-1}$$
$$\frac{x^n-y^n}{x+y}=x^{n-1}-x^{n-2}y+x^{n-3}y^2-x^{n-4}y^3+\ldots\ldots-y^{n-1}$$
$$\frac{x^n+y^n}{x+y}=x^{n-1}-x^{n-2}y+x^{n-3}y^2-x^{n-4}y^3+\ldots\ldots+y^{n-1}.$$

Put $x^n=a$; then $x=\sqrt[n]{a}$; $x^{n-1}=\sqrt[n]{a^{n-1}}$; $x^{n-2}=\sqrt[n]{a^{n-2}}$, &c.;
$y^n=b$; then $y=\sqrt[n]{b}$; $y^2=\sqrt[n]{b^2}$; $y^3=\sqrt[n]{b^3}$, &c.;

hence, by substitution in the three preceding equalities, we have

$$\frac{a-b}{\sqrt[n]{a}-\sqrt[n]{b}}=\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}+\sqrt[n]{a^{n-4}b^3}+\ldots+\sqrt[n]{b^{n-1}}.\ (1)$$

$$\frac{a-b}{\sqrt[n]{a}+\sqrt[n]{b}}=\sqrt[n]{a^{n-1}}-\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}-\sqrt[n]{a^{n-4}b^3}+\ldots-\sqrt[n]{b^{n-1}}.\ (2)$$

$$\frac{a+b}{\sqrt[n]{a}+\sqrt[n]{b}}=\sqrt[n]{a^{n-1}}-\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}-\sqrt[n]{a^{n-4}b^3}+\ldots+\sqrt[n]{b^{n-1}}.\ (3)$$

Now, the dividend being the product of the divisor and quotient, it is obvious that a binomial surd of the form $\sqrt[n]{a}-\sqrt[n]{b}$ will be rendered rational by multiplying it by n terms of the second side of equation (1), and a binomial surd of the form $\sqrt[n]{a}+\sqrt[n]{b}$ will be rationalized by employing n terms of the second side of equality (2), or (3), according as n is even or odd, the product in the former case being $a-b$, and in the latter $a-b$ or $a+b$.

Note.—When n is an even number, employ equation (2), and when it is an odd number, equation (3), in order to rationalize $\sqrt[n]{a}+\sqrt[n]{b}$.

EXAMPLES.

(1) Find a multiplier to rationalize $\sqrt[3]{11}-\sqrt[3]{7}$.

Employing equation (1), we have $a=11$, $b=7$, and $n=3$; hence required multiplier $=\sqrt[3]{11^2}+\sqrt[3]{11.7}+\sqrt[3]{7^2}=\sqrt[3]{121}+\sqrt[3]{77}+\sqrt[3]{49}$.

And,
$$\begin{array}{llll} \sqrt[3]{121} & +\sqrt[3]{77} & +\sqrt[3]{49} & \\ \sqrt[3]{11} & -\sqrt[3]{7} & & \\ \hline \sqrt[3]{1331} & +\sqrt[3]{847} & +\sqrt[3]{539} & \\ & -\sqrt[3]{847} & -\sqrt[3]{539} & -\sqrt[3]{343} \\ \hline 11 & * & * & -\ 7 \end{array} =4, \text{ a rational product.}$$

(2) Rationalize the binomial surd $\sqrt[3]{5}+\sqrt[3]{4}$.

Here we have $a=5$, $b=4$, $n=3$, an odd number; hence by equation (3) we have multiplier required, $=\sqrt[3]{25}-\sqrt[3]{20}+\sqrt[3]{16}$; and, by multiplication, $(\sqrt[3]{5}+\sqrt[3]{4})(\sqrt[3]{25}-\sqrt[3]{20}+\sqrt[3]{16})=5+4=9=$ a rational number.

(3) What multiplier will render the denominator of the fraction $\dfrac{1}{\sqrt[5]{7}-\sqrt[5]{2}}$ a rational quantity?

(4) Change $\dfrac{5}{\sqrt[3]{4}-\sqrt[3]{2}}$ into a fraction that shall have a rational denominator.

(5) Change $\dfrac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}\pm\sqrt[3]{xy}+\sqrt[3]{y^2}}$ into a fraction that shall have a rational denominator.

(6) Change $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$ into a fraction that shall have a rational denominator.

ANSWERS.

(3) $\sqrt[5]{7^4}+\sqrt[5]{7^3.2}+\sqrt[5]{7^2.2^2}+\sqrt[5]{7.2^3}+\sqrt[5]{2^4}$.

(4) $\dfrac{5(\sqrt[3]{16}+\sqrt[3]{8}+\sqrt[3]{4})}{2}$.

(5) $\dfrac{\sqrt[3]{x^2}(\sqrt[3]{x}\mp\sqrt[3]{y})}{x\pm y}=\dfrac{x\mp\sqrt[3]{x^2y}}{x\mp y}$

(6) $\dfrac{a+\sqrt{a^2-x^2}}{x}$.

104. *To extract the square root of a binomial surd.*

Before commencing the investigation of the formula for the extraction of the square root of a binomial surd, it will be necessary to premise two or three lemmas.

Lemma 1. The square root of a quantity can not be partly rational and partly irrational.

For, if $\sqrt{a}=b+\sqrt{c}$, then, by squaring, we have

$$a=b^2+c+2b\sqrt{c}; \text{ therefore, } \sqrt{c}=\frac{a-b^2-c}{2b};$$

that is, an irrational equal to a rational quantity, which is absurd.

Lemma 2. If $a \pm \sqrt{b} = x \pm \sqrt{y}$ be an equation consisting of rational and irrational quantities, then $a = x$, and $\sqrt{b} = \sqrt{y}$; *i. e.*, the rational and irrational parts of the two members of an equation must be separately equal.

For, if a be not equal to x, let $a - x = d$; then we have

$$\pm \sqrt{y} \mp \sqrt{b} = a - x; \text{ but } a - x = d; \text{ therefore}$$
$$\pm \sqrt{y} \mp \sqrt{b} = d, \text{ which is impossible;}$$
$$\therefore a = x, \text{ and, taking away these equals, } \sqrt{b} = \sqrt{y}.$$

Lemma 3. If $\sqrt{a + \sqrt{b}} = x + y$, then $\sqrt{a - \sqrt{b}} = x - y$; where x and y are supposed to be one or both irrational quantities.

For, since $a + \sqrt{b} = x^2 + y^2 + 2xy$; and since x^2 and y^2 are both rational, $2xy$ must be irrational. By Lemma 2, we have

$$a = x^2 + y^2; \quad \sqrt{b} = 2xy$$
$$\therefore a - \sqrt{b} = x^2 - 2xy + y^2$$
$$\text{and } \sqrt{a - \sqrt{b}} = x - y.$$

Let it now be required to extract the square root of $a + \sqrt{b}$.

Assume $\sqrt{a + \sqrt{b}} = x + y$; then $\sqrt{a - \sqrt{b}} = x - y$

$$\therefore a + \sqrt{b} = x^2 + y^2 + 2xy$$
$$a - \sqrt{b} = x^2 + y^2 - 2xy$$

$\therefore$ By addition, $2a = 2(x^2 + y^2)$, or $a = x^2 + y^2$.

Again, $\sqrt{a + \sqrt{b}} \times \sqrt{a - \sqrt{b}} = x^2 - y^2$, or $\sqrt{a^2 - b} = x^2 - y^2$.

Hence
$$x^2 + y^2 = a$$
$$x^2 - y^2 = \sqrt{a^2 - b} = c, \text{ suppose.}$$

Therefore, by addition and subtraction, we have

$$x^2 = \frac{a + c}{2} \text{ and } y^2 = \frac{a - c}{2}$$

$$\therefore x = \sqrt{\frac{a + c}{2}} \text{ and } y = \sqrt{\frac{a - c}{2}}.$$

Hence $$\sqrt{a + \sqrt{b}} = \sqrt{\frac{a + c}{2}} + \sqrt{\frac{a - c}{2}} \quad \ldots\ldots\ldots \quad (1)$$

$$\sqrt{a - \sqrt{b}} = \sqrt{\frac{a + c}{2}} - \sqrt{\frac{a - c}{2}} \quad \ldots\ldots\ldots \quad (2)$$

where $c = \sqrt{a^2 - b}$; and, therefore, $a^2 - b$ must be a perfect square; and this is the *test* by which we discover the possibility of the operation proposed.*

* When the quantity $a^2 - b$ is not a square, the values of a and b are no longer rational; but it is clear that the formulas (1) and (2) will still give true results. As, however, these are more complicated than the original expressions themselves, they are rarely employed; yet, when $\sqrt{b}$ is imaginary, the result merits attention.

In order to examine this case, change b into $-b^2$; $a + \sqrt{b}$ becomes $a + b\sqrt{-1}$. The remarkable circumstance just alluded to is this, that the square root of $a + b\sqrt{-1}$ has the same form as this quantity itself.

This is shown from the formula (1), for since $c = \sqrt{a^2 + b^2}$, when b is changed into $-b^2$, the second member becomes $\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} + \sqrt{\frac{a - \sqrt{a^2 + b^2}}{2}}$. The quantity under the first radical is positive, and that under the second negative, since $\sqrt{a^2 + b^2}$ is greater than

EXAMPLES.

(1) What is the square root of $11+\sqrt{72}$, or $11+6\sqrt{2}$?

Here $a=11$; $b=72$; $c=\sqrt{a^2-b}=\sqrt{121-72}=7$

$$\therefore \sqrt{11+6\sqrt{2}}=\sqrt{\frac{a+c}{2}}+\sqrt{\frac{a-c}{2}}=3+\sqrt{2}.$$

(2) What is the square root of $23-8\sqrt{7}$?

Here $a=23$; $b=8^2\times 7=448$; $c=\sqrt{a^2-b}=\sqrt{529-448}=9$

$$\therefore \sqrt{23-8\sqrt{7}}=\sqrt{\frac{a+c}{2}}-\sqrt{\frac{a-c}{2}}=4-\sqrt{7}.$$

(3) What is the square root of $14\pm 6\sqrt{5}$? Ans. $3\pm\sqrt{5}$.

(4) What is the square root of $18\pm 2\sqrt{77}$? Ans. $\sqrt{11}\pm\sqrt{7}$.

(5) What is the square root of $94+42\sqrt{5}$? Ans. $7+3\sqrt{5}$.

(6) To what is $\sqrt{np+2m^2-2m\sqrt{np+m^2}}$ equal? Ans. $\sqrt{np+m^2}-m$.

(7) Simplify the expression $\sqrt{16+30\sqrt{-1}}+\sqrt{16-30\sqrt{-1}}$. Ans. 10.

(8) To what is $\sqrt{28+10\sqrt{3}}$ equal? Ans. $5+\sqrt{3}$.

(9) $\sqrt{bc+2b\sqrt{bc-b^2}}+\sqrt{bc-2b\sqrt{bc-b^2}}=\pm 2b$

(10) $\sqrt{ab+4c^2-d^2+2\sqrt{4abc^2-abd^2}}=\sqrt{ab}+\sqrt{4c^2-d^2}$.

(11) What is the square root of $-2\sqrt{-1}$? Ans. $1-\sqrt{-1}$.

(12) What is the square root of $3-4\sqrt{-1}$? Ans. $2-\sqrt{-1}$.

(13) What is square root of $\frac{3\sqrt{3}+2\sqrt{6}}{4}\cdot\frac{112+20\sqrt{12}}{\sqrt{3}}$?

Ans. $(1+\sqrt{2})\cdot(5+\sqrt{3})$.

BINOMIAL THEOREM.

105. It is manifest, from what has been said above, that algebraic polynomials may be raised to any power merely by applying the rules of multiplication. We can, however, in all cases obtain the desired result without having recourse to this operation, which would frequently prove exceedingly tedious. When a binomial quantity of the form $x+a$ is raised to any power, the successive terms are found in all cases to bear a certain relation to each other. This law, when expressed generally in algebraic language, constitutes what is called the "Binomial Theorem." It was discovered by Sir Isaac Newton, who seems to have arrived at the general principle by examining the results of actual multiplication in a variety of particular cases, a method which we shall here pursue, and give a rigorous demonstration of the proposition in a subsequent article of this treatise.

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a; representing the quantity under the first radical by a^2, and that under the second by $-\beta^2$, the expression takes the form $a+\beta\sqrt{-1}$; hence

$$\sqrt{a+b\sqrt{-1}}=a+\beta\sqrt{-1}.$$

Q. E. D.

Let us form the successive powers of $x+a$ by actual multiplication.

$$
\begin{array}{l}
x+a \\
x+a \\
\hline
x^2+xa \\
\quad +xa+a^2 \\
\hline
x^2+2xa+a^2 \quad \ldots\ldots\ldots\ldots \text{2d power.} \\
x+a \\
\hline
x^3+2x^2a+xa^2 \\
\quad +x^2a+2xa^2+a^3 \\
\hline
x^3+3x^2a+3xa^2+a^3 \quad \ldots\ldots\ldots\ldots \text{3d power.} \\
x+a \\
\hline
x^4+3x^3a+3x^2a^2+xa^3 \\
\quad +x^3a+3x^2a^2+3xa^3+a^4 \\
\hline
x^4+4x^3a+6x^2a^2+4xa^3+a^4 \quad \ldots\ldots\ldots\ldots \text{4th power.} \\
x+a \\
\hline
x^5+4x^4a+6x^3a^2+4x^2a^3+xa^4 \\
\quad +x^4a+4x^3a^2+6x^2a^3+4xa^4+a^5 \\
\hline
x^5+5x^4a+10x^3a^2+10x^2a^3+5xa^4+a^5 \quad \ldots\ldots\ldots\ldots \text{5th power.} \\
x+a \\
\hline
x^6+5x^5a+10x^4a^2+10x^3a^3+5x^2a^4+xa^5 \\
\quad +x^5a+5x^4a^2+10x^3a^3+10x^2a^4+5xa^5+a^6 \\
\hline
x^6+6x^5a+15x^4a^2+20x^3a^3+15x^2a^4+6xa^5+a^6 \quad \ldots\ldots \text{6th power.} \\
x+a \\
\hline
x^7+6x^6a+15x^5a^2+20x^4a^3+15x^3a^4+6x^2a^5+xa^6 \\
\quad +x^6a+6x^5a^2+15x^4a^3+20x^3a^4+15x^2a^5+6xa^6+a^7 \\
\hline
x^7+7x^6a+21x^5a^2+35x^4a^3+35x^3a^4+21x^2a^5+7xa^6+a^7 \quad \ldots \text{7th power.}
\end{array}
$$

In order that these results may be more clearly exhibited to the eye, we shall arrange them in a table.

TABLE OF THE POWERS OF $x+a$.

$(x+a)$	$x+a$
$(x+a)^2$	$x^2+2xa+a^2$
$(x+a)^3$	$x^3+3x^2a+3xa^2+a^3$
$(x+a)^4$	$x^4+4x^3a+6x^2a^2+4xa^3+a^4$
$(x+a)^5$	$x^5+5x^4a+10x^3a^2+10x^2a^3+5xa^4+a^5$
$(x+a)^6$	$x^6+6x^5a+15x^4a^2+20x^3a^3+15x^2a^4+6xa^5+a^6$
$(x+a)^7$	$x^7+7x^6a+21x^5a^2+35x^4a^3+35x^3a^4+21x^2a^5+7xa^6+a^7$
$(x+a)^8$	$x^8+8x^7a+28x^6a^2+56x^5a^3+70x^4a^4+56x^3a^5+28x^2a^6+8xa^7+a^8$.

In the above table, the quantities in the left-hand column are called the *expressions* for a binomial raised to the *first*, *second*, *third*, &c., *power*; the corresponding quantities in the right-hand column are called the *expansions*, or *developments*, of those in the left.

106. The developments of the successive powers of $x-a$ are precisely the same with those of $x+a$, with this difference, that the signs of the terms are alternately $+$ and $-$; thus,

$$(x-a)^5=x^5-5x^4a+10x^3a^2-10x^2a^3+5xa^4-a^5,$$

and so for all the others.

107. On considering the above table, we shall perceive that,

I. In each case the first term of the expansion is the first term of the binomial raised to the given power, and the last term of the expansion is the second term of the binomial raised to the given power. Thus, in the expansion of $(x+a)^4$ the first term is x^4, and the last term is a^4, and so for all the other expansions.

II. The quantity a does not enter into the first term of the expansion, but appears in the second term with the exponent unity. The powers of x decrease by unity, and the powers of a increase by unity in each successive term. Thus, in the expansion of $(x+a)^6$ we have x^6, x^5a, x^4a^2, x^3a^3, x^2a^4, xa^5, a^6.

III. The coefficient of the first term is unity, and the coefficient of the second term is, in every case, the exponent of the power to which the binomial is to be raised. Thus, the coefficient of the second term of $(x+a)^2$ is 2, of $(x+a)^6$ is 6, of $(x+a)^7$ is 7.

IV. The coefficient of any term after the second may be found by multiplying the coefficient of the preceding term by the index of x in that term, and dividing by the number of terms preceding the required term. Thus, in the expansion of $(x+a)^4$ the coefficient of the second term is 4; this multiplied by 3, the index of x in that term, gives 12, which, when divided by 2, the number of terms preceding the third term, gives 6, the coefficient of the third term. Again, 6, the coefficient of the third term multiplied by 2, the exponent of x in that term, gives 12, which, when divided by 3, the number of terms preceding the fourth term, gives 4, the coefficient of the fourth term. So, also, 35, the coefficient of the fifth term in the expansion of $(x+a)^7$, when multiplied by 3, the index of x in that term, gives 105, which, when divided by 5, the number of terms preceding the sixth, gives 21, the coefficient of that term.

By attending to the above observations we can always raise a binomial of the form $(x+a)$ to any required power, without the process of actual multiplication.

EXAMPLE I.

Raise $x+a$ to the 9th power.

The first term is . . . . x^9a^0;

The second term is . . . . $9x^8a^1$;

The third term is . . . . $\frac{9\times 8}{2}x^7a^2= 36x^7a^2$;

The fourth term is . . . . $\frac{36\times 7}{3}x^6a^3= 84x^6a^3$;

The fifth term is . . . . $\frac{84\times6}{4}x^5a^4=126x^5a^4$;

The sixth term is . . . . $\frac{126\times5}{5}x^4a^5=126x^4a^5$;

The seventh term is . . . . $\frac{126\times4}{6}x^3a^6=84x^3a^6$;

The eighth term is . . . . $\frac{84\times3}{7}x^2a^7=36x^2a^7$;

The ninth term is . . . . $\frac{36\times2}{8}x^1a^8=9x^1a^8$;

The tenth term is . . . . $\frac{9\times1}{9}x^0a^9=x^0a^9$.

Hence,

$(x+a)^9=x^9+9x^8a+36x^7a^2+84x^6a^3+126x^5a^4+126x^4a^5+84x^3a^6+36x^2a^7+9xa^8+a^9$.

EXAMPLE II.

In like manner,

$(x-a)^{10}=x^{10}-10x^9a+45x^8a^2-120x^7a^3+210x^6a^4-252x^5a^5+210x^4a^6-120x^3a^7+45x^2a^8-10xa^9+a^{10}$.

108. The labor of determining the coefficients may be much abridged by attending to the following additional considerations:

V. The number of terms in the expanded binomial is always greater by unity than the index of the binomial. Thus, the number of terms in $(x+a)^4$ is $4+1$, or 5; in $(x+a)^{10}$ is $10+1$, or 11.

VI. Hence, when the exponent is an even number, the number of terms in the expansion will be odd, and it will be observed, on examining the examples already given, that after we pass the middle term the coefficients are repeated in a reverse order; thus,

The coefficients of $(x+a)^4$ are 1, 4, $\underline{6}$, 4, 1.

The coefficients of $(x+a)^6$ are 1, 6, 15, $\underline{20}$, 15, 6, 1.

The coefficients of $(x+a)^8$ are 1, 8, 28, 56, $\underline{70}$, 56, 28, 8, 1.

VII. When the exponent is an odd number, the number of terms in the expansion will be even, and there will be two middle terms, or two contiguous terms, each of which is equally distant from the corresponding extremities of the series; in this case the coefficient of the two middle terms is the same, and then the coefficients of the preceding terms are reproduced in a reverse order; thus,

The coefficients of $(x+a)^3$ are 1, 3, 3, 1.

The coefficients of $(x+a)^5$ are 1, 5, $\underline{10}$, $\underline{10}$, 5, 1.

The coefficients of $(x+a)^7$ are 1, 7, 21, $\underline{35}$, $\underline{35}$, 21, 7, 1.

The coefficients of $(x+a)^9$ are 1, 9, 36, 84, $\underline{126}$, $\underline{126}$, 84, 36, 9, 1.

109. If the terms of the given binomial be affected with coefficients or exponents, they must be raised to the required powers, according to the principles already established for the involution of monomials; thus.

EXAMPLE III.*

Raise $(2x^3+5a^2)$ to the 4th power.

The first term will be . . . . . $(2x^3)^4 \qquad\qquad =16x^{12}$;

The second term will be . . . . $4(2x^3)^3\times(5a^2) \qquad =4\times8\times5x^9a^2$;

The third term will be . . . . . $\frac{4\times3}{2}\times(2x^3)^2\times(5a^2)^2=6\times4\times25x^6a^4$;

The fourth term will be . . . . $\frac{6\times2}{3}(2x^3)^1\times(5a^2)^3 \qquad =4\times2\times125x^3a^6$;

The fifth term will be . . . . . $\frac{4}{4}(2x^3)^0\times(5a^2)^4 \qquad =625a^8$;

$$\therefore\ (2x^3+5a^2)^4=16x^{12}+160x^9a^2+600x^6a^4+1000x^3a^6+625a^8.$$

EXAMPLE IV.*

In like manner,

$$(a^3+3ab)^9=(a^3)^9+9(a^3)^8\times(3ab)+36(a^3)^7\times(3ab)^2+84(a^3)^6\times(3ab)^3$$
$$+126(a^3)^5\times(3ab)^4+126(a^3)^4\times(3ab)^5+84(a^3)^3\times(3ab)^6$$
$$+36(a^3)^2\times(3ab)^7+9a^3\times(3ab)^8+(3ab)^9$$
$$=a^{27}+27a^{25}b+324a^{23}b^2+2268a^{21}b^3+10206a^{19}b^4+30618a^{17}b^5$$
$$+61236a^{15}b^6+78732a^{13}b^7+59049a^{11}b^8+19683a^9b^9.$$

110. We shall now proceed to exhibit the binomial theorem in a general form. Let it be required to raise any binomial $(x+a)$ to the power represented by the general algebraic symbol n. Then, by the preceding principles, we shall have

The first term . . . . x^n;

The second term . . . . $nx^{n-1}a$;

The third term . . . . $\frac{n(n-1)}{1.2}x^{n-2}a^2$;

The fourth term . . . . $\frac{n(n-1)(n-2)}{1.2.3}x^{n-3}a^3$;

The fifth term . . . . $\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}x^{n-4}a^4$,

&c. . . . . &c.

The last term . . . . a^n.

The whole number of terms will be $n+1$, and the coefficients be repeated in a reverse order after the $\left(\frac{n+1}{2}\right)^{\text{th}}$, or $\left(\frac{n}{2}+1\right)^{\text{th}}$ term, according as n is odd or even; moreover, the terms will all have the sign $+$, if the quantity to be expanded be of the form $x+a$, and they will have the sign $+$ and $-$ alternately, if the quantity be of the form $x-a$. Hence, generally,

$$(x+a)^n=x^n+nx^{n-1}a+\frac{n(n-1)}{1.2}x^{n-2}a^2+\frac{n(n-1)(n-2)}{1.2.3}x^{n-3}a^3+\ldots\ldots$$
$$+\frac{n(n-1)(n-2)}{1.2.3}x^3a^{n-3}+\frac{n(n-1)}{1.2}x^2a^{n-2}+nxa^{n-1}+a^n$$

$$(x-a)^n=x^n-nx^{n-1}a+\frac{n(n-1)}{1.2}x^{n-2}a^2 \ldots\ldots\ldots\ldots\ \pm a^n.$$

* The best method of proceeding in these examples is to raise $(y+z)$ to the fourth and ninth powers, and then, in the expansions thus obtained, to substitute $2x^3$ for y, and $5a^2$ for z in the first, and a^2 for y, and $3ab$ for z in the second.

In this last case, if n be an even number, the last term, being one of the odd terms, will have the sign $+$; and if n be an odd number, the last term, being one of the even terms, will have the sign $-$.

Both forms may be included in one by employing the double sign.

Thus,

$$(x \pm a)^n = x^n \pm nx^{n-1}a + \frac{n(n-1)}{1.2}x^{n-2}a^2 \pm \frac{n(n-1)(n-2)}{1.2.3}x^{n-3}a^3 + \ldots\ldots, \&c.$$

EXAMPLE V.

To exemplify the application of the theorem in this form, let it be required to raise $x+a$ to the power 5.

Here we have $n=5$, $n-1=4$, $n-2=3$, &c.

Hence,

x^n	is x^5	$= x^5$
$nx^{n-1}a$	is $5x^4a$	$= 5x^4a$
$\frac{n(n-1)}{1.2}x^{n-2}a^2$	is $\frac{5.4}{1.2}x^3a^2$	$=10x^3a^2$
$\frac{n(n-1)(n-2)}{1.2.3}x^{n-3}a^3$	is $\frac{5.4.3}{1.2.3}x^2a^3$	$=10x^2a^3$
$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}x^{n-4}a^4$	is $\frac{5.4.3.2}{1.2.3.4}xa^4$	$= 5xa^4$
$\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}x^0a^5$	is $\frac{5.4.3.2.1}{1.2.3.4.5}x^0a^5$	$= x^0a^5$

$$(x+a)^5 = x^5 + 5x^4a + 10x^3a^2 + 10x^2a^3 + 5xa^4 + a^5.$$

EXAMPLE VI.

Raise $5c^2-2yz$ to the 4th power.

Here,

$x=5c^2$, $a=2yz$, $n=4$:

$\therefore x^n$	becomes	$(5c^2)^4$	$= 625c^8$
$nx^{n-1}a$	becomes	$4(5c^2)^3 \times (2yz)$	$=1000c^6yz$
$\frac{n(n-1)}{1.2}x^{n-2}a^2$	becomes	$\frac{4.3}{1.2}(5c^2)^2 \times (2yz)^2$	$= 600c^4y^2z^2$
$\frac{n(n-1)(n-2)}{1.2.3}x^{n-3}a^3$ . . .	becomes	$\frac{4.3.2}{1.2.3}(5c^2)^1 \times (2yz)^3$	$= 160c^2y^3z^3$
$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}x^{n-4}a^4$	becomes	$\frac{4.3.2.1}{1.2.3.4}(5c^2)^0 \times (2yz)^4$	$= 16y^4z^4$

$$\therefore (5c^2-2yz)^4 = 625c^8 - 1000c^6yz + 600c^4y^2z^2 - 160c^2y^3z^3 + 16y^4z^4.$$

111. We have sometimes occasion to employ a particular term in the expansion of a binomial, while the remainder of the series does not enter into our calculations. Our labor will, in a case like this, be much abridged, if we can at once determine the term sought, without reference either to those which precede, or to those which follow it. This object will be attained by finding what is called the *general term* of the series.

If we examine the general formula, we shall soon perceive that a certain relation subsists between the coefficients and exponents of each term in the expanded binomial, and the place of the term in the series; thus, The first term is

x^n, which may be put under the form x^{n-1+1};

The second term is

$nx^{n-1}a \qquad = nx^{n-2+1}a^{2-1}$;

The third term is

$$\frac{n(n-1)}{1.2}x^{n-2}a^2 = \frac{n(n-3+2)}{1.(3-1)}x^{n-3+1}a^{3-1};$$

The fourth term is

$$\frac{n(n-1)(n-2)}{1.2.3}x^{n-3}a^3 = \frac{n(n-1)(n-4+2)}{1.2.(4-1)}x^{n-4+1}a^{4-1};$$

The fifth term is

$$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}x^{n-4}a^4 = \frac{n(n-1)(n-2)(n-5+2)}{1.2.3.(5-1)}x^{n-5+1}a^{5-1};$$

The sixth term is

$$\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}x^{n-5}a^5 = \frac{n(n-1)(n-2)(n-3)(n-6+2)}{1.2.3.4.(6-1)}x^{n-6+1}a^{6-1}.$$

$$\ldots\ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots$$

Observing the connection between the numerical quantities, it is manifest, that if we designate the place of any term by the general symbol p, the p^{th} term is

$$\frac{n(n-1)(n-2)(n-3) \ldots\ldots\ldots\ldots (n-p+2)}{1.2.3.4 \ldots\ldots\ldots\ldots (p-1)}x^{n-p+1}a^{p-1}.$$

This is called the *general term*, because by giving to p the values 1, 2, 3, 4, ... we can obtain in succession the different terms of the series for $(x+a)^n$.

EXAMPLE VII.

Required the 7^{th} term of the expansion of $(x+a)^{12}$.

Here $\left.\begin{matrix} n=12 \\ p=7 \end{matrix}\right\} \therefore n-p+2=7,\ n-p+1=6$

$\qquad\qquad p-1=6.$

Substituting these values in the general expression, we find that the term sought is

$$\frac{12.11.10.9.8.7}{1.2.3.4.5.6}x^6a^6,\text{*} \text{ or } 924x^6a^6.$$

EXAMPLE VIII.

Required the 5^{th} term of $(2c^4-4h^5)^9$.

Here $n=9,\ p=5,\ x=2c^4,\ a=4h^5$;

$$\therefore n-p+2=6,\ n-p+1=5,\ p-1=4;$$

$$\therefore \text{the } 5^{th} \text{ term is } \frac{9.8.7.6}{1.2.3.4}(2c^4)^5\times(4h^5)^4, \text{ or } 126\times32\times256c^{20}h^{20}.$$

Since the second term of the proposed binomial has the sign —, all the even terms of the expansion will have the sign —, and all the odd terms the sign $+$; therefore the 5^{th} term is

$$+1032192c^{20}h^{20}.$$

EXAMPLE IX.

Required the middle term of the expansion of $(x-a)^{18}$.

Since the exponent is 18, the whole number of terms will be 19, and hence

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* The operation here to be performed is best effected by canceling the factors.

the middle term will be the 10^{th}; and since it is an even term, it will have the sign —; hence it will be

$$-\frac{18.17.16.15.14.13.12.11.10}{1.2.3.4.5.6.7.8.9}x^9a^9, \text{ or } -48620x^9a^9.$$

EXAMPLE X.

Required the third and the last terms of the expansion of $\left(\frac{1}{2}x+2y\right)^7$

Ans. $\frac{21}{8}x^5y^2$ and $128y^7$

TO EXTRACT THE n^{th} ROOT OF A NUMBER.

112. The n^{th} power of 10 is 1 with n ciphers, and the n^{th} power of any number below 10 must be less, and can, therefore, be composed of no more than n figures. The n^{th} power of 100 is 1 with $2n$ ciphers, and the n^{th} power of any number between 10 and 100 can not, therefore, contain more than $2n$ figures, nor less than n. For a like reason, the n^{th} power of three figures can not contain more than $3n$, nor less than $2n$. That of four figures can not contain more than $4n$, nor less than $3n$, &c. The n^{th} root of a number being required, it is evident from the above that there will be as many figures in the root as there are periods of n figures in the given number, counting from right to left, and one more if any figures remain on the left. The root may be divided into units and tens, and the n^{th} power of it, or the given number, will be equal, according to the Binomial Theorem, to the n^{th} power of the tens plus n times the $n-1$ power of the tens into the units plus a number of other terms which need not be considered. Tens have one cipher on the right, and hence the n^{th} power of tens has n ciphers on the right; the n right-hand significant figures, therefore, make no part of the n^{th} power of the tens; to find the tens of the root, then, the n^{th} root of those figures which remain, after rejecting n on the right, must be sought by an independent operation; but if there are more than n of these remaining figures, the tens of the root are expressed by a number containing more than one figure, which number may be separated into its units and tens, the n^{th} power of the tens of which does not contain the n significant figures on the right of that number upon which the independent operation is now performing, and in consequence these n figures are also rejected. After rejecting periods of n figures successively, beginning on the right until there remains but one period and part or the whole of another period on the left, let these be considered an independent number, its root will contain two figures, tens and units; the n^{th} root of the tens is to be sought in what is left after rejecting the right-hand period; the $n-1$ power of the tens has $n-1$ ciphers on the right; so, also, has any multiple of this, and, therefore, n times the $n-1$ power of the tens into the units; which last quantity, therefore, is not to be sought in the $n-1$ right-hand significant figures; after subtracting the n^{th} power of the tens just found, only one figure of the next period, therefore, is to be placed on the right of the remainder, which is then divided by n times the $n-1$ power of the tens; the quotient will not be exactly the units, for the dividend contains also a part of the other terms of the power of the binomial which were not considered; this quotient may be greater than the units of the root, but never can be less; it must be diminished till the n^{th} power of the two figures found is equal to or less than

the independent number under consideration. Annex now to this independent number the next period on the right of it, and consider what is thus obtained as a new independent number; the two figures of the root already found will be the tens of the root of the new number; bringing down one figure of the right-hand period of it to the remainder after subtracting the n^{th} power of the two figures of the root just found from the first independent number, and dividing by n times the $n-1$ power of the tens, now composed of two figures, a third figure of the root is obtained; proceeding in this manner, the entire root of the given number will at length be extracted.*

EXAMPLES.

(1) $\sqrt[6]{504321,\ 2366}=8{,}921.$

(2) $\sqrt[8]{1164532,\ 07234}.$

(3) $\sqrt[9]{233416517309451}.$

(4) $\sqrt[12]{282429536481}.$

113. By employing the binomial theorem, we can raise any polynomial to any power, without the process of actual multiplication.

For example, let it be required to raise $x+a+b$ to the power 4.

Put

$$a+b \qquad =y;$$

Then,

$$\begin{aligned}(x+a+b)^4 &= (x+y)^4,\\ &= x^4+4x^3y+6x^2y^2+4xy^3+y^4, \text{ or putting for } y \text{ its value,}\\ &= x^4+4x^3(a+b)+6x^2(a+b)^2+4x(a+b)^3+(a+b)^4.\end{aligned}$$

Expanding $(a+b)^2$, $(a+b)^3$, $(a+b)^4$, by the binomial theorem, and performing the multiplications indicated, we shall arrive at the expansion of $(x+a+b)^4$.

It is manifest that we may apply a similar process to any polynomial.

The following is a demonstration of a general formula for the

POWER OF A POLYNOMIAL.

In the expression

$$(a+b+c+d\ldots.)^m$$

make $x=b+c+d\ldots$ the above power will be equal to $(a+x)^m$, and by the binomial theorem the term which contains a^n in the development of this may be written

$$\frac{1.2.3.4\ldots.m\times a^n x^{m-n}}{1.2.3\ldots n\times 1.2.3\ldots(m-n)}.\dagger \qquad [a]$$

Making $y=c+d\ldots$ we have $x^{m-n}=(b+y)^{m-n}$, and developing this last power, the term containing $b^{n'}$ may be put under the form

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* If there be decimals in the given number, ciphers must be annexed, if necessary, to make exact periods of decimals, on a principle similar to that explained in (Art. 90).

If the index of the root to be extracted be composed of factors, it can be extracted by means of the successive roots, the degrees of which are expressed by these factors. For if the $\sqrt[mnp]{a^{mnp}}$ be required, we have $\sqrt[m]{a^{mnp}}=a^{np}$, $\sqrt[n]{a^{np}}=a^p$, and $\sqrt[p]{a^p}=a$.

The best way to extract roots of numbers of a degree higher than the square is by means of logarithms.

† This may be obtained from the ordinary form of the general term of the binomial formula

$$\frac{m(m-1)\ \ldots.\ (m-n+1)a^n x^{m-n}}{1\,.\,2\,.\,3\ \ldots.\ n},$$

by multiplying both numerator and denominator by $1\,.\,2\,.\,3\ \ldots\ (m-n)$.

$$\frac{1.2.3.4....(m-n)\times b^{n'}y^{m-n-n'}}{1.2.3...n'\times 1.2.3....(m-n-n')}.$$

It is evident that if this quantity be put in the place of x^{m-n} in the expression [a], the result will represent the assemblage of the terms which contain $a^n b^{n'}$ in the power of the given polynomial. This result, after canceling common factors, will be

$$\frac{1.2.3.4....m\times a^n b^{n'}y^{m-n-n'}}{1.2.3...n\times 1.2.3...n'\times 1.2.3...(m-n-n')}. \qquad [b]$$

Making $z=d+...$ we shall have $y^{m-n-n'}=(c+z)^{m-n-n'}$, and the term containing $c^{n''}$ will be

$$\frac{1.2.3...(m-n-n')\times c^{n''}z^{m-n-n'-n''}}{1.2.3....n''\times 1.2.3....(m-n-n'-n'')};$$

substituting this expression for $y^{m-n-n'}$ in [b], we have

$$\frac{1.2.3...m\times a^n b^{n'}c^{n''}z^{m-n-n'-n''}}{1.2.3...n\times 1.2.3...n'\times 1.2.3...n''\times 1.2.3...(m-n-n'-n'')}.$$

It is evident now, without carrying the reasoning farther, that if V be the general term of the development of

$$(a+b+c+d...)^m,$$

this term may be represented thus,

$$V=\frac{1.2.3.4....m\times a^n b^{n'}c^{n''}...}{1.2.3...n\times 1.2.3...n'\times 1.2.3...n''\times ..}$$

n, n', n''... being any positive whole numbers at pleasure, subjected only to the condition that their sum shall be equal to m. So that all the terms of the required development may be obtained by giving in this formula to n, n', n''... all the entire positive values which satisfy the condition

$$n+n'+n''....=m.$$

When one of these numbers is made zero, V takes an illusory form. If, for example, $n=0$, the series $1.2.3...n$ placed in the denominator is nonsensical, because factors increasing from one will never present us with a factor zero. To relieve this difficulty, let us recur to the general term [a] in the development of $(a+x)^m$, and observe that the hypothesis $n=0$ reduces it to $\frac{x^m}{1.2.3...0}$. But the hypothesis $n=0$ ought to give in this development the term which does not contain a, and this term is x^m. Then, in order that this term shall be deduced from the formula [a], it is sufficient to consider the series $1.2.3...n$ as equivalent to 1 in this particular case of $n=0$. The same observation should be extended to the other series of factors contained in the denominator of V, and then V will give, without any exception, all the terms of the power of the polynomial $a+b+c+$, &c.

TO EXTRACT THE m^{th} ROOT OF A POLYNOMIAL.

The problem is, *having given a polynomial*, P, *which is the* m*th* *power of another polynomial*, p, *to find the latter*.

Let us consider the two polynomials as arranged according to the decreasing exponents of some letter, x, and call a, b, c, ... the unknown terms of the root p. They must be such that, in raising $a+b+c...$ to the power m, we obtain all the terms which compose P. But if we imagine that we have formed this power by successive multiplications, it is clear that, in the result, the term in which x has the highest exponent is the m^{th} power of a; then *we shall know the first term of the root sought*, p, *by extracting the* mth *root of the first term of the given polynomial*, P.

The first term of the root being found, it will be easy to obtain the second; but I prefer to show at once how, when we know several successive terms of the root setting out from the first, we can determine the term which comes immediately after.

Let u represent the sum of the known terms, and v that of the unknown; then $P=(u+v)^m$, or, developing,

$$P=u^m+mu^{m-1}v+ku^{m-2}v^2+k'u^{m-3}v^3+, \&c.$$

I have not exhibited the composition of the coefficients k, k'.., this not being necessary, as will appear. From this equality, by subtracting u^m from both the equals, we obtain

$$P-u^m=mu^{m-1}v+ku^{m-2}v^2+k'u^{m-3}v^3+, \&c.$$

The first of these equals, $P-u^m$, is a quantity which we can calculate by forming the m^{th} power of the known quantity u, and subtracting it from the polynomial P. The second is a sum of products, by means of which we can easily assign the composition of the first term of the remainder $P-u^m$, and, consequently, discover the first term of the unknown part, v.

First, if we develop u^{m-1}, it is clear, by the rules of multiplication alone, that the first term of the development, that is, the one which contains x, with the highest exponent, will be a^{m-1}; then, if we call f the first term of v, the first term of the product $mu^{m-1}v$ will be $ma^{m-1}f$. By a similar course of reasoning, we perceive that the first terms in the developments of the other products will be respectively $ka^{m-2}f^2$, $k'a^{m-3}f^3$, ... These terms, abstraction being made of the coefficients which have no influence upon the degree of x, can be deduced from the term $ma^{m-1}f$, by suppressing in it one or more factors equal to a, and replacing them by as many factors equal to f. But f being of a degree inferior to a with respect to x, these changes can give only terms of a degree inferior to $ma^{m-1}f$. Then, after having subtracted from the given polynomial P the m^{th} power of the part u of the root already found, the first term of the remainder is equal to the product of m times the power $m-1$ of the first term a of the root by the first of those terms which remain still to be found. Therefore, dividing the first term of the remainder by m times the power $m-1$ of the first term of the root, the quotient will be a new term of this root. This conclusion furnishes the means of discovering successively all the terms of the root as soon as the first is known. *To have the second term*, b, *subtract from the given polynomial* P *the* mth *power of the first term of the root, then divide the first term of the remainder by* ma^{m-1}; *to have the third term*, c, *of the root, subtract from* P *the* mth *power of* a+b, *then divide the first term of this remainder by* ma^{m-1}, and so on.

If in any part of the process, the remainder being arranged according to the powers of x, its first term is not divisible by m times the $m-1$ power of the first term of the root, the given polynomial will not have an exact root of the degree m.

Observe, also, that the m^{th} root of the last term of the given polynomial ought to be the last term of the root. Therefore, if the process leads to a term in the root of a less degree than this, the given polynomial is not an exact power of the order m.

We may arrange according to the ascending powers of a letter, x, as was remarked at (Art. 80, III.), when treating of the square root, and the above observation will undergo the same modification as there stated.

It would be superfluous to speak of the case where the letter of arrangement, x, enters, with the same exponent, into several terms. The method of proceeding in such a case has been already sufficiently indicated in previous articles.

EXAMPLES.

(1) Extract the 5th root of $32x^5-80x^4+80x^3-40x^2+10x-1$.

(2) Extract the 5th root of $729-2916x^2+4860x^4-4320x^6-576x^{10}+64x^{12}$.

Ans. $3-2x^2$.

(3) Extract the fifth root of $x^{-20}+15x^{-16}-5x^{-14}+90x^{-12}-60x^{-10}+282x^{-8} - 252x^{-6}+505x^{-4}-496x^{-2}+495-465x^2 +275x^4-80x^6+15x^8-x^{10}$.

Ans. $x^{-4}+3-x^2$.

114. In the observations made upon the expansion of $(x+a)^n$, we have supposed n to be a positive integer. The binomial theorem, however, is applicable, whatever may be the nature of the quantity n, whether it be positive or negative, integral or fractional.* When n is a positive integer, the series consists of $n+1$ terms; in every other case the series never terminates, and the development of $(x+a)^n$ constitutes what is called *an infinite series.*

Before proceeding to consider this extension of the theorem, we may remark, that in all our reasonings with regard to a quantity such as $(x+a)^n$, we may confine our attention to the more simple form $(1+a)^n$, to which the former may always be reduced. For,

$$(x+a)=x\left(1+\frac{a}{x}\right)$$

$$\therefore (x+a)^n=\left\{x\left(1+\frac{a}{x}\right)\right\}^n$$

$$=x^n\left(1+\frac{a}{x}\right)^n, \text{ or } x^n(1+u)^n, \text{ if we put } \frac{a}{x}=u$$

$$=x^n\left\{1+n.\frac{a}{x}+\frac{n(n-1)}{1.2}.\frac{a^2}{x^2}+\frac{n(n-1)(n-2)}{1.2.3}.\frac{a^3}{x^3}+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}.\frac{a^4}{x^4}+,\ \&c.\right\}\dagger$$

Suppose $n=\frac{r}{s}$, where r and s are any whole numbers whatever,

Then $(x+a)^n$ becomes $(x+a)^{\frac{r}{s}}$, and substituting $\frac{r}{s}$ for n in the series,

$$(x+a)^{\frac{r}{s}}=x^{\frac{r}{s}}\left(1+\frac{a}{x}\right)^{\frac{r}{s}}$$

$$=x^{\frac{r}{s}}\left\{1+\frac{r}{s}.\frac{a}{x}+\frac{\frac{r}{s}\left(\frac{r}{s}-1\right)}{1.2}.\frac{a^2}{x^2}+\frac{\frac{r}{s}\left(\frac{r}{s}-1\right)\left(\frac{r}{s}-2\right)}{1.2.3.}.\frac{a^3}{x^3}+\frac{\frac{r}{s}\left(\frac{r}{s}-1\right)\left(\frac{r}{s}-2\right)\left(\frac{r}{s}-3\right)}{1.2.3.4}\frac{a^4}{x^4}+,\ \&c.\right\}$$

* A perfectly rigorous demonstration of the binomial theorem for any exponent whatever, integral or fractional, positive or negative, will be found towards the close of this treatise.

† This expansion may be obtained by substituting, in the general form (Art. 110), 1 for x, and $\frac{a}{x}$ for a.

Or, reduced,

$$[a] \qquad = x^{\frac{r}{s}}\Big(1+\frac{r}{s}\cdot\frac{a}{x}+\frac{r(r-s)}{1.2.s^2}\cdot\frac{a^2}{x^2}+\frac{r(r-s)(r-2s)}{1.2.3.s^3}\cdot\frac{a^3}{x^3}$$
$$+\frac{r(r-s)(r-2s)(r-3s)}{1.2.3.4.s^4}\cdot\frac{a^4}{x^4}+, \&c.\Big\}$$

The binomial theorem, under this form, is extensively employed in analysis for developing algebraic expressions in series.

EXAMPLE I.

Expand $\sqrt{x+a}$ in a series.

$$\sqrt{x+a}=(x+a)^{\frac{1}{2}}$$
$$=x^{\frac{1}{2}}\left(1+\frac{a}{x}\right)^{\frac{1}{2}}. \quad \text{Here } r=1,\ s=2.$$
$$=x^{\frac{1}{2}}\left\{1+\frac{1}{2}\cdot\frac{a}{x}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1.2}\cdot\frac{a^2}{x^2}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1.2.3}\cdot\frac{a^3}{x^3}\right.$$
$$\left.+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{1.2.3.4}\cdot\frac{a^4}{x^4}+\ldots\ldots\ldots\right\}$$
$$=x^{\frac{1}{2}}\left\{1+\frac{1}{2}\cdot\frac{a}{x}+\frac{\frac{1}{2}\times-\frac{1}{2}}{1.2}\cdot\frac{a^2}{x^2}+\frac{\frac{1}{2}\times-\frac{1}{2}\times-\frac{3}{2}}{1.2.3}\cdot\frac{a^3}{x^3}\right.$$
$$\left.+\frac{\frac{1}{2}\times-\frac{1}{2}\times-\frac{3}{2}\times-\frac{5}{2}}{1.2.3.4}\cdot\frac{a^4}{x^4}+\ldots\ldots\ldots\right\}$$
$$=x^{\frac{1}{2}}\left\{1+\frac{1}{2}\cdot\frac{a}{x}-\frac{1}{1.2.4}\cdot\frac{a^2}{x^2}+\frac{1.3}{1.2.3.8}\cdot\frac{a^3}{x^3}-\frac{1.3.5}{1.2.3.4.16}\right.$$
$$\left.\frac{a^4}{x^4}+\ldots\ldots\ldots\right\}$$

which last may be derived at once from [a], and put under the form

$$x^{\frac{1}{2}}\left\{1+\frac{1}{2}\cdot\frac{a}{x}-\frac{1}{2.4}\cdot\frac{a^2}{x^2}+\frac{1.3}{2.4.6}\cdot\frac{a^3}{x^3}-\frac{1.3.5}{2.4.6.8}\cdot\frac{a^4}{x^4}\right.$$
$$\left.+\frac{1.3.5.7}{2.4.6.8.10}\cdot\frac{a^5}{x^5}-, \&c. \ldots\ldots\right\}$$

where the law of the series is evident.

EXAMPLE II.

Expand $\sqrt{a^2-a^2e^2}$ in a series.

$$\sqrt{a^2-a^2e^2}=(a^2-a^2e^2)^{\frac{1}{2}}$$
$$=a(1-e^2)^{\frac{1}{2}} \quad \text{Here, } r=1,\ s=2,\ \frac{a}{x}=-e^2$$
$$=a\left\{1-\frac{1}{2}\cdot e^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1.2}\cdot e^4-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1.2.3}\cdot e^6\right.$$
$$\left.+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{1.2.3.4}e^8-, \&c. \ldots\right\}$$
$$=a\left\{1-\frac{1}{2}e^2-\frac{1}{2.4}e^4-\frac{1.3}{2.4.6}e^6-\frac{1.3.5}{2.4.6.8}e^8-, \&c.\right\}$$

EXAMPLE III.

Expand $\frac{m}{\sqrt{b^2+c^4}}$ in a series.

$$\frac{m}{\sqrt{b^2+c^4}}=m(b^2+c^4)^{-\frac{1}{2}}$$

$$=mb^{-1}\left(1+\frac{c^4}{b^2}\right)^{-\frac{1}{2}} \quad \text{Here } r=1, s=-2, \frac{a}{x}=\frac{c^4}{b^2}.$$

$$=\frac{m}{b}\left\{1-\frac{1}{2}\cdot\frac{c^4}{b^2}+\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{1.2}\cdot\frac{c^8}{b^4}\right.$$
$$+\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{1.2.3}\cdot\frac{c^{12}}{b^6}$$
$$\left.+\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}-3\right)}{1.2.3.4}\cdot\left(\frac{c^4}{b^2}\right)^4 \&c.\right\}$$

$$=\frac{m}{b}\left\{1-\frac{1}{2}\cdot\frac{c^4}{b^2}+\frac{-\frac{1}{2}\times-\frac{3}{2}}{1.2}\cdot\frac{c^8}{b^4}+\frac{-\frac{1}{2}\times-\frac{3}{2}\times-\frac{5}{2}}{1.2.3}\right.$$
$$\left.\cdot\frac{c^{12}}{b^6}+\frac{-\frac{1}{2}\times-\frac{3}{2}\times-\frac{5}{2}\times-\frac{7}{2}}{1.2.3.4}\cdot\frac{c^{16}}{b^8}+, \&c. \ldots\right\}$$

$$=\frac{m}{b}\left\{1-\frac{1}{2}\cdot\frac{c^4}{b^2}+\frac{1.3}{2.4}\cdot\frac{c^8}{b^4}-\frac{1.3.5}{2.4.6}\cdot\frac{c^{12}}{b^6}+, \&c. \ldots\right\}$$

which last expression might be derived immediately from formula [a]. The same remark will apply in the following examples.

EXAMPLE IV.

Expand $\frac{n}{\sqrt{b^2-c^2e^2}}$ in a series.

$$\frac{n}{\sqrt{b^2-c^2e^2}}=n(b^2-c^2e^2)^{-\frac{1}{2}}$$

$$=nb^{-1}\left(1-\frac{c^2e^2}{b^2}\right)^{-\frac{1}{2}} \quad \text{Here } r=-1, s=2, \frac{a}{x}=-\frac{c^2e^2}{b^2}.$$

$$=\frac{n}{b}\left\{1+\frac{1}{2}\cdot\frac{c^2e^2}{b^2}+\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{1.2}\cdot\left(\frac{c^2e^2}{b^2}\right)^2\right.$$
$$-\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{1.2.3}\cdot\left(\frac{c^2e^2}{b^2}\right)^3$$
$$\left.+\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}-3\right)}{1.2.3.4}\left(\frac{c^2e^2}{b^2}\right)^4-, \&c.\right\}$$

$$=\frac{n}{b}\left\{1+\frac{1}{2}\cdot\frac{c^2e^2}{b^2}+\frac{1.3}{2.4}\cdot\frac{c^4e^4}{b^4}+\frac{1.3.5}{2.4.6}\cdot\frac{c^6e^6}{b^6}\right.$$
$$\left.+\frac{1.3.5.7}{2.4.6.8}\cdot\frac{c^8e^8}{b^8}+, \&c. \ldots\ldots\ldots\ldots\right\}$$

EXAMPLE V.

Expand $\frac{p+q}{\sqrt[4]{(m^3+n^5)^3}}$ in a series.

$$\frac{p+q}{\sqrt[4]{(m^3+n^5)^3}}=(p+q)(m^3+n^5)^{-\frac{3}{4}}$$

$$=m^{-\frac{9}{4}}(p+q)\left(1+\frac{n^5}{m^3}\right)^{-\frac{3}{4}} \qquad \text{Here, } r=-3,\ s=4,\ \frac{a}{x}=\frac{n^5}{m^3}.$$

$$=\frac{(p+q)}{m^{\frac{9}{4}}}\left\{1-\frac{3}{4}\cdot\frac{n^5}{m^3}+\frac{-\frac{3}{4}\left(-\frac{3}{4}-1\right)}{1.2}\cdot\left(\frac{n^5}{m^3}\right)^2 + \frac{-\frac{3}{4}\left(-\frac{3}{4}-1\right)\left(-\frac{3}{4}-2\right)}{1.2.3}\cdot\left(\frac{n^5}{m^3}\right)^3\ \ldots\ldots\right\}$$

$$=\frac{(p+q)}{m^{\frac{9}{4}}}\left\{1-\frac{3}{4}\cdot\frac{n^5}{m^3}+\frac{3.7}{1.2.4^2}\frac{n^{10}}{m^6}-\frac{3.7.11}{1.2.3.4^3}\frac{n^{15}}{m^9}+\frac{3.7.11.15}{1.2.3.4.4^4}\frac{n^{20}}{m^{12}}-,\ \&c.\ \ldots\ldots\ldots\right\}$$

EXAMPLE VI.

$$\frac{1}{(c+x)^2}=(c+x)^{-2}=\frac{1}{c^2}\left\{1-\frac{2x}{c}+\frac{3x^2}{c^2}-\frac{4x^3}{c^3}+,\ \&c.\right\}$$

EXAMPLE VII.

$$(c^2-x^2)^{\frac{3}{4}}=c^{\frac{3}{2}}\left\{1-\frac{3}{2^2}\cdot\frac{x^2}{c^2}-\frac{3}{2^5}\cdot\frac{x^4}{c^4}-\frac{5}{2^7}\cdot\frac{x^6}{c^6}-\frac{7}{2^9}\cdot\frac{x^8}{c^8}-,\ \&c.\right\}$$

EXAMPLE VIII.

$$(a^2-ax)^{-\frac{3}{10}}=\frac{1}{a^{\frac{3}{5}}}\left\{1+\frac{3}{10}\cdot\frac{x}{a}+\frac{3.13}{10^2}\cdot\frac{x^2}{1.2.a^2}+\frac{3.13.23}{10^3}\cdot\frac{x^3}{1.2.3.a^3}+\frac{3.13.23.33}{10^4}\cdot\frac{x^4}{1.2.3.4.a^4}+,\ \&c.\ \ldots\ldots\ldots\right\}$$

EXAMPLE IX.

$$\frac{1}{(1+x)^{\frac{1}{5}}}=1-\frac{x}{5}+\frac{6x^2}{5.10}-\frac{6.11.x^3}{5.10.15}+\frac{6.11.16.x^4}{5.10.15.20}-,\ \&c.\right\}$$

EXAMPLE X.

The eleventh term of the series for $(a^3-x^3)^{\frac{7}{3}}$ is $-\frac{2618}{4782969}\cdot\frac{x^{30}}{a^{23}}$.

115. The binomial theorem is also employed to determine approximate values of the roots of numbers.

In the formula

$$(x+a)^n=x^n\left(1+n\cdot\frac{a}{x}+\frac{n(n-1)}{1.2}\cdot\frac{a^2}{x^2}+\frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{a^3}{x^3}+----\right).$$

Let us put $n=\frac{1}{r}$, the expression becomes $(x+a)^{\frac{1}{r}}$ or $\sqrt[r]{x+a}$, and we have

$$\sqrt[r]{x+a}=\sqrt[r]{x}(1+\frac{1}{r}.\frac{a}{x}+\frac{\frac{1}{r}\left(\frac{1}{r}-1\right)}{1.2}.\frac{a^2}{x^2}+\frac{\frac{1}{r}\left(\frac{1}{r}-1\right)\left(\frac{1}{r}-2\right)}{1.2.3}.\frac{a^3}{x^3}+\ldots\ldots.)$$

$$=\sqrt[r]{x}(1+\frac{1}{r}.\frac{a}{x}-\frac{1}{r}.\frac{r-1}{2r}.\frac{a^2}{x^2}+\frac{1}{r}.\frac{r-1}{2r}.\frac{2r-1}{3r}.\frac{a^3}{x^3}-, \&c.\ldots\ldots.)$$

If we wished to form a new term, it would manifestly be obtained by multiplying the fourth by $\frac{3r-1}{4r}$ and $\frac{a}{x}$, then changing the sign, and so on for the rest, the terms after the first being alternately positive and negative.

This being premised, let it be required to extract the cube root of 31. The greatest cube contained in 31 is 27; in the above formula let us make $r=3$, $x=27$, $a=4$, and we shall then have

$$\sqrt[3]{31}=\sqrt[3]{27+4}$$

$$=\sqrt[3]{27}\left(1+\frac{4}{27}\right)^{\frac{1}{3}}$$

$$=3\,(1+\frac{1}{3}\cdot\frac{4}{27}-\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{16}{729}+\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{5}{9}\cdot\frac{64}{19683}-, \&c.)$$

$$=3+\frac{4}{27}-\frac{16}{2187}+\frac{320}{531441}-, \&c.$$

The succeeding term will be found by multiplying $\frac{320}{531441}$ by $\frac{3r-1}{4r}\cdot\frac{a}{x}$, or $\frac{2}{3}\cdot\frac{4}{27}$, and then changing the sign, which will give us $-\frac{2560}{43046721}$.

In like manner, we shall find the next term by multiplying $\frac{2560}{43046721}$ by $\frac{4r-1}{5r}\cdot\frac{a}{x}$, it will, therefore, be $\frac{2560}{43046721}\times\frac{11}{15}\times\frac{4}{27}=\frac{112640}{17433922005}$, and so on for any number of terms.

Let us, however, confine our attention to the first five terms of the series, and reduce them to decimals; we shall have, for the sum of the additive terms,

$$\left\{\begin{array}{r} 3=3.00000 \\ \frac{4}{27}=0.14815 \\ \frac{320}{531441}=0.00060 \end{array}\right\}=3.14875.$$

And for the sum of the subtractive terms,

$$\left\{\begin{array}{r} -\frac{16}{2187}=-0.00731 \\ -\frac{2560}{43046721}=-0.00006 \end{array}\right\}=-0.00737.$$

Hence

$$\sqrt[3]{31}=3.14138\ldots\ldots\ldots\ldots.$$

a result which we shall proceed to show is within 0.00001 of the truth.

116. When the expression for a number is expanded in a series of terms, the numerical values of which go on decreasing continually, we easily perceive

that the greater the number of terms which we take, the more nearly shall we approach to the real value of the proposed expression. Such a series is called *converging*. If we suppose the terms of the series alternately positive and negative, we can, upon stopping at any particular term, determine precisely the degree of approximation at which we have arrived.

Let there be a series $a-b+c-d+e-f+g-h+k-l+m$ ---- composed of an indefinite number of terms, in which we suppose that the quantities a, b, c, d go on diminishing in succession, and let us designate by N the number represented by this series, we shall prove that *the numerical value of* N *lies between any two consecutive sums of any number of the terms of the above series.*

For let us take any two consecutive sums,

$$a-b+c-d+e-f, \text{ and } a-b+c-d+e-f+g.$$

Upon considering the first of these, we perceive that the terms which follow $-f$ are $+(g-h)+(k-l)+$-----; but since the series is a decreasing one, the positive differences $g-h$, $k-l$, &c., are all positive numbers; hence it follows that, in order to obtain the complete value of N, we must add to the sum $a-b+c-d+e-f$ some positive number. Hence

$$a-b+c-d+e-f<\text{N}.$$

With regard to the second sum, the terms which follow $+g$ are $-(h-k)$, $-(l-m)$, ----; but the partial differences, $h-k$, $l-m$, &c., are positive; hence $-(h-k)$, $-(l-m)$ ----, are all negative, and, therefore, in order to obtain the complete value of N, we must subtract some positive number from the sum $a-b+c-d+e-f+g$. Hence

$$a-b+c-d+e-f+g>\text{N},$$

and it has been shown that

$$a-b+c-d+e-f \quad <\text{N};$$

therefore N lies between these two sums.

From this it follows that, since g is the numerical value of the difference of these two sums, *the error committed when we assume a certain number of terms* $a-b+c-d+e-f$ *for the value of* N *is numerically less than the term which immediately follows that at which we stopped.*

In the preceding example, all the terms after the first being alternately positive and negative, we may conclude that the numerical value of the first five terms

$$3+\frac{4}{27}-\frac{16}{2187}+\frac{320}{531441}-\frac{2560}{43046721}$$

differs from the true value of $\sqrt[3]{31}$ by a quantity less than the value of the sixth term, which was found to be equal to $\frac{112640}{17433922005}$; but this fraction is by mere inspection less than $\frac{1}{100000}$, therefore, when we assume that $\sqrt[3]{31}=3.14138$, the result is within 0.00001 of the truth.

117. From what has been said above it will be seen that, in order to obtain an approximate value of the n^{th} root of any number N by the method of series, we may make use of the following

RULE.

Resolve the given number N *into two parts of the form* $\text{p}^{\text{n}}+\text{q}$, *where* p^{n} *is the highest* n^{th} *power contained in* N, *and in the development of* $(\text{x}+\text{a})^{\frac{1}{\text{n}}}$ *make*

$x=p^n$, $a=q$. *The number of terms to be taken in the resulting series will depend upon the degree of accuracy required, and can be determined by the principle just explained. Convert all the terms of which account is taken into decimals, and then effect the reduction between the additive and subtractive terms.*

This method can not be employed with advantage except when $\frac{q}{p^n}$ is a small fraction; for unless this be the case, the terms of the series will not diminish with sufficient rapidity, and it will be necessary to take account of a great number of terms in order to arrive at a near approximation.

It may happen that p^n is $<q$; we must then modify the above process, for then $\frac{p^n}{q}$ or $\frac{a}{x}$ is greater than unity, and therefore all the powers of $\frac{a}{x}$ will increase in numerical value as the degree of the power increases.

Suppose, for example, that the cube root of 56 is sought, 27 being the greatest cube contained in 56, we shall have

$$x=27,\ a=29 \text{ and } \therefore \frac{a}{x}=\frac{29}{27},$$

and the terms of the series will go on increasing instead of diminishing (we do not speak of the coefficients, which are fractions differing but little from unity).

But we may resolve 56 into $64-8$, or 4^3-8; but $\frac{8}{64}$, or $\frac{1}{8}$, is a small fraction.

On the other hand, if we substitute $-a$ for a in the expression for $\sqrt[n]{x+a}$, we have

$$\sqrt[n]{x-a}=x^{\frac{1}{n}}(1-\frac{1}{n}\cdot\frac{a}{x}-\frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{a^2}{x^2}-\frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{2n-1}{3n}\cdot\frac{a^3}{x^3}-\text{---}.$$

If we put $x=64$, $a=8$, we shall obtain a series of terms which will decrease with great rapidity.

Here all the terms, with the exception of the first, are negative, and we can not apply to this series the criterion established in (Art. 116) for fixing the degree of approximation. But we shall approach very nearly to the required degree of approximation if we take into account such a number of terms that the first which we neglect shall be less, by one tenth, for example, than the decimal place to which we wish to limit the approximation.

The student may take the following examples as exercises:

(1) $\sqrt[5]{39}=\sqrt[5]{32+7}=2.0807\ldots$ true to 0.0001.

(2) $\sqrt[3]{65}=\sqrt[3]{64+1}=4.02073\ldots$ true to 0.00001.

(3) $\sqrt[4]{260}=\sqrt[4]{256+4}=4.01553\ldots$ true to 0.00001.

(4) $\sqrt[7]{108}=\sqrt[7]{128-20}=1.95204\ldots$ true to 0.00001.

118. Roots of imaginary expressions of the form $a\pm b\sqrt{-1}$ are extracted by putting the expression under the form $(a\pm b\sqrt{-1})^{\frac{1}{n}}$, and developing by the binomial theorem; a series of terms will thus be obtained, which may be put under the form $A+B\sqrt{-1}$, A representing the algebraic sum of the rational terms, and B the algebraic sum of the coefficients of $\sqrt{-1}$. Algebra furnishes no other general method for this transformation, but when n is a power of 2, it can be effected without the aid of series.

Let us consider, first, the two radicals $\sqrt{a+b\sqrt{-1}}$ and $\sqrt{a-b\sqrt{-1}}$. Placing

$$[1]\quad \sqrt{a+b\sqrt{-1}}+\sqrt{a-b\sqrt{-1}}=x$$

$$[2]\quad \sqrt{a+b\sqrt{-1}}-\sqrt{a-b\sqrt{-1}}=y,$$

and squaring both, there results

$$2a+2\sqrt{a^2+b^2}=x^2$$
$$2a-2\sqrt{a^2+b^2}=y^2.$$

Whatever may be the sign of a, the value of x^2 is positive, but that of y^2 is negative. From these equalities we derive

$$[3]\quad x=\sqrt{2a+2\sqrt{a^2+b^2}},\ y=\left(\sqrt{-2a+2\sqrt{a^2+b^2}}\right)\sqrt{-1}.$$

But the equalities [1] and [2] give

$$\sqrt{a+b\sqrt{-1}}=\frac{x+y}{2},\quad \sqrt{a-b\sqrt{-1}}=\frac{x-y}{2}.$$

Then, finally, putting for x and y the values [3], we shall have

$$[4]\quad \sqrt{a+b\sqrt{-1}}=\frac{1}{2}\sqrt{2a+2\sqrt{a^2+b^2}}+\frac{1}{2}\sqrt{-2a+2\sqrt{a^2+b^2}}\sqrt{-1}$$

$$[5]\quad \sqrt{a-b\sqrt{-1}}=\frac{1}{2}\sqrt{2a+2\sqrt{a^2+b^2}}-\frac{1}{2}\sqrt{-2a+2\sqrt{a^2+b^2}}\sqrt{-1}.$$

Now, if we consider the radical expressions

$$\sqrt[4]{a\pm b\sqrt{-1}},\ \sqrt[8]{a\pm b\sqrt{-1}},\ \sqrt[16]{a\pm b\sqrt{-1}},\ \&c.,$$

we observe that the extraction of a single root which is some power of two, can be replaced by successive extractions of the square root; consequently, the repetition of the formulas [4] and [5] will always reduce the above expressions to expressions of the form $A\pm B\sqrt{-1}$.

REMARK.—In each of these formulas the first member, by reason of the radicals which it contains, may have four different values, and the same is true of the second member. In both, the four values of the first member are the same, and this is the case evidently with the second member; so that the two formulas make really but one. They present no difference except when we use them simultaneously in the same algebraical calculation, because then we ought to regard the terms into which $\sqrt{-1}$ enters as affected with contrary signs. But then it is necessary to remark besides, that, by the very manner in which we have arrived at these formulas, $\sqrt{a^2+b^2}$ in them represents the product of $\sqrt{a+b\sqrt{-1}}\sqrt{a-b\sqrt{-1}}$; consequently, the determinations of these two radicals ought always to be supposed associated in such a manner that their product should have the sign which is given to $\sqrt{a^2+b^2}$ in the second member. Without attention to this the formulas might lead to false results.

Another remark of importance may be added here.

The methods of proceeding in certain operations upon imaginary expressions, exhibited at (Art. 66), were suited to the restrictions which in ordinary cases would be understood as pertaining to the radical sign. If, however, this sign have its most general signification, it must be used in its ambiguous sense, that is, as having $\pm$ before it. Then $\sqrt{-a} \times \sqrt{-a}$ would have a more extended sense than simply the square of $\sqrt{-a}$. It would have, in fact, four values,

$$+\sqrt{-a} \times +\sqrt{-a}, \quad -\sqrt{-a} \times +\sqrt{-a}, \quad +\sqrt{-a} \times -\sqrt{-a},$$
$$-\sqrt{-a} \times -\sqrt{-a},$$

or

$$-a, \qquad +a, \qquad +a, \qquad -a.$$

These four, in fact, amount to but two, $+a$ and $-a$, which are the values obtained by the ordinary rule of multiplication, $\sqrt{-a} \times \sqrt{-a} = \sqrt{a^2} = \pm a$.

If the quantities under the radical are different, the reasoning will be a little varied. Let the product be required of

$$\sqrt{-a} \times \sqrt{-b}.$$

The first of these factors $\sqrt{-a}$ may be put under the form $a'\sqrt{-1}$, and the second under the form $b'\sqrt{-1}$. The product will then be expressed by

$$a'b'\sqrt{-1} \times \sqrt{-1}.$$

But after what has just been said, if there be no restriction in the meaning of the sign $\sqrt{}$, we have $\sqrt{-1} \times \sqrt{-1} = \pm 1$. Hence

$$a'b'\sqrt{-1} \times \sqrt{-1} = \pm a'b'.$$

But since the square of $a'b'$ is $a'^2b'^2$, or ab, we have $a'b' = \sqrt{ab}$, and, therefore,

$$\sqrt{-a} \times \sqrt{-b} = \pm \sqrt{ab},$$

the result which we should obtain by the ordinary rule for the multiplication of radicals. We thus perceive that this rule gives us the true product in its most general form when there is no restriction in the sense of the radical sign.

RATIOS AND PROPORTION.

119. Numbers may be compared in two ways.

When it is required to determine by how much one number is greater or less than another, the answer to this question consists in stating the *difference* between these two numbers. This difference is called the *Arithmetical Ratio* of the two numbers. Thus, the arithmetical ratio of 9 to 7 is $9-7$, or 2, and if a, b designate two numbers, their arithmetical ratio is represented by $a-b$.

When it is required to determine how many times one number contains, or is contained in another, the answer to this question consists in stating the *quotient* which arises from dividing one of these numbers by the other. This quotient is called the *Geometrical Ratio* of the two numbers. The term *Ratio*, when used without any qualification, is always understood to signify a geometrical ratio, and we shall, at present, confine our attention to ratios of this description.

120. By the *ratio* of two numbers, then, we mean the *quotient* which arises from dividing one of these numbers by the other. Thus, the ratio of 12 to 4 is represented by $\frac{12}{4}$ or 3, the ratio of 5 to 2 is $\frac{5}{2}$ or 2.5, the ratio of 1 to 3 is $\frac{1}{3}$ or .333... We here perceive that the value of a ratio can not always be expressed exactly, except in the form of a vulgar fraction, but that, by taking a sufficient number of terms of the decimal, we can approach as nearly as we please to the true value.

121. If a, b designate two numbers, the ratio of a to b is the quotient arising from dividing a by b, and will be represented by writing them $a:b$, or $\frac{a}{b}$.

122. A ratio being thus expressed, the first term, or a, is called the *antecedent* of the ratio; the last term, or b, is called the *consequent* of the ratio.

123. It appears, therefore, that, in arithmetic and algebra, the theory of ratios becomes identified with the theory of fractions, and a ratio may be defined as a fraction whose numerator is the antecedent, and whose denominator is the consequent of the ratio.

124. When the antecedent of a ratio is greater than the consequent, the ratio is called *a ratio of greater inequality*; when the antecedent is less than the consequent, it is called *a ratio of less inequality*; and when the antecedent and consequent are equal, it is called *a ratio of equality*. Thus, $\frac{12}{4}$ is a ratio of greater inequality, $\frac{12}{144}$ is a ratio of less inequality, $\frac{3}{3}$ or 1 is a ratio of equality. It is manifest that a ratio of equality may always be represented by unity.

125. When the antecedents of two or more ratios are multiplied together to form a new antecedent, and their consequents multiplied together to form a new consequent, the several ratios are said to be *compounded*, and the resulting ratio is called the *sum* of the compounding ratios. Thus, the ratio $\frac{a}{b}$ is compounded with the ratio $\frac{c}{d}$ by multiplying the antecedents a, c for a new antecedent, and the consequents b, d for a new consequent, and the resulting ratio $\frac{ac}{bd}$ is called the sum of the ratios $\frac{a}{b}$ and $\frac{c}{d}$.

In like manner, the ratios $\frac{m}{n}, \frac{p}{q}, \frac{r}{s}, \frac{t}{w}$ are compounded by multiplying all the antecedents together for a new antecedent, and all the consequents for a new consequent, and the resulting ratio, $\frac{mprt}{nqsw}$, is called the sum of the ratios $\frac{m}{n}, \frac{p}{q}, \frac{r}{s}, \frac{t}{w}$.

126. When a ratio is compounded with itself the resulting ratio is called the *duplicate ratio*, or *double ratio* of the primitive. Thus, if we compound the ratio $\frac{a}{b}$ with $\frac{a}{b}$, the resulting ratio, $\frac{a^2}{b^2}$, is called the duplicate ratio of $\frac{a}{b}$.

Similarly, $\frac{a^3}{b^3}$ is called the *triplicate ratio*, or *triple ratio* of $\frac{a}{b}$.

And, generally, $\frac{a^n}{b^n}$ is called the sum of the n ratios $\frac{a}{b}$.

According to the same principle, the ratio $\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}$ is called the *subduplicate ratio*, or *half ratio* of $\frac{a}{b}$; for the duplicate ratio of $\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}$ is $\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}\times\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}=\frac{a}{b}$.

So, also, the ratio $\frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$ is called the *subtriplicate ratio*, or *one third of the ratio* of $\frac{a}{b}$. For the triple ratio of $\frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$ is $\frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}\times\frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}\times\frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}=\frac{a}{b}$.

And, in general, $\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$ is called *one* n^{th} *of the ratio* $\frac{a}{b}$; for n times the ratio $\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$ is $\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}\times\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}\times\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}\times\ldots$ to n terms $=\frac{a}{b}$.

NOTE.—The ratio $\frac{a^{\frac{3}{2}}}{b^{\frac{3}{2}}}$ is called the sesquiplicate ratio of $\frac{a}{b}$, for it is compounded of the simple and subduplicate ratio; thus, $\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}\times\frac{a}{b}=\frac{a^{\frac{3}{2}}}{b^{\frac{3}{2}}}$.

127. *If the terms of a ratio be both multiplied, or both divided, by the same quantity, the value of the ratio remains unchanged.*

The ratio of a to b is represented by the fraction $\frac{a}{b}$; and since the value of a fraction is not changed, if we multiply, or divide, both numerator and denominator by the same quantity, the truth of the proposition is evident. Thus,

$$\frac{a}{b}=\frac{ma}{mb}=\frac{\frac{a}{n}}{\frac{b}{n}},\text{ or } a:b=ma:mb=\frac{a}{n}:\frac{b}{n}.$$

128. *Ratios are compared with each other by reducing the fractions, by which they are represented, to a common denominator.*

If we wish to ascertain whether the ratio of 2 to 7 is greater or less than that of 3 to 8, since these ratios are represented by the fractions $\frac{2}{7}$ and $\frac{3}{8}$, which are equivalent to $\frac{16}{56}$ and $\frac{21}{56}$; and since the latter of these is greater than the former, it appears that the ratio of 2 to 7 is less than the ratio of 3 to 8.

129. *A ratio of greater inequality is diminished, and a ratio of a less inequality is increased, by adding the same quantity to both terms.*

Let $\frac{a}{b}$ represent any ratio, and let x be added to each of its terms. The two ratios will then be

$$\frac{a}{b}, \frac{a+x}{b+x},$$

which, reduced to a common denominator, become

$$\frac{ab+ax}{b(b+x)}, \frac{ab+bx}{b(b+x)}.$$

If $a>b$, *i. e.*, if $\frac{a}{b}$ be a ratio of greater inequality, then

$$\frac{ab+ax}{b(b+x)}>\frac{ab+bx}{b(b+x)};$$

and $\therefore \frac{a}{b}$ is diminished by the addition of the same quantity to each of its terms.

Again, if $a<b$, *i. e.*, if $\frac{a}{b}$ be a ratio of less inequality, then

$$\frac{ab+ax}{b(b+x)}<\frac{ab+bx}{b(b+x)};$$

and $\therefore \frac{a}{b}$ is increased by the addition of the same quantity to each of its terms.

130. *If there be any number of ratios in which the consequent of the first ratio is the antecedent of the second, and the consequent of the second the antecedent of the third, and so on, the sum of any number of said ratios is the ratio of the first antecedent to the last consequent.*

Let the proposed ratios be

$$\frac{a}{b}, \frac{b}{c}, \frac{c}{d}, \frac{d}{e}, \frac{e}{f}, ----.$$

Then, by (Art. 125), their sum is

$$\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}\times\frac{d}{e}\times\frac{e}{f}---;$$

or

$$\frac{abcde----}{bcdef----},$$

$$\text{i. e., } \frac{a}{f}.$$

131. Proportion is an equality of ratios.

Thus, if a, b, c, d be four quantities, such that a, when divided by b, gives the same quotient as c when divided by d, then a, b, c, d are said to be *in proportion*, or to be *proportionals*; the numbers 20, 5, 36, 9 are proportionals, for $\frac{20}{5}=4$, and $\frac{36}{9}=4$.

When four quantities are proportionals, it is usually enunciated by saying that *the first is to the second as the third is to the fourth*. Thus, if a, b, c, d are proportionals, we say that a is to b as c is to d, and this is expressed by writing them

$$a:b::c:d, \text{ or } a:b=c:d,$$

or as fractions,

$$\frac{a}{b}=\frac{c}{d}.$$

The first or second form of notation is usually employed in geometry, the last in analytical investigations. The signs :: and = have precisely the same meaning. The sign : is the sign of division.

132. The expression $a:b::c:d$, or $\frac{a}{b}=\frac{c}{d}$, is called a proportion, and a, b, c, d are severally called the *terms* of the proportion. The first and last are called the *extreme terms*, the second and third the *mean terms*. The first term is called the *first antecedent*, the second term the *first consequent*, the third term the *second antecedent*, and the fourth term the *second consequent*.

133. When the second and third terms of a proportion are identical, the quantity which forms these terms is called a *mean proportional* between the other two; thus, if we have three quantities a, b, c, such that

$$a:b::b:c, \text{ or } \frac{a}{b}=\frac{b}{c},$$

then b is said to be a mean proportional to a and c, and c is called a *third proportional* to a and b.

If, in a series of proportional magnitudes, each consequent be identical with the next antecedent, these quantities are said to be in *continued proportion*; thus, if we have a series of quantities, a, b, c, d, e, f, g, h, such that

$$a:b::b:c::c:d::d:e::e:f::f:g::g:h,$$

or

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}=\frac{e}{f}=\frac{f}{g}=\frac{g}{h},$$

then the quantities a, b, c, d, e, f, g, h are in continued proportion.

A continued proportion is called a progression.

The following are the most important propositions connected with the subject of proportion.

I. *If four quantities be proportionals, the product of the extreme terms will be equal to the product of the mean terms.*

Let

$$a:b::c:d,$$

or

$$\frac{a}{b}=\frac{c}{d}.$$

Multiplying these equals by bd, the expression becomes

$$ad=bc.$$

II. Conversely, *If the product of any two quantities be equal to the product of any other two, these four quantities will constitute a proportion, the terms of one of the products being the means, and the terms of the other the extremes.*

Let

$$ad=bc.$$

Dividing these equals by bd, the expression becomes

$$\frac{a}{b}=\frac{c}{d}, \text{ or } \frac{c}{d}=\frac{a}{b};$$

i. e., $a:b::c:d$, or $c:d::a:b$.

In the first, a and b are the extremes, and b and c the means; in the second, b and c are the extremes, and a and d the means.

III. *If three quantities be in continued proportion, the product of the extreme terms is equal to the square of the mean.*

This follows immediately from I.; for let a, b, c be three quantities in continued proportion, then

$$a:b::b:c, \text{ or } \frac{a}{b}=\frac{b}{c}$$

$$\therefore\ ac=b\times b \text{ by I.}$$
$$=b^2.$$

IV. Conversely, *If the product of any two quantities be equal to the square of a third, the last quantity will be a mean proportional between the other two.*

Thus, if $ac=b^2$, b is a mean proportional between a and c; for, since

$$ac=b^2,$$

dividing these equals by bc,

$$\frac{a}{b}=\frac{b}{c}, \text{ or } a:b::b:c.$$

V. *Quantities which have the same ratio to the same quantity are equal to one another, and those to which the same quantity has the same ratio are equal to one another.*

First, let a and b have the same ratio to the same quantity c, then $a=b$. Since

$$a:c::b:c,$$

or

$$\frac{a}{c}=\frac{b}{c};$$

multiply these equals by c $\therefore$ $a=b$.

Again, let c have the same ratio to each of the quantities a and b, then $a=b$. Since

$$c:a::c:b,$$

or

$$\frac{c}{a}=\frac{c}{b};$$

dividing these equals by c,

$$\frac{1}{a}=\frac{1}{b}$$

$$\therefore\ a=b.$$

VI. *Ratios that are equal to the same are equal to one another.*

Let $a:b::x:y$ \
And $c:d::x:y$ } Then $a:b::c:d$.

This is an axiom.

VII. *If four quantities be proportionals, they will be proportionals also* alternando, *that is, the first will have the same ratio to the third that the second has to the fourth.*

Let $a:b::c:d$, then, also, $a:c::b:d$.

Since $\frac{a}{b}=\frac{c}{d}$, divide each of these equals by c, and multiply each by b.

Then $\frac{a}{c}=\frac{b}{d}$; i. e., $a:c::b:d$.

VIII. *If four quantities be proportionals, they will be proportionals also* invertendo, *that is, the second will have to the first the same ratio that the fourth has to the third.*

Let $a:b::c:d$, then, also, $b:a::d:c$.

Since $\frac{a}{b}=\frac{c}{d}$, divide unity by each of these equals.

We have

$$\frac{1}{\left(\frac{a}{b}\right)}=\frac{1}{\left(\frac{c}{d}\right)},$$

or

$$\frac{b}{a}=\frac{d}{c}\text{; i. e., } b:a::d:c.$$

IX. *If four quantities be proportionals, they will be proportionals also* componendo, *that is, the first, together with the second, will have to the second the same ratio that the third, together with the fourth, has to the fourth.*

Let $a:b::c:d$, then, also, $a+b:b::c+d:d$.

Since $\frac{a}{b}=\frac{c}{d}$, add 1 to each of these equals, then

$$\frac{a}{b}+1=\frac{c}{d}+1,$$

or

$$\frac{a+b}{b}=\frac{c+d}{d}\text{; i. e., } a+b:b::c+d:d.$$

X. *If four quantities be proportionals, they will be proportionals also* dividendo, *that is, the difference of the first and second will have to the second the same ratio that the difference of the third and fourth has to the fourth.*

Let $a:b::c:d$, then, also, $a-b:b::c-d:d$.

Since $\frac{a}{b}=\frac{c}{d}$, subtract unity from each of these equals, then

$$\frac{a}{b}-1=\frac{c}{d}-1,$$

or

$$\frac{a-b}{b}=\frac{c-d}{d}\text{; i. e., } a-b:b::c-d:d.$$

XI. *If four quantities be proportionals, they will be proportionals also* convertendo, *that is, the first will have to the difference of the first and second the same ratio that the third has to the difference of the third and fourth.*

Let $a:b::c:d$, then, also, $a:a-b::c:c-d$.

Since $\frac{a}{b}=\frac{c}{d}$, then, by prop. VIII., $\frac{b}{a}=\frac{d}{c}$; and hence, subtracting these equal quantities from unity,

$$1-\frac{b}{a}=1-\frac{d}{c},$$

or

$$\frac{a-b}{a}=\frac{c-d}{c},$$

or

$$\frac{a}{a-b}=\frac{c}{c-d}\text{; i. e., } a:a-b::c:c-d.$$

XII. *If four quantities be proportionals, the sum of the first and second will have to their difference the same ratio that the sum of the third and fourth has to their difference.*

Let $a:b::c:d$, then, also, $a+b:a-b::c+d:c-d.$

Since $\frac{a}{b}=\frac{c}{d}$, we have,

by prop. IX., $\frac{a+b}{b}=\frac{c+d}{d}$;

and, by prop. X., $\frac{a-b}{b}=\frac{c-d}{d}$;

dividing these equals by each other,

$$\frac{\frac{a+b}{b}}{\frac{a-b}{b}}=\frac{\frac{c+d}{d}}{\frac{c-d}{d}}$$

or

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}; \text{ i. e., } a+b:a-b::c+d:c-d.$$

XIII. *If there be any number of quantities more than two, and as many others, which, taken two and two in order, are proportionals* (ex æquali), *the first will have to the last of the first rank the same ratio that the first of the second rank has to the last.*

Let

a, b, c, d . . . . be any number of quantities,

and

e, f, g, h . . . . as many others.

Let

$$\left.\begin{matrix} a:b::e:f \\ b:c::f:g \\ c:d::g:h \end{matrix}\right\} \text{Then, also, } a:d::e:h.$$

For, since

$$\frac{a}{b}=\frac{e}{f}$$
$$\frac{b}{c}=\frac{f}{g}$$
$$\frac{c}{d}=\frac{g}{h};$$

multiplying the first column together, and also the second,

$$\frac{abc}{bcd}=\frac{efg}{fgh},$$

or

$$\frac{a}{d}=\frac{e}{h}; \text{ i. e., } a:d::e:h.$$

XIV. *If there be any number of quantities more than two, and as many others, which, taken two and two in a cross order, are proportionals* (ex æquali perturbatâ), *the first will have to the last of the first rank the same ratio that the first of the second rank has to the last.*

Let

$$a,\ b,\ c,\ d\ \ldots .\ \text{be any number of quantities,}$$

and

$$e,\ f,\ g,\ h\ \ldots .\ \text{as many others.}$$

Let

$$\left.\begin{array}{l} a:b::g:h \\ b:c::f:g \\ c:d::e:f \end{array}\right\} \text{Then, also, } a:d::e:h.$$

For, since

$$\frac{a}{b}=\frac{g}{h}$$

$$\frac{b}{c}=\frac{f}{g}$$

$$\frac{c}{d}=\frac{e}{f},$$

$$\frac{abc}{bcd}=\frac{gfe}{hgf},$$

or

$$\frac{a}{d}=\frac{e}{h};\ \text{i. e., } a:d::e:h.$$

XV. *If four quantities be proportionals, any powers or roots of these quantities will also be proportionals.*

Let $a:b::c:d$; then, also, $a^n:b^n::c^n:d^n$.

Since

$\frac{a}{b}=\frac{c}{d}$, raising each of these equals to the nth power, $\left(\frac{a}{b}\right)^n=\left(\frac{c}{d}\right)^n$,

or

$\frac{a^n}{b^n}=\frac{c^n}{d^n}$; i. e., $a^n:b^n::c^n:d^n$,

where n may be either integral or fractional.

XVI. *If there be any number of proportional quantities, the first will have to the second the same ratio that the sum of all the antecedents has to the sum of all the consequents.*

Let a, b, c, d, e, f, g, h be any number of proportional quantities, such that

$$a:b::c:d::e:f::g:h.$$

Then

$$a:b::a+c+e+g:b+d+f+h.$$

Since

$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h},$$

we have

$$ab=ba$$

$$ad=bc$$

$$af=be$$

$$ah=bg,$$

and

$$\therefore a(b+d+f+h)=b(a+c+e+g)$$

$$\therefore \frac{a}{b}=\frac{a+c+e+g}{b+d+f+h}$$

or

$$a:b::a+c+e+g:b+d+f+h.$$

XVII. *If three quantities be in continued proportion, the first will have to the third the duplicate ratio of that which it has to the second.*

Let $a:b::b:c$, then $a:c::a^2:b^2$.

Since

$$\frac{a}{b}=\frac{b}{c}, \text{ multiply each of these equals by } \frac{a}{b}; \text{ then}$$

$$\frac{a}{b}\times\frac{a}{b}=\frac{b}{c}\times\frac{a}{b}, \text{ or } \frac{a^2}{b^2}=\frac{a}{c}; \text{ i. e., } a:c::a^2:b^2.$$

XVIII. *If four quantities be in continued proportion, the first will have to the fourth the triplicate ratio of that which it has to the second.*

Let a, b, c, d be four quantities in continued proportion, so that

$a:b::b:c::c:d$; then, also, $a:d::a^3:b^3$.

Since

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}, \text{ we have}$$

$$\frac{a}{b}=\frac{b}{c}$$

$$\frac{a}{b}=\frac{c}{d}$$

$$\frac{a}{b}=\frac{a}{b}.$$

Multiplying these equals together,

$$\frac{a^3}{b^3}=\frac{bca}{cdb},$$

or

$$\frac{a^3}{b^3}=\frac{a}{d}; \text{ i. e., } a:d::a^3:b^3.$$

XIX. *If two proportions be multiplied together, term by term, the products will form a proportion.*

Let $a:b::c:d$,

and $e:f::g:h$;

then $ae:bf::cg:dh$,

for $\frac{a}{b}=\frac{c}{d}$, and $\frac{e}{f}=\frac{g}{h}$;

hence, multiplying equals,

$$\frac{ae}{bf}=\frac{cg}{dh}, \text{ or } ae:bf::cg:dh.$$

The compatibility of any change in the order of the terms of a proportion may be tested by forming the product of the extremes and means in both the original and changed proportion, when, if they agree, the change is correct. Thus, $a:b::c:d$ may be written $d:b::c:a$, for we have $ad=bc$ in both.

EXAMPLES IN PROPORTION.

(1) The mercurial barometer stands at a height of 30 inches, and the specific gravity of quicksilver is $13\frac{23}{40}$. How high would a water barometer stand?

Ans. 33 feet $11\frac{1}{4}$ inches.

(2) The weights of a lever have the same ratio as the lengths of the opposite arms. The ratio of the weights is 5, and the longer arm 10 inches. What is the length of the shorter arm? Ans. 2 inches.

(3) The weights of a lever are 6 and 8 pounds, and the length of the shorter arm 18 inches. What is that of the longer? Ans. 24 inches.

(4) At the end of an arm of a lever 5 inches long, what weight can be supported by $2\frac{1}{3}$ pounds acting at the end of an arm $4\frac{2}{3}$ inches long?

Ans. $2\frac{8}{15}$ pounds.

(5) Triangles are to each other as the products of their bases by their altitudes. The bases of two triangles are to each other as 17 and 18, and their altitudes as 21 and 23. What is the ratio of the triangles themselves?

Ans. 119:138.

(6) The force of gravitation is inversely as the square of the distance. At the distance 1 from the centre of the earth this force is expressed by the number 32.16. By what is it expressed at the distance 60?

Ans. 0.0089.

(7) The motion of a planet about the sun for a short space is proportional to unity divided by the duplicate of the distance. If the motion be represented by v when the distance is r, by what will it be expressed when the distance is r'?

Ans. $\frac{r^2v}{r'^2}$.

(8) The times of revolution of the planets about the sun are in the sesquiplicate ratio of their mean distances. The mean distance of the earth from the sun being expressed by 1, that of Jupiter will be 5.202776; the time of revolution of the earth is 365.2563835 days. What is the time of revolution of Jupiter? Ans. 4332.5848212 days.

EQUATIONS.

PRELIMINARY REMARKS.

134. An *equation*, in the most general acceptation of the term, is composed of two algebraic expressions which are equal to each other, connected by the sign of equality.

Thus, $ax=b$, $cx^2+dx=e$, $cx^3+gx^2=hx+k$, $mx^4+nx^3+px^2+qx+r=0$, are equations.

The two quantities separated by the sign $=$ are called the *members* of the equation, the quantity to the left of the sign $=$ is called the *first member*, the quantity to the right the *second member*. The quantities separated by the signs $+$ and $-$ are called the *terms* of the equation.

135. Equations are usually composed of certain quantities which are known and given, and others which are unknown. The known quantities are in general represented either by numbers, or by the first letters in the alphabet, a, b, c, &c.; the unknown quantities by the last letters, s, t, x, y, z, &c.

136. Equations are of different kinds.

1°. An equation may be such that one of the members is a repetition of the other; as, $2x-5=2x-5$.

2°. One member may be merely the result of certain operations *indicated* in the other member; as, $5x+16=10x-5-(5x-21)$, $(x+y)(x-y)=x^2-y^2$, $\frac{x^3-y^3}{x-y}=x^2+xy+y^2$.

3°. All the quantities in each member may be known and given; as, $25=10+15$, $a+b=c-d$, in which, if we substitute for a, b, c, d the known quantities which they represent, the equality subsisting between the two members will be self-evident.

In each of the above cases the equation is called *an identical equation.*

4 . Finally, the equation may contain both known and unknown quantities, and be such that the equality subsisting between the two members can not be made manifest, until we substitute for the unknown quantity or quantities certain other numbers, the value of which depends upon the known numbers which enter into the equation. The discovery of these unknown numbers constitutes what is called the *solution of the equation.*

When found and put in the place of the letters which represent them, if they make the equality of the two members evident, the equation is said to be *verified*, or *satisfied*.

The word *equation*, when used without any qualification, is always understood to signify an equation of this last species; and these alone are the objects of our present investigations.

$x+4=7$ is an equation properly so called, for it contains an unknown quantity x, combined with other quantities which are known and given, and the equality subsisting between the two members of the equation can not be made manifest until we find a value for x, such that, when added to 4, the result will be equal to 7. This condition will be satisfied if we make $x=3$; and this value of x being determined, the equation is solved.

The value of the unknown quantity thus discovered is called the *root* of the equation, being the radix out of which the equation is formed; the term *root* here has a different sense from that in which we have hitherto used it, viz., that of the base of a power.

137. Equations are divided into *degrees* according to the highest power of the unknown quantity which they contain. Those which involve the simple or first power only of the unknown quantity are called *simple equations*, or *equations of the first degree;* those into which the square of the unknown quantity enters are called *quadratic equations*, or *equations of the second degree:* so we have *cubic equations*, or *equations of the third degree; biquadratic equations*, or *equations of the fourth degree; equations of the fifth, sixth, ... n^{th} degree.* Thus,

$ax+b=cx+d$ is a simple equation.

$4x^2-2x=5-x^2$ is a quadratic equation.

$x^3+px^2=2q$ is a cubic equation.

$x^n+px^{n-1}+qx^{n-2}+$, &c., $=r$, is an equation of the n^{th} degree.

138. *Numerical equations* are those which contain numbers only, in addition to the unknown quantities. Thus, $x^3+5x^2=3x+17$ and $4x=7y$ are numerical equations.

139. *Literal equations* are those in which the known quantities are represented by letters only, or by both letters and numbers. Thus, $x^3+px^2+qx=r$, $x^4-3px^3y+5qx^2y^2+rxy^3=5$ are literal equations.

140. Let us now pass on to consider the solution of equations, it being understood that *to solve an equation is to find the value of the unknown quantity, or to find a number which, when substituted for the unknown quantity in the equation, renders the first member identical with the second.*

The difficulty of solving equations depends upon the degree of the equations and the number of unknown quantities. We first consider the most simple case.

ON THE SOLUTION OF SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

141. The various operations which we perform upon equations, in order to arrive at the value of the unknown quantities, are founded upon the following axioms:

If to two equal quantities the same quantity be added, the sums will be equal.

If from two equal quantities the same quantity be subtracted, the remainders will be equal.

If two equal quantities be multiplied by the same quantity, the products will be equal.

If two equal quantities be divided by the same quantity, the quotients will be equal.

These axioms, when applied to the two equal quantities which constitute the two members of every equation, will enable us to deduce from them new equations, which are all satisfied by the same value of the unknown quantity, and which will lead us to discover that value.

142. The unknown quantity may be combined with the known quantities in the given equation by the operations of addition, subtraction, multiplication, and division. We shall consider these different cases in succession.

I. Let it be required to solve the equation

$$x+a=b.$$

If, from the two equal quantities $x+a$ and b, we subtract the same quantity a, the remainders will be equal, and we shall have

$$x+a-a=b-a,$$

or

$$x=b-a, \text{ the value of } x \text{ required.}$$

So, also, in the equation

$$x+6=24.$$

Subtracting 6 from each of the equal quantities $x+6$ and 24, the result is

$$\begin{aligned} x&=24-6 \\ &=18, \text{ the value of } x \text{ required.} \end{aligned}$$

II. Let the equation be

$$x-a=b.$$

If, to the two equal quantities $x-a$ and b, the same quantity a be added, the sums will be equal; then we have

$$x-a+a=b+a,$$

or

$$x=b+a, \text{ the value of } x \text{ required.}$$

So, also, in the equation

$$x-6=24.$$

Adding 6 to each of these equal quantities, the result is

$$\begin{aligned} x&=24+6 \\ &=30, \text{ the value of } x \text{ required.} \end{aligned}$$

It follows from (I.) and (II.) that

*We may transpose any term of an equation from one member to the other by changing the sign of that term.**

We may change the signs of every term in each member of the equation without altering the value of the expression.†

If the same quantity appear in each member of the equation affected with th same sign, it may be suppressed.

III Let the equation be

$$ax=b.$$

Dividing each of these equals by a, the result is

$$x=\frac{b}{a}, \text{ the value of } x \text{ required.}$$

So, also, in the equation

$$6x=24.$$

Dividing each of these equals by 6, the result is

$$x=4, \text{ the value of } x \text{ required.}$$

From this it follows that,

When one member of an equation contains the unknown quantity alone, affected with a coefficient, and the other member contains known quantities only, the value of the unknown quantity is found by dividing each member of the equation by the coefficient of the unknown quantity

IV. Let the equation be

$$\frac{x}{a}=b.$$

Multiplying each of these equals by a, the result is

$$x=ab, \text{ the value of } x \text{ required.}$$

So, also, in the equation

$$\frac{x}{6}=24.$$

Multiplying each of these equals by 6, the result is

$$x=144.$$

From this it follows that,

When one member of the equation contains the unknown quantity alone, divided by a known quantity, and the other member contains known quantities only, the value of the unknown quantity is found by multiplying each member of the equation by the quantity which is the divisor of the unknown quantity.

V. Let the equation be

$$\frac{ax}{b}-c=\frac{dx}{e}-\frac{m}{n}.$$

In order to solve this equation, we must clear it of fractions; to effect this, reduce the fractions to equivalent ones, having a common denominator (Art. 41), the equation becomes

$$\frac{aenx}{ben}-\frac{bcen}{ben}=\frac{bdnx}{ben}-\frac{bem}{ben}.$$

Multiply these equal quantities by the same quantity ben, or, which is evi-

* If we transpose a plus term, it subtracts this term from both members; and if we transpose a minus term, it adds this term to both.

† This is, in fact, the same thing as transposing every term in each member of the equation, or multiplying throughout by —1.

dently the same thing, suppress the denominator ben in each of the fractions, the result is

$$aenx - bcen = bdnx - bem,$$ an equation clear of fractions.

So, also, in the equation

$$\frac{2x}{3} - \frac{3}{4} = 11 + \frac{x}{5}.$$

Reducing the fractions to a common denominator

$$\frac{40x}{60} - \frac{45}{60} = \frac{660}{60} + \frac{12x}{60}.$$

Multiplying both members of the equation by 60, the result is

$$40x - 45 = 660 + 12x,$$ an equation clear of fractions.

If the denominators have common factors, we can simplify the above operation by reducing them to their least common denominator, which is done (see Art. 44) by finding the least common multiple of the denominators. Thus, in the equation

$$\frac{5x}{12} - \frac{4x}{3} - 13 = \frac{7}{8} - \frac{13x}{6}.$$

The least common multiple of the numbers 12, 3, 8, 6 is 24, which is, therefore, the least common denominator of the above fractions, and the equation will become

$$\frac{10x}{24} - \frac{32x}{24} - \frac{312}{24} = \frac{21}{24} - \frac{52x}{24}.$$

Multiplying both members of the equation by 24, the result is

$$10x - 32x - 312 = 21 - 52x,$$ an equation clear of fractions.

Hence it appears that,

In order to clear an equation of fractions, reduce the fractions to a common denominator, and then multiply each term by this common denominator. In the fractional terms the common denominator will be simply suppressed.

143. From what has been said above, we deduce the following general

RULE FOR THE SOLUTION OF A SIMPLE EQUATION CONTAINING ONE UNKNOWN QUANTITY.

1°. *Clear the equation of fractions, and perform in both members all the algebraic operations indicated.*

2°. *Transpose all the terms containing the unknown quantity to one member of the equation, and all the terms containing known quantities only to the other member, and reduce each member to its most simple form.*

3°. *We thus obtain an equation, one member of which contains the unknown quantity alone, affected with a coefficient, and the other member contains known quantities only; the value of the unknown quantity will be found by dividing the member composed of the known quantities by the coefficient of the unknown quantity.*

The terms containing the unknown quantity are usually collected in the *first member* of the equation, though they may often be more conveniently collected in the second; the second being afterward written as the first member, and the first as the second.

Sometimes an equation presents itself as one of a degree higher than the first, but both members are divisible by such a power of the unknown quantity as to reduce the equation to one of the first degree.

In other cases, clearing an equation of fractions reduces it, by the canceling of those terms which contain the higher powers of the unknown quantity, to the first degree.

A proportion containing an unknown quantity in any of its terms can be thrown into the form of an equation by multiplying the extremes, and also the means, and setting the two products thus formed equal to each other.

EXAMPLE I.

Given, $19x+13=59-4x.$

Transposing, $19x+4x=59-13.$

Reducing, $23x=46.$

Dividing by 23, $x=2.$

Verification.—Substitute 2 for x in the given equation, it becomes

$$19\times 2+13=59-4\times 2, \text{ or}$$

$$38+13=59-8, \text{ an identity.}$$

Let this process be repeated in some of the following examples.

EXAMPLE II.

Given, $\frac{x}{6}-\frac{x}{4}+10=\frac{x}{3}-\frac{x}{2}+11.$

Reducing to least common denominator 12,

$$\frac{2x}{12}-\frac{3x}{12}+10=\frac{4x}{12}-\frac{6x}{12}+11.$$

Multiplying both members by 12,

$$2x-3x+120=4x-6x+132.$$

Transposing, $2x-3x-4x+6x=132-120.$

Reducing, $x=12.$

EXAMPLE III.

Given, $\frac{5x+3}{4}+7=\frac{4x-10}{10}+10.$

Reducing to least common denominator 20,

$$\frac{25x+15}{20}+7=\frac{8x-20}{20}+10.$$

Multiplying both members by 20,

$$25x+15+140=8x-20+200.$$

Transposing, $25x-8x=200-20-15-140.$

Reducing, $17x=25.$

Dividing by 17, $x=\frac{25}{17}.$

EXAMPLE IV.

Given, $\frac{2x-5}{4}-\frac{7x+10}{3}=16-\frac{12x-10}{5}.$

Reducing to common denominator,

$$\frac{30x-75}{60}-\frac{140x+200}{60}=16-\frac{144x-120}{60}.$$

Multiplying both members by 60,

$$30x-75-140x-200=960-144x+120.$$

Transposing, $30x-140x+144x=960+75+200+120.$

Reducing, $34x=1355$.

Dividing by 34, $x=\frac{1355}{34}$.

It is unnecessary to write the common denominator.

EXAMPLE V.

Given, $\frac{12-4x}{10}-\frac{2x+5}{5} = 3+\frac{7x+60}{2}-50$.

Reducing to least common denominator, 10, and neglecting it, we have

$$12-4x-4x-10 =30+ 35x+300-500.$$

Transposing, $-4x-4x-35x=30+300 - 12+10-500$.

Reducing, $-43x=-172$.

Changing the signs of both members,*

$$43x= 172.$$

Dividing by 43, $x= 4$.

EXAMPLE VI.

Given, $ax+b =cx+d$.

Transposing, $ax-cx= d-b$.

Simplifying, $(a-c)x= d-b$.

Dividing by $(a-c)$, $x=\frac{d-b}{a-c}$.

EXAMPLE VII.

$$\frac{ax}{b}+\frac{cx}{d}+e=fx+\frac{gx}{h}+m.$$

Reducing to a common denominator,

$$\frac{adhx}{bdh}+\frac{bchx}{bdh}+e=fx+\frac{bdgx}{bdh}+m.$$

Multiplying by bdh,

$$adhx+bchx+bdeh=bdfhx+bdgx+bdhm.$$

Transposing, $adhx+bchx-bdfhx-bdgx=bdhm-bdeh$.

Simplifying, $(adh+bch-bdfh-bdg)x=bdhm-bdeh$.

Dividing by coefficient of x, $x=\frac{bdhm-bdeh}{adh+bch-bdfh-bdg}$

$$=\frac{bdh(m-e)}{adh+bch-bdfh-bdg}.$$

EXAMPLE VIII.

Given, $\frac{x}{a}-1-\frac{dx}{c}+3ab=0$.

Reducing to common denominator and neglecting it,

$$cx-ac-adx+3a^2bc=0.$$

Transposing and simplifying, $(c-ad)x=ac-3a^2bc$.

Dividing by coefficient of x, $x=\frac{ac(1-3ab)}{c-ad}$.

Verification.

$$\frac{\frac{ac(1-3ab)}{c-ad}}{a}-1-\frac{acd(1-3ab)}{c(c-ad)}+3ab=0;$$

* Or dividing both members by -43, gives $x=4$.

or

$$\frac{c(1-3ab)}{c-ad}-1-\frac{ad(1-3ab)}{c-ad}+3ab=0;$$

or

$$c-3abc-c+ad-ad+3a^2bd+3abc-3a^2bd=0.$$

EXAMPLE IX.

Given, $x+18=3x-5.$

Transposing, $18+5=3x-x$

$$23=2x$$

$$x=\frac{23}{2}=11\tfrac{1}{2}.$$

EXAMPLE X.

Given, $\frac{a}{x}=\frac{b}{c}+\frac{d}{e}.$

Clearing of fractions, $ace=bex+cdx$

$$ace=(be+cd)x$$

$$x=\frac{ace}{be+cd}.$$

EXAMPLE XI.

Given, $3x^2-10x=8x+x^2.$

Dividing by x, $3x-10=8+x$

$$x=9.$$

EXAMPLE XII.

Given, $x^m=ax^{m-1}.$

Dividing by x^{m-1}, $x=a.$

EXAMPLE XIII.

Given, $\frac{ax^m-a'}{x^m}=a-\frac{a''}{x^{m-1}}.$

Multiplying by x^m, $ax^m-a'=ax^m-a''x.$

Canceling ax^m in both members,

$$-a'=-a''x \therefore x=\frac{a'}{a''}.$$

EXAMPLE XIV.

Given, $a:bx::c:d \therefore bcx=ad \therefore x=\frac{ad}{bc}.$

144. In addition to the axioms in (Art. 141) we may subjoin the following: *If two equal quantities be raised to the same power, the results will be equal. If the same root of two equal quantities be extracted, the results will be equal.* Hence any equation may be cleared of a single radical quantity by transposing all the other terms to the opposite side, and then raising each member to the power denoted by the index of the radical. If there be more than one radical, the operation must be repeated. Thus:

EXAMPLE XV.

Given, $\sqrt{3x+7}=10.$

Squaring each member of the equation,

$$3x+7=100.$$

Transposing, $3x=100-7.$

Reducing, and dividing by 3, $x=31.$

EXAMPLE XVI.

Given, $\sqrt{4x+2}=\sqrt{4x}+5.$

Squaring both sides of the equation,

$$4x+2=4x+10\sqrt{4x}+25.$$

Reducing, $-10\sqrt{4x}=23.$

Squaring both sides, $400x=529.$

$$x=\frac{529}{400}.$$

EXAMPLE XVII.

Given, $\frac{\sqrt{x}+28}{\sqrt{x}+4}=\frac{\sqrt{x}+38}{\sqrt{x}+6}.$

Clearing the equation of fractions,

$$x+28\sqrt{x}+6\sqrt{x}+168=x+38\sqrt{x}+4\sqrt{x}+152.$$

Transposing and reducing, $16=8\sqrt{x}.$

Dividing both members by 8, $2=\sqrt{x}.$

Squaring both members, $4=x.$

EXAMPLE XVIII.

Given, $\sqrt[m]{a+x}=\sqrt[2m]{x^2+5ax+b^2}.$

Raising both members to the m^{th} power,

$$a+x=\sqrt{x^2+5ax+b^2}.$$

Squaring both members, $a^2+2ax+x^2=x^2+5ax+b^2.$

Transposing and reducing, $-3ax=b^2-a^2.$

Changing the signs, $3ax=a^2-b^2.$

Dividing by $3a$, $x=\frac{a^2-b^2}{3a}.$

EXAMPLE XIX.

Given, $\frac{\sqrt[m]{x}-a^2}{\sqrt[2m]{x}-a}=\frac{\sqrt[2m]{x}-a}{b}.$

Since $\sqrt[m]{x}$ is the square of $\sqrt[2m]{x}$, and a^2 is the square of a, we can perform the division indicated in the first fraction, and have for a quotient

$$\sqrt[2m]{x}+a=\frac{\sqrt[2m]{x}-a}{b},$$

$$\therefore (b-1)\sqrt[2m]{x}=-(b+1)a,$$

$$\therefore \sqrt[2m]{x}=-\frac{(b+1)a}{(b-1)},$$

$$\therefore x=\left(\frac{(b+1)a}{b-1}\right)^{2m}$$

(20) Given $4x+36=5x+34.$ Ans. $x=2.$

(21) Given $4x-12+3x+1=2x+4.$ Ans. $x=3.$

(22) Given $3a+x-5b+2=7b-a+c+6$. Ans. $x=12b-4a+c+4$.

(23) Given $13\frac{3}{4}-\frac{x}{2}=2x-8\frac{3}{4}$. Ans. $x=9$.

(24) Given $12\frac{1}{4}+3x-6-\frac{7x}{3}=\frac{3x}{4}-5\frac{3}{8}$. Ans. $x=139\frac{1}{2}$.

(25) Given $\frac{x}{2}+\frac{x}{3}=\frac{x}{4}+7$. Ans $x=12$.

(26) Given $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=13$. Ans. $x=12$.

(27) Given $x+\frac{x}{2}-\frac{x}{3}=4x-17$. Ans. $x=6$.

(28) Given $5-\frac{x+4}{11}=x-3$. Ans. $x=7$.

(29) Given $x+\frac{3x-5}{2}=12-\frac{2x-4}{3}$. Ans. $x=5$.

(30) Given $\frac{x+1}{3}+\frac{x+3}{4}=\frac{x+4}{5}+16$. Ans. $x=41$.

(31) Given $5x-\frac{5x}{2}+12=\frac{4x}{3}+26$. Ans. $x=12$.

(32) Given $7x+13\frac{3}{4}-\frac{x}{2}=\frac{4x}{5}-8\frac{3}{4}+\frac{41x}{5}$. Ans. $x=9$.

(33) Given $8x-7\frac{1}{4}-\frac{3}{4}x\pm10-5x-2\frac{3}{4}=0$. Ans. $x=0$, or $8\frac{3}{9}$.

(34) Given $4(5x+7-\frac{2}{3})=\frac{2}{3}(3x+9-4)$. Ans. $x=-1\frac{2}{9}$.

(35) Given $\frac{x+\frac{1}{2}x+\frac{1}{3}x}{21-\frac{4}{5}}=\frac{20x-25}{101}$. Ans. $x=2\frac{4}{13}$.

(36) Given $\frac{x-5}{4}+6x=\frac{284-x}{5}$. Ans. $x=9$.

(37) Given $x+\frac{11-x}{3}=\frac{19-x}{2}$. Ans. $x=5$.

(38) Given $3x+\frac{2x+6}{5}=5+\frac{11x-37}{2}$. Ans. $x=7$.

(39) Given $\frac{6x-4}{3}-2=\frac{18-4x}{3}+x$. Ans. $x=4$.

(40) Given $21+\frac{3x-11}{16}=\frac{5x-5}{8}+\frac{97-7x}{2}$. Ans. $x=9$

(41) Given $3x-\frac{x-4}{4}-4=\frac{5x+14}{3}-\frac{1}{12}$. Ans. $x=7$

(42) Given $\frac{x-1}{7}+\frac{23-x}{5}=7-\frac{4+x}{4}$ Ans. $x=8$

(43) Given $\frac{7x+5}{3}-\frac{16+4x}{5}+6=\frac{3x+9}{2}$. Ans. $x=1$

(44) Given $\frac{3x+4}{5}-\frac{7x-3}{2}=\frac{x-16}{4}$. Ans. $x=2$

(45) Given $\frac{17-3x}{5}-\frac{4x+2}{3}=5-6x+\frac{7x+14}{3}$. Ans. $x=4$

(46) Given $x-\frac{3x-3}{5}+4=\frac{20-x}{2}-\frac{6x-8}{7}+\frac{4x-4}{5}$. Ans. $x=6$

(47) Given $\frac{4x-21}{9}+3\frac{3}{4}+\frac{57-3x}{4}=241-\frac{5x-96}{12}-11x$. Ans. $x=21$.

(48) Given $\frac{6x+18}{13}-4\frac{5}{6}-\frac{11-3x}{36}=5x-48-\frac{13-x}{12}-\frac{21-2x}{18}$.

Ans. $x=10$.

(49) Given $21+\frac{3x-11}{16}=\frac{5x-5}{8}+\frac{97-7x}{2}$. Ans. $x=9$.

(50) Given $\frac{bx}{a}-\frac{d}{c}=\frac{a}{b}-\frac{cx}{d}$. Ans. $x=\frac{ad}{bc}$.

(51) Given $23+\frac{5x-1}{11}+\frac{3x-2}{5}-\frac{11x-3}{12}=\frac{13x-15}{3}-\frac{8x-2}{7}$.

Ans. $x=9$.

(52) Given $4x+\frac{1}{10}-\frac{3x-13}{16}-\frac{12+7x}{9}=7x-33-\frac{9+5x}{10}-\frac{11x-17}{8}$.

Ans. $x=15$.

(53) Given $\frac{ace}{d}-\frac{(a+b)^2x}{a}-bx=ae-3bx$. Ans. $x=\frac{a^2e(c-d)}{(a^2+b^2)d}$.

(54) Given $\frac{a+3x}{4a}-\frac{7a-5x}{6b}+3-\frac{9x}{4}=\frac{x}{ab}+\frac{5x}{6b}$.

Ans. $x=\frac{39ab-14a^2}{27ab-9b+12}$.

(55) Given $\frac{bx}{2b-a}-\frac{(3bc+ad)x}{2ab(a+b)}-\frac{5ab}{3c-d}=\frac{(3bc-ad)x}{2ab(a-b)}-\frac{5a(2b-a)}{a^2-b^2}$.

Ans. $x=\frac{5a(2b-a)}{3c-d}$.

(56) Given $ax+c=bx+d$. Ans. $x=\frac{d-c}{a-b}$.

(57) Given $2ax-bx+2ab=4a^2-ab-3ax$. Ans. $x=\frac{4a^2-3ab}{5a-b}$.

(58) Given $(3a-x)(a-b)+2ax=4b(a+x)$. Ans. $x=\frac{7ab-3a^2}{a-3b}$.

(59) Given $\frac{1}{2}ax+\frac{1}{3}bx=c$. Ans. $x=\frac{6c}{3a+2b}$.

(60) Given $\frac{x}{a}-1-\frac{dx}{c}+3ab=0$. Ans. $x=\frac{ac(1-3ab)}{c-ad}$.

(61) Given $\frac{a^2x}{b-c}+dc=bx-ac$. Ans. $x=\frac{abc-ac^2+bcd-c^2d}{b^2-bc-a^2}$.

(62) Given $\frac{ax}{b}-c=\frac{mx}{n}+d$. Ans. $x=\frac{bcn+bdn}{an-bm}$.

(63) Given $\frac{ax}{a-b}+4b=\frac{cx}{3a+b}$. Ans. $x=\frac{8ab^2+4b^3-12a^2b}{3a^2+ab-ac+bc}$.

(64) Given $\frac{3bx}{2a^2}-\frac{x-b}{a+b}=\frac{bx-a^2}{a^2-b^2}-\frac{x}{4a}$. Ans. $x=\frac{4a^2(a^2+ab-b^2)}{3a^3-6a^2b+ab^2+6b^3}$.

(65) Given $\frac{(a+b)(x-b)}{a-b}-3a=\frac{4ab-b^2}{a+b}-2x+\frac{a^2-bx}{b}$.

Ans. $x=\frac{a^4+3a^3b+4a^2b^2-6ab^3+2b^4}{b(4a^2+2ab-2b^2)}$

(66) Given $\frac{ax}{m}+\frac{b}{m}+\frac{cx}{k}=\frac{px}{m}-\frac{q}{m}-\frac{rx}{k}$. Ans. $x=-\frac{kb+kq}{ka-kp+mc+rm}$
$=-\frac{k(b+q)}{k(a-p)+m(c+r)}$.

(67) Given $\frac{x}{k}+\frac{ax}{2k}+\frac{bx}{3k}+\frac{cx}{4k}=\frac{mx}{k}+p$.

Ans. $x=\frac{12pk}{12(1-m)+2(3a+2b)+3c}$.

(68) Given $r(ax+b-c)=c(px+q-r)$. Ans. $x=\frac{cq-rb}{ra-cp}$.

(69) Given $\frac{x+px-qx}{p-q}=\frac{mx-n}{m}$. Ans. $x=\frac{n(q-p)}{m}$.

(70) Given $(\frac{3}{4}m+p)(\frac{2}{7}x-3r)=(\frac{5}{8}m+2p)(\frac{3}{8}x-7r)$.

Ans. $x=\frac{r(952m+4928p)}{9m+208p}$.

(71) Given $\frac{m^2x}{n}-\frac{h^2}{g}+5nx=\frac{4m^2x-h^2n-8n^2gx}{5ng}$.

Ans. $x=\frac{4nh^2}{5m^2g-4m^2+33n^2g}$.

(72) Given $\frac{13(5ax-22\frac{3}{7}b)}{9ck^2}=\frac{24(3ax-20\frac{2}{5}b)}{7c^2k}$.

Ans. $x=\frac{(2041c-4406\frac{2}{5}k)b}{(455c-648k)a}$.

(73) Given $\frac{13m-7x}{m+p}+\frac{4m-x}{m-p}=\frac{m+p}{m-p}-kx$.

Ans. $x=\frac{11mp-16m^2+p^2}{6p-8m+k(m^2-p^2)}$.

(74) Given $\frac{3abc}{a+b}+\frac{(2a+b)b^2x}{a(a+b)^2}+\frac{a^2b^2}{(a+b)^3}=3cx+\frac{bx}{a}$.

Ans. $x=\frac{3a^2bc(a+b)^2+a^3b^2}{(3ac+b)(a+b)^3-(2a+b)(a+b)b^2}$.

(75) Given $ax-\frac{a^2-3bx}{a}-ab^2=bx+\frac{6bx-5a^2}{2a}-\frac{bx+4a}{4}$.

Ans. $x=\frac{4ab^2-10a}{4a-3b}$.

(76) Given $ax^2+bx=cx^2+dx$. Ans. $x=\frac{d-b}{a-c}$.

(77) Given $Ax^m+Bx^{m-1}=Cx^m-Dx^{m-1}$. Ans. $x=\frac{D+B}{C-A}$.

(78) Given $\frac{14a^3-2a^2bx}{17mx}-31c^5=\frac{21a^3+5a^2bx}{33mx}+20c^5$.

Ans. $x=\frac{105a^3}{151a^2b+28611c^5m}$

(79) Given $\frac{4m(K^2-5x^2)}{8x}=7mp+\frac{5m(g^2-2x)}{4}$. Ans. $x=\frac{2K^2}{28p+5g^2}$.

(80) Given $\frac{24x^8}{3-5x}=\frac{5x^8}{7-3x}$. Ans. $x=3\frac{12}{47}$.

(81) Given $\frac{ax^n}{b+cx}=\frac{mx^n}{p+qx}$. Ans. $x=\frac{bm-ap}{aq-cm}$.

(82) Given $12-x:\frac{x}{2}::4:1$. Ans. $x=4$.

(83) Given $\frac{5x+4}{2}:\frac{18-x}{4}::7:4$. Ans. $x=2$.

(84) Given $2\odot:1::1:3.1416$. Ans. $\odot=0.1591$.

(85) Given $a:t::\frac{b}{c}:7c$. Ans. $t=\frac{7ac^2}{b}$.

(86) Given $r:1::c:3.1416$. Ans. $r=\frac{c}{3.1416}$.

(87) Given $\sqrt{4x+16}=12$. Ans. $x=32$.

(88) Given $\sqrt[3]{2x+3}+4=7$. Ans. $x=12$.

(89) Given $\sqrt{12+x}=2+\sqrt{x}$. Ans. $x=4$.

(90) Given $\sqrt{x+40}=10-\sqrt{x}$. Ans. $x=9$.

(91) Given $\sqrt{x-16}=8-\sqrt{x}$. Ans. $x=25$.

(92) Given $\sqrt{x-24}=\sqrt{x}-2$. Ans. $x=49$.

(93) Given $\sqrt{x-a}=\sqrt{x}-\frac{1}{2}\sqrt{a}$. Ans. $x=\frac{25a}{16}$.

(94) Given $\sqrt{5}\times\sqrt{x+2}=\sqrt{5x}+2$. Ans. $x=\frac{9}{20}$.

(95) Given $\sqrt{4a+x}=2\sqrt{b+x}-\sqrt{x}$. Ans. $x=\frac{(b-a)^2}{2a-b}$.

(96) Given $x+a+\sqrt{2ax+x^2}=b$. Ans. $x=\frac{(b-a)^2}{2b}$.

(97) Given $\frac{x-ax}{\sqrt{x}}=\frac{\sqrt{x}}{x}$. Ans. $x=\frac{1}{1-a}$.

(98) Given $\frac{\sqrt{x}+28}{\sqrt{x}+4}=\frac{\sqrt{x}+38}{\sqrt{x}+6}$. Ans. $x=4$

(99) Given $\frac{\sqrt{x}+2a}{\sqrt{x}+b}=\frac{\sqrt{x}+4a}{\sqrt{x}+3b}$. Ans. $x=\left(\frac{ab}{a-b}\right)^2$.

(100) Given $\frac{3x-1}{\sqrt{3x}+1}=1+\frac{\sqrt{3x}-1}{2}$. Ans. $x=3$

(101) Given $\frac{ax-b^2}{\sqrt{ax}+b}=c+\frac{\sqrt{ax}-b}{c}$. Ans. $x=\frac{1}{a}\left(b+\frac{c^2}{c-1}\right)^2$

(102) Given $x=\sqrt{a^2+x\sqrt{b^2+x^2}}-a$. Ans. $x=\frac{b^2-4a^2}{4a}$

(103) Given $\sqrt{5+x}+\sqrt{x}=\frac{15}{\sqrt{5+x}}$. Ans. $x=4$

(104) Given $\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}=\frac{3}{2}\sqrt{\frac{x}{x+\sqrt{x}}}$. Ans. $x=\frac{25}{16}$.

(105) Given $\frac{1}{x}+\frac{1}{a}=\sqrt{\frac{1}{a^2}+\sqrt{\frac{4}{a^2x^2}+\frac{9}{x^4}}}$. Ans. $x=2a$.

(106) Given $\sqrt{10x+3}=7$. Ans. $x=\frac{46}{10}$.

(107) Given $\sqrt{x-32}=16-\sqrt{x}$ Ans. $x=81$.

(108) Given $\frac{5x-9}{\sqrt{5x}+3}-1=\frac{\sqrt{5x}-3}{2}$. Ans. $x=5$.

(109) Given $h\sqrt[3]{ax-b}=k\sqrt[3]{cx+dx-f}$. Ans. $x=\frac{bh^3-fk^3}{ah^3-(c+d)k^3}$.

(110) Given $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=\sqrt{m}$. Ans. $x=\frac{2a\sqrt{m}}{1+m}$.

(111) Given $\sqrt[3]{a^2+c}=\sqrt[4]{\frac{a^2+c}{d(x+9)}}$. Ans. $x=\frac{1}{d\sqrt[3]{a^2+c}}-9$.

(112) Given $\frac{mx}{n}\sqrt{p^2x^2+q^2}+\frac{mpx^2}{n}=rx$. Ans. $x=\frac{(nr+mq)(nr-mq)}{2mnpr}$.

When an equation can never be verified, whatever value we put in the place of the unknown quantity, it is said to be *impossible;* and when an equation is always verified, whatever value be put for the unknown quantity, it is said to be indeterminate.

CASES OF IMPOSSIBILITY AND INDETERMINATION IN EQUATIONS OF THE FIRST DEGREE.

I. PROBLEM.—To find a number such that the third of it, augmented by 75, and five twelfths of it, diminished by 35, shall make three quarters of it, added to 49.

The equation is

$$\frac{x}{3}+75+\frac{5x}{12}-35=\frac{3x}{4}+49, \qquad [1]$$

$$\therefore \frac{x}{3}+\frac{5x}{12}-\frac{3x}{4}=9$$

$$\therefore 4x+5x-9x=108$$

$$\therefore 0=108.$$

An absurdity. There is, therefore, no value of x which can satisfy the equation [1].

The impossibility may be rendered evident in the equation [1] itself by reducing the similar terms in the first member; thus,

$$\frac{3x}{4}+40=\frac{3x}{4}+49.$$

It is evident that the two members will always differ by 9, whatever be the value of x.

II. PROBLEM.—To find a number such that, adding together the half of it increased by 10, two thirds of it increased by 20, and five sixths of it diminished by 34, the sum shall be equal to twice the excess of this number over 5.

$$\frac{x+10}{2}+\frac{2(x+20)}{3}+\frac{5(x-34)}{6}=2(x-5), \qquad [2]$$

$$\therefore 3x+30+4x+80+5x-170=12x-60$$

$$\therefore 3x+4x+5x-12x=170-30-80-60$$

$$i.\ e.,\ 0=0.$$

The unknown x is, therefore, altogether indeterminate; that is to say, it may be taken equal to 2 or 3, or any number whatever.

ON THE SOLUTION OF SIMPLE EQUATIONS, CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

145. A single equation, containing two unknown quantities, admits of an infinite number of solutions; for if we assign any arbitrary value to one of the unknown quantities, the equation will determine the corresponding value of the other unknown quantity. Thus, in the equation $y=x+10$, each value which we may assign to x will, when augmented by 10, furnish a corresponding value of y. Thus, if $x=2$, $y=12$; if $x=3$, $y=13$, and so on. An equation of this nature is called an *indeterminate equation*, and since the value of y depends upon that of x, y is said to be a *function* of x.

In general, every quantity, whose value depends upon one or more quantities, is said to be a FUNCTION *of these quantities.*

Thus, in the equation $y=ax+b$, we say that y is a function of x, and that y is *expressed in terms of* x, and the known quantities a, b.

If, however, we have *two* equations between two unknown quantities, and if these equations hold good together, then it will be seen presently that we can combine them in such a manner as to obtain determinate values for each of the unknown quantities; that is to say, each of the unknown quantities will have but a single value, which will satisfy the equations. The equations in this case are called determinate.

In general, in order that questions may admit of determinate solutions, we must have *as many separate equations as there are unknown quantities;* a group of equations of this nature is called *a system of simultaneous equations.*

If the number of equations exceed the number of unknown quantities, unless the equations in excess conform to the values of the unknown quantities determined by the others, the equations are said to be incompatible. Thus, if we have $x+y=10$ and $x-y=6$, the only values of x and y which will satisfy both these equations are 8 for x, and 2 for y. Now, if we were to add another equation to these, it must conform to these values, and could not be written in any form at pleasure. Thus, we might for a third equation say $xy=16$; but we could not write $xy=100$, for this third equation would be incompatible with the other two.*

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* Equations may be incompatible when the number does not exceed the number of unknowns, as the following problem will show:

A sportsman was asked how many birds he had taken. He replied, if 5 be added to the third of those I took last year, it will make the half of the number taken this year. But if from three times this last half 5 be taken, you will have precisely the number taken last year. How many did he take in each year?

Let $x=$ the number this year, and $y=$ the number last year.

$$\frac{x}{2}=\frac{y}{3}+5,\quad y=\frac{3x}{2}-5.$$

Substituting in the first the value of y in the second,

$$\frac{x}{2}=\frac{x}{2}-\frac{5}{3}+5$$

$$\therefore\ 3x-3x=30-10$$

$$0=20;$$

an absurd equality, whence we conclude that there exist no values of x and y which satisfy the two equations.

This is because the conditions of the problem are inconsistent with each other. When, however, the two equations are derived from the same problem, and its conditions are not contradictory, values for x and y will always be found to satisfy them.

146. In order to solve a system of two simple equations containing two unknown quantities, we must endeavor to deduce from them a single equation containing only one unknown quantity; we must, therefore, make one of the unknown quantities disappear, or, as it is termed, we must *eliminate* it. The equation thus obtained, containing one unknown quantity only, will give the value of the unknown quantity which it involves, and, substituting the value of this unknown quantity in either of the equations containing the two unknown quantities, we shall arrive at the value of the other unknown quantity.

The process which most naturally suggests itself for the *elimination* of one of the unknown quantities, is to derive from one of the two equations an expression for that unknown quantity *in terms* of the other unknown quantity, and then substitute this expression in the other equation. We shall see that the *elimination* may be effected by different methods, which are more or less simple according to the nature of the question proposed.

EXAMPLE I.

Let it be proposed to solve the system of equations

$$\left.\begin{aligned} y-x&=6 \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. (1) \\ y+x&=12 \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. (2) \end{aligned}\right\}$$

147. FIRST METHOD.—From equation (1) we find the value of y in terms of x, which gives $y=x+6$; substituting the expression $x+6$ for y in equation (2), it becomes $x+6+x=12$, from which we find the determinate value $x=3$; since we have already seen that $y=x+6$, we find also the determinate value $y=3+6$ or 9.

Thus it appears, that although each of the above equations, considered separately, admits of an infinite number of solutions, yet the *system* of equations admits only one *common solution*, $x=3$, $y=9$.

148. SECOND METHOD.—Derive from each equation an expression for y in terms of x, we shall then have

$$y=x+6$$
$$y=12-x.$$

These two values of y must be equal to one another, and, by *comparing* them, we shall obtain an equation involving only one unknown quantity, viz.,

$$x+6=12-x.$$

Whence

$$x=3.$$

Substituting the value of x in the expression $y=x+6$, we find $y=9$.

The substitution of 3, the value of x, in the second expression, $y=12-x$, leads necessarily to the same value of y; thus, $12-3=9$, for we derived the value of x from the equation $x+6=12-x$.

149. THIRD METHOD.—Since the coefficients of y are equal in the two equations, it is manifest that we may eliminate y by *subtracting the two equations from each other*, which gives

$$(y+x)-(y-x)=12-6.$$

Whence

$$2x=6$$
$$x=3.$$

Having thus obtained the value of x, we may deduce that of y by making $x=3$ in either of the proposed equations; we can, however, determine the value of y directly, by observing that, since the coefficients of x in the proposed equations are equal, and have opposite signs, we may eliminate x by *adding the two equations together*, which gives

$$(y-x)+(y+x)=12+6.$$

Whence

$$2y=18$$
$$y=9.$$

If we examine the three above methods, we shall perceive that they consist in expressing that *the unknown quantities have the same values in both equations.*

These methods have derived their names from the processes employed to effect the elimination of the unknown quantities.

The first is called the *method of elimination by substitution.*

The second is called the *method of elimination by comparison.*

The third is called the *method of elimination by addition and subtraction.*

The rule for the first is to *find the value of one of the unknown quantities in one of the equations, and substitute it in the other equation.*

For the second, is to *find the value of the same unknown quantity in each of the two given equations, and set these values equal.*

And for the third, is to *make the coefficient of the unknown quantity to be eliminated the same in the two equations, and add or subtract as the case may require. Add, if the signs of the equal terms are different, and if they are alike, subtract.*

By either of these rules a single equation, containing but one unknown quantity, is obtained.

EXAMPLE II.

Take the equations

$$\left.\begin{array}{ll} 2x+3y=13 \ldots\ldots\ldots\ldots & (1) \\ 5x+4y=22 \ldots\ldots\ldots\ldots & (2) \end{array}\right\}$$

1°. *Eliminating by substitution.*

From equation (1), we find

$$y=\frac{13-2x}{3}.$$

Substituting the value of y in terms of x in equation (2), it becomes

$$5x+4\times\frac{13-2x}{3}=22;$$

an equation containing x alone, which, when solved, gives

$$x=2.$$

This value of x, substituted in either of the equations (1) or (2), gives

$$y=3.$$

2°. *Eliminating by comparison.*

From equation (1) $y=\frac{13-2x}{3}$.

From equation (2) $y=\frac{22-5x}{4}$.

Equating these values of y, $\frac{13-2x}{3}=\frac{22-5x}{4}$; an equation containing x only

Whence

$$x=2.$$

Substituting this value for x in either of the preceding expressions for y, we find

$$y=3.$$

3°. *Eliminating by subtraction.*

In order to eliminate y, we perceive that if we could deduce from the proposed equations two other equations in x and y, in which the coefficients of y should be equal, the elimination of y would be effected by *subtracting* one of these new equations from the other.

It is easily seen that we shall obtain two equations of the form required if we multiply all the terms of each equation by the coefficient of y in the other. Multiplying, therefore, all the terms of equation (1) by 4, and all the terms of equation (2) by 3, they become

$$8x+12y=52$$
$$15x+12y=66.$$

Subtracting the former of these equations from the latter, we find

$$7x=14.$$

Whence

$$x=2.$$

In like manner, in order to eliminate x, multiply the first of the proposed equations by 5, and the second by 2, they will then become

$$10x+15y=65$$
$$10x+\ 8y=44.$$

Subtracting the latter of these two equations from the former,

$$7y=21.$$

Whence

$$y=3.$$

In order to solve a system of three simple equations between three unknown quantities, we must first eliminate one of the unknown quantities by one of the methods explained above; this will lead to a system of two equations, containing only two unknown quantities; the value of these two unknown quantities may be found by any of the methods described in the last article, and substituting the value of these two unknown quantities in any one of the original equations, we shall arrive at an equation which will determine the value of the third unknown quantity.

EXAMPLE III.

Take the system of equations

$$\left.\begin{aligned} 3x+2y+\ z&=16 \ \ldots\ldots\ldots\ (1)\\ 2x+2y+2z&=18 \ \ldots\ldots\ldots\ (2)\\ 2x+2y+\ z&=14 \ \ldots\ldots\ldots\ (3)\end{aligned}\right\}$$

1°. *Eliminating by substitution.*

From equation (1), we find

$$z=16-3x-2y \ \ldots\ldots\ldots\ (4).$$

Substituting this value of z in equations (2) and (3), they become

$$\left.\begin{aligned} 2x+2y+2(16-3x-2y)&=18 \ \ldots\ (5)\\ 2x+2y+\ (16-3x-2y)&=14 \ \ldots\ (6)\end{aligned}\right\}$$

these last two equations contain x and y only, and, if treated according to any of the above methods, will give us

$$x=2,\ y=3.$$

Substituting these values of x and y in any one of the equations (1), (2), (3), (4), we find

$$z=4.$$

2°. *Eliminating by comparison.*

In order to eliminate z, derive from each of the three proposed equations a value of z in terms of x and y; we then have

$$z=16-3x-2y$$
$$z=9-x-y$$
$$z=14-2x-2y;$$

equating the first of these values of z with the second and with the third in succession, we arrive at a system of two equations:

$$\left.\begin{aligned}16-3x-2y&=9-x-y\\16-3x-2y&=14-2x-2y\end{aligned}\right\}$$

containing x and y only; these equations give

$$x=2,\ y=3;$$

these values of x and y, when substituted in any of the three expressions for z, give

$$z=4.$$

3°. *Eliminating by subtraction.*

In order to eliminate z between equations (1) and (2),

$$3x+2y+z=16$$
$$2x+2y+2z=18;$$

we perceive that, in order to reduce these equations to two others in which the coefficients of z shall be the same, it will be sufficient to divide the two members of the second equation by 2, for we thus have

$$x+y+z=9.$$

Subtracting this from the first equation,

$$3x+2y+z=16,$$

we find an equation between two unknown quantities,

$$2x+y=7 \quad \ldots\ldots\ldots\ldots\ldots \quad (a).$$

In order to eliminate z between equations (1) and (3),

$$3x+2y+z=16$$
$$2x+2y+z=14.$$

Subtract the latter from the former, which gives

$$x=2;$$

the substitution of this value of x in equation (a) gives

$$y=3,$$

and the substitution of these values of x and y in any of the proposed equations gives

$$z=4.$$

The particular form of the proposed equations enables us to simplify the above calculation; for if we subtract equation (3) from equations (1) and (2) in succession, we have

$$(3x+2y+z)-(2x+2y+z)=16-14, \text{ whence } x=2$$
$$(2x+2y+2z)-(2x+2y+z)=18-14, \text{ whence } z=4;$$

and substituting these values of x and z in any of the proposed equations, we find

$$y=3.$$

In order to solve a system of four equations between four unknown quantities, we reduce this case to the last by eliminating one of the unknown quantities. We thus arrive at a system of three equations between three unknown quantities, from which the value of these three unknown quantities may be found. Substituting these values in any one of the equations which involve the other unknown quantity, we deduce from it the value of that unknown quantity.

EXAMPLE IV.

Take the system of equations

$$\left.\begin{array}{ll} x+y+z+t=14 \ldots\ldots\ldots\ldots & (1) \\ x+y+z-t=4 \ldots\ldots\ldots\ldots & (2) \\ x+y-z+2t=11 \ldots\ldots\ldots\ldots & (3) \\ x-y+z+3t=18 \ldots\ldots\ldots\ldots & (4) \end{array}\right\}$$

The first equation gives

$$t=14-x-y-z \ldots\ldots\ldots\ldots (5).$$

Substituting this expression for t in the three other equations, we find

$$x+y+z=9 \ldots\ldots\ldots\ldots (6)$$
$$x+y+3z=17 \ldots\ldots\ldots\ldots (7)$$
$$x+2y+z=12 \ldots\ldots\ldots\ldots (8).$$

In order to solve these three equations between x, y, z, we find from the first

$$z=9-x-y \ldots\ldots\ldots\ldots (9);$$

and substituting this value of z in the two other equations, they become

$$x+y=5 \ldots\ldots\ldots\ldots (10)$$
$$y=3 \ldots\ldots\ldots\ldots (11)$$

Whence $x=2 \ldots\ldots\ldots\ldots (12).$

Substituting the values of x and y in equation (8), we find

$$z=4 \ldots\ldots\ldots\ldots (13).$$

Substituting these values of x, y, z in any of the first five equations, we find

$$t=5.$$

We can arrive at the same result more simply by subtracting equation (1) from the three following in succession; we shall thus find

$$2t=14-4,\ 2z-t=14-11,\ 2y-2t=14-18;$$

the first of these three new equations gives $t=5$; this value of t, substituted in the two other equations, gives $z=4$, $y=3$; and substituting these values of y, z, t in any one of the original equations, we find $x=2$.

By following a process of reasoning analogous to the above, we shall be able to resolve a system of any number of equations of the first degree, provided there be as many equations as unknown quantities.

It frequently happens that each of the proposed equations do not involve all the unknown quantities. In this case, a little dexterity will enable us to effect the elimination very quickly.

EXAMPLE V.

Take the system of equations

$$\left.\begin{array}{ll} 2x-3y+2z=13 \ldots\ldots & (1) \\ 4t-2x=30 \ldots\ldots & (2) \\ 4y+2z=14 \ldots\ldots & (3) \\ 5y+3t=32 \ldots\ldots & (4) \end{array}\right\}$$

Upon examining these equations, we perceive that the elimination of z between equations (1) and (3) will give an equation in x and y, and that the elimination of t between equations (2) and (4) will give a second equation in x and y. These two unknown quantities may thus be easily determined:

The elimination of z between (1) and (3) gives . . . . $7y-2x=1$

The elimination of t between (2) and (4) gives . . . . $20y+6x=38$

Multiply the first of these equations by 3, and then add them, we have . . . . $41y=41$

Whence . . . . $y=1$

Substituting the value of y in $7y-2x=1$, we have . . . $x=3$

Substitute this value of x in (2), we have . . . . $4t-6=30$

Whence . . . . $t=9$

Finally, the substitution of the value of y in (3) gives . . $z=5$

The following general *rule* may be given for a system of any number of equations:

Eliminate one of the unknown quantities by combining the first equation with each of the others, or by combining them all in any way in separate pairs. The number of equations and of unknown quantities is thus made one less. Proceed with these in the same way till there is but one equation and one unknown quantity. Having found the value of this, substitute it in a preceding equation containing but two unknown quantities, which will then have but one, whose value may be found. Substitute the values of the two unknown quantities thus found in an equation immediately preceding, containing only three, and so on, till all the values of the unknown quantities are obtained.

We have seen in the method of elimination by subtraction that, in order to render the coefficients of the unknown quantity the same in both equations, we must multiply each of the equations by the coefficient of the unknown quantity, which it is required to eliminate, in the other. If the coefficients of the unknown quantity have a common factor, this operation may be simplified; thus

EXAMPLE VI.

Take the system of equations

$$\left.\begin{array}{l} 12x+32y=340 \ldots\ldots (1) \\ 8x+24y=254 \ldots\ldots (2) \end{array}\right\}$$

In order to render the coefficients of y equal, observe that 32 and 24 have a common factor, 8; it will suffice then to multiply equation (1) by 3, and equation (2) by 4; they then become

$$36x+96y=1020$$
$$32x+96y=1016.$$

Subtracting the latter from the former,

$$4x=4$$
$$x=1.$$

Again, in order to eliminate x, since 12 and 8 have a common factor, 4, it will suffice to multiply equation (1) by 2, and equation (2) by 3; we then have

$$24x+64y=680$$
$$24x+72y=762.$$

Subtracting the former of these two equations from the latter, we have

$$8y=82$$
$$y=10\tfrac{1}{4}.$$

(7) Given $x+y=15$ . . . . (1)
$x-y=\ 7$ . . . . (2)

Ans. $x=11,\ y=4$.

(8) Given $x+y\ =10$ . . . . (1)
$2x-3y=\ 5$ . . . . (2)

Ans. $x=7,\ y=3$.

(9) Given $2x+3y=13$ . . . . (1)
$5x+4y=22$ . . . . (2)

Ans. $x=2,\ y=3$.

(10) Given $x=4y$ . . . . (1)
$2x+3y=44$ . . . . (2)

Ans. $x=16,\ y=4$.

(11) Given $2x+3y=\ 70$ . . . . (1)
$4x+5y=130$ . . . . (2)

Ans. $x=20,\ y=10$.

(12) Given $3x-5y=13$ . . . . (1)
$2x+7y=81$ . . . . (2)

Ans. $x=16,\ y=7$.

(13) Given $11x+3y=100$ . . . . (1)
$4x-7y=\ \ 4$ . . . . (2)

Ans. $x=8,\ y=4$.

(14) Given $\frac{x}{2}+\frac{y}{3}=7$ . . . . (1)
$\frac{x}{3}+\frac{y}{2}=8$ . . . . (2)

Ans. $x=6,\ y=12$.

(15) Given $\frac{x}{7}+7y=99$ . . . . (1)
$\frac{y}{7}+7x=51$ . . . . (2)

Ans. $x=7,\ y=14$

(16) Given $3t+\frac{7u}{2}=22$ . . . . (1)
$11u-\frac{2t}{5}=20$ . . . . (2)

Ans. $t=5,\ u=2$.

(17) Given $x+1:y::5:3$ . . . . (1)
$\frac{7+x}{4}-\frac{5-y}{2}=\frac{42}{12}-\frac{2x-1}{4}$ . . . . (2)

Ans. $x=4,\ y=3$.

(18) Given $\frac{2r}{3}+\frac{4s}{5}=64$ . . . . (1)
$\frac{5r}{6}+\frac{9s}{10}=77$ . . . . (2)

Ans. $r=60,\ s=30$

(19) Given $5\rho+\frac{2}{3}\sigma=131\frac{1}{4}$ . . . . (1)
$13\rho-\ \sigma=142\frac{1}{3}$ . . . . (2)

Ans. $\rho=16\frac{269}{492},\ \sigma=72\frac{331}{492}$

(20) Given $6\frac{2}{7}\chi-14\psi=5\psi+119\frac{2}{5}$ . . . . (1)

$7\chi+140=2\psi$ . . . . (2)

Ans. $\chi=-24.06$, $\psi=-14.2$.

(21) Given $9x=4x'$ . . . . (1)

$x+x'=26$ . . . . (2)

Ans. $x=8$, $x'=18$.

(22) Given $\dfrac{21}{14+z_1}=\dfrac{14}{84+z_2}$ . . . . (1)

$21z_1+28z_2=334$ . . . . (2)

Ans. $z_1=61\frac{15}{119}$, $z_2=-33\frac{100}{119}$.

(23) Given $2x-\dfrac{y+3}{4}=7$ . . . . (1)

$4y-\dfrac{8-x}{3}=24\frac{1}{2}-\dfrac{2y+1}{2}$ . . . . (2)

Ans. $x=5$, $y=5$.

(24) Given $5+\dfrac{y-8}{5}=\dfrac{3x+4y+3}{10}-\dfrac{2x+7-y}{15}$ . . . . (1)

$\dfrac{7x+6}{11}=\dfrac{9y+5x-8}{12}-\dfrac{x+y}{4}$ . . . . . . (2)

Ans. $x=7$, $y=9$.

(25) Given $(x+5)(y+7)=(x+1)(y-9)+112$ . . . . (1)

$2x+10=3y+1$ . . . . (2)

Ans. $x=3$, $y=5$.

(26) Given $\dfrac{2x}{3}-4+\dfrac{y}{2}+x=8-\dfrac{3y}{4}+\dfrac{1}{12}$ . . . . (1)

$\dfrac{y}{6}-\dfrac{x}{2}+2=\dfrac{1}{6}-2x+6$ . . . . (2)

Ans. $x=2$, $y=7$.

(27) Given $\dfrac{x-2}{5}-\dfrac{10-x}{3}=\dfrac{y-10}{4}$ . . . . (1)

$\dfrac{2y+4}{3}-\dfrac{2x+y}{8}=\dfrac{x+13}{4}$ . . . . (2)

Ans. $x=7$, $y=10$.

(28) Given $\dfrac{2y}{18}-\dfrac{8x-2}{36}=1-\dfrac{4+y}{3}+\dfrac{x-y}{6}$ . . . . (1)

$x:3y::4:7$ . . . . (2)

Ans. $x=12$, $y=7$.

(29) Given $x-\dfrac{3y-2+x}{11}=1+\dfrac{15x+\frac{4y}{3}}{33}$ . . . . (1)

$\dfrac{3x+2y}{6}-\dfrac{y-5}{4}=\dfrac{11x+152}{12}-\dfrac{3y+1}{2}$ . . . . (2)

Ans. $x=8$, $y=9$.

(30) Given $1+\dfrac{25+5y}{6}-\dfrac{7x-6}{3}=10-\dfrac{3x-10+7y}{12}$ . . (1)

$\dfrac{12-x}{9}:5x-\dfrac{14+y}{3}::1:8$. . . . . (2)

Ans. $x=3$, $y=7$.

(31) Given* $\frac{4x}{x^2}+\frac{5y}{y^2}=\frac{9}{y}-1$ . . . . (1)

$\frac{5}{x}+\frac{4}{y}=\frac{7}{x}+\frac{3}{2}$ . . . . (2)

Ans. $x=4$, $y=3$

(32) Given $5x+7y=43$ . . . . (1)

$11x+9y=69$ . . . . (2)

Ans. $x=3$, $y=4$

(33) Given $8x-21y=33$ . . . . (1)

$6x+35y=177$ . . . . (2)

Ans. $x=12$, $y=3$.

(34) Given $\frac{2x}{3}-4+\frac{y}{2}+x=8-\frac{3y}{4}+\frac{1}{12}$ . . . . (1)

$\frac{y}{6}-\frac{x}{2}+2=\frac{1}{6}-2x+6$ . . . . (2)

Ans. $x=2$, $y=7$.

(35) Given $x-\frac{3x+5y}{17}+17=5y+\frac{4x+7}{3}$ . . . . (1)

$\frac{22-6y}{3}-\frac{5x-7}{11}=\frac{x+1}{6}-\frac{8y+5}{18}$ . . . . . (2)

Ans. $x=8$, $y=2$.

(36) Given $ax+by=c$ . . . . (1)

$fx+gy=h$ . . . . (2)

Ans. $x=\frac{cg-bh}{ag-bf}$, $y=\frac{ah-cf}{ag-bf}$.

(37) Given $x+y=s$ . . . . (1)

$x-y=d$ . . . . (2)

Ans. $x=\frac{s+d}{2}$, $y=\frac{s-d}{2}$.

(38) Given $x+y=s$ . . . . (1)

$bx=ay$ . . . . (2)

Ans. $x=\frac{as}{a+b}$, $y=\frac{bs}{a+b}$.

(39) Given $ax+by=c$ . . . . (1)

$mx-ny=d$ . . . . (2)

Ans. $x=\frac{nc+bd}{na+mb}$, $y=\frac{mc-ad}{na+mb}$.

(40) Given $7ax=4b$ . . . . (1)

$2cx+3dy=4c$ . . . . (2)

Ans. $x=\frac{4b}{7a}$, $y=\frac{28ac-8bc}{21ad}$.

(41) Given $bcx=cy-2b$ . . . . (1)

$b^2y+\frac{a(c^3-b^3)}{bc}=\frac{2b^3}{c}+c^3x$ . . . . (2)

Ans. $x=\frac{a}{bc}$, $y=\frac{a+2b}{c}$.

---

* These equations should not be cleared of fractions, but the unknown fractions be eliminated by making them alike, and subtracting.

(42) Given $\frac{a}{b+y}=\frac{b}{3a+x}$ . . . . (1)

$ax+2by=d$ . . . . (2)

Ans. $x=\frac{d-6a^2+2b^2}{3a}$, $y=\frac{3a^2+d-b^2}{3b}$.

(43) Given $x-\frac{y-a}{b}=c$ . . . . (1)

$y-\frac{a-x}{b}=d$ . . . . (2)

Ans. $x=\frac{a-ab+b^2c+bd}{b^2+1}$, $y=\frac{a+ab-bc+b^2d}{b^2+1}$.

(44) Given $\frac{a+4b}{m+x}=\frac{2a-3b}{3m-y}$ . . . . (1)

$5ax-2by=c$ . . . . (2)

(45) Given $b^2y+\frac{g(c^3-b^3)}{bc}-\frac{2b^3}{c}=c^3x$ . . . . (1)

$b(cx+2)=cy$ . . . . (2)

Ans. $x=\frac{g}{bc}$, $y=\frac{g+2b}{c}$.

(46) Given $17x-\frac{33b}{b+f}+(b+10f)y=f^2x$ . . . . (1)

$4x+5y=\frac{9b-2f}{b^2-f^2}$ . . . . (2)

(47) Given $\frac{a}{x}+\frac{b}{y}=m$ . . . . (1)

$\frac{c}{x}+\frac{d}{y}=m$ . . . . (2)

Ans. $x=\frac{bc-ad}{nb-md}$, $y=\frac{bc-ad}{mc-na}$.

(48) Given $x+y=s$ . . . . (1)

$x^2-y^2=d$ . . . . (2)

Ans. $x=\frac{s^2+d}{2s}$, $y=\frac{s^2-d}{2s}$.

(49) Given $x+y:a::x-y:b$ . . . . (1)

$x^2-y^2=c$ . . . . (2)

Ans. $x=\frac{a+b}{2}\sqrt{\frac{c}{ab}}$, $y=\frac{a-b}{2}\sqrt{\frac{c}{ab}}$.

(50) Given $x+\sqrt{x^2+y}=a$ . . . . (1)

$x+\sqrt{x^2-y}=b$ . . . . (2)

Ans. $x=\frac{a^2+b^2}{2(a+b)}$, $y=\frac{ab(a-b)}{a+b}$.

(51) Given $x^2+xy=a$ . . . . (1)

$y^2+xy=b$ . . . . (2)

Ans. $x=\frac{a}{\sqrt{a+b}}$, $y=\frac{b}{\sqrt{a+b}}$.

(52) Given $2x+3y+4z=16$ (1)

$3x+2y-5z=8$ (2)

$5x-6y+3z=6$ . . . . (3)

Ans. $x=3$; $y=2$; $z=1$.

(53) Given $5x-6y+4z=15$ . . . . (1)

$7x+4y-3z=19$ . . . . (2)

$2x+y+6z=46$ . . . . (3)

Ans. $x=3$; $y=4$; $z=6$.

(54) Given* $\frac{1}{x}+\frac{1}{y}=a$ . . . . (1)

$\frac{1}{x}+\frac{1}{z}=b$ . . . . (2)

$\frac{1}{y}+\frac{1}{z}=c$ . . . . (3)

Ans. $x=\frac{2}{a+b-c}$; $y=\frac{2}{a-b+c}$; $z=\frac{2}{b+c-a}$.

(55) Given $x+y=36$; $x+z=49$; $y+z=53$.

Ans. $x=16$; $y=20$; $z=33$

(56) Given $v+w+z=30$; $v+w-z=18$; $v-w+z=14$.

Ans. $v=16$; $w=8$; $z=6$

(57) Given $u+\frac{1}{2}v=164$; $v+\frac{1}{4}w=82$; $u+\frac{1}{5}w=136$.

Ans. $u=128$; $v=72$; $w=40$.

(58) Given $ax+by=c$; $my+nz=p$; $fx+gz=q$.

Ans. $x=\frac{bnq+cgm-bgp}{agm+bfn}$;

$y=\frac{agp+cfn-anq}{agm+bfn}$;

$z=\frac{amq+bfp-cfm}{agm+bfn}$

(59) Given $3(ax+by)=z$; $5y=7(x+3a)$; $11x=\frac{3}{8}z+121$.

Ans. $x=\frac{4840+189ab}{440-45a-63b}$;

$y=\frac{6776+1848a-189a^2}{440-45a-63b}$;

$z=\frac{14520a+5544ab+20328b}{440-45a-63b}$.

(60) Given $\frac{7}{x-5}=\frac{5}{z}$; $\frac{9}{y}=\frac{11}{z-9}$; $\frac{13}{x}=\frac{15}{y-13}$.

Ans. $x=-40\frac{509}{570}$; $y=-34\frac{7}{38}$; $z=-32\frac{89}{114}$.

(61) Given $\frac{a+b}{a+x}=\frac{b-c}{b-y}$; $\frac{b+c}{b+y}=\frac{c-d}{c-z}$; $\frac{d+k}{d+z}=\frac{k-h}{k-x}$.

---

* Do not clear of fractions, but make $\frac{1}{x}$, $\frac{1}{y}$, &c., the unknown quantities.

(62) Given $2x-\frac{3y}{4}=93-\frac{1}{2}x-\frac{1}{4}y$ . . . . (1)

$7x-5z=y+x-86$ . . . . (2)

$\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=58$ . . . . (3)

Ans. $x=48$; $y=54$; $z=64$.

(63) Given $6x-4y+5z=2\frac{11}{12}$ . . . . (1)

$4x+3y-7z=1\frac{1}{4}$ . . . . (2)

$12x-6y-3z=3\frac{1}{4}$ . . . . (3)

Ans. $x=\frac{1}{2}$; $y=\frac{1}{3}$; $z=\frac{1}{4}$.

(64) Given $18x-7y-5z=11$ . . . . (1)

$4y-\frac{2}{3}x+2\frac{2}{3}z=108$ . . . . (2)

$3\frac{1}{2}z+2y+\frac{3}{4}x=80$ . . . . (3)

Ans. $x=12$; $y=25$; $z=6$.

(65) Given $y+\frac{z}{3}=\frac{x}{5}+5$ . . . . (1)

$\frac{x-1}{4}-\frac{y-2}{5}=\frac{z+3}{10}$ . . . . (2)

$x-\frac{2y-5}{3}=1\frac{3}{4}-\frac{z}{12}$ . . . . (3)

Ans. $x=5$; $y=7$; $z=-3$.

(66) Given $\frac{x}{3}+\frac{y}{5}+\frac{2z}{7}=58$ . . . . (1)

$\frac{5x}{4}+\frac{y}{6}+\frac{z}{3}=76$ . . . . (2)

$\frac{x}{2}+\frac{3z}{8}+\frac{u}{5}=79$ . . . . (3)

$y+z+u=248$ . . . . (4)

Ans. $x=12$; $y=30$; $z=168$; $u=50$.

(67) Given $7x-2z+3u=17$ . . . . (1)

$4y-2z+t=11$ . . . . (2)

$5y-3x-2u=8$ . . . . (3)

$4y-3u+2t=9$ . . . . (4)

$3z+8u=33$ . . . . (5)

Ans. $x=2$; $y=4$; $z=3$; $u=3$; $t=1$.

Elimination may be effected in a general form, and particular cases be resolved by substitution in this form.

We shall illustrate this with a system of three equations.

Given
$$ax+by+cz+k=0,$$
$$a'x+b'y+c'z+k'=0,$$
$$a''x+b''y+c''z+k''=0.$$

Eliminating among these three equations by any of the foregoing methods, we find

$$x=\frac{(b''c'-b'c'')k+(bc''-b''c)k'+(b'c-bc')k''}{(a'b''-a''b')c+(a''b-ab'')c'+(ab'-a'b)c''},$$

$$y=\frac{(a'c''-a''c')k+(a''c-ac'')k'+(ac'-a'c)k''}{\text{The same denominator as in the value of } x},$$

$$z=\frac{(a''b'-a'b'')k+(ab''-a''b)k'+(a'b-ab')k''}{\text{The same denominator as in the value of } x}.$$

To apply this general form to a particular case, take (Example 53) above.

$$x=\frac{(1\times-3-4\times6)(-15)+(-6\times6-1\times4)(-19)+[4\times4-(-6\times-3)](-46)}{(7\times1-2\times4)4+(2\times-6-5\times1)(-3)+(5\times4-7\times-6)6}=\frac{1257}{419}=3,$$

$$y=\frac{(42+6)(-15)+(8-30)(-19)+(-15-28)(-46)}{419}=\frac{1676}{419}=4,$$

$$z=\frac{(1)(-15)+(+17)(-19)+(-62)(-46)}{419}=\frac{2514}{419}=6.$$

Changing the signs of k, k', k'', in order that they may be positive in the second member of the three proposed equations, and performing the multiplications indicated in the general values of x, y, and z, they may be written as follows:

$$x=\frac{kb'c''-kc'b''+ck'b''-bk'c''+bc'k''-cb'k''}{ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''},$$

$$y=\frac{ak'c''-ac'k''+ca'k''-ka'c''+kc'a''-ck'a''}{\text{The same denominator as that of } x},$$

$$z=\frac{ab'k''-ak'b''+ka'b''-ba'k''+bk'a''-kb'a''}{\text{The same denominator as before}}.$$

By observing carefully the composition of the formulas for two and three equations, we may discover general rules by means of which we can calculate the formulas suitable for any number of equations.

First Rule.—To find the common denominator in the values of all the unknown quantities. With the two letters a and b form the arrangements ab and ba, then interpose the sign — between them, thus:

$$ab-ba.$$

If there are but two equations to resolve, place an accent on the 2° letter of each term, and the result, $ab'-ba'$, will be the common denominator of the values of x and y.

If there are three equations, pass the letter c through all the places in each term of the expression $ab-ba$, taking care to alternate the signs; ab will thus give $abc-acb+cab$; also, $-ba$ will give $-bac+bca-cba$, and the whole

$$abc-acb+cab-bac+bca-cba;$$

then place one accent on the 2° letter of each term, and two on the 3°, and the resulting expression will be the common denominator of the values of x, y, and z.

If there are four equations, take the letter d, which is the coefficient of the fourth unknown u, and pass it through all the places in each term of the sexinomial above formed, taking care to alternate the signs of the terms furnished by each of them, beginning with + for those which result from a term preceded by the sign +, and with — for those resulting from a term affected with the sign —; finally, place one accent on the 2° letter, two on the 3°, and three on the 4°. The resulting polynomial is the common denominator of the four unknown quantities x, y, z, u.

$$\begin{aligned}&ab'c''d'''-ab'd''c'''+ad'b''c'''-da'b''c'''\\&-ac'b''d'''+ac'd''b'''-ad'c''b'''+da'c''b'''\\&+ca'b''d'''-ca'd''b'''+cd'a''b'''-dc'a''b'''\end{aligned}$$

$$
\begin{array}{l}
-ba'c''d'''+ba'd''c'''-bd'a''c'''+db'a''c''' \\
+bc'a''d'''-bc'd''a'''+bd'c''a'''-db'c''a''' \\
-cb'a''d'''+cb'd''a'''-cd'b''a'''+dc'b''a'''.
\end{array}
$$

If there be a greater number of equations, proceed in the same manner.

SECOND RULE.—The numerators may be derived from the common denominator. For this purpose, it is only necessary to replace, without touching the accents, the letter which serves for coefficient of the unknown quantity we wish to find, by the letter k, which represents the known term in the second member. Thus: change a into k, to have the numerator of x; b into k, to have that of y; and so on.

There remains still a method of elimination to be mentioned, which alone is applicable to equations of higher degrees, as well as to those of the first. It is called the method of the common divisor. It consists, where two equations are given, in dividing one by the other (after transferring all the terms to the first member in both), that divisor by the remainder, and so on till the letter of arrangement, which must be one of the unknown quantities, is exhausted from the remainders. The last remainder containing but the other unknown quantity, being put equal to zero, will present an equation from which the first unknown quantity is eliminated.

If there be three or more equations, eliminate one of the unknown quantities in this way between the first and second, then between the first and third, and so on.

The reason which may be given for this rule here, though a better one will be furnished hereafter, is, that the dividend being zero and the divisor zero, the quotient must be zero and the remainder zero.

Let us apply this method to Example (8) above. The two given equations are

$$
\begin{array}{l}
x+y-10=0 \\
2x-3y-5=0.
\end{array}
$$

Elimination,

$$
\begin{array}{r|l}
2x-3y-5 & x+y-10 \\
\underline{2x+2y-20} & 2 \\
-5y+15 \div 5. & \\
-y+3=0 \therefore y=3. &
\end{array}
$$

Substituting this value in $x+y-10=0$, we obtain $x=7$.

EXAMPLE II.

Given $x^3+3yx^2+3y^2x-98=0$ . . . . (1)

$x^2+4yx-2y^2-10=0.$

Elimination,

$$
\begin{array}{r|l}
x^3+3yx^2+3y^2x-98 & x^2+4yx-2y^2-10 \\
\underline{x^3+4yx^2-2y^2x-10x} & x-y \\
-yx^2+5y^2x+10x-98 & \\
\underline{-yx^2-4y^2x+2y^3+10y} &
\end{array}
$$

$$
\begin{array}{r|l}
x^2+4yx-2y^2-10 & 9y^2x+10x-2y^3-10y-98, \text{ or,} \\
9y^2+10 & (9y^2+10)x-2y^3-10y-98 \\
\hline
(9y^2+10)x^2+(36y^3+40y)x-18y^4-110y^2-100 & x+19y^3+25y+49 \\
\underline{(9y^2+10)x^2-(2y^3+10y+98)x} & \\
(38y^3+50y+98)x-18y^4-110y^2-100 \div 2 & \\
(19y^3+25y+49)x-9y^4-55y^2-50 & \\
\underline{9y^2+10} & \\
(9y^2+10)(19y^3+25y+49)x-81y^6-585y^4-1000y^2-500 & \\
\underline{(9y^2+10)(19y^3+25y+49)x-38y^6-240y^4-1960y^3-250y^2-2940y-4802} & \\
-43y^6-345y^4+1960y^3-750y^2+2940y+4802. &
\end{array}
$$

This last remainder, put equal to zero, will make an equation from which x is eliminated, and which contains only y. It is called the *final* equation.

ON THE SOLUTION OF PROBLEMS WHICH PRODUCE SIMPLE EQUATIONS.

150. Every problem which can be solved by Algebra includes in its enunciation a certain number of conditions of such a kind that, in taking at pleasure values for the unknown quantities, it is always easy to see whether or not they will verify these conditions. In the greater part of questions in Algebra, these verifications consist in this, that, after having effected certain operations upon the values of the known and unknown quantities, we ought to arrive at equalities. This being understood, if the unknown quantities be represented by letters, algebraic expressions may be formed in which shall be indicated, by means of signs, all the calculations necessary to be made, as well upon the unknown numbers as upon the known, to find the quantities which ought to be equal. Consequently, joining these expressions by the sign of equality, we shall have one or more equations, which will be satisfied when the true values of the unknown quantities are substituted in the place of the letters which represent them.

Reciprocally, when all the conditions of the problem are expressed in the equations, the values of the unknown quantities which satisfy these equations must certainly satisfy the enunciation of the problem.

It is impossible to give a general rule which will enable us to translate every problem into algebraic language; this is an art which can be acquired by reflection and practice alone. Two rules which may be of some service are the following: 1. *Indicate upon the unknown quantities represented by letters, and upon the known quantities represented either by letters or numbers, the same operations as would be necessary to verify them if they were known.* 2. *Form two different expressions of the same quantity, and set them equal.* We shall give a few examples, which will serve to initiate the student, and the rest must be left to his own ingenuity.

PROBLEM 1.

To find two numbers such that their sum shall be 40, and their difference 16.

Let x denote the least of the two numbers required,

Then will $x+16=$ the greater,

And $x+x+16=40$ by the question;

That is, $2x=40-16=24$;

Or $x=\frac{24}{2}=12=$ less number,

And $x+16=12+16=28=$ greater number required.

PROBLEM 2.

What number is that, whose $\frac{1}{3}$ part exceeds its $\frac{1}{4}$ part by 16?

Let $x=$ number required,

Then will its $\frac{1}{3}$ part be $\frac{1}{3}x$, and its $\frac{1}{4}$ part $\frac{1}{4}x$;

And, therefore, $\frac{1}{3}x-\frac{1}{4}x=16$ by the question,

Or, clearing of fractions, $4x-3x=192$;

Hence $x=192$, the number required.

PROBLEM 3.

Divide £1000 among A, B, and C, so that A shall have £72 more than B and C £100 more than A.

Let $x=$ B's share of the given sum,

Then will $x+72=$ A's share,

And $x+172=$ C's share,

And the sum of all their shares, $x+x+72+x+172$,

Or $3x+244=1000$ by the question;

That is, $3x=1000-244=756$,

Or $=\frac{756}{3}=$ £252 $=$ B's share;

Hence $x+72=252+72=$ £324 $=$ A's share,

And $x+172=252+172=$ £424 $=$ C's share;

B's share £252

A's share 324

C's share 424

Sum of all . . £1000, the proof.

PROBLEM 4.

Out of a cask of wine, which had leaked away $\frac{1}{3}$, 21 gallons were drawn, and then, being gauged, it appeared to be half full: how much did it hold?

Let it be supposed to have held x gallons,

Then it would have leaked $\frac{1}{3}x$ gallons;

Consequently, there had been taken away $21+\frac{1}{3}x$ gallons.

But $21+\frac{1}{3}x=\frac{1}{2}x$ by the question,

Or $126+2x=3x$;

Hence $3x-2x=126$,

Or $x=126=$ number of gallons required.

PROBLEM 5.

A hare, pursued by a greyhound, is 60 of her own leaps in advance of the dog. She makes 9 leaps during the time that the greyhound makes only 6; but 3 leaps of the greyhound are equivalent to 7 leaps of the hare. How many leaps must the greyhound make before he overtakes the hare?

It is manifest, from the enunciation of the problem, that the space which must be traversed by the greyhound is composed of the 60 leaps which the hare is in advance, together with the space which the hare passes over from the time that the greyhound starts in pursuit until he overtakes her.

Let $x=$ the whole number of leaps made by the greyhound. Since the hare makes 9 leaps during the time that the greyhound makes 6, it follows that the hare will make $\frac{9}{6}$ or $\frac{3}{2}$ leaps during the time that the greyhound makes 1, and she will consequently make $\frac{3x}{2}$ leaps during the time that the greyhound makes x leaps.

We might here suppose that, in order to obtain the equation required, it would be sufficient to put x equal to $60+\frac{3x}{2}$; in doing this, however, we should commit a manifest mistake, for the leaps of the greyhound are greater than the leaps of the hare, and we should thus be equating two heterogeneous numbers; that is to say, numbers related to a different unit. In order to remove this difficulty, we must express the leaps of the hare in terms of the leaps of the greyhound, or the contrary.

According to the conditions of the problem, 3 leaps of the greyhound are equal to 7 leaps of the hare; hence 1 leap of the greyhound is equal to $\frac{7}{3}$ leaps of the hare, and, consequently, x leaps of the greyhound are equal to $\frac{7x}{3}$ leaps of the hare; hence we have at length the equation

$$\frac{7x}{3} = 60 + \frac{3x}{2};$$

Clearing of fractions, $$14x = 360 + 9x$$
$$x = 72.$$

Hence the greyhound will make 72 leaps before he reaches the hare, and in that time the hare will make $72 \times \frac{3}{2}$, or 108 leaps.

PROBLEM 6.

Find a number such, that when it is divided by 3 and by 4, and the quotients afterward added, the sum is 63.

Let x be the number; then, by the conditions of the problem, we have

$$\frac{x}{3} + \frac{x}{4} = 63;$$

Clearing of fractions, $$7x = 63 \times 12$$
$$x = 108.$$

If we wished to find a number such that, when divided by 5 and by 6, the sum of the quotients is 22, we must again translate the problem into algebraic language, and then solve the equation; in this case we have

$$\frac{x}{5} + \frac{x}{6} = 22;$$

Clearing of fractions $$11x = 22 \times 30$$
$$x = 60.$$

If, however, we desire to solve both these problems at once, and all others of the same class, which differ from the above in the numerical values only, we must substitute for these particular numbers the symbols a, b, c, ----, which may represent any numbers whatever, and then solve the following question.

Find a number such that, when it is divided by a and by b, and the quotients afterward added, the sum is p. We have

$$\frac{x}{a} + \frac{x}{b} = p;$$
$$(a+b)x = abp$$
$$x = \frac{abp}{a+b}.$$

151. This expression is not, strictly speaking, the value of the unknown quantity in our problems, but it presents to our view the calculations which are requisite for the solution of them all. An expression of this nature is called a *formula*. This formula points out to us that the unknown quantity is obtained by multiplying together the three numbers involved in the question, and then dividing their product, abp, by $a+b$, the sum of the two divisors; or we should rather say, that our formula is a concise method of enunciating the above rule.* Algebra, then, may be considered as a language whose object is to express various processes of reasoning, as also the results or conclusions to which they lead.

Such is the advantage of the above formula, that, by aid of it, the most ignorant arithmetician could solve either of the proposed problems as readily as the most expert algebraist. The former, however, could only arrive at the result by a blind reliance on the rule which the formula expresses; but different kinds of problems require different formulæ, and the algebraist alone possesses the secret by which they can be discovered.

PROBLEM 7.

A laborer engaged to serve 40 days upon these conditions: that for every day he worked he was to receive 80 cents, but for every day he was idle he was to forfeit 32 cents. Now at the end of the time he was entitled to receive $15.20. It is required to find how many days he worked and how many he was idle.

Let x be the number of days he worked;
Then will $40-x$ be the number of days he was idle;
Also $x\times 80=80x=$ the sum earned,
And $(40-x)\times 32=1280-32x=$ sum forfeited;
Hence $80x-(1280-32x)=1520$ by the question;
That is, $80x-1280+32x=1520$,
Or $112x=1520+1280=2800$;
Hence $x=\frac{2800}{112}=25=$ number of days he worked,
And $40-x=40-25=15=$ number of days he was idle.

We may generalize the above problem in the following manner:

Let $n=$ the whole number of days for which he is hired,
$a=$ the wages for each day of work,
$b=$ the forfeit for each day of idleness,
$c=$ the sum which he receives at the end of n days,
$x=$ the number of days of work;
Then $n-x=$ the number of days of idleness,
$ax=$ the sum due to him for the days of work,
$b(n-x)=$ the sum he forfeits for the days of idleness.

We thus find for the equation of the problem,

$$ax-b(n-x)=c;$$

Whence
$$ax-bn+bx=c$$
$$(a+b)x=c+bn$$
$$x=\frac{c+bn}{a+b}, \text{ the number of days of work,}$$

---

* Let the student try this rule upon a variety of numbers; he will see that the general formula embraces as many particular examples as he chooses to imagine.

And $\therefore$
$$n-x=n-\frac{c+bn}{a+b}$$
$$=\frac{an+bn-c-bn}{a+b}$$
$$=\frac{an-c}{a+b},$$* the number of days of idleness.

By substituting in these general expressions, for the number of days of work and number of days of idleness, the particular numerical values of the letters, the same result will be obtained as before.

PROBLEM 8.

A can perform a piece of work in 6 days, B can perform the same work in 8 days: in what time will they finish it if both work together?

Let $x=$ the time required.

Since A can perform the whole work in 6 days, $\frac{1}{6}$ will denote the quantity he can perform in 1 day, and therefore $\frac{x}{6}$ the quantity he can perform in x days; for the same reason, $\frac{x}{8}$ will be the quantity which B can perform in x days; and we shall thus have

$$\frac{x}{6}+\frac{x}{8}=1\dagger$$
$$14x=48$$
$$x=3\tfrac{3}{7} \text{ days.}$$

Let us generalize the above problem.

A can perform a piece of work in a days, B in b days, C in c days, D in d days: in what time will they perform it if they all work together?

Let $x=$ the time;

Then, since A can perform the whole work in a days, $\frac{1}{a}$ will denote the quantity he can perform in 1 day, and, consequently, $\frac{x}{a}$ will be the quantity he can perform in x days; for the same reason, $\frac{x}{b}, \frac{x}{c}, \frac{x}{d}$ will be the quantities which B, C, D can perform respectively in x days; we thus have

$$\frac{x}{a}+\frac{x}{b}+\frac{x}{c}+\frac{x}{d}=\text{(whole work)},$$
$$=1;$$
$$\therefore x=\frac{abcd}{abc+abd+acd+bcd}.$$

What is the rule expressed by this formula?

---

* Let the student translate the formula for the number of days of idleness, and that for the number of days of work, into a rule.

† We might represent the piece of work by p; then $\frac{p}{6}$ and $\frac{p}{8}$ would express the quantities which A and B can perform in one day, and the equation would be

$$\frac{px}{6}+\frac{px}{8}=p,$$

which, divided throughout by p, gives the equation in the text. When the value of a quantity is immaterial, as in this case, it is best represented by 1.

PROBLEM 9.

A courier, who traveled at the rate of $31\frac{1}{2}$ miles in 5 hours, was dispatched from a certain city; 8 hours after his departure, another courier was sent to overtake him. The second courier traveled at the rate of $22\frac{1}{2}$ miles in 3 hours. In what time did he overtake the first, and at what distance from the place of departure?

Let $x=$ number of hours that the second courier travels.

Then, since the first courier travels at the rate of $31\frac{1}{2}$ miles in 5 hours, that is, $\frac{63}{10}$ miles in 1 hour, he will travel $\frac{63}{10}x$ miles in x hours, and, since he started 8 hours before the second courier, the whole distance traveled by him will be $(8+x)\frac{63}{10}$.

Again, since the second courier travels at the rate of $22\frac{1}{2}$ miles in 3 hours that is, $\frac{45}{6}$ miles in one hour, he will hence travel $\frac{45}{6}x$ miles in x hours.

The couriers are supposed to be together at the end of the time x, and therefore the distance traveled by each must be the same; hence

$$\frac{45}{6}x=(8+x)\frac{63}{10}$$

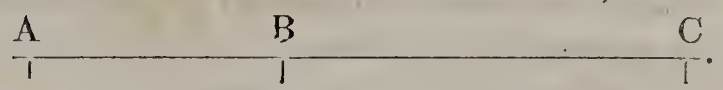
$$450x=(8+x)378;$$

$$\therefore 72x=3024$$

$$x=42.$$

Hence the second courier will overtake the first in 42 hours, and the whole distance traveled by each is $\frac{45}{6}\times 42=315$ miles.

To generalize the above,



Let a courier, who travels at the rate of m miles in t hours, be dispatched from B in the direction C; and n hours after his departure, let a second courier, who travels at the rate of m' miles in t' hours, be sent from A, which is distant d miles from B, in order to overtake the first. In what time will he come up with him, and what will be the whole distance traveled by each?

Let $x=$ number of hours that the second courier travels.

Then, since the first courier travels at the rate of m miles in t hours, that is, $\frac{m}{t}$ miles in 1 hour, he will travel $\frac{m}{t}x$ miles in x hours, and, since he started n hours before the second courier, the whole distance traveled by him will be $(n+x)\frac{m}{t}$.

Again, since the second courier travels at the rate of m' miles in t' hours, that is, $\frac{m'}{t'}$ miles in 1 hour, he will travel $\frac{m'}{t'}x$ miles in x hours; but since he started from A, which is distant d miles from B, the whole distance traveled by the second courier, or $\frac{m'}{t'}x$, will be greater than the whole distance traveled by the first courier, by this quantity d; hence

$$\frac{m'}{t'}x - d = (n+x)\frac{m}{t}$$

$$\left(\frac{m'}{t'} - \frac{m}{t}\right)x = \frac{mn}{t} + d;$$

$$\therefore\ x = \frac{(mn+td)t'}{m't - mt'}.$$

The whole distance traveled by first courier, $= \frac{m}{t} \cdot \left\{ \frac{(mn+td)t'}{m't-mt'} + n \right\}$

The whole distance traveled by second courier, $= \frac{m'}{t'} \cdot \frac{(mn+td)t'}{m't-mt'}$.

PROBLEM 10.

A father, who has three children, bequeaths his property by will in the following manner: To the eldest son he leaves a sum, a, together with the n^{th} part of what remains; to the second he leaves a sum, $2a$, together with the n^{th} part of what remains after the portion of the eldest and $2a$ have been subtracted; to the third he leaves a sum, $3a$, together with the n^{th} part of what remains after the portions of the two other sons and $3a$ have been subtracted. The property is found to be entirely disposed of by this arrangement. Required the amount of the property.

Let $x =$ the property of the father.

If we can, by means of this quantity, find algebraic expressions for the portions of the three sons, we must subtract their sums from the whole property x, and, putting this remainder $=0$, we shall determine the equation of the problem.

Let us endeavor to discover these three portions.

Since x represents the whole property of the father, $x-a$ is the remainder after subtracting a; hence,

Portion of eldest son, $= a + \frac{x-a}{n}$

$$= \frac{an+x-a}{n} \quad \ldots\ldots\ldots\ldots \quad (1)$$

Portion of second son, $= 2a + \frac{x - 2a - \frac{an+x-a}{n}}{n}$

$$= 2a + \frac{nx - 3an - x + a}{n^2}$$

$$= \frac{2an^2 + nx - 3an - x + a}{n^2} \quad \ldots\ldots\ldots \quad (2)$$

Portion of third son, $= 3a + \frac{x - 3a - \frac{an+x-a}{n} - \frac{2an^2+nx-3an-x+a}{n^2}}{n}$

$$= 3a + \frac{n^2x - 6an^2 - 2nx + 4an + x - a}{n^3}$$

$$= \frac{3an^3 + n^2x - 6an^2 - 2nx + 4an + x - a}{n^3} \quad \ldots\ldots \quad (3)$$

According to the conditions of the problem, the property is entirely disposed of. Hence, when the sum of the three portions is subtracted from x, the difference must be equal to zero; this gives us the equation

$$x-\frac{an+x-a}{n}-\frac{2an^2+nx-3an-x+a}{n^2}-\frac{3an^3+n^2x-6an^2-2nx+4an+x-a}{n^3}=0;$$

clearing the equation of fractions, and reducing,

$$n^3x-6an^3-3n^2x+10an^2+3nx-5an-x+a=0$$

$$\therefore (n^3-3n^2+3n-1)x=6an^3-10an^2+5an-a$$

$$x=\frac{6an^3-10an^2+5an-a}{n^3-3n^2+3n-1}=\frac{(6n^3-10n^2+5n-1)a}{(n-1)^3}.$$

By reflecting upon the conditions of the problem, we may obtain an equation much more simple than the preceding. It is stated that the portion of the third son is $3a$, together with the n^{th} of what remains, and that the property is thus entirely disposed of; in other words, the portion of the third son is $3a$, and the remainder just mentioned is nothing.

We found the expression for that remainder* to be

$$\frac{n^2x-6an^2-2nx+4an+x-a}{n^2}.$$

Equating this quantity to zero, we have

$$\frac{n^2x-6an^2-2nx+4an+x-a}{n^2}=0$$

$$\therefore n^2x-6an^2-2nx+4an+x-a=0$$

$$(n^2-2n+1)x=6an^2-4an+a$$

$$x=\frac{6an^2-4an+a}{n^2-2n+1}$$

$$=\frac{(6n^2-4n+1)a}{(n-1)^2}.$$

This result is, moreover, more simple than the former. We can easily prove that the two expressions are *numerically identical*, for, applying to the two polynomials $(6n^3-10n^2+5n-1)a$, and (n^3-3n^2+3n+1), the process for finding the greatest common measure, we shall find that these two expressions have a common factor $n-1$; dividing, therefore, both terms of the first result by this common factor, we arrive at the second.

The above problem will point out to the student the importance of examining with great attention the enunciation of any proposed question, in order to discover those circumstances which may tend to facilitate the solution; he will otherwise run the risk of arriving at results more complicated than the nature of the case demands.

The above problem admits of a solution less direct, but more simple and elegant than those already given. It is founded on the observation that, after having subtracted $3a$ from the former portions, nothing ought to remain.

Let us represent by r_1, r_2, r_3 the three remainders mentioned in the enunciation; the algebraic expressions for the three portions must be

$$a+\frac{r_1}{n},\ 2a+\frac{r_2}{n},\ 3a+\frac{r_3}{n}.$$

1°. By the conditions of the problem, we have $r_3=0$.
Hence the third portion is $3a$.

---

* Next above (3).

2°. The remainder, after the second son has received $2a+\frac{r_2}{n}$, may be represented by $r_2-\frac{r_2}{n}$, or $\frac{(n-1)r_2}{n}$.

But this is the portion of the third son; hence we have

$$\frac{(n-1)r_2}{n}=3a$$

$$\therefore r_2=\frac{3an}{n-1}$$

Hence the portion of the second son is $2a+\frac{3an}{n-1}\div n=2a+\frac{3a}{n-1}$, or, reducing,

$$\frac{2an+a}{n-1}.$$

3°. The remainder, after the eldest son has received $a+\frac{r_1}{n}$, may be represented by $r_1-\frac{r_1}{n}$, or $\frac{(n-1)r_1}{n}$.

But this remainder forms the portion of the other two sons; hence we have

$$\frac{(n-1)r_1}{n}=\frac{2an+a}{n-1}+3a$$

$$\therefore r_1=\frac{5an^2-2an}{(n-1)^2}$$

Hence the portion of the eldest son is $a+\frac{5an^2-2an}{(n-1)^2}\div n=a+\frac{5an-2a}{(n-1)^2}$, or, reducing,

$$\frac{an^2+3an-a}{n^2-2n+1}$$

Hence the whole property is

$$3a+\frac{2an+a}{n-1}+\frac{an^2+3an-a}{n^2-2n+1};$$

reducing the whole to a common denominator,

$$\frac{3a(n^2-2n+1)+(2an+a)(n-1)+an^2+3an-a}{n^2-2n+1};$$

performing the operations indicated, and reducing,

$$\frac{(6n^2-4n+1)a}{n^2-2n+1},$$

the result obtained above.

This solution is more complete than the former, for we obtain at the same time the property of the father and the expressions for the portions of his three sons.

We shall now solve one or two problems in which it is either necessary or convenient to employ more than one unknown quantity.

PROBLEM 11.

Required two numbers whose sum is 70 and whose difference is 16.

Let x and y be the two numbers.

Then, by the conditions of the problem,

$$x+y=70 \ldots\ldots\ldots\ldots (1)$$
$$x-y=16 \ldots\ldots\ldots\ldots (2),$$

which are the two equations required for its solution.

Adding the two equations,

$$2x=86$$
$$x=43.$$

Subtracting the second from the first,

$$2y=54$$
$$y=27.$$

Hence 43 and 27 are the two numbers.

PROBLEM 12.

A person has two kinds of gold coin, 7 of the larger, together with 12 of the smaller, make 288 shillings; and 12 of the larger, together with 7 of the smaller, make 358 shillings. Required the value of each kind of coin.

Let x be the value of the larger coin expressed in shillings, y that of the smaller.

Then, by the conditions of the problem,

$$7x+12y=288 \ldots\ldots\ldots (1)$$

And

$$12x+7y=358 \ldots\ldots\ldots (2).$$

Multiplying equation (1) by 7, and equation (2) by 12, and subtracting the former product from the latter, $95x=2280$

$\therefore x=24.$

Substituting this value of x in equation (1), it becomes $168+12y=288$

$\therefore y=10.$

The larger of the two coins is worth 24 shillings, the smaller 10 shillings.

PROBLEM 13.

An individual possesses a capital of $30,000, for which he receives interest at a certain rate; he owes, however, $20,000, for which he pays interest at a certain rate. The interest he receives exceeds that which he pays by $800. Another individual possesses a capital of $35,000, for which he receives interest at the second of the above rates; he owes, however, $24,000, for which he pays interest at the first of the above rates. The interest which he receives exceeds that which he pays by $310. Required the two rates of interest.

Let x and y denote the two rates of interest for $100.

In order to find the interest of $30,000 at the rate x, we have the proportion,

$$100:30{,}000::x:\frac{30{,}000x}{100}=300x.$$

In like manner, to find the interest of $20,000 at the rate of y,

$$100:20{,}000::y:\frac{20{,}000y}{100}=200y.$$

But, by the enunciation of the problem, the difference of these two sums is $800; hence we shall have, for the first equation,

$$300x-200y=800 \ldots\ldots (1).$$

Translating, in like manner, the second condition of the problem into algebraic language, we arrive at the second equation,

$$350y-240x=310 \quad \ldots\ldots\ldots \quad (2)$$

The two members of the first equation are divisible by 100, and those of the second by 10; they may therefore be replaced by the following:

$$3x-\ 2y=\ 8 \quad \ldots\ldots\ldots \quad (3)$$
$$35y-24x=31 \quad \ldots\ldots\ldots \quad (4)$$

In order to eliminate x, multiply equation (3) by 8, and then add equation (4); hence

$$19y=95$$
$$\therefore\ y=\ 5.$$

Substituting this value of y in equation (3), we have

$$3x-10=8$$
$$\therefore\ x=6.$$

Then the first rate of interest is 6 per cent., and the second 5 per cent.

PROBLEM 14.

An artisan has three ingots composed of different metals melted together. A pound of the first contains 7 oz. of silver, 3 oz. of copper, and 6 oz. of tin. A pound of the second contains 12 oz. of silver, 3 oz. of copper, and 1 oz. of tin. A pound of the third contains 4 oz. of silver, 7 oz. of copper, and 5 oz. of tin. How much of each of these three ingots must he take in order to form a fourth, each pound of which shall contain 8 oz. of silver, $3\frac{3}{4}$ oz. of copper, and $4\frac{1}{4}$ oz. of tin?

Let x, y, and z be the number of ounces which he must take in each of the ingots respectively, in order to form a pound of the ingot required.

Since, in the first ingot, there are 7 oz. of silver in a pound of 16 oz., it follows that in 1 oz. of the ingot there are $\frac{7}{16}$ oz. of silver, and, consequently, in x oz. of the ingot there must be $\frac{7x}{16}$ oz. of silver. In like manner, we shall find that $\frac{12y}{16}$, $\frac{4z}{16}$ represent the number of ounces of silver taken in the second and third ingots in order to form the fourth; but, by the conditions of the problem, the fourth ingot is to contain 8 oz. of silver; we shall thus have

$$\frac{7x}{16}+\frac{12y}{16}+\frac{4z}{16}=8 \quad \ldots\ldots\ldots \quad (1)$$

And reasoning precisely in the same manner for the copper and tin, we find

$$\frac{3x}{16}+\frac{3y}{16}+\frac{7z}{16}=\frac{15}{4} \quad \ldots\ldots\ldots \quad (2)$$

$$\frac{6x}{16}+\frac{y}{16}+\frac{5z}{16}=\frac{17}{4} \quad \ldots\ldots\ldots \quad (3)$$

which are the three equations required for the solution of the problem.

Clearing them of fractions, they become

$$7x+12y+4z=128 \quad \ldots\ldots\ldots \quad (4)$$
$$3x+\ 3y+7z=\ 60 \quad \ldots\ldots\ldots \quad (5)$$
$$6x+\ \ y+5z=\ 68 \quad \ldots\ldots\ldots \quad (6)$$

In these three equations the coefficients of y are most simple; it will, therefore, be convenient to eliminate this unknown quantity first.

Multiply equation (5) by 4, and subtract equation (4) from the product, we have . . . . . $5x+24z=112$ . . (7)

Multiply equation (6) by 3, and subtract equation (5) from the product, we have . . . . . $15x+\ 8z=144$ . . (8)

Multiply equation (8) by 3, and subtract equation (7) from the product, we have . . . . . $40x=320$

$\therefore\ x=\ 8$

Substitute this value of x in equation (8); it becomes . . . . $120+\ 8z=144$

$\therefore\ z=\ 3$

Substitute these values of x and z in equation (6); it becomes . . . . $48+y+15=68$

$\therefore\ y=\ 5$

Hence, in order to form a pound of the fourth ingot, he must take 8 ounces of the first, 5 ounces of the second, and 3 ounces of the third.

PROBLEM 15.

There are three workmen, A, B, C. A and B together can perform a certain piece of labor in a days; A and C together in b days; and B and C together in c days. In what time could each, singly, execute it, and in what time could they finish it if all worked together?

Let $x=$ time in which A alone could complete it.
$y=$ time in which B alone could complete it.
$z=$ time in which C alone could complete it.

Since A and B together can execute the whole in a days, the quantity which they perform in one day is $\frac{1}{a}$; and since A alone could do the whole in x days, the quantity he could perform in one day is $\frac{1}{x}$; for the same reason, the quantity which B could perform in one day is $\frac{1}{y}$; the sum of what they could do singly must be equal to the quantity they can do together; hence

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{a} \quad \ldots\ldots\ldots\ldots (1)$$

In like manner, we shall have

$$\frac{1}{x}+\frac{1}{z}=\frac{1}{b} \quad \ldots\ldots\ldots\ldots (2)$$

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{c} \quad \ldots\ldots\ldots\ldots (3)$$

Subtract equation (3) from (1),

$$\frac{1}{x}-\frac{1}{z}=\frac{1}{a}-\frac{1}{c} \quad \ldots\ldots\ldots\ldots (4)$$

Add equations (2) and (4),

$$\frac{2}{x}=\frac{1}{a}+\frac{1}{b}-\frac{1}{c};$$

$$\therefore\ x=\frac{2abc}{ac+bc-ab}.$$

In like manner,

$$y=\frac{2abc}{ab+bc-ac}$$

$$z=\frac{2abc}{ab+ac-bc}.$$

Let t be the time in which they could finish it if all worked together; then, by Prob. 8,

$$t\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=1;$$

$$\therefore t\left(\frac{1}{a}+\frac{1}{z}\right)=1$$

$$t\left(\frac{1}{a}+\frac{ab+ac-bc}{2abc}\right)=1;$$

$$\therefore t=\frac{2abc}{ab+ac+bc}.$$

(16) What two numbers are those whose difference is 7 and sum 33?

Ans. 13 and 20.

(17) To divide the number 75 into two such parts that three times the greater may exceed 7 times the less by 15.

Ans. 54 and 21.

(18) In a mixture of wine and cider, $\frac{1}{2}$ of the whole *plus* 25 gallons was wine, and $\frac{1}{3}$ part *minus* 5 gallons was cider; how many gallons were there of each?

Ans. 85 of wine, and 35 of cider.

(19) A bill of $34 was paid in half dollars and dimes, and the number of pieces of both sorts that were used was just 100; how many were there of each?

Ans. 60 half dollars and 40 dimes.

(20) Two travelers set out at the same time from New York and Albany, whose distance is 150 miles; one of them goes 8 miles a day, and the other 7; in what time will they meet?

Ans. In 10 days.

(21) At a certain election 375 persons voted, and the candidate chosen had a majority of 91; how many voted for each?

Ans. 233 for one, and 142 for the other.

(22) What number is that from which, if 5 be subtracted, $\frac{2}{3}$ of the remainder will be 40?

Ans. 65.

(23) A post is $\frac{1}{4}$ in the mud, $\frac{1}{3}$ in the water, and 10 feet above the water; what is its whole length?

Ans. 24 feet.

(24) There is a fish whose tail weighs 9 pounds, his head weighs as much as his tail and half his body, and his body weighs as much as his head and his tail; what is the whole weight of the fish?

Ans. 72 pounds.

(25) After paying away $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I had 66 guineas left in my purse; what was in it at first?

Ans. 120 guineas.

(26) A's age is double of B's, and B's is triple of C's, and the sum of all their ages is 140; what is the age of each?

Ans. A's =84, B's =42, and C's =14.

(27) Two persons, A and B, lay out equal sums of money in trade; A gains \$630, and B loses \$435, and A's money is now double of B's; what did each lay out?

Ans. \$1500.

(28) A person bought a chaise, horse, and harness, for \$450; the horse came to twice the price of the harness, and the chaise to twice the price of the horse and harness; what did he give for each?

Ans. \$100 for the horse, \$50 for the harness, and \$300 for the chaise.

(29) Two persons, A and B, have both the same income: A saves $\frac{1}{5}$ of his yearly, but B, by spending \$250 per annum more than A, at the end of 4 years finds himself \$500 in debt; what is their income?

Ans. \$625.

(30) A person has two horses, and a saddle worth \$250; now, if the saddle be put on the back of the first horse, it will make his value double that of the second; but if it be put on the back of the second, it will make his value triple that of the first; what is the value of each horse?

Ans. One \$150, and the other \$200.

(31) To divide the number 36 into three such parts that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third may be all equal to each other?

Ans. The parts are 8, 12, and 16.

(32) A footman agreed to serve his master for £8 a year and a livery, but was turned away at the end of 7 months, and received only £2 13*s.* 4*d.* and his livery; what was its value?

Ans. £4 16*s.*

(33) A person was desirous of giving 3*d.* a piece to some beggars, but found that he had not money enough in his pocket by 8*d.*; he therefore gave them each 2*d.*, and had then 3*d.* remaining; required the number of beggars?

Ans. 11.

(34) A person in play lost $\frac{1}{4}$ of his money, and then won 3*s.*; after which, he lost $\frac{1}{3}$ of what he then had, and then won 2*s.*; lastly, he lost $\frac{1}{7}$ of what he then had; and this done, found he had but 12*s.* remaining; what had he at first?

Ans. 20*s.*

(35) To divide the number 90 into 4 such parts that if the first be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the sum, difference, product, and quotient shall be all equal to each other?

Ans. The parts are 18, 22, 10, and 40 respectively.

(36) The hour and minute hand of a clock are exactly together at 12 o'clock: when are they next together?

Ans. 1 hour $5\frac{5}{11}$ minutes.

(37) There is an island 73 miles in circumference, and three footmen all start together to travel the same way about it: A goes 5 miles a day, B 8, and C 10; when will they all come together again?

Ans. 73 days.

(38) How much foreign brandy at 8*s*. per gallon, and domestic spirits at 3*s*. per gallon, must be mixed together, so that, in selling the compound at 9*s*. per gallon, the distiller may clear 30 per cent.?

Ans. 51 gallons of brandy, and 14 of spirits.

(39) A man and his wife usually drank out a cask of beer in 12 days; but when the man was from home, it lasted the woman 30 days; how many days would the man alone be in drinking it?

Ans. 20 days.

(40) If A and B together can perform a piece of work in 8 days; A and C together in 9 days; and B and C in 10 days: how many days will it take each person to perform the same work alone?

Ans. A $14\frac{34}{49}$ days, B $17\frac{23}{41}$, and C $23\frac{7}{31}$.

(41) A book is printed in such a manner that each page contains a certain number of lines, and each line a certain number of letters. If each page were required to contain 3 lines more, and each line 4 letters more, the number of letters in a page would be greater by 224 than before; but if each page were required to contain 2 lines less, and each line 3 letters less, the number of letters in a page would be less by 145 than before. Required the number of lines in each page, and the number of letters in each line.

Ans. 29 lines, 32 letters.

(42) Hiero, king of Syracuse, had given a goldsmith 10 pounds of gold with which to make a crown. The work being done, the crown was found to weigh 10 pounds; but the king, suspecting that the workman had alloyed it with silver, consulted Archimedes. The latter, knowing that gold loses in water 52 thousandths of its weight, and silver 99 thousandths, ascertained the weight of the crown, plunged in water, to be 9 pounds 6 ounces. This discovered the fraud. Required the quantity of each metal in the crown.

Ans. 7 pounds $12\frac{12}{47}$ ounces of gold, 2 pounds $3\frac{35}{47}$ ounces of silver.

(43) To divide a number a into two parts which shall have to each other the ratio of m to n.

$$\text{Ans. } \frac{ma}{m+n}, \frac{na}{m+n}.$$

(44) To divide a number a into three parts which shall be to each other as $m:n:p$.

$$\text{Ans. } \frac{ma}{m+n+p}, \frac{na}{m+n+p}, \frac{pa}{m+n+p}.$$

(45) A banker has two kinds of change; there must be a pieces of the first to make a crown, and b pieces of the second to make the same: now a person wishes to have c pieces for a crown. How many pieces of each kind must the banker give him?

Ans. $\frac{a(b-c)}{b-a}$ of the first kind, $\frac{b(c-a)}{b-a}$ of the second.

(46) An innkeeper makes this bargain with a sportsman: every day that the latter brings a certain quantity of game he is to receive a sum a, but every day that he fails to bring it he is to pay a sum b. After a number n of days it may happen that neither owes the other, or that the first owes the second, or that the second owes the first a sum c. Required a formula which

shall express in all three cases the number of days that the sportsman brought the game.

Ans. $x=\frac{bn \pm c}{a+b}$.

In the first case $c=0$, in the second case we must take the positive sign, in the third case the negative sign.

(47) If one of two numbers be multiplied by m, and the other by n, the sum of the products is p; but if the first be multiplied by m', and the second by n', the sum of the products is p'. Required the two numbers.

Ans. $\frac{n'p-np'}{mn'-m'n}$, $\frac{mp'-m'p}{mn'-m'n}$.

(48) An ingot of metal which weighs n pounds loses p pounds when weighed in water. This ingot is itself composed of two other metals, which we may call M and M′; now n pounds of M loses q pounds when weighed in water, and n pounds of M′ loses r pounds when weighed in water. How much of each metal does the original ingot contain?

Ans. $\frac{n(r-p)}{r-q}$ pounds of M, $\frac{n(p-q)}{r-q}$ pounds of M′.

REMARKS UPON EQUATIONS OF THE FIRST DEGREE.

152. Algebraic formulæ can offer no distinct ideas to the mind unless they represent a succession of numerical operations which can be actually performed. Thus, the quantity $b-a$, when considered by itself alone, can only signify an absurdity when $a>b$. It will be proper for us, therefore, to review the preceding calculations, since they sometimes present this difficulty.

Every equation of the first degree may be reduced to one which has all its signs positive, such as

$$ax+b=cx+d \quad \dots\dots\dots\dots \quad (1)^*$$

Subtracting $cx+b$ from each member, we then have

$$ax-cx=d-b.$$

Whence

$$x=\frac{d-b}{a-c} \quad \dots\dots\dots\dots \quad (2)$$

This being premised, three different cases present themselves;

1°. $d>b$ and $a>c$.

2°. One of these conditions only may hold good.

3°. $b>d$ and $c>a$.

In the first case the value of x in equation (2) resolves the problem without giving rise to any embarrassment; in the second and third cases it does not, at first, appear what signification we ought to attach to the value of x; and it is this that we propose to examine.

In the second case one of the subtractions, $d-b$, $a-c$, is impossible; for example, let $b>d$ and $a>c$; it is manifest that the proposed equation (1) is absurd, since the two terms ax and b of the first member are respectively greater than the two terms cx and d of the second. Hence, when we encounter a difficulty of this nature, we may be assured that the proposed prob-

* We can always change the negative terms of an equation into positive ones by transposing them from the member in which they are found to the other member.

lem is absurd, since the equation is merely a faithful expression of its conditions in algebraic language.

In the third case we suppose $b>d$ and $c>a$; here both subtractions are impossible; but let us observe that, in order to solve equation (1), we subtracted from each member the quantity $cx+b$, an operation manifestly impossible, since each member $<cx+b$. This calculation being erroneous, let us subtract $ax+d$ from each member; we then have

$$b-d=cx-ax.$$

Whence

$$x=\frac{b-d}{c-a} \quad \ldots\ldots\ldots\ldots\ldots (3)$$

This value of x, when compared with equation (2), differs from it in this only, that the signs of both terms of the fraction have been changed, and the solution is no longer obscure. We perceive that, when we meet with this third case, it points out to us that, instead of transposing all the terms involving the unknown quantity to the first member of the equation, we ought to place them in the second; and that it is unnecessary, in order to correct this error, to recommence the calculation; it is sufficient to change the signs of both numerator and denominator.

When the equation is absurd, as in the second case, we may nevertheless make use of the negative solution obtained in this case; for if we substitute $-x$ for $+x$, the proposed equation becomes

$$-ax+b=-cx+d.$$

Whence $$x=\frac{b-d}{a-c},$$

a value equal to that in (2), but positive. If, then, we modify the question in such a manner as to agree with this new equation, this second problem, which will bear a marked resemblance to the first, will no longer be absurd, and, with the exception of the sign, will have the same solution.

Let us take, for example, the following problem:

A father, aged 42 years, has a son aged 12; *in how many years will the age of the son be one fourth of that of the father?*

Let $x=$ the number of years required.

Then $$\frac{42+x}{4}=12+x;$$

$$\therefore x=-2.$$

Thus the problem is absurd. But if we substitute $-x$ for $+x$, the equation becomes

$$\frac{42-x}{4}=12-x$$

and the conditions corresponding to this equation change the problem to the following:

A father, aged 42 years, has a son aged 12; *how many years have elapsed since the age of the son was one fourth of that of the father?**

Here $$x=2.$$

* As a problem is translated into algebraic language by means of an equation, so an equation may be translated back into a problem, provided the general nature of the problem be known.

Take another example.

What number of dollars is that, the sum of the third and fifth parts of which, *diminished* by 7, is equal to the original number?

Here $$\frac{x}{3}+\frac{x}{5}-7=x.$$

Whence $$x=-15.$$

The problem is absurd; but, substituting $-x$ for $+x$,

$$-\frac{x}{3}-\frac{x}{5}-7=-x;$$

or

$$\frac{x}{3}+\frac{x}{5}+7=x,$$

which gives

$$x=15;$$

and the problem should read, What number of dollars is that, the third and fifth parts of which, *when increased by* 7, give the original number?

153. With regard to the interpretation of negative results in the solution of problems, then, we may, from what is seen above, establish the following general principle:

When we find a negative value for the unknown quantity in problems of the first degree, it points out an absurdity in the conditions of the problem proposed; provided the equation be a faithful representation of the problem, and of the true meaning of all the conditions.

The value so obtained, neglecting its sign, may be considered as the answer to a problem which differs from the one proposed in this only, that certain quantities which were additive in the first have become subtractive in the second, and reciprocally.

154. The equation (2) presents still two varieties. If $a=c$, we have

$$x=\frac{d-b}{0};$$

in this case the original equation becomes

$$ax+b=ax+d,$$

whence $b=d$; if, therefore, b be not equal to d, the problem would seem absurd.*

But the expression $\frac{d-b}{0}$, or, in general, $\frac{m}{0}$, where m may be any quantity, represents a number *infinitely great.* For, if we take a fraction $\frac{m}{n}$, the smaller we make n, the greater will the number represented by $\frac{m}{n}$ become; thus, for $n=\frac{1}{2}, \frac{1}{100}, \frac{1}{1000}$, the results are 2, 100, 1000 times m. The limit is *infinity*, which corresponds to $n=0$. Or, we may say, to prove $\frac{m}{0}$ infinite, that

* The absurdity is removed by considering that finite quantities have no effect when added to infinite ones; that, in comparison with infinities, finite quantities are all equal to one another, and all equal to zero.

a finite quantity evidently contains an infinite number of zeros. The symbol for the value of x in this case is

$$x=\infty.^*$$

By clearing the expression $\frac{m}{0}=\infty$ of fractions, we have $m=0\times\infty$, from which it appears that the product of zero by infinity is finite. So, also, $\frac{m}{\infty}=0$, or the quotient of a finite quantity by infinity, is zero.

155. If, in equation (2), $a=c$, and $b=d$, we have

$$x=\frac{0}{0};$$

in this case the original equation becomes

$$ax+b=ax+b.$$

Here the two members of the equation are equal, whatever may be the value of x, which is altogether arbitrary, and may have any value at pleasure. We perceive, then, that *a problem is indeterminate, and is susceptible of an infinite number of solutions, when the value of the unknown quantity appears under the form* $\frac{0}{0}$.

It is, however, highly important to observe, that the expression $\frac{0}{0}$ does not always indicate that the problem is indeterminate, but merely the existence of a factor common to both terms of the fraction, which factor becomes 0 under a particular hypothesis.

Suppose, for example, that the solution of a problem is exhibited under the form $x=\frac{a^3-b^3}{a^2-b^2}$.

If, in this formula, we make $a=b$, then $x=\frac{0}{0}$.

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* This infinite value of expressions like $\frac{m}{0}$ may be sometimes positive, sometimes negative, and sometimes indifferently positive or negative.

1°. Let there be the formula $x=\frac{m}{(n-z)^2}$, in which m and n are two invariable numbers, which we suppose positive, and different from zero, while z can have all possible values. Making $z=n$, we have $x=\frac{m}{0}$. But as the denominator, $(n-z)^2$, is always positive, whatever z may be, the infinity here should be regarded as designating the positive infinity.

2°. By analogous reasoning, we see that if we have the formula $x=\frac{-m}{(n-z)^2}$ and $z=n$, we should have the negative infinity $z=-\infty$.

3°. Let there be the formula $x=\frac{m}{n-z}$. The hypothesis $z=n$ gives still $x=\frac{m}{0}$, but here the infinity will have an ambiguous sign. Suppose, at first, $z<n$, and cause z to increase, the formula will give increasing values, which will be all positive. On the contrary, taking $z>n$, then diminishing z till it becomes equal to n, the formula gives increasing values, which are negative. Therefore, the hypothesis $z=n$ ought to be considered as causing the formula to take two infinite values, the one positive and the other negative. This is indicated by writing $x=\pm\infty$. The ∞ is here the transition value between $+$ and $-$. Zero is also a transition value between $+$ and $-$. For, let $x=n-z$: if $z<n$, and z increase till $z>n$, the value of x in changing from $+$ to $-$ passes through 0. Quantities in changing sign must always pass through 0 or ∞. They may, however, pass through 0 or ∞ without changing sign, as in $x=(n-z)^2$, and $\frac{m}{(n-z)^2}$.

But we must remark, that a^3-b^3 may be put under the form $(a-b)(a^2+ab+b^2)$, and that a^2-b^2 is equivalent to $(a-b)(a+b)$; hence the above value of x will be

$$x=\frac{(a-b)(a^2+ab+b^2)}{(a-b)(a+b)}.$$

Now if, before making the hypothesis $a=b$, we suppress the common factor $a-b$, the value of x becomes

$$x=\frac{a^2+ab+b^2}{a+b},$$

an expression which, under the hypothesis that $a=b$, is reduced to

$$x=\frac{3a^2}{2a}=\frac{3a}{2}.$$

Take, as a second example, the expression

$$x=\frac{a^2-b^2}{(a-b)^2}=\frac{(a+b)(a-b)}{(a-b)(a-b)};$$

making $a=b$, the value of x becomes $x=\frac{0}{0}$, in consequence of the existence of the common factor $a-b$; but if, in the first instance, we suppress the common factor $a-b$, the value of x becomes

$$x=\frac{a+b}{a-b};$$

an expression which, under the hypothesis that $a=b$, is reduced to

$$x=\frac{2a}{0}=\infty.$$

From this it appears that *the symbol $\frac{0}{0}$ in algebra sometimes indicates the existence of a factor common to the two terms of the fraction which is reduced to that form.* Hence, before we can pronounce with certainty upon the true value of such a fraction, we must ascertain whether its terms involve a common factor. If none such be found to exist, then we conclude that the equation in question is really *indeterminate*. If a common factor be found to exist, we must suppress it, and then make anew the particular hypothesis. This will now give us the true value of the fraction, which may present itself under one of the three forms $\frac{A}{B}$, $\frac{A}{0}$, $\frac{0}{0}$.

In the first case, the equation is *determinate;* in the second, it is *impossible in finite numbers;* in the third, it is *indeterminate.*

There are other forms of indetermination besides $\frac{0}{0}$; for, whatever be the values of P and Q, we have

$$\frac{P}{Q}=P\times\frac{1}{Q}=\frac{\frac{1}{Q}}{\frac{1}{P}}.$$

The first of these equivalents of $\frac{P}{Q}$, where P and Q both equal zero, becomes $0\times\infty$, and the second becomes $\frac{\infty}{\infty}$, which symbols must, therefore, be considered as having the same meaning with $\frac{0}{0}$.

DISCUSSION OF FORMULAS FURNISHED BY THE GENERAL EQUATIONS OF THE FIRST DEGREE, WITH TWO OR MORE UNKNOWN QUANTITIES.

When the common denominator of the general values of the unknown quantities reduces to zero, it is not readily seen how the given equations are to be verified. We shall examine here the particular cases of this kind which may occur.

Resume the two equations,

$$ax+by=k \qquad [1]$$
$$a'x+b'y=k' \qquad [2]$$

from which we derive the formulas

$$x=\frac{kb'-bk'}{ab'-ba'},\quad y=\frac{ak'-ka'}{ab'-ba'}.$$

First particular Case.—Suppose the denominators to be zero and the numerators not; then we have

$$ab'-ba'=0,\ x=\frac{kb'-bk'}{0},\ y=\frac{ak'-ka'}{0}.$$

The values of x and y are then infinite; that is to say, in order to satisfy the two given equations, they must surpass every assignable magnitude.

From the equality $ab'-ba'=0$, we derive $a'=\frac{ab'}{b}$, and, consequently, the equation [2], by putting in it this value, becomes

$$\frac{ab'}{b}x+b'y=k',\ \therefore\ b'(ax+by)=bk'.$$

The first member is the first member of [1] multiplied by b'; the same relation must subsist between the second members, in order that the value of x and y may verify at the same time equations [1] and [2]. Hence $bk'=kb'$, or, $kb'-bk'=0$; *i. e.*, the numerator of x would be equal to zero, which is contrary to hypothesis.*

In this way the impossibility of finding values of x and y, which satisfy at the same time the two given equations, is made apparent; but this impossibility is still better characterized by the infinite values, which, at the same time that they indicate the impossibility, show besides that it arises from the fact that the values of the unknown quantities are too great to be assigned.

If we suppose $ab'-ba'$ to be at first a very small quantity, the values of x and y will be very great, but they will always satisfy the equations until the instant $ab'-ba'$ reduces to zero, when, if we can not effect in a direct manner the verification of the equations, it is solely because x and y then surpass all assignable magnitude.†

Second particular Case.—Suppose the denominator to be zero at the same time as one of the numerators; for example, that we have

$$ab'-ba'=0,\ kb'-bk'=0.$$

I maintain that the other numerator will be also equal to zero; for the two equalities above give

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* The note to art. 154 explains this anomaly. The finite quantities kb' and bk' are equal when compared with infinity.

† Considered in relation to the question, the conditions of which are expressed by the problem, infinite values may be sometimes a true solution of the question. The application of algebra to geometry furnishes numerous examples of this kind; among others may be cited that where an angle is unknown, and we find for its tangent an infinite value. It is clear, then, that the angle must be right.

$$a'=\frac{ab'}{b},\ k'=\frac{kb'}{b},$$

and, consequently, the other numerator becomes

$$ak'-ka'=\frac{akb'}{b}-\frac{akb'}{b}=0.$$

If at first we had supposed this numerator equal to zero, we could have proved in a similar manner that of x to be so also.

The present hypothesis then gives

$$x=\frac{0}{0},\ y=\frac{0}{0}.$$

Of themselves these symbols indicate indetermination; I shall prove, by going back to the equations, that they ought, in fact, to be indeterminate.

For this purpose, substitute in equation [2] the values of a' and k', found above, and it becomes

$$\frac{ab'}{b}x+b'y=\frac{kb'}{b},\ \therefore \frac{b'}{b}(ax+by)=\frac{b'}{b}k.$$

Thus we see that it can be formed by multiplying the two members of equation [1] by $\frac{b'}{b}$; then all values of x and y which satisfy one of the two equations will also satisfy the other. But if we give to x values at pleasure in equation [1], we can, by resolving it afterward, find corresponding values of y; and as these same values satisfy the second equation, we conclude that the proposed equations admit an infinite number of solutions.

Let it, however, be observed, that the indetermination in this case does not permit us to take whatever value of y, and, at the same time, of x, we please, because the above explication shows that, when one of these unknown quantities is assumed, the value of the other is determined.

The case before us comprehends that in which $k=0$, $k'=0$, $ab'-ba'=0$, because then x and y become $\frac{0}{0}$. If we return to the equations proposed, they reduce to these,

$$ax+by=0,\ a'x+b'y=0.$$

They give respectively

$$y=-\frac{a}{b}x,\ y=-\frac{a'}{b'}x.$$

But upon the hypothesis of $ab'-ba'=0$, we derive $\frac{a}{b}=\frac{a'}{b'}$; then the two values of y are equal, whatever be that of x, and there is veritable indetermination.

Yet it is to be observed, that, if we take the relation of y to x, this relation is determinate, because we have

$$\frac{y}{z}=-\frac{a}{b}=-\frac{a'}{b'}.$$

If the condition $\frac{a}{b}=\frac{a'}{b'}$ had not existed, the two values of y above could not have been equal, except we suppose $x=0$; y would have been then zero, and the relation of x and y no longer determinate, but indeterminate.

A similar discussion to the above might be given to a system of three or more equations, with as many unknown quantities. It would, however, be more difficult to investigate the cases of impossibility and indetermination, and it is not worth while to delay upon them. We shall content ourselves with setting down here some observations intended to caution the student against certain hasty conclusions to which he might naturally be led.

We have seen, in the case of two equations with two unknown quantities, that x and y become infinite and indeterminate simultaneously.

The first error which might be committed would be that of supposing from analogy that, in the case of several equations, the unknown quantities would all become infinite or indeterminate together. Suppose, for example, there are under consideration the three equations

$$ax+by+cz=k,$$
$$a'x+b'y+c'z=k',$$
$$a''x+b''y+c''z=k''.$$

The common denominator of the values of x, y, z, is

$$R=ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a'',$$

and it may be written in three ways:

$$R=a(b'c''-c'b'')+a'(cb''-bc'')+a''(bc'-cb'),$$
$$R=b(c'a''-a'c'')+b'(ac''-ca'')+b''(ca'-ac'),$$
$$R=c(a'b''-b'a'')+c'(ba''-ab'')+c''(ab'-ba').$$

Place

$$b'c''=c'b'',\ cb''=bc''.$$

From these equations we deduce $bc'=cb'$, and, consequently, R becomes zero. Then the numerator of x, which is formed from R by changing a, a', a'' into k, k', k'', becomes zero also. But as the numerator of y is formed by placing k, k', k'' in R instead of b, b', b'', there is no reason why this numerator should become zero, unless we make some new hypothesis. The same may be said of that of z. Thus the value of x can take the indeterminate form $\frac{0}{0}$, where the values of y and z are infinite.

But with regard to this indeterminate form, another error still is to be avoided, because it may be that the indetermination is only apparent (see Art. 155). In order to judge better of it, we shall have regard only to the single relation

$$b'c''=c'b'',\ \therefore\ c''=\frac{c'b''}{b'}.$$

Substituting this value of c'' in the general value of x, it will be seen that $bc'-cb'$ becomes a common factor of both numerator and denominator. But by hypothesis this factor is zero; it is its presence, then, which produces the appearance of indetermination. Suppressing it, we have the true value of x, which appears no longer indeterminate, unless some new hypothesis be joined to those already made.*

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* An important observation should be made before quitting the subject of indetermination.

When the two terms of a fraction decrease so as to become less than any assignable quantity, if the suppositions which cause one of them to decrease indefinitely are entirely independent of those which cause the other to do so, the values of these terms may be taken as near zero as we please, and such that their relation, which is the value of the fraction, may be equal to any quantity whatever; consequently, the symbol $\frac{0}{0}$, at which we arrive when the two terms shall have attained the limit of their decrease, will express complete indetermination. But it may happen that the two terms of the fraction are connected together in such a way, that to a very small value of one there corresponds always

156. We shall conclude this discussion with the following problem, which will serve as an illustration of the various singularities which may present themselves in the solution of a simple equation.

PROBLEM.

Two couriers set off at the same time from two points, A and B, in the same straight line, and travel in the same direction, A C. The courier who sets out from A travels m miles an hour, the courier who sets out from B travels n miles an hour; the distance from A to B is a miles. At what distance from the points A and B will the couriers be together?

Let C be the point where they are together, and let x and y denote the distances A C and B C, expressed in miles.

We have manifestly for the first equation

$$x-y=a \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

Since m and n denote the number of miles traveled by each in an hour, that is, the respective velocities of the two couriers, it follows that the time required to traverse the two spaces, x and y, must be designated by $\frac{x}{m}$, $\frac{y}{n}$; these two periods, moreover, are equal; hence we have for our second equation

$$\frac{x}{m}=\frac{y}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

The values of x and y, derived from equations (1) and (2), are

$$x=\frac{am}{m-n},\ y=\frac{an}{m-n}.$$

1°. So long as we suppose $m>n$, or $m-n$ positive, the problem will be solved without embarrassment. For, in that case, we suppose the courier who starts from A to travel faster than the courier who starts from B; he must, therefore, overtake him eventually, and a point C can always be found where they will be together.

2°. Let us now suppose $m<n$, or $m-n$ negative, the values of x and y are both negative, and we have

$$x=-\frac{am}{n-m},\ y=-\frac{an}{n-m}.$$

The solution, therefore, in this case, points out that some absurdity must exist in the conditions of the problem. In fact, if we suppose $m<n$, we suppose that the courier who sets out from A travels slower than the courier who sets out from B; hence the distance between them augments every instant, and it is impossible that the couriers can ever be together if they travel in the direction A C. Let us now substitute $-x$ for $+x$, and $-y$ for $+y$, in equations (1) and (2); when modified in this manner, they become

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a very small value of the other; and that, when they converge toward zero, their relation converges toward a determinate limit, which it does not attain till the moment that the two terms vanish, and the fraction presents itself under the form $\frac{0}{0}$.* A particular example of this last case is the vanishing of a common factor of the numerator and denominator. The same remark is applicable to the symbol $\frac{\infty}{\infty}$.

* This principle is fully exemplified in the differential calculus

$$\left.\begin{array}{r} y-x=a \\ \frac{x}{m}=\frac{y}{n} \end{array}\right\}$$

equations which, when resolved, give

$$x=\frac{am}{n-m},\ y=\frac{an}{n-m},$$

in which the values of x and y are positive.

These values of x and y give the solution, not of the proposed problem, which is absurd under the supposition that $m<n$, but of the following, which is the translation of the changed equations.

Two couriers set out at the same time from the points A and B, and *travel in the direction* B C′, &c. (the rest as before) ; the values of x and y mark the distances A C′, B C′, of the point C′, where the couriers are together, from the points of departure A and B.

From this problem, as well as that of the father and son above, may be deduced the following rule, when the value of the unknown quantity is found to be negative :

Change the sign of the unknown quantity in the first equation, or the one derived immediately from the problem ; this changed equation, translated into common language, will furnish the problem which will give a positive solution.

*If the problem be at first enunciated in a general manner, then negative values of the unknown quantity may be regarded as furnishing a true solution, but are to be interpreted in a contrary sense. Thus, if positive values represent distance to the right, negative will represent distance to the left ; if positive express distance upward, negative distance downward ; if the former indicate time future, the latter must indicate time past ; if the one gain, the other loss ; if the one a rate of increase, the other a rate of decrease, &c.**

3°. Let us next suppose $m=n$; the values of x and y in this case become

$$x=\frac{am}{0},\ y=\frac{an}{0},$$

or

$$x=\infty,\ y=\infty;$$

that is to say, x and y each represent *infinity.* In fact, if we suppose $m=n$, we suppose the courier who sets out from A to travel exactly at the same rate as the courier who sets out from B ; consequently, the original distance, a, by which they are separated will always remain the same, and if the couriers travel *forever*, they can never be together.†

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* Applications of this use of positive and negative quantities constantly occur in trigonometry and analytical geometry.

† Since $m=n$, equation (2) gives $x=y$, and equation (1), in consequence, $a=0$. To understand this, we must recur to the principle stated in (Art. 154). We may here extend a little the statement there made. All zeros are equal when compared with finite quantities, but not when compared with one another. Thus, $2x$ is twice as great as x, though x be 0; but $2x+a=x+a=a$, if $x=0$. In the first of these cases one zero, $2x$, is compared with another, and then they are not equal; in the second, both zeros, $2x$ and x, are compared with the finite quantity, a, and then are equal.

Again, $x+a=x+10a=x+0=x$, if $x=\infty$; but $10a$ is ten times as great as a, when un connected with infinity. Finite quantities are, therefore, all equal to one another, and all equal to zero when compared with infinite ones, but not when simply compared with one another. It is rare that algebra can be employed to demonstrate moral or religious truth;

4°. Let us suppose $m=n$, and also $a=0$; the values of x and y in this case become

$$x=\frac{0}{0}, y=\frac{0}{0};$$

that is to say, the problem is *indeterminate*, and admits of an infinite number of solutions. In fact, if we suppose $a=0$, we suppose that the couriers start from the same point, and if we at the same time suppose $m=n$, or that they travel equally fast, it is manifest *that they must always be together*, and consequently *every point* in the line A C satisfies the conditions of the problem.

5°. Finally, if we suppose $a=0$, and m *not* $=n$, the values of x and y in this case become

$$x=0, y=0.$$

In fact, if we suppose the couriers to set out from the same point, and to travel with different velocities, it is manifest that the point of departure is the only point in which they can be together.

ADDITIONAL PROBLEMS.

(1) The rent of an estate is greater than it was last year by 8 per cent. of the rent of that year; this year's rent is 1890. What was last year's?

Ans. 1750.

(2) A company of 90 persons consists of men, women, and children; the men are 4 in number more than the women, and the children 10 more than the men and women together. How many of each?

Ans. 22 men, 18 women, and 50 children.

(3) From the first of two mortars in a battery 36 shells are thrown before the second is ready for firing. Shells are then thrown from both in the proportion of 8 from the first to 7 of the second, the second mortar requiring as much powder for 3 charges as the first does for 4. It is required to determine after how many discharges of the second mortar the quantity of powder consumed by it is equal to the quantity consumed by the first.

Ans. 189 discharges of the second mortar.

(4) The fore wheels of a carriage are $5\frac{1}{4}$ feet and the hind wheels $7\frac{1}{8}$ feet in circumference; the difference of the number of revolutions of the wheels is 2000. What is the length of the journey?

Ans. 39900 feet, or $7\frac{49}{88}$ miles.

(5) Three brothers, A, B, and C, buy a house for £2000; C can pay the whole price if B give him half his money; B can pay the whole price if A give him one third of his money; A can pay the whole price if C give him one fourth of his money. How much has each?

Ans. A £1680, B £1440, C £1280.

(6) The passengers of a ship were $\frac{1}{4}$ Germans, $\frac{1}{5}$ French, $\frac{1}{6}$ English, $\frac{1}{8}$

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but the objection to the doctrine of the special and immediate superintendence of Providence in the affairs of men, that it implies an incredible degree of condescension in an infinite being, finds in the principle above stated a satisfactory refutation. As compared with infinity, the smallest portion of matter is equal to the greatest, and it is therefore no more an act of condescension on the part of God to charge himself with the care of an individual than of a nation—with the revolutions of a satellite than with the movements of a system.

Dutch, and the residue, amounting to 31, Americans. How many were there in the whole?

Ans. 120.

(7) Suppose the sound of a bell to be heard at the distance of 1142 feet in a second in a still atmosphere, and that a wind is blowing sufficient to occasion a delay of $\frac{1}{5}$ in time. In how many seconds will the sound reach a distance of 6000 feet?

Ans. 6.304.

(8) Quicksilver expands, for each degree of the centigrade thermometer, $\frac{1}{5550}$ of its volume. According to this, how high would the barometer stand when the temperature is 0°, if, when the temperature is 21°, it stands at a height of 27 inches $8\frac{1}{2}$ lines?

Ans. 27 in. $7\frac{458}{1857}$ lines.

(9) What degree of heat in a centigrade thermometer would be required to cause the barometer to rise to 26 inches 8 lines, if 0° raised it to 26 inches 4 lines?

Ans. $70\frac{20}{79}$.

(10) A piece of silver, the specific gravity of which is $10\frac{1}{2}$, weighs 84 oz. How much weight will it lose in water?

Ans. 8 oz.

(11) In a mass of zinc and copper, weighing 100 pounds, 8 parts are of the former and 3 of the latter. How much zinc must be added, that the proportions may be as 14 : 5?

Ans. $3\frac{35}{55}$.

(12) At the extremities of two arms of a balanced lever, whose lengths are 16 and 21 feet, two weights are suspended, which together amount to $65\frac{2}{5}$ pounds. How much is suspended at each arm?

Ans. $37\frac{22}{185}$ and $28\frac{52}{185}$.

(13) The range of temperature of a thermometer during the year was $44\frac{3}{10}$°. The ratio of the degrees at which it stood at the extreme points above and below zero was 7 : 4. What were the points?

Ans. $28\frac{21}{110}$ above, $16\frac{6}{55}$ below.

(14) In 4000 pounds of gunpowder there are 3240 less of sulphur than of charcoal and saltpetre, 2760 less of charcoal than of sulphur and saltpetre. How much of each of these?

Ans. Sulphur 380, charcoal 620, saltpetre 3000.

(15) It is required to divide the number 99 into five such parts that the first may exceed the second by 3, be less than the third by 10, greater than the fourth by 9, and less than the fifth by 16.

Ans. The parts are 17, 14, 27, 8, and 33.

(16) A and B began trade with equal stocks. In the first year A tripled his stock, and had £27 to spare; B doubled his, and had £153 to spare. Now the amount of both their gains was five times the stock of either. What was that stock?

Ans. £90.

(17) What two numbers are as 2 to 3; to each of which, if 4 be added, the sums will be as 5 to 7?

Ans. 16 and 24.

(18) Four places are situated in the order of the letters A B, C, D. The

distance from A to D is 34 miles. The distance from A to B is to the distance from C to D as 2 is to 3; and one fourth of the distance from A to B, added to half the distance from C to D, is three times the distance from B to C. What are the respective distances?

Ans. AB=12, BC=4, CD=18.

(19) A field of wheat and oats, which contained 20 acres, was put out to a laborer to reap for 6 guineas (of 21*s.* each), the wheat at 7 shillings an acre and the oats at 5 shillings. The laborer, falling ill, reaped only the wheat. How much money ought he to receive, according to the bargain?

Ans. £4 11*s.*

(20) A general having lost a battle, found that he had only half his army +3600 men left, fit for action, one eighth of his men +600 being wounded, and the rest, which were one fifth of the whole army, either slain, taken prisoners, or missing. Of how many men did his army consist?

Ans. 24000.

(21) A shepherd in time of war was plundered by a party of soldiers, who took $\frac{1}{4}$ of his flock and $\frac{1}{4}$ of a sheep; another party took from him $\frac{1}{3}$ of what he had left, and $\frac{1}{3}$ of a sheep more; then a third party took $\frac{1}{2}$ of what now remained, and $\frac{1}{2}$ a sheep. After which he had but 25 sheep left. How many had he at first?

Ans. 103.

(22) A trader maintained himself for three years at the expense of £50 a year, and in each of those years augmented his stock by $\frac{1}{3}$ of what remained unexpended. At the end of 3 years his original stock was doubled. What was that stock?

Ans. 740.

(23) There is a certain number consisting of two digits, the sum of these digits is 5, and if 9 be added to the number, the digits are transposed. What is the number?

Ans. 23.

(24) A coach has 4 more outside than inside passengers. Seven outsides could travel at 2*s.* less expense than 4 insides. The fare of the whole amounted to £9; but at the end of half the journey the coach took up 3 more outside and one more inside passenger, in consequence of which the fare of the whole became increased in the proportion of 19 to 15. Required the number of passengers, and the fare of each kind.

Ans. 5 inside, 9 outside; fares, 18 and 10 shillings.

(25) The hands of a clock are together at 12: at what times will they be together during the next 12 hours?

Ans. $5\frac{5}{11}$ minutes past 1, $10\frac{10}{11}$ minutes past 2, and so on, in each successive hour $5\frac{5}{11}$ later.

(26) A person sets out from a certain place, and goes at the rate of 11 miles in 5 hours; and 8 hours after another person sets out from the same place, and goes after him at the rate of 13 miles in 3 hours. How far must the latter travel to overtake the former?

Ans. $35\frac{3}{4}$ miles.

(27) A reservoir which is full of water may be emptied at two cocks. One is opened, and $\frac{1}{4}$ of the water runs out; another is opened, and the two run-

ning together, empty the vessel in $\frac{5}{4}$ of an hour more than was required for the first cock alone to empty the fourth part. If the two cocks had been opened at the commencement, the reservoir would have been emptied in $\frac{1}{4}$ of an hour sooner. How long would it have taken the first cock, running alone, to empty the reservoir?

Ans. 4 hours.

INDETERMINATE ANALYSIS OF THE FIRST DEGREE.

157. If there be proposed for solution *one* equation of the first degree, containing *two* unknown quantities, any value at pleasure may be given to one of the unknown quantities, and the equation will make known a corresponding value for the other; from which it appears that the equation admits of an infinite number of solutions. The number of solutions will, however, not be so unlimited, if it be required that the values of x and y shall be whole numbers; and still less so, if they must be both entire and positive.

Let there be the equation

$$ax+by=c,$$

a, b, c being any whole numbers whatever, either positive or negative; and as all the factors common to these three numbers could be suppressed, suppose this to have been done.

And first, let it be observed, that if there should remain now a common factor in a and b, the equation could not admit of a solution in whole numbers; for whatever values might be substituted for x and y, the first member would be divisible by this common factor of a and b, while the second member would not, and the equality would therefore be impossible: a and b must therefore be supposed prime to each other.

158. Take, for example, the equation

$$24x+65y=243 \quad \ldots\ldots\ldots\ldots \quad (1)$$

in which the coefficients 24 and 65 are prime to each other.

Resolving it, with respect to x,

$$x=\frac{243-65y}{24}=10-2y+\frac{3-17y}{24}.$$

In order that x and y may both be whole numbers, and, at the same time, satisfy the given equation, it is necessary that $\frac{3-17y}{24}$ should be a whole number.

Representing this by t, we have

$$\frac{3-17y}{24}=t \quad \ldots\ldots\ldots\ldots \quad (2)$$

and

$$x=10-2y+t \quad \ldots\ldots \quad (3)$$

The solution of the given equation in whole numbers then reduces itself to the solution of the equation (2).

We resolved the given equation with respect to the unknown quantity which had the least coefficient: doing the same with (2),

$$y=\frac{3-24t}{17}=-t+\frac{3-7t}{17};$$

and proceeding as before,

$$\frac{3-7t}{17}=t' \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

$$y=-t+t' \quad \ldots\ldots\ldots\ldots \quad (5)$$

The solution of (2) in whole numbers depends on that of (4), which, resolved with respect to t, gives

$$t=\frac{3-17t'}{7}=-2t'+\frac{3-3t'}{7}$$

$$\frac{3-3t'}{7}=t'' \quad \ldots\ldots\ldots\ldots\ldots \quad (6)$$

$$t=-2t'+t'' \quad \ldots\ldots\ldots \quad (7)$$

Continuing in the same way,

$$t'=\frac{3-7t''}{3}=1-2t''-\frac{t''}{3}$$

$$\frac{t''}{3}=t''' \quad \ldots\ldots\ldots\ldots\ldots \quad (8)$$

$$t'=1-2t''-t''' \quad \ldots\ldots\ldots \quad (9)$$

Equation (8) gives

$$t''=3t''' \quad \ldots\ldots\ldots\ldots\ldots \quad (10)$$

The solutions of the given equation in whole numbers are therefore obtained by giving to the indeterminate quantity t''' all possible values in whole numbers, positive or negative; and for each of these values of t''', the equations (10), (9), (7), (5), and (3), determine successively the values of the indeterminate quantities t'', t', t, and of the unknown quantities y and x. The equation is therefore resolved in the manner required.

Formulas may be obtained which give immediately the values of x and y in terms of t'''. For, substituting the value $3t'''$ of t'' in (9), we find $t'=1-7t'''$; substituting this value of t' and that of t'' in (7), we find $t=-2+17t'''$; substituting this last value and that of t' in equation (5), we find $y=3-24t'''$, and from (3), $x=2+65t'''$.

These last two expressions give all the entire solutions of the proposed equations by attributing successively to t''' all possible values in entire numbers, positive or negative.

159. The same process with the general form

$$ax+by=c$$

would run thus,

$$y=\frac{c-ax}{b} \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

Dividing a by b, and calling q the quotient, r the remainder,

$$y=\frac{c-(bq+r)x}{b}=-qx+\frac{c-rx}{b},$$

make

$$\frac{c-rx}{b}=t, \therefore x=\frac{c-bt}{r} \quad \ldots\ldots \quad (2)$$

Calling q' the quotient of b by r, and r' the remainder,

$$x=-q't+\frac{c-r't}{r},$$

make

$$\frac{c-r't}{r}=t', \therefore t=\frac{c-rt'}{r'} \quad \ldots\ldots \quad (3)$$

And calling q'' the quotient of r by r', and r'' the remainder,

$$t=-q''t'+\frac{c-r''t'}{r'},$$

make

$$\frac{c-r''t'}{r'}=t'' \quad \ldots\ldots\ldots\ldots \quad (4)$$

and so on. The process is now evident, and it will be perceived that the coefficients r, r', r'', which enter into the equations (2), (3), (4), are the successive remainders which would be obtained in operating as if to find the com mon divisor of a and b. We must at length arrive at a remainder 1, because a and b are supposed prime to each other.

For the sake of being more definite, let r'' be supposed to be this remainder then equation (4) gives

$$t'=-r't''+c \quad \ldots\ldots\ldots\ldots \quad (5)$$

By means of equations (2), (3), (4), and (5), the values of y, x, t, and t' may be written as follows:

$$\begin{aligned} y&=-qx+t \\ x&=-q't+t' \\ t&=-q''t'+t'' \\ t'&=-r't''+c. \end{aligned}$$

This series of equations shows that any entire value being assumed for t'', the resulting value of t' substituted in that of t, the values of t, t' in that of x, and the values of x, t in that of y, the proposed equation is resolved in whole numbers.

160. The success of the method is founded on the progressive diminution which division effects upon the coefficients of the indeterminates; there is no reason, however, why the constant term, found in the successive equations, should not also be divided. In this way the calculation will involve smaller numbers, an advantage which is not to be neglected.

For example, take the equation

$$3x-8y=43.$$

As the multiplier of x is less than that of y, resolve the equation with reference to x,

$$x=\frac{8y+43}{3}.$$

Dividing 8 by 3, the quotient is 2, and the remainder 2; and dividing 43 by 3, the quotient is 14, remainder 1; then

$$x=2y+14+\frac{2y+1}{3}=2y+14+t$$

$$2y+1=3t$$

$$y=\frac{3t-1}{2}=t+\frac{t-1}{2}=t+t'$$

$$t-1=2t'$$

$$t=2t'+1,$$

in which last equality t' may receive all possible entire values. By means of this value may be found

$$y=t+t'=2t'+1+t'=3t'+1$$

$$x=2y+14+t=2(3t'+1)+14+2t'+1=8t'+17.$$

Giving to t' the values 0, 1, 2, 3,... we find

$$y = 1, \ 4, \ 7, 10, \ldots$$
$$x = 17, 25, 33, 41, \ldots$$

t' may also receive the negative values

$$-1, -2, -3, \ldots$$

161. In the above example, the values of y and x form two arithmetical progressions, the first of which has the common difference 3, the coefficient of x in the proposed equation; and the second the common difference 8, the coefficient of y taken with the contrary sign. This proposition may be seen to be general by effecting the successive substitutions in the general solution, but the following demonstration is preferable.

It appears, from the general investigation already made, that the equation

$$ax + by = c \quad \ldots\ldots\ldots\ldots \quad (1)$$

admits of an infinite number of solutions in whole numbers, whatever may be the signs of a and b, provided they are prime to each other. Suppose one of these solutions to be

$$x = \text{A}, \ y = \text{B}.$$

These values must satisfy the given equation (1), thus,

$$a\text{A} + b\text{B} = c.$$

Subtracting this equality from (1), we have

$$a(x - \text{A}) + b(y - \text{B}) = 0$$

$$\therefore y = \text{B} + \frac{a(\text{A} - x)}{b}.$$

The values of x are to be whole numbers, and such that y shall also be a whole number. Then the product $a(\text{A}-x)$ must be divisible by b; but a is prime with b, $(\text{A}-x)$ is, therefore, a multiplier of b (see Art. 84, Note), hence we may write

$$\text{A} - x = bt;$$

t being some whole number. From whence

$$x = \text{A} - bt, \ y = \text{B} + at.$$

These formulas exhibit the law of the values to be obtained for x and y, when there are given to t all entire values successively. If t be taken equal to 0, 1, 2, 3, ... there results

$$x = \text{A}, \ \text{A} - b, \ \text{A} - 2b, \ \text{A} - 3b, \ \&\text{c}.$$
$$y = \text{B}, \ \text{B} + a, \ \text{B} + 2a, \ \text{B} + 3a, \ \&\text{c}.$$

In general, when t increases by unity, y increases by a, and x by $-b$. *The solutions in whole numbers, then, of the equation* ax+by=c, *are the corresponding terms of two progressions by differences. In the progression belonging to each of the indeterminates,* x *and* y, *the common difference is equal to the coefficient of the other indeterminate. But it is necessary to be careful to take one of the coefficients with the same sign that it has in the equation, and the other with the contrary sign.*

It is immaterial which of the coefficients is taken with the contrary sign, because in the formulas which express x and y the signs of bt and $-at$ may be changed, since t can receive all possible values, positive and negative.

162. In the general equation, if $c=0$, so that

$$ax + by = 0,$$

as one solution is evidently $x=0$, $y=0$, the general formulas become

$$x = bt, \ y = -at.$$

163. Again, suppose c to be a multiple of a or b. Let $c=bd$, then

$$ax+by=bd.$$

One solution is evidently $x=0$, $y=d$; hence the general values are

$$x=bt,\ y=d-at.$$

Example, $$5x-7y=21.$$

The evident solution is $x=0$, $y=-3$, and the general values

$$x=7t,\ y=-3+5t.$$

164. We shall point out two simplifications which may sometimes be made in the calculations. An example will explain them.

$$80x-17y=39.$$

Resolving it with respect to y,

$$y=\frac{80x-39}{17}.$$

If 80 be divided by 17, $80=17\times4+12$; but as the remainder, 12, exceeds half the divisor, 17, we observe that we may write

$$80=17\times(4+1)+12-17=17\times5-5;$$

that is, augmenting the quotient by unity, we have a negative remainder less than half the divisor, which causes a more rapid reduction in the numbers. The 39, divided by 17, leaves a remainder $+5$, which it is unnecessary to change. We have then

$$y=\frac{(17\times5-5)x-17\times2-5}{17}=5x-2-\frac{5x+5}{17}.$$

But another simplification now presents itself, from the fact that 5 is a factor of $5x+5$, and this numerator may be written $5(x+1)$. In order to render $5(x+1)$ divisible by 17, it is only necessary to take $x+1$, any multiple whatever of 17. Whence the auxiliary equation

$$x+1=17t;$$

$$\therefore\ x=17t-1,\ y=80t-7.$$

RESOLUTION OF THE EQUATION $ax+by=c$ IN NUMBERS BOTH ENTIRE AND POSITIVE.

165. We begin as if the values of x and y were required to be entire only, and thus derive, as before, expressions of the form

$$x=A-bt,\ y=B+at.$$

But now, instead of attributing to t all possible values in whole numbers, we choose only those which will render x and y positive. Hence there result for t certain limitations which are always easy to determine.

First, let us consider the case where a and b have the same sign in the equation

$$ax+by=c\ \ldots\ldots\ldots\ldots\ (1)$$

Suppose a and b positive; for if they were both negative, they might be rendered positive by changing all the signs of the equation. We must also suppose c to be positive, otherwise the equation would be impossible in positive whole numbers.

Write the general values of x and y under the following form:

$$x=b\left(\frac{A}{b}-t\right),\ y=a\left(t-\frac{-B}{a}\right).$$

Then we perceive that, to render x positive, it is necessary, and is sufficient, to take $t<\frac{A}{b}$, and likewise, in order that y may be positive, to take $t>\frac{-B}{a}$.

The signs $>$ and $<$ do not exclude equality; that is to say, if the first limit were a number n, we might make $t=n$. The corresponding value of x would be $x=0$.

166. Since t must be an entire number between two limits, it follows that the number of solutions of the equation is also limited.

And this is evident from the equation itself; for a and b being positive, if we substitute for x and y positive numbers, the two terms $ax+by$ will be always positive; and as their sum has to remain constantly equal to c, it is impossible that either of these terms should increase indefinitely.

It may happen that there is no whole number between the limits assigned above for t; then we conclude that the equation is impossible. Such a case would happen if the limits should be embraced between two consecutive whole numbers like these, $t>4\frac{1}{3}$ and $t<4\frac{5}{7}$; or, again, if they were contradictory, as, for example, $t>4\frac{1}{3}$ and $t<3\frac{5}{7}$.

167. In the second place, consider the case in which a and b are of contrary signs. Suppose the equation in question to be

$$ax-by=c \quad \ldots\ldots\ldots\ldots \quad (2)$$

in which a and b represent two positive numbers. Then the general values of x and y are of the form

$$x=A+bt,\ y=B+at.$$

But we can write them

$$x=b\left(t-\frac{-A}{b}\right),\ y=a\left(t-\frac{-B}{a}\right).$$

And we perceive at once that to have x and y positive, we must have, at the same time,

$$t>\frac{-A}{b} \text{ and } t>\frac{-B}{a};$$

that is to say, we may attribute to t all entire values above the greatest of these limits without excluding equality, if this limit is an entire number.

By this we perceive that the equation $ax-by=c$ admits always of an infinite number of solutions, while the equation $ax+by=c$ admits of but a limited number, and even may not have any.

Let us apply what precedes to some problems.

168. Problem I.—*A company of men and women expend at a feast* 1000 *francs. The men pay each* 19 *francs, and the women* 11 *francs. How many men and how many women are there?*

Let x represent the number of men and y the number of women. We have to resolve in entire numbers the equation

$$19x+11y=1000 \quad \ldots\ldots\ldots \quad (3)$$

In making the calculation, as in (160), and profiting by the simplifications indicated by (Art. 164), we have successively,

$$y=\frac{1000-19x}{11}=91-2x+\frac{3x-1}{11}=91-2x+t$$

$$3x-1=11t$$

$$x=\frac{11t+1}{3}=4t+\frac{1-t}{3}=4t+t'$$

$$1-t=3t'$$

$$t=1-3t'.$$

Arrived at this point, we return to x and y, and they become

$$x=4t+t'=4(1-3t')+t'=4-11t'$$
$$y=91-2x+t=91-2(4-11t')+(1-3t')=84+19t'.$$

Thus, the general formulas which express x and y in terms of t' are

$$x=4-11t',\ y=84+19t'.$$

In order that x may be positive, it is necessary and sufficient that we have $11t'<4$, or $t'<\frac{4}{11}$; and in order that y should be also positive, it is necessary and sufficient that we have $19t'>-84$, or $t'>-4\frac{8}{19}$. Then we must take t', one of the series of values,

$$t'=0,\ -1,\ -2,\ -3,\ -4.$$

To these values correspond

$$x=4,\ 15,\ 26,\ 37,\ 48$$
$$y=84,\ 65,\ 46,\ 27,\ 8.$$

The number of solutions is limited, as we ought to expect, since, in the equation (3), the terms containing x and y are of the same sign.

There are five solutions in all, to wit;

1st solution, 4 men and 84 women.
2d solution, 15 men and 65 women.
3d solution, 26 men and 46 women.
4th solution, 37 men and 27 women.
5th solution, 48 men and 8 women.

Remark.—From what has been said at (161), it is sufficient to procure a single solution of the equation (3) to form immediately the general values of x and y. Thus, after having found above $t=1-3t'$, we make $t'=0$; and if we calculate the corresponding values $t=1$, $x=4$, $y=84$, it is evident that the values $x=4$, $y=84$, ought to form one solution of the equation; then we can place immediately $x=4-11t'$, $y=84+19t'$.

169. Problem II.—*With two measuring rods of different lengths, the one* 5 *feet, and the other* 7, *it is required to make, by placing them the one after the other, a length of* 23 *feet.*

This problem requires the solution in whole numbers of the equation

$$5x+7y=23.$$

We derive from it successively

$$x=\frac{23-7y}{5}=5-y-\frac{2+2y}{5}=5-y-2t$$
$$1+y=5t$$
$$y=5t-1$$
$$x=6-7t$$

In order that y may be positive, we must make $t>\frac{1}{5}$; and that x may be positive, $t<\frac{6}{7}$. As no whole number falls between $\frac{1}{5}$ and $\frac{6}{7}$, we conclude that the problem is impossible.

Remark.—The equation would have had an infinite number of solutions if negative values had been admitted. For example, if $t=0$, we have $x=6$, $y=-1$. This solution indicates that by placing one of the rods, that of 5 feet, 6 times in succession, and placing afterward the rod of 7 feet, so as to cut off its length from the end of the distance thus obtained, the remainder would be the required length, 23 feet.

170. Problem III.—*A person purchased some hares and sheep. Each hare cost him* 8 *shillings, and each sheep* 27. *He found that he had paid for*

the hares 97 shillings more than for the sheep. How many hares did he purchase, and how many sheep?

$$8x-27y=97$$

$$x=\frac{27y+97}{8}=3y+12+\frac{3y+1}{8}=3y+12+t$$

$$3y+1=8t$$

$$y=\frac{8t-1}{3}=3t-\frac{t+1}{3}=3t-t'$$

$$t+1=3t'$$

$$t=3t'-1.$$

By making $t'=0$, we have $t=-1$, $y=-3$, $x=2$. And the general values are

$$x=27t'+2,\ y=8t'-3.$$

The values of x and y having to be positive, these formulas show that t' ought also to be positive, and large enough to cause $8t'>3$, or $t'>\frac{3}{8}$. We may then give to t' all the values $t'=1, 2, 3,$ &c., to infinity; and we form, consequently, the table,

$$t'=\ 1,\ 2,\ 3,\ 4,\ \text{\&c.}$$
$$x=29,\ 56,\ 83,\ 110,\ \text{\&c.}$$
$$y=\ 5,\ 13,\ 21,\ 29,\ \text{\&c.}$$

The problem admits of an infinite number of solutions; and the answer is, that there are 29 hares and 5 sheep, or 56 hares and 13 sheep, or 83 hares and 21 sheep, &c.

171. PROBLEM IV.—*To find a number such that, in dividing it by* 11, *there remains* 3, *and dividing it by* 17, *there remains* 10.

Let the number be represented by N, then

$$N=11x+3 \text{ and } N=17y+10$$

$$\therefore\ 11x+3=17y+10 \quad \ldots\ldots\ (6)$$

Proceeding as before,

$$x=\frac{17y+7}{11}=y+\frac{6y+7}{11}=y+t$$

$$6y+7=11t$$

$$y=\frac{11t-7}{6}=2t-1-\frac{t+1}{6}=2t-1-t'$$

$$t+1=6t'$$

$$t=6t'-1.$$

The hypothesis $t'=0$ gives $t=-1$, $y=-3$, $x=-4$; and then we conclude immediately that

$$x=17t'-4,\ y=11t'-3.$$

We can not take t' negative, nor even $t'=0$, because x and y would become negative; but we may take $t'=1, 2, 3,$ &c., to infinity.

If we wish formulas in which we can give to the indeterminate all entire positive values setting out from zero, all that is necessary is to change t' into $1+\theta$, θ being the new indeterminate. Then we have

$$x=13+17\theta,\ y=8+11\theta.$$

By means of these values, we find

$$N=11x+\ 3=11(13+17\theta)+\ 3=146+187\theta$$
$$N=17y+10=17(\ 8+11\theta)+10=146+187\theta.$$

These two expressions are equal, and they should be, since equation (6) has been formed by equating the values of N. We perceive that there is an infinity of numbers which fulfill the two conditions enunciated, and that they are all represented by the formula

$$N=146+187\theta,$$

in which θ is an indeterminate, which may receive all positive values beginning with zero.

It is easy to show that this number N satisfies the enunciation; that is to say, that if we divide it by 11, the remainder will be 3, and if by 17, the remainder will be 10; for we have

$$\frac{N}{11}=17\theta+13+\frac{3}{11}, \text{ and } \frac{N}{17}=11\theta+8+\frac{10}{17}.$$

172. Problem V.—*To find a number such that, dividing it by* 11, *there remains* 3; *dividing by* 17, *there remains* 10; *and dividing it by* 37, *there remains* 13.

In the preceding problem we have found the numbers which fulfill the first two conditions. Putting x for θ, which we may do, since θ can be any positive whole number, this formula becomes

$$N=146+187x \quad \ldots\ldots\ldots \quad (8)$$

But in order that the number N may fulfill the third condition, we must have $N=37y+13$. Then we have the equation

$$37y+13=146+187x.$$

Then

$$y=\frac{187x+133}{37}=5x+3+\frac{2x+22}{37}=5x+3+2t$$

$$x+11=37t$$

$$x=37t-11.$$

In order that x may be positive, we must give to t only positive values above zero. But in making $t=1+\theta$, we can attribute to θ all the entire positive values beginning by zero. By this change x becomes

$$x=26+37\theta.$$

And by substituting this value in formula (8), we obtain

$$N=5008+6919\theta.$$

Such is the general formula of the numbers which satisfy the three conditions enunciated.

173. The determination of the limits led to the necessity of finding (165) the values of the final indeterminate t, which render positive expressions of the form $A+bt$, or, in other terms, which are such as to make

$$A+bt>0.$$

Transposing the term A,

$$bt>-A.$$

If b is positive, dividing by b,

$$t>-\frac{A}{b}.$$

But if b is negative, the division by b changes the signs of the inequality, and the two members are unequal in the contrary sense; *i. e.*,

$$t<-\frac{A}{b}.$$

Suppose, more generally, that we have the inequality

$$at+b>ct+d.$$

By the transposition of the terms,

$$(a-c)t>d-b.$$

Then, according as $a-c$ is a positive or negative quantity, we derive

$$t>\frac{d-b}{a-c}, \text{ or } t<\frac{d-b}{a-c}.$$

This process is called resolution of inequalities. The whole subject of inequalities will be found treated in a subsequent article.

174. Resolution in whole numbers of several equations of the first degree, when the number of equations is less than that of the unknown quantities.

Let there be for resolution the equations

$$2x+14y-7z=341 \ldots\ldots (1)$$
$$10x+\ 4y+9z=473 \ldots\ldots (2)$$

If we multiply the first equation by 5, and afterward subtract the second, we shall have

$$66y-44z=1232.$$

Or, dividing by 22,

$$3y-2z=56 \ldots\ldots\ldots\ldots (3)$$

But the entire values of y and z, which suit the proposed equations, ought also to satisfy this; consequently, applying to it the method already known, we have

$$y=2t,\ z=3t-28.$$

If we had but equation (3), we should have its solutions in whole numbers, by giving to t all the whole-number values possible. But this equation takes the place of only one of the proposed, so that it is necessary that the values of y and z should be such that, in adding to them certain values of x, which must also be entire, one of these proposed equations shall be verified. For this reason we substitute the preceding values of y and z in equation (1), and seek for the entire values of x and t, which belong to the resulting equation.

The substitution gives

$$2x+7t=145;$$

and from this we obtain, designating by t' any whole number whatever,

$$x=69+7t',\ t=1-2t'.$$

Then place the value $t=1-2t'$ in those of y and z, and you find the unknown quantities x, y, z expressed in terms of t', to wit:

$$x=69+7t',\ y=2-4t',\ z=-25-6t'.$$

These formulas make known all the entire values which satisfy the equations proposed.

If it be desired besides that these values should be positive, t must be so chosen that

$$69+7t'>0, \text{ whence } t'>-9\tfrac{6}{7};$$
$$2-4t'>0, \text{ whence } t'<\ \ \tfrac{1}{2};$$
$$-25-6t'>0, \text{ whence } t'<-4\tfrac{1}{6}.$$

From this we find the only values which can be attributed to t' are $t'=-5$, -6, -7, -8, -9. By substituting these numbers, we shall have five solutions in positive whole numbers:

$$x=34,\ 27,\ 20,\ 13,\ \ 6$$
$$y=22,\ 26,\ 30,\ 34,\ 38$$
$$z=\ 5,\ 11,\ 17,\ 23,\ 29.$$

175. The preceding example shows sufficiently the method to be pursued in resolving equations of the first degree in positive whole numbers, when the number of equations exceeds that of the unknown quantities. But, to leave nothing to be desired, I shall indicate the method to be pursued in the case of three equations.

Let there be, then, between the unknowns x, y, z, u three equations of the 1st degree, which I will name collectively the equations [A].

By the elimination of x we shall find between y, z, and u two equations of the 1st degree : I shall name them [B].

By the elimination of y we shall deduce from these last an equation of the 1st degree between z and u: I shall name it [C].

From the equation [C] we derive z and u expressed in function of an auxiliary indeterminate t.

These values being substituted in one of the equations [B], we derive from it an equation between y and t, and from this the values of y and t in function of a new indeterminate t'; consequently, we can also express z and u in terms of t'.

Finally, these values of y, z, u being carried into one of the equations [A], there will result an equation between x and t', which will enable us to find x and t', and, consequently, y, z, and u, in function of a new indeterminate t''.

When the equation is to be resolved in whole numbers of any sign whatever, we may attribute to the final indeterminate t'' all possible values in whole numbers. But when the solutions are to be restricted to such as are at the same time entire and positive, there will exist for t'' limitations which it will be always easy to assign.

176. When we have two more unknowns than equations, or several more, the indetermination is still greater; but the condition of having values which shall be at the same time entire and positive, may limit considerably the number of solutions. We shall confine ourselves to two examples, which will suffice to show how the method explained above should be modified in such cases.

Given to resolve in positive whole numbers the equation

$$10x+9y+7z=58 \quad . \; . \; . \; . \; . \; . \; . \quad (4)$$

As the unknown z has the smallest coefficient, I derive

$$z=\frac{58-9y-10x}{7};$$

and, effecting the division as far as possible,

$$z=8-y-x+\frac{2-2y-3x}{7}.$$

The numerator $2-2y-3x$ must be a whole number, divisible by 7 ; therefore I place

$$2-2y-3x=7t;$$

$$\therefore y=\frac{2-3x-7t}{2}=1-x-3t-\frac{x+t}{2};$$

and, $x+t$ being obliged to be a whole number divisible by 2, I place, also,

$$x+t=2t' \;\therefore\; x=-t+2t';$$

and, going back to y and z, we express these unknowns in function of t and t'. We have thus the three formulas

$$x=-t+2t', \; y=1-2t-3t', \; z=7+4t+t' \quad . \; . \; . \; . \quad (5)$$

In order to have the entire and positive solutions of the proposed equation (4), we must give to t and t' all the entire values, which satisfy simultaneously the three conditions

$$-t+2t'>0,\ 1-2t-3t'>0,\ 7+4t+t'>0 \quad . \ . \ . \ . \quad (6)$$

From hence result limitations for t and t', which will be discovered by employing for these inequalities operations altogether analogous to those of elimination. For greater neatness, suppose the signs $>$ exclude equality; that is to say, that none of the three unknowns, x, y, and z, can be zero.

First, if we multiply the 1st by 3 and the 2d by 2, they become

$$-3t+6t'>0,\ 2-4t-6t'>0;$$

adding, t' disappears, and we have

$$2-7t>0 \ \therefore t<\tfrac{2}{7}.$$

A similar elimination between the second inequality and the third gives

$$22+10t>0 \ \therefore t>-2\tfrac{1}{5}.$$

We see that the indeterminate t is embraced between the limits $-2\frac{1}{5}$ and $\frac{2}{7}$; then we should take only

$$t=-2,\ -1,\ 0.$$

Let us consider each of these values successively.

1°. If we make $t=-2$ in the three inequalities (6), they become

$$2+2t'>0,\ 5-3t'>0,\ -1+t'>0;$$

$$\therefore t'>-1,\ t'<1\tfrac{2}{3},\ t'>1.$$

As there is no whole number between 1 and $1\frac{2}{3}$, it follows that the value $t=-2$, which furnishes these limits for t', ought to be rejected.

2°. If we make $t=-1$, the three inequalities (6) become

$$1+2t'>0,\ 3-3t'>0,\ 3+t'>0;$$

$$\therefore t'>-\tfrac{1}{2},\ t'<+1,\ t'>-3.$$

Between $-\frac{1}{2}$ and $+1$ there is no other entire number except 0; then we can take $t=-1$ and $t'=0$.

3°. If we make $t=0$, the inequalities become

$$2t'>0,\ 1-3t'>0,\ 7+t'>0;$$

$$\therefore t'>0,\ t'<\tfrac{1}{3},\ t'>-7.$$

Between 0 and $\frac{1}{3}$ there is no whole number; consequently, the value $t=0$ ought also to be rejected.

The only values of t and t' to which positive values in whole numbers of x, y, and z correspond are, then, $t=-1$ and $t'=0$. By substituting them in the formulas (5), we obtain

$$x=1,\ y=3,\ z=3,$$

and this solution is the only one admissible.

177. For a second example, I propose the two equations

$$6x+\ 7y+3z+2u=100$$
$$24x+12y+7z+3u=200.$$

Eliminating u, we have

$$30x+3y+5z=100.$$

As in this equation the terms $30x$ and 100 are divisible by 5, it will be best to take the value of z: this is

$$z=20-6x-\frac{3y}{5}.$$

From which we see that y ought to be a multiple of 5; consequently, we have

$$y=5t$$
$$z=20-6x-3t;$$

then, by substituting these values in the first of the two proposed equations it becomes

$$6x+35t+60-18x-9t+2u=100;$$

or, rather,

$$-12x+26t+2u=40;$$
$$\therefore\ u=20+6x-13t.$$

The three unknowns, y, z, u, are thus found expressed in functions of x, and of the indeterminate auxiliary t.

In order to resolve the two proposed equations in positive numbers, it is evidently necessary to take x and t positive, since x is one of the primitive unknowns, and since $y=5t$. But it is necessary to satisfy also the inequalities

$$20-6x-3t>0,\quad 20+6x-13t>0.$$

In adding them, x disappears, and there remains

$$40-16t>0\ \therefore\ t<2\tfrac{1}{2};$$

then the values which we ought to give to t are $t=0$, 1, 2.

With the value $t=0$ we should have

$$y=0,\ z=20-6x,\ u=20+6x;$$

and we see that we can make $x=0$, 1, 2, 3. From whence result for the proposed equations

$$\begin{cases}x=0\\y=0\\z=20\\u=20\end{cases}\qquad\begin{cases}x=1\\y=0\\z=14\\u=26\end{cases}\qquad\begin{cases}x=2\\y=0\\z=8\\u=32\end{cases}\qquad\begin{cases}x=3\\y=0\\z=2\\u=38.\end{cases}$$

With the value $t=1$ we should have

$$y=5,\ z=17-6x,\ u=7+6x;$$

and the only admissible values of x are $x=0$, 1, 2. Thence result the three solutions

$$\begin{cases}x=0\\y=5\\z=17\\u=7\end{cases}\qquad\begin{cases}x=1\\y=5\\z=11\\u=13\end{cases}\qquad\begin{cases}x=2\\y=5\\z=5\\u=19.\end{cases}$$

Finally, with the value $t=2$ we should have

$$y=10,\ z=14-6x,\ u=-6+6x.$$

The only admissible values of x are $x=1$, 2; and from thence result the two further solutions

$$\begin{cases}x=1\\y=10\\z=8\\u=0\end{cases}\qquad\begin{cases}x=2\\y=10\\z=2\\u=6.\end{cases}$$

In all, nine solutions. There would be but three if those were excluded in which one of the unknowns is zero.

EXAMPLES.

1°. Two countrymen have together 100 eggs. The one says to the other, If I count my eggs by eights, there is a surplus of 7. The second answers, If I count mine by tens, I find the same surplus of 7. How many eggs had each?

Ans. Number of eggs of the first, $=63$ or 23; of the second, $=37$ or 77.

2°. To find three whole numbers such that, if we multiply the first by 3, the second by 5, and the third by 7, the sum of the products shall be 560;

and such, moreover, that if the first be multiplied by 9, the second by 25, and the third by 49, the sum of the products shall be 2920.

Ans. First number, $=15$ or 50.
Second number, $=82$ or 40.
Third number, $=15$ or 30.

3°. A person purchased 100 animals at 100 dollars; sheep at $3\frac{1}{2}$ dollars a piece; calves at $1\frac{1}{3}$ dollars; and pigs at $\frac{1}{2}$ a dollar. How many animals had he of each kind?

Ans. Sheep, 5, 10, 15.
Calves, 42, 24, 6.
Pigs, 53, 66, 79.

4°. In a foundry two kinds of cannon are cast; each cannon of the first sort weighs 1600 lbs., and each of the second 2500 lbs.; and yet for the second there are used 100 lbs. of metal less than for the first. How many cannons are there of each kind?

Ans. Of the first, 11, 36...; of the second, 7, 23....
Or, of the first, $11+25t$; of the second, $7+16t$.

5°. A farmer purchased 100 head of cattle for 4000 francs, to wit: oxen at 400 francs apiece, cows at 200, calves at 80, and sheep at 20. How many had he of each?

Ans. In excluding the solutions which contain a zero the problem admits of the ten following:

Oxen,	1,	1,	1,	1,	1,	1,	1,	1,	4,	4.
Cows,	1,	2,	3,	4,	5,	6,	7,	8,	1,	2.
Calves,	24,	21,	18,	15,	12,	9,	6,	3,	5,	2.
Sheep,	74,	76,	78,	80,	82,	84,	86,	88,	90,	92.

QUADRATIC EQUATIONS.

178. *Quadratic equations*, or *equations of the second degree*, are divided into two classes.

I. Equations which involve the square only of the unknown quantity. These are termed *incomplete* or *pure quadratics*. Of this description are the equations

$$ax^2=b\,;\ 3x^2+12=150-x^2\,;\ \frac{x^2}{3}-\frac{5}{12}+3x^2=\frac{7}{24}+2x^2+\frac{259}{24}\,;$$

they are sometimes called *quadratic equations of two terms*, because, by transposition and reduction, they can always be exhibited under the general form

$$ax^2=b.$$

Thus the third of the equations given above,

$$\frac{x^2}{3}-\frac{5}{12}+3x^2=\frac{7}{24}+2x^2+\frac{259}{24},$$

when cleared of fractions, becomes

$$8x^2-10+72x^2=7+48x^2+259,$$

or, transposing and reducing,

$$32x^2=276,$$

which is of the form

$$ax^2=b.$$

II. Equations which involve both the square and the simple power of the unknown quantity. These are termed *adfected* or *complete quadratics*. Of this description are the equations

$$ax^2+bx=c\,;\ x^2-10x=7\,;\ \frac{5x^2}{6}-\frac{x}{2}+\frac{3}{4}=8-\frac{2x}{3}-x^2+\frac{273}{12}\,;$$

they are sometimes called *quadratic equations of three terms*, because, by transposition and reduction, they can always be exhibited under the general form

$$ax^2+bx=c.$$

Thus, the third of the equations given above,

$$\frac{5x^2}{6}-\frac{x}{2}+\frac{3}{4}=8-\frac{2x}{3}-x^2+\frac{273}{12},$$

when cleared of fractions, becomes

$$10x^2-6x+9=96-8x-12x^2+273,$$

or, transposing and reducing,

$$22x^2+2x=360,$$

which is of the form

$$ax^2+bx=c.$$

SOLUTION OF PURE QUADRATICS CONTAINING ONE UNKNOWN QUANTITY.

179. The solution of the equation

$$ax^2=b$$

presents no difficulty. Dividing each member by a, it becomes

$$x^2=\frac{b}{a},$$

whence

$$x=\pm\sqrt{\frac{b}{a}}.$$

If $\frac{b}{a}$ be a particular number, either integral or fractional, we can extract its square root, either exactly or approximately, by the rules of arithmetic. If $\frac{b}{a}$ be an algebraic expression, we must apply to it the rules established for the extraction of the square root of algebraic quantities.

It is to be remarked, that since the square both of $+m$ and $-m$ is $+m^2$, so, in like manner, both $\left(+\sqrt{\frac{b}{a}}\right)^2$ and $\left(-\sqrt{\frac{b}{a}}\right)^2$ is $+\frac{b}{a}$. Hence the above equation is susceptible of two solutions, or has *two roots*; that is, there are two quantities which, when substituted for x in the original equation, will render the two members identical; these are

$$x=+\sqrt{\frac{b}{a}}\ \text{and}\ x=-\sqrt{\frac{b}{a}};$$

for, substitute each of these values in the original equation $ax^2=b$, it becomes

$$a\times\left(+\sqrt{\frac{b}{a}}\right)^2=b,\ \text{or}\ a\times\frac{b}{a}=b,\ \text{i. e.,}\ b=b,$$

and

$$a\times\left(-\sqrt{\frac{b}{a}}\right)^2=b,\ \text{or}\ a\times\frac{b}{a}=b,\ \text{i. e.,}\ b=b.$$

Hence it appears that in pure quadratics the two values of the unknown quantity are equal with contrary signs.*

EXAMPLE I.

Find the values of x which satisfy the equation

$$4x^2-7=3x^2+9.$$

Transposing and reducing, $x^2=16$

$$\therefore x=\pm\sqrt{16}$$
$$=\pm 4;$$

hence the two values of x are $+4$ and -4, and either of these, if substituted for x in the original equation, will render the two members identical.

EXAMPLE II.

$$\frac{x^2}{3}-3+\frac{5x^2}{12}=\frac{7}{24}-x^2+\frac{299}{24}.$$

Clearing of fractions, $8x^2-72+10x^2=7-24x^2+299$

Transposing and reducing, $42x^2=378$

$$x^2=\frac{378}{42}$$
$$=9$$
$$\therefore x=\pm 3,$$

and the two values of x are $+3$ and -3.

EXAMPLE III.

$$3x^2=5$$
$$x^2=\frac{5}{3}$$
$$x=\pm\sqrt{\frac{5}{3}}$$
$$=\frac{\pm\sqrt{15}}{3}$$

Since 15 is not a perfect square, we can only approximate to the two values of x. We find the approximate values to be

$$x=1.290994, \text{ or } -1.290994.$$

EXAMPLE IV.

$$\frac{x}{\sqrt{r^2+x^2}-x}=m.$$

Clearing of fractions, $x=m\sqrt{r^2+x^2}-mx,$

$$\therefore (m+1)x=m\sqrt{r^2+x^2}.$$

Squaring, $(m^2+2m+1)x^2=m^2(r^2+x^2),$

$$\therefore (2m+1)x^2=m^2r^2$$
$$x=\pm\frac{mr}{\sqrt{2m+1}}.$$

* One might suppose that in extracting the square root of both members of such an equation as $x^2=b$, the double sign should be prefixed to x, the root of x^2, also. But it is to be observed, that it is the value of $+x$ that is required. Besides, suppose we were to write $\pm x=\pm\sqrt{b}$; combining these signs in all possible ways, there result the four equations,

$$+x=+\sqrt{b},\ +x=-\sqrt{b},\ -x=+\sqrt{b},\ -x=-\sqrt{b},$$

the last two of which may be deduced from the first two by changing the signs of the two members; the equation $\pm x=\pm\sqrt{b}$ expresses nothing more, therefore, than the equation $x=\pm\sqrt{b}$. We might always omit $\pm$, since it is implied before $\sqrt{\ }$.

EXAMPLE V.

$$\frac{m+x+\sqrt{2mx+x^2}}{m+x-\sqrt{2mx+x^2}}=n.$$

Render the denominator rational by multiplying both terms of the fraction by the numerator, the equation then becomes

$$\frac{(m+x+\sqrt{2mx+x^2})^2}{m^2}=n.$$

Extracting the root,

$$m+x+\sqrt{2mx+x^2}=\pm m\sqrt{n}.$$

Transposing, $\sqrt{2mx+x^2}=\pm m\sqrt{n}-(m+x)$.

Squaring, $2mx+x^2=m^2n\mp 2m\sqrt{n}(m+x)+(m+x)^2$.

Transposing and reducing,

$$\pm 2m\sqrt{n}(m+x)=m^2(1+n),$$

$$\therefore m+x=\frac{m(1+n)}{\pm 2\sqrt{n}}$$

$$x=\frac{m(1+n)}{\pm 2\sqrt{n}}-m$$

$$=\pm m\cdot\frac{(\sqrt{n}\pm 1)^2}{2\sqrt{n}}.$$

(6) $11(x^2-4)=5(x^2+2)$. Ans. $x=\pm 3$.

(7) $\frac{x+7}{x^2-7x}-\frac{x-7}{x^2+7x}-\frac{7}{x^2-73}=0$. Ans. $x=\pm 9$.

(8) $\frac{m+\sqrt{m^2-x^2}}{x}=\frac{x}{n}$. Ans. $x=\pm\sqrt{2mn-n^2}$.

(9) $\frac{\sqrt{m^2-x^2}-\sqrt{n^2+x^2}}{\sqrt{m^2-x^2}+\sqrt{n^2+x^2}}=\frac{p}{q}$. Ans. $x=\pm\sqrt{\frac{m^2(p-q)^2-n^2(p+q)^2}{2(p^2+q^2)}}$.

(10) $\frac{\sqrt{p+x}+\sqrt{p-x}}{\sqrt{x}}-\sqrt{\frac{x}{q}}=0$. Ans. $x=\pm 2\sqrt{pq-q^2}$.

180. In the same manner we may solve all equations whatsoever, of any degree, which involve only one power of the unknown quantity; that is, all equations which are included under the general form

$$ax^n=b,$$

or equations of two terms.

For, dividing each member of the equation by a, it becomes

$$x^n=\frac{b}{a}.$$

Extracting the n^{th} root on both sides,

$$x=\sqrt[n]{\frac{b}{a}}.$$

If n be an even number, then the radical must be affected with the double sign $\pm$, for, in that case, both $\left(+\sqrt[n]{\frac{b}{a}}\right)^n$ and $\left(-\sqrt[n]{\frac{b}{a}}\right)^n$ will equally produce $\frac{b}{a}$.

EXAMPLE XI.

$$5x^6-57=2x^6+135$$
$$3x^6=192$$
$$x^6=64$$
$$x=\sqrt[6]{64}=\sqrt[3]{\sqrt{64}}=\sqrt[3]{\pm 8}=\pm 2.$$

Here $+2$ and -2 are two of the roots of the above equation

EXAMPLE XII.

$$\frac{\sqrt{p+x}}{p}+\frac{\sqrt{p+x}}{x}=\frac{\sqrt{x}}{q},$$
$$(p+x)\sqrt{p+x}=\frac{px\sqrt{x}}{q}.$$

Or, $$(p+x)^{\frac{3}{2}}=x^{\frac{3}{2}}.\frac{p}{q}.$$

Squaring, $$(p+x)^3=x^3.\frac{p^2}{q^2}.$$

Extracting the cube root, $$p+x=x\sqrt[3]{\frac{p^2}{q^2}}.$$

$$\therefore x=\frac{p}{\sqrt[3]{\frac{p^2}{q^2}}-1}.$$

EXAMPLE XIII.

$$\frac{p}{q}x^{\frac{p}{q}-1}=\frac{r}{s}x^{\frac{r}{s}-1}.$$ Ans. $x=\left(\frac{qr}{ps}\right)^{\frac{qs}{ps-qr}}.$

EXAMPLE XIV.

$$64y^6-48y^4+12y^2-1=64.$$

Extracting the cube root, we have

$$4y^2-1=4 \therefore y=\sqrt{\frac{5}{4}}=\frac{\sqrt{5}}{2}.$$

EXAMPLE XV.

$$x^3-y^3=117 \ldots\ldots\ldots\ldots (1)$$
$$x-y=\ 3 \ldots\ldots\ldots\ldots (2)$$

Cubing the latter equation,

$$x^3-3x^2y+3xy^2-y^3=27,$$

but $x^3 \quad -y^3=117.$

$\therefore$ by subtraction, $3x^2y-3xy^2=90,$

and $xy(x-y)=30;$

dividing by (2), we have $\therefore xy=10.$

Now from (2) $x^2-2xy+y^2=9,$

and $4xy=40.$

$\therefore$ by addition, $x^2+2xy+y^2=49,$

and $x+y=\pm 7,$

but (2) $x-y=3.$

By addition, $2x=10$, or -4,

$\therefore x=5$, or -2,

and by subtraction $2y=4$, or -10,

$\therefore y=2$, or -5.

(16) $4x^2-2=2x^2+26$. Ans. $x=\pm\sqrt{14}$.

(17) $x^2:(18-x)^2::25:16$. Ans. $x=10$.

(18) $\frac{x}{14-x}:\frac{14-x}{x}::16:9$. Ans. $x=8$.

(19) $\frac{75(x-7)}{x-4}=\frac{48(x-4)}{x-7}$. Ans. $x=19$.

(20) $x^2-xy=40$, $xy-y^2=15$. Ans. $x=\pm 8$, $y=\pm 3$.

(21) $(x-y)x=91$, $(x-y)^2=49$. Ans. $x=\pm 13$, $y=\pm 6$.

(22) $(x-y)\frac{x}{y}=24$, $(x-y)\frac{y}{x}=6$. Ans. $x=24$, or -8, $y=12$, or 4.

(23) $x^2y=48$, $xy^2=36$. Ans. $x=4$, $y=3$.

(24) $\frac{1}{2}xy=\sqrt{x^2+y^2}+x+y$, $x^2+y^2=(x+y)^2-\frac{1}{4}xy^2$. Ans. $x=6$, $y=8$.

(25) $\frac{x^3+y^3}{xy}=x+y$, $\frac{yx^2+y^2x}{xy}=4$. Ans. $x=2$, $y=2$.

(26) $x^a+y^a=a$, $x^a-y^a=b$.

(27) $x^5-5x^4+10x^3-10x^2+5x-1=32$. Ans. $x=3$.

(28) $t^8-2t^4-1=25$. Ans. $t=\pm\sqrt[4]{6}$.

(29) $\sqrt[4]{x}-\sqrt[4]{y}=3$, $\sqrt[4]{x}+\sqrt[4]{y}=7$. Ans. $x=625$, $y=16$.

(30) $x^4-y^4=369$, $x^2-y^2=9$. Ans. $x=\pm 5$, $y=\pm 4$.

(31) $x^3-y^3=56$, $x-y=\frac{16}{xy}$. Ans. $x=4$ or -2, $y=2$ or -4.

(32) $x^2y+y^2=116$, $xy^{\frac{1}{2}}+y=14$. Ans. $x=5$ or $2\sqrt{\frac{2}{5}}$, $y=4$ or 10.

(33) $\sqrt[3]{x}+\sqrt[3]{y}=6$, $x+y=72$. Ans. $x=64$ or 8, $y=8$ or 64.

(34) $x^{\frac{4}{3}}+y^{\frac{2}{5}}=20$, $x^{\frac{2}{3}}+y^{\frac{1}{5}}=6$. Ans. $x=\pm 8$ or $\pm\sqrt{8}$, $y=32$ or 1024.

(35) $x^4+2x^2y^2+y^4=1296-4xy(x^2+xy+y^2)$, $x-y=4$. Ans. $x=5$ or -1, $y=1$ or -5.

181. We have seen that an equation of the form $ax^2=b$ has *two roots*, or that there are two quantities which, when substituted for x in the original equation, will render the two members identical. In like manner, we shall find that every equation which involves x in the third power has *three roots*; an equation which contains x^4 has *four roots*; and it is a general proposition in the theory of equations that *an equation has as many roots as it has dimensions.*

182. The above method of solving the equation $ax^n=b$ will give us only *one* of the n roots of the equation if n be an odd number, and *two* roots if n be an even number. Such a solution must, therefore, be considered imperfect, and we must have recourse to different processes to obtain the remaining roots. This, however, is a subject which we must postpone for the present.

SOLUTION OF COMPLETE QUADRATICS, CONTAINING ONE UNKNOWN QUANTITY.

183. In order to solve the general equation

$$ax^2+bx=c,$$

let us begin by dividing both members by a, the coefficient of x^2; the equation then becomes

$$x^2+\frac{b}{a}x=\frac{c}{a},$$

or,

$$x^2+px=q,$$

putting, for the sake of simplicity,

$$\frac{b}{a}=p,\ \frac{c}{a}=q.$$

This form of the quadratic equation may be produced by multiplying together two simple equations. Suppose

$$x-a=0,\ x-b=0;$$
$$\therefore\ (x-a)(x-b)=0,$$

which is satisfied by making $x=a$, or $x=b$.

Multiplying the two factors $(x-a)$ and $(x-b)$, the equation becomes

$$x^2-(a+b)x+ab=0\ \ldots\ldots\ (1)$$

Substituting first a, and then b, for x, this may be written either

$$a^2-(a+b)a+ab=0,$$

or

$$b^2-(a+b)b+ab=0,$$

which are identical.

Putting in equation (1) above p, in place of $-(a+b)$, and $-q$ in place of ab, it assumes the form

$$x^2+px-q=0.$$

But

$$p^2=a^2+2ab+b^2$$
$$-4q=\qquad 4ab$$

By subtraction, $p^2+4q=a^2-2ab+b^2=(a-b)^2$;

$$\therefore\ a-b=\sqrt{p^2+4q}.$$
$$a+b=-p.$$

By addition and subtraction, $a=-\frac{1}{2}p+\frac{1}{2}\sqrt{p^2+4q}$

$$b=-\tfrac{1}{2}p-\tfrac{1}{2}\sqrt{p^2+4q}.$$

As a and b are the values of x, and differ only in the sign of the radical part, both may be written together thus:

$$x=-\tfrac{1}{2}p\pm\tfrac{1}{2}\sqrt{p^2+4q}.$$

Hence the following rule for resolving a complete or adfected quadratic equation.

Reduce the given equation to the form $x^2+px-q=0$ *by clearing of fractions, transposing all the terms to the first member, and dividing throughout by the coefficient of the square of the unknown quantity. The equation being thus prepared, the value of the unknown quantity will be equal to* $\frac{1}{2}$ *the coefficient of its first power with the sign changed,* $\pm\frac{1}{2}$ *the square root of the square of this coefficient* -4 *times the known terms of the equation.*

The expression $x=-\frac{1}{2}p\pm\frac{1}{2}\sqrt{p^2+4q}$ may, by passing the $\frac{1}{2}$ under the radical, be written $x=-\frac{1}{2}p\pm\sqrt{(\frac{1}{2}p)^2+q}$, which, translated into a rule, is often the more convenient form.

EXAMPLES.

(1) $x^2-\frac{11}{3}x+2=0.$

By the rule,

$$x=\frac{11}{6}\pm\tfrac{1}{2}\sqrt{\left(\frac{11}{3}\right)^2-4\times 2}=\frac{11}{6}\pm\tfrac{1}{2}\sqrt{\frac{121}{9}-8}=\frac{11}{6}\pm\tfrac{1}{2}\sqrt{\frac{49}{9}}=\frac{11}{6}\pm\frac{7}{6};$$

$$\therefore\ x=3\ \text{or}\ \frac{2}{3},$$

according as we use the upper or lower sign.

(2) $3x-x^2=2$; changing all the signs,

$x^2-3x=-2$, or $x^2-3x+2=0$.

By the rule,

$$x=\frac{3}{2}\pm\tfrac{1}{2}\sqrt{9-2\times 4}=2 \text{ or } 1.$$

Either of these values of x will satisfy the given equation. First substituting 2, we have

$$3\times 2-4=2;$$

and substituting 1, we have

$$3\times 1-1=2.$$

(3) $x^2+6x=16$.

By the form,

$$x=-\tfrac{1}{2}p\pm\sqrt{(\tfrac{1}{2}p)^2+q}$$

$$x=-3\pm\sqrt{9+16}=2 \text{ or } -8.$$

(4) $x^2-10x=-21$

$$x=5\pm\sqrt{25-21}$$

$$x=7 \text{ or } 3.$$

(5) $acx^2+bcx-adx-bd=0$.

Dividing by ac,

$$x^2+\left(\frac{b}{a}-\frac{d}{c}\right)x=\frac{bd}{ac}.$$

$\therefore$ by the rule,

$$x=-\tfrac{1}{2}\left(\frac{b}{a}-\frac{d}{c}\right)\pm\tfrac{1}{2}\sqrt{\left(\frac{b}{a}-\frac{d}{c}\right)^2+\frac{4bd}{ac}};$$

$$\therefore x=\frac{d}{c}, \text{ or } -\frac{b}{a}.$$

(6) $x^2+6x=27$. Ans. $x=3$, or -9.

(7) $x^2-7x+3\tfrac{1}{4}=0$ Ans. $x=6\tfrac{1}{2}$, or $\tfrac{1}{2}$.

(8) $x^2+\frac{10x}{3}=19$. Ans. $x=3$, or $-6\tfrac{1}{3}$.

(9) $x=\frac{5}{3}+\frac{x^2}{12}$. Ans. $x=10$, or 2.

(10) $x^2-6x+8=80$. Ans. $x=12$, or -6.

(11) $x^2-10x+17=1$ Ans. $x=8$, or 2.

(12) $x^2-x-40=170$. Ans. $x=15$, or -14.

(13) $3x^2-9x-4=80$ Ans. $x=7$, or -4.

(14) $7x^2-21x+13=293$. Ans. $x=8$, or -5.

(15) $\frac{x^2}{3}+\frac{4x}{5}-19=15\tfrac{1}{5}$. Ans. $x=9$, or $-\frac{57}{5}$.

(16) $\frac{2x^2}{3}+3\tfrac{1}{2}=\frac{x}{2}+8$. Ans. $x=3$, or $-\frac{9}{4}$.

(17) $x+4+\frac{7x-8}{x}=13$. Ans. $x=4$, or -2.

(18) $4u-\frac{36-u}{u}=46$. Ans. $u=12$, or $-\tfrac{3}{4}$.

(19) $16-\frac{5-\rho}{2}=\frac{9-3\rho}{\rho}+3\rho$. Ans. $\rho=6$, or $\tfrac{3}{5}$.

(20) $\frac{\psi+3}{2}+\frac{16-2\psi}{2\psi-5}=5\tfrac{1}{5}$. Ans. $\psi=5$, or $\frac{69}{10}$.

(21) $14+4\,\square-\frac{\square+7}{\square-7}=3\,\square+\frac{9+4\,\square}{3}$. Ans. $\square=9$, or 28.

(22) $\frac{7\Delta^2}{11}-\frac{2\Delta}{3}=\frac{11\Delta+18}{33}$. Ans. $\Delta=2$, or $-\frac{3}{7}$.

(23) $\frac{<+22}{3}-\frac{4}{<}=\frac{9<-6}{2}$. Ans. $<\,=2$, or $\frac{12}{25}$.

(24) $\frac{\phi}{\phi+1}+\frac{\phi+1}{\phi}=\frac{13}{6}$. Ans. $\phi=2$, or -3.

(25) $\frac{X}{X+60}=\frac{7}{3X-5}$. Ans. $X=14$, or -10.

(26) $\frac{8v}{v+2}-6=\frac{20}{3v}$. Ans. $v=10$, or $-\frac{2}{3}$.

(27) $\frac{48}{v+3}=\frac{165}{v+10}-5$. Ans. $v=5\frac{2}{5}$, or 5.

(28) $x^2-8x=14$. Ans. $x=9.4772+$, or $-1.4772+$.

(29) $3x^2+x=7$. Ans. $x=1.3699+$, or $-1.7032+$.

(30) $6x-30=3x^2$. Ans. $x=1\pm3\sqrt{-1}$.

(31) $(x-\sqrt{142.334})(x+\sqrt{142.334})=27.22x$.

Ans. $x=13.61\pm\sqrt{327.566}$.

(32) $23:(140+x)=(240+x):1041$. Ans. $x=-27.4$ or -352.6.

(33) $(x+6):(3x+12)=(3x-12):(x-6)$. Ans. $x=\pm\frac{\sqrt{54}}{2}$.

(34) $21z^2-1617z+20748=0$. Ans. $z=60.72$, or 16.27.

(35) $3.5g^2-11.75g-41.25=0$. Ans. $g=5.4$, or -2.11.

(36) $(3x+1)(4x-2)=(13x+7)(5x-3)$. Ans. $x=\frac{1\pm\sqrt{1008}}{53}$.

(37) $\frac{x}{x+60}-\frac{7}{3x-5}=0$. Ans. $x=14$, or -10.

(38) $\frac{w+4}{3}-\frac{7-w}{w-3}=\frac{4w+7}{9}-1$. Ans. $w=21$, or 5.

(39) $\frac{15-p}{4}-\frac{12-3p}{4p-5}=7p-\frac{23p+60}{7}$. Ans. $p=3$, or $\frac{229}{148}$.

(40) $\frac{x+11}{x}+\frac{9+4x}{x^2}=7$. Ans. $x=3$, or $-\frac{1}{2}$.

(41) $\frac{2q+9}{9}+\frac{4q-3}{4q+3}=3+\frac{3q-16}{18}$. Ans. $q=6$, or $-\frac{19}{4}$.

(42) $\frac{2x-1}{3-x}=\frac{8-x^2}{2x-2}+\frac{x}{2}$. Ans. $x=2$, or $-\frac{11}{3}$.

(43) $\frac{3}{6x-x^2}+\frac{6}{x^2+2x}=\frac{11}{5x}$. Ans. $x=3$, or $\frac{26}{11}$.

(44) $\frac{4x^2+7x}{19}+\frac{5x-x^2}{3+x}=\frac{4x^2}{9}$. Ans. $x=3$, or $-\frac{37}{10}$.

(45) $\frac{x^4+2x^3+8}{x^2+x-6}=x^2+x+8$. Ans. $x=4$, or $-\frac{14}{3}$.

(46) $cx-\frac{ac}{a+b}=(a+b)x^2$. Ans. $x=\frac{c\pm\sqrt{c^2-4ac}}{2(a+b)}$

(47) $(1+ax):(1-bx)=(1+bx):(1-ax)$. Ans. $x=\pm\sqrt{a^2-b^2}$

(48) $2(b-c)y\sqrt{2}+a^2=(b-c)^2+ay^2$.

Ans. $y=\dfrac{\sqrt{2}(b-c)\pm\sqrt{a^3+(2-a)(b-c)^2}}{a}$.

184. If $b=a$ in the general form $(x-a)(x-b)=0$, it assumes the particular form $(x-a)^2=x^2-2ax+a^2=0$.

If the two values of x be $+a$ and $-a$, the form $(x-a)(x+a)=x^2-a^2=0$.

185. Recollecting that the value of the unknown quantity is called the *root* of the equation, it is seen that every equation of the second degree has two roots. and, by the general form (1), $x^2-(a+b)x+ab=0$, that their sum is equal to the coefficient of the second term with the contrary sign, and that their product is equal to the absolute term or known quantity, when transposed to the first member. Thus, in Example 4, above, the sum of the two roots 3 and -9 is -6, and the product -27. The same may be seen in other examples.

The general form $ax^2+bx=c$ is capable of producing all the particular forms by the supposition of particular values for the coefficients. Thus, if $b=0$, it assumes the form of pure equations. If $c=0$, it may be written

$$x(ax+b)=0,$$

which we perceive may be verified by making $x=0$, or $ax+b=0 \therefore x=-\frac{b}{a}$.

The roots are, therefore, in this case, 0 and $-\frac{b}{a}$. Whenever an equation is divisible throughout by the unknown quantity, one of its roots is zero.

When we know that the two roots of the equation of the second degree are real, the above relations make known at once the nature of these roots; for example, admitting that those of the equation $x^2-2x-7=0$ are real, we conclude immediately that they are of different signs, because their product is equal to the absolute term -7, and, moreover, that the greater is positive, because their sum is $+2$, the coefficient of x taken with the contrary sign.

186. Another mode of solution may be derived as follows:

If we can, by any transformation, render the first member of the equation $x^2+px=q$ the perfect square of a binomial, a simple extraction of the square root will reduce the equation in question to a simple equation.

But $(x+\frac{1}{2}p)^2$ is $x^2+px+\frac{1}{4}p^2$.

In order, therefore, that the first member may be transformed to a perfect square, we must add to it the square of $\frac{1}{2}p$; that is, *the square of half the coefficient of the second term, or simple power of* x; it thus becomes

$$x^2+px+\frac{p^2}{4},$$

which is the square of $x+\frac{p}{2}$. But since we have added $\frac{p^2}{4}$ to the left-hand member of the equation, in order that the equality between the two members may not be destroyed we must add the same quantity to the right-hand member also; the equation thus transformed will be

$$x^2+px+\frac{p^2}{4}=\frac{p^2}{4}+q,$$

or

$$\left(x+\frac{p}{2}\right)^2=\frac{p^2}{4}+q.$$

Extracting the root, $x+\frac{p}{2}=\pm\sqrt{\frac{p^2}{4}+q}.$

Transposing, $x=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}+q}$

$$=\frac{-p\pm\sqrt{p^2+4q}}{2},$$

the same form for the value of x as we obtained by the first method.

We affix the sign $\pm$ to $\sqrt{\frac{p^2}{4}+q}$, because the square both of $+\sqrt{\frac{p^2}{4}+q}$, and also of $-\sqrt{\frac{p^2}{4}+q}$, is $+\left(\frac{p^2}{4}+q\right)$, and every quadratic equation must, therefore, have two roots.

From what has just been said, we deduce the following general

RULE FOR THE SOLUTION OF A COMPLETE QUADRATIC EQUATION.

1. *Transpose all the known quantities, when necessary, to one side of the equation, arrange all the terms involving the unknown quantity on the other side, and reduce the equation to the form* $ax^2+bx=c$.

2. *Divide each side of the equation by the coefficient of* x^2.

3. *Add to each side of the equation the square of half the coefficient of the simple power of* x.

That member of the equation which involves the unknown quantity will thus be rendered a perfect square, and, extracting the root on both sides, the equation will be reduced to one of the first degree, which may be solved in the usual manner.

EXAMPLE I.

$$12x-210=205-3x^2+5.$$

Transposing and reducing,

$$3x^2+12x=420.$$

Dividing by the coefficient of x^2,

$$x^2+4x=140.$$

Completing the square by adding to each side the square of half the coefficient of the second term,

$$x^2+4x+4=140+4,$$

or

$$(x+2)^2=144.$$

Extracting the root, $x+2=\pm\sqrt{144}$

$$=\pm 12$$

$$\therefore x=-2\pm 12.$$

Hence

$$\begin{cases} x=-2+12=10 \\ x=-2-12=-14. \end{cases}$$

Either of these two numbers, when substituted for x in the original equation, will render the two members identical.

EXAMPLE II.

$$2x^2+34=20x+2.$$

Transposing and reducing,

$$2x^2-20x=-32.$$

Dividing by 2, $\qquad x^2-10x=-16.$

Completing the square,

$$x^2-10x+25=25-16,$$

or $\qquad (x-5)^2=9.$

Extracting the root, $\qquad x-5=\pm\sqrt{9}.$

$$x=5\pm 3.$$

Hence

$$\begin{cases} x=5+3=8 \\ x=5-3=2. \end{cases}$$

EXAMPLE III.

$$3x^2-2x=65.$$

Dividing by 3, $\qquad x^2-\frac{2}{3}x=\frac{65}{3}.$

Completing the square,

$$x^2-\frac{2}{3}x+\left(\frac{1}{3}\right)^2=\frac{65}{3}+\left(\frac{1}{3}\right)^2,$$

or

$$\left(x-\frac{1}{3}\right)^2=\frac{196}{9}.$$

$$\therefore x-\frac{1}{3}=\pm\sqrt{\frac{196}{9}}$$

$$=\pm\frac{14}{3}$$

$$x=\frac{1}{3}\pm\frac{14}{3}.$$

Hence

$$\begin{cases} x=\frac{1+14}{3}=5 \\ x=\frac{1-14}{3}=-4\frac{1}{3}. \end{cases}$$

EXAMPLE IV.

$$x^2+x-2=0.$$

Transposing, $\qquad x^2+x \quad =2.$

The coefficient of x in this case is 1; $\therefore$ in order to complete the square, we must add to each side $\left(\frac{1}{2}\right)^2$, or $\frac{1}{4}$.

$$\therefore x^2+x+\frac{1}{4}=2+\frac{1}{4}$$

$$\left(x+\frac{1}{2}\right)^2=\frac{9}{4}$$

$$x+\frac{1}{2}=\pm\frac{3}{2}$$

$$\therefore x=1, \text{ and } x=-2.$$

EXAMPLE V.

$$6x-30=3x^2.$$

Transposing, $-3x^2+6x=30.$

Changing the sign on both sides,

$$3x^2-6x=-30.$$

Dividing by 3, $x^2-2x=-10.$

Completing the square, $x^2-2x+1=1-10,$

or

$$(x-1)^2=-9.$$

$$\therefore\ x-1=\pm\sqrt{-9}.$$

Hence

$$\begin{cases} x=1+\sqrt{-9} \\ x=1-\sqrt{-9}. \end{cases}$$

In the above example, the values of x contain imaginary quantities, and the roots of the equation are, therefore, said to be impossible.

EXAMPLE VI.

$$\frac{5}{6}x^2-\frac{1}{2}x+\frac{3}{4}=8-\frac{2}{3}x-x^2+\frac{273}{12}.$$

Clearing of fractions,

$$10x^2-6x+9=96-8x-12x^2+273.$$

Transposing and reducing,

$$22x^2+2x=360.$$

Dividing both members by 22,

$$x^2+\frac{2}{22}x=\frac{360}{22}.$$

Adding $\left(\frac{1}{22}\right)^2$ to both members,

$$x^2+\frac{2}{22}x+\left(\frac{1}{22}\right)^2=\frac{360}{22}+\left(\frac{1}{22}\right)^2.$$

Extracting the root,

$$x+\frac{1}{22}=\pm\sqrt{\frac{360}{22}+\left(\frac{1}{22}\right)^2}$$

$$=\pm\sqrt{\frac{7921}{(22)^2}}$$

$$=\pm\frac{89}{22}.$$

Hence

$$\begin{cases} x=-\frac{1}{22}+\frac{89}{22}=4 \\ x=-\frac{1}{22}-\frac{89}{22}=-\frac{45}{11}. \end{cases}$$

EXAMPLE VII.

$$ax^2-\frac{ac}{a+b}=cx-bx^2.$$

Transposing, $(a+b)x^2-cx=\frac{ac}{a+b}.$

Dividing by $a+b$, $x^2-\frac{c}{a+b}\cdot x=\frac{ac}{(a+b)^2}$.

Completing the square,

$$x^2-\frac{c}{a+b}\cdot x+\frac{c^2}{4(a+b)^2}=\frac{ac}{(a+b)^2}+\frac{c^2}{4(a+b)^2},$$

or

$$\left\{x-\frac{c}{2(a+b)}\right\}^2=\frac{c^2+4ac}{4(a+b)^2}.$$

Extracting the root,

$$x-\frac{c}{2(a+b)}=\pm\frac{\sqrt{c^2+4ac}}{2(a+b)}.$$

$$\therefore x=\frac{c\pm\sqrt{c^2+4ac}}{2(a+b)}.$$

The two values of x here are

$$x=\frac{c+\sqrt{c^2+4ac}}{2(a+b)},\ x=\frac{c-\sqrt{c^2+4ac}}{2(a+b)}.$$

EXAMPLE VIII.

$$a^2+b^2-2bx+x^2=\frac{m^2x^2}{n^2}.$$

Transposing, $(n^2-m^2)x^2-2bn^2x=-n^2(a^2+b^2)$.

Dividing by the coefficient of x^2,

$$x^2-\frac{2bn^2x}{n^2-m^2}=-n^2\cdot\frac{a^2+b^2}{n^2-m^2}.$$

Completing the square,

$$x^2-\frac{2bn^2x}{n^2-m^2}+\left(\frac{bn^2}{n^2-m^2}\right)^2=\left(\frac{bn^2}{n^2-m^2}\right)^2-\frac{n^2(a^2+b^2)}{n^2-m^2},$$

or

$$\left\{x-\frac{bn^2}{n^2-m^2}\right\}^2=\frac{n^2}{n^2-m^2}\left\{\frac{b^2n^2}{n^2-m^2}-(a^2+b^2)\right\}$$

$$=\frac{n^2}{(n^2-m^2)^2}\left\{m^2(a^2+b^2)-n^2a^2\right\}$$

Extracting the root,

$$x-\frac{bn^2}{n^2-m^2}=\pm\frac{n}{n^2-m^2}\sqrt{m^2(a^2+b^2)-n^2a^2}$$

$$x=\frac{n}{n^2-m^2}\left\{bn\pm\sqrt{m^2(a^2+b^2)-n^2a^2}\right\}$$

The two values of x are

$$x=\frac{n}{n^2-m^2}\left\{bn+\sqrt{m^2(a^2+b^2)-n^2a^2}\right\}$$

$$x=\frac{n}{n^2-m^2}\left\{bn-\sqrt{m^2(a^2+b^2)-n^2a^2}\right\}$$

(9) $x^2+4x=21$. Ans. $x=3$, $x=-7$.

(10) $x^2-9x+4\frac{1}{4}=0$. Ans. $x=8\frac{1}{2}$, $x=\frac{1}{2}$.

(11) $622x-15x^2=6384$. Ans. $x=22\frac{4}{5}$, $x=18\frac{2}{3}$.

(12) $8x^2-7x+34=0$. Ans. $x=\frac{7+\sqrt{-1039}}{16}$, $x=\frac{7-\sqrt{-1039}}{16}$.

(13) $3x^2+x=11$. Ans. $x=\frac{-1+\sqrt{133}}{6}$, $x=\frac{-1-\sqrt{133}}{6}$

(14) $\frac{x}{3}-4-x^2+2x-\frac{4x^2}{5}=45-3x^2+4x$.

Ans. $x=7.12\ldots$, $x=-5.73\ldots\ldots$

(15) $3x-\frac{6x^2-40}{2x-1}-\frac{3x-10}{9-2x}=2$. Ans. $x=\frac{23}{2}$, $x=4$.

(16) $\frac{90}{x}-\frac{90}{x+1}-\frac{27}{x+2}=0$. Ans. $x=4$, $x=-\frac{5}{3}$.

(17) $abx^2+\frac{3a^2x}{c}=\frac{6a^2+ab-2b^2}{c^2}-\frac{b^2x}{c}$. Ans. $x=\frac{2a-b}{ac}$, $x=-\frac{3a+2b}{bc}$.

(18) $mx^2-2mx\sqrt{n}=nx^2-mn$. Ans. $x=\frac{\sqrt{mn}}{\sqrt{m}+\sqrt{n}}$, $x=\frac{\sqrt{mn}}{\sqrt{m}-\sqrt{n}}$.

(19) $4a^2x^2+4a^2c^2x+4abd^2x-9cd^2x^2+(ac^2+bd^2)^2=0$.

Ans. $x=-\frac{ac^2+bd^2}{2a+3d\sqrt{c}}$, $x=-\frac{ac^2+bd^2}{2a-3d\sqrt{c}}$.

(20) $\frac{5a+10ab^2}{9b^2-3a^2b^2}x^2-\left(\frac{5\sqrt{a+b}}{3b^3}+\frac{(1+2b^2)cd\sqrt{c}}{3-a^2}\right)x+\frac{cd}{ab}\sqrt{(a+b)c}=0$.

Ans. $x=\frac{(3-a^2)\sqrt{a+b}}{ab(1+2b^2)}$, $x=\frac{3b^2cd\sqrt{c}}{5a}$.

187. The above rule will enable us to solve, not only quadratic equations, but all equations which can be reduced to the form

$$x^{2n}+px^n=q;$$

that is, all equations which contain only two powers of the unknown quantity, and in which one of these powers is double of the other.

For if, in the above equation, we assume $y=x^n$, then $y^2=x^{2n}$, and it becomes

$$y^2+py=q.$$

Solving this according to the rule,

$$y=\frac{-p\pm\sqrt{p^2+4q}}{2}.$$

Putting for y its value,

$$x^n=\frac{-p\pm\sqrt{p^2+4q}}{2}.$$

Extracting the nth root on both sides,

$$x=\sqrt[n]{\frac{-p\pm\sqrt{p^2+4q}}{2}}.$$

EXAMPLE I.

$$x^4-25x^2=-144.$$

Assume $x^2=y$, the above becomes

$$y^2-25y=-144.$$

Whence $y=16$, $y=9$.

But since $x^2=y\ \therefore x=\pm\sqrt{y}$;

$$\therefore x=\pm\sqrt{16},\ x=\pm\sqrt{9}.$$

Thus the four values of x are $+4$, -4, $+3$, -3.

EXAMPLE II.

$$x^4-7x^2=8.$$

Assume $x^2=y$, $\qquad y^2-7y=8.$

Whence $\qquad y=8, \quad y=-1$

And since $\qquad x^2=y \therefore x=\pm\sqrt{y}.$

Whence the four roots of the equation are $\pm\sqrt{8}$, $\pm\sqrt{-1}$, the last two of which are impossible roots.

EXAMPLE III.

Let $\qquad x^6-2x^3=48.$

Assume $x^3=y$, the above becomes

$$y^2-2y=48.$$

Whence $\qquad y=8, \text{ or } -6.$

But since $\qquad x^3=y \therefore x=\sqrt[3]{y}.$

Hence two of the roots of the above equation are $+\sqrt[3]{8}$ and $-\sqrt[3]{6}$; the remaining four roots can not be determined by this process.

EXAMPLE IV.

Let $\qquad 2x-7\sqrt{x}=99,$

or $\qquad 2x-7x^{\frac{1}{2}}=99.$

This equation manifestly belongs to this class, for the exponent of x in the first term is 1, and in the second term half as great, or $\frac{1}{2}$.

In this case assume $\sqrt{x}=y$, the equation becomes

$$2y^2-7y=99.$$

Whence $\qquad y=9, \quad y=-\frac{11}{2}.$

But since $\qquad \sqrt{x}=y \therefore x=y^2$

$$\therefore x=81, \quad x=\frac{121}{4}.$$

To account for the two values of x in this equation, it must be observed that one belongs to $+\sqrt{x}$, the other to $-\sqrt{x}$.

This will appear clearly in the following example.

EXAMPLE V.

$$ax=b+\sqrt{cx} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

Solving this equation in the same manner as the preceding, we shall find

$$x=\frac{2ab+c+\sqrt{4abc+c^2}}{2a^2}, \quad x=\frac{2ab+c-\sqrt{4abc+c^2}}{2a^2}.$$

If we substitute these two values of x in the original equation, we shall find that the first only will verify it; the second belongs to the equation

$$ax=b-\sqrt{cx} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

These two equations, multiplied together, produce the complete quadratic equation

$$a^2x^2-(2ab+c)x+b^2=0,$$

whose roots are the two values of x given above.

The explication of this matter is, that $\sqrt{x}$ is always supposed to have the double sign $\pm$, and therefore the general form expressed by equation (1) involves covertly that expressed by equation (2). It is necessary, therefore, in examples of this kind, to try the answers obtained, by substituting them, in order to see which belongs to the given form.

188. Many other equations of degrees higher than the second may be solved by completing the square; although, it must be remarked, we can seldom obtain *all* the roots in this manner. The transformations to which we subject equations of this nature, in order that the rule may become applicable, depend upon various algebraic artifices, for which no general rule can be given. The following examples will serve to give the student some idea of the course he must pursue; a little practice will soon render him dextrous in the employment of such devices.

EXAMPLE VI.

Let $$\sqrt{x+12}+\sqrt[4]{x+12}=6$$

Assume $x+12=y$, the equation then becomes

$$y^{\frac{1}{2}}+y^{\frac{1}{4}}=6,$$

which evidently belongs to the same class as the previous examples; completing the square, we shall have

$$y^{\frac{1}{4}}=2, \text{ or } -3.$$

Raising both sides of the equation to the power of 4,

$$y=16, \text{ or } 81$$

$$\therefore x, \text{ or } y-12= 4, \text{ or } 69.$$

EXAMPLE VII.

Let $$2x^2+\sqrt{2x^2+1}=11.$$

Add 1 to each member of the equation, it becomes

$$2x^2+1+\sqrt{2x^2+1}=12.$$

Assume $2x^2+1=y$, then

$$y+y^{\frac{1}{2}}=12.$$

Completing the square, and solving, we find

$$y^{\frac{1}{2}}, \text{ or } \sqrt{2x^2+1}=3, \text{ and } -4$$

$$2x^2+1=9, \text{ and } 16$$

$$x^2=4, \text{ and } \frac{15}{2}.$$

Hence $$x=+2, -2, +\sqrt{\frac{15}{2}}, -\sqrt{\frac{15}{2}}.$$

It may be remarked, that it is in general unnecessary to substitute y, which has been done in the above examples for the sake of perspicuity alone.

EXAMPLE VIII.

Let $$\left(x+\frac{8}{x}\right)^2+x=42-\frac{8}{x}.$$

Transposing $$\left(x+\frac{8}{x}\right)^2+\left(x+\frac{8}{x}\right)=42.$$

Considering $x+\frac{8}{x}$ as one quantity, and completing the square.

$$\left(x+\frac{8}{x}\right)^2+\left(x+\frac{8}{x}\right)+\frac{1}{4}=\frac{169}{4},$$

$$\therefore x+\frac{8}{x}=-\frac{1}{2}\pm\frac{13}{2}$$

$$=6, \text{ and } -7.$$

Hence we have the two equations

$$x^2-6x=-8$$

$$x^2+7x=-8.$$

Solving the first in the usual manner, we find

$$x=4, \text{ and } 2,$$

and by the second, we have

$$x=\frac{-7+\sqrt{17}}{2}, \text{ and } \frac{-7-\sqrt{17}}{2},$$

which are the four roots of the proposed equation. If we had reduced this equation by performing the operations indicated, instead of employing the above artifice, it would have become

$$x^4+x^3-26x^2+8x+64=0,$$

a complete equation of the fourth degree.

The roots of equations of the fourth degree, reducible to the second as above, present themselves ordinarily under the form $\sqrt{a\pm\sqrt{b}}$, and frequently afford an application of the process exhibited at (Art. 104).

(9) $x^4+4x^2=12$. Ans. $x=\pm\sqrt{2}$, or $\pm\sqrt{-6}$.

(10) $x^6-8x^3-513=0$. Ans. $x=3$, or $-\sqrt[3]{19}$.

(11) $x^4-13x^2+36=0$. Ans. $x=\pm 2$, $x=\pm 3$.

(12) $(x^2-2)^2=\frac{1}{4}(x^2+12)$. Ans. $x=\pm 2$, $x=\pm\frac{1}{2}$.

(13) $(x^2-1)(x^2-2)+(x^2-3)(x^2-4)=x^4+5$. Ans. $x=\pm 1$, $x=\pm 3$.

(14) $x^{2n}-mx^n=p$. Ans. $x=\left(\frac{m\pm\sqrt{m^2+4p}}{2}\right)^{\frac{1}{n}}$.

(15) $\frac{\sqrt{4x}+2}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}$. Ans. $x=4$.*

(16) $\frac{\sqrt{a^2x}+b}{a+\sqrt{x}}=\frac{a-\sqrt{x}}{\sqrt{x}}$. Ans. $x=\left(\frac{-b\pm\sqrt{4a^3+4a^2+b^2}}{2(a+1)}\right)^2$.

(17) $\sqrt{x^3}-2\sqrt{x}-x=0$. Ans. $x=4$.

(18) $\sqrt{x^5}+\sqrt{x^3}=6\sqrt{x}$. Ans. $x=2$.

(19) $\frac{x}{2}=22\frac{1}{6}+\frac{\sqrt{x}}{3}$. Ans. $x=49$.

(20) $\frac{\frac{3\sqrt{x}}{5}-2}{x-5}-\frac{1}{20}=0$. Ans. $x=25$.

(21) $x^{\frac{6}{5}}+x^{\frac{3}{5}}=756$. Ans. $x=243$, or $(-28)^{\frac{5}{3}}$.

(22) $x^3-x^{\frac{3}{2}}=56$. Ans. $x=4$, or $(-7)^{\frac{2}{3}}$.

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* In this and some of the following examples another value, $x=\frac{64}{9}$, is also found, but it will not satisfy the equation, and is, therefore, to be rejected. [See Ex. 5, p. 214.]

(23) $3x^{\frac{5}{3}}+x^{\frac{5}{6}}=3104$. Ans. $x=64$, or $\left(\frac{-97}{3}\right)^{\frac{6}{5}}$.

(24) $ax^{\frac{3}{2}}+bx^{\frac{3}{4}}=c$. Ans. $x=\left(\frac{\pm\sqrt{b^2+4ac}-b}{2a}\right)^{\frac{4}{3}}$.

(25) $3x^{\frac{4}{3}}-\frac{5x^{\frac{8}{3}}}{2}=-592$. Ans. $x=8$, or $\left(-\frac{74}{5}\right)^{\frac{3}{4}}$.

(26) $x^n-2ax^{\frac{n}{2}}=b$. Ans. $x=(a\pm\sqrt{a^2+b})^{\frac{2}{n}}$.

(27) $\frac{123+41\sqrt{x}}{5\sqrt{x}-x}=\frac{4(5\sqrt{x}+x)}{3-\sqrt{x}}-\frac{2x^2}{(5\sqrt{x}-x)(3-\sqrt{x})}$. Ans. $x=3$.

(28) $\frac{x}{\sqrt{x}+\sqrt{a-x}}+\frac{x}{\sqrt{x}-\sqrt{a-x}}=\frac{b}{\sqrt{x}}$. Ans. $x=\frac{b\pm\sqrt{b^2-2ab}}{2}$.

(29) $\frac{x+\sqrt{x^2-9}}{x-\sqrt{x^2-9}}=(x-2)^2$. Ans. $x=5$, or 3, or $\frac{8\pm\sqrt{-11}}{5}$.

(30) $x+5=\sqrt{x+5}+6$. Ans. $x=4$.

(31) $x+16-7\sqrt{x+16}=10-4\sqrt{x+16}$. Ans. $x=9$.

(32) $\sqrt{x+12}+\sqrt[4]{x+12}=6$. Ans. $x=4$.

(33) $x^2-2x+6\sqrt{x^2-2x+5}=11$. Ans. $x=1$, or $1\pm2\sqrt{15}$.

(34) $2x^2+3x-5\sqrt{2x^2+3x+9}+3=0$. Ans. $x=3$, or $-\frac{9}{2}$.

(35) $[(x-2)^2-x]^2-(x-2)^2=88-(x-2)$. Ans. $x=6$, or -1, or $\frac{5\pm3\sqrt{-3}}{2}$.

(36) $(x+6)^2+2x^{\frac{1}{2}}(x+6)=138+x^{\frac{1}{2}}$. Ans. $x=4$.

(37) $x-1=2+\frac{2}{x^{\frac{1}{2}}}$. Ans. $x=4$.

(38) $x^4-2x^3+x=132$. Ans. $x=4$, or -3, or $\frac{1\pm\sqrt{-43}}{2}$.

(39) $9x+\sqrt{16x^2+36x^3}=15x^2-4$. Ans. $x=\frac{4}{3}$, or $-\frac{1}{3}$, or $x=\frac{9\pm\sqrt{481}}{50}$.

(40) $x=\frac{12+8x^{\frac{1}{2}}}{x-5}$. Ans. $x=9$, or $\frac{-3\pm\sqrt{-7}}{2}$.

(41) $\frac{49x^2}{4}+\frac{48}{x^2}-49=9+\frac{6}{x}$. Ans. $x=2$, or $-\frac{8}{7}$, or $\frac{-3\pm\sqrt{93}}{7}$.

(42) $\frac{x^4}{2}+\frac{17x^3}{4}-17x=8$. Ans. $x=\pm2$, or -8, or $-\frac{1}{2}$.

(43) $\left(x^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}+\left(a^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}=\frac{x^2}{a}$. Ans. $x=\pm a\sqrt{\frac{1\pm\sqrt{5}}{2}}$.

(44) $x^4-(2bc+4a^2)x^2+b^2c^2=0$. Ans. $x=\pm\sqrt{bc+2a^2\pm2a\sqrt{bc+a^2}}$.

(45) $x^2-x+5\sqrt{2x^2-5x+6}=\frac{3x+33}{2}$. Ans. $x=\frac{5\pm\sqrt{1329}}{4}$, and $x=3$, and $-\frac{1}{2}$.

(46) $\frac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}=\frac{x}{a}$. Ans. $x=\frac{a}{8}(\pm\sqrt{-7}-3)$.

NOTE.—In some of the above examples we have given answers which will not satisfy the equation unless the double sign be understood before the radical. In some cases this sign is understood, in others not; but whether it is or not will always be known from the problem from which the equation is derived.

ON THE SOLUTION OF QUADRATIC EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

189. An equation containing two unknown quantities is said to be of the *second degree* when it involves terms in which *the sum of the exponents of the unknown quantities is equal to* 2, *but never exceeds* 2. Thus,

$$3x^2-4x+y^2-xy-5y+6=0,\ 7xy-4x+y=0,$$

are equations of the second degree.

It follows from this that every equation of the second degree containing two unknown quantities is of the form

$$ay^2+bxy+cx^2+dy+ex+f=0,$$

where a, b, c, ... represent known quantities, either numerical or algebraical; *i. e.*, the equation contains the second power of each of the unknown quantities, the first power of each, and the product of the two. Not that every equation of the second degree contains all these, but when any one of them is wanting the coefficient of that term, in the general form, is said to be zero.

Let it be required to determine the values of x and y, which satisfy the equations.

$$\left.\begin{array}{l} a\,y^2+b\,xy+c\,x^2+d\,y+e\,x+f\,=0 \ \ldots\ldots\ (1) \\ a'y^2+b'xy+c'x^2+d'y+e'x+f'=0 \ \ldots\ldots\ (2) \end{array}\right\}$$

Arranging these two equations according to the powers of y, they become

$$\left.\begin{array}{l} a\,y^2+(b\,x+d\,)y+(c\,x^2+e\,x+f\,)=0 \ \ldots\ldots \\ a'y^2+(b'x+d')y+(c'x^2+e'x+f')=0 \ \ldots\ldots \end{array}\right\}$$

Put $\qquad b\,x+d\,=h\,;\ \ c\,x^2+e\,x+f\,=k$

$\qquad\qquad b'x+d'=h'\,;\ \ c'x^2+e'x+f'=k'.$

$$\therefore\ a\,y^2+h\,y+k\,=0 \ \ldots\ldots\ldots\ (3)$$

$$a'y^2+h'y+k'=0 \ \ldots\ldots\ldots\ (4)$$

Multiply (3) and (4) by a' and a respectively, and also by k' and k; then

$$aa'y^2+a'hy+a'k=0 \ \ldots\ldots\ldots\ (5)$$

$$aa'y^2+ah'y+ak'=0 \ \ldots\ldots\ldots\ (6)$$

$$ak'y^2+hk'y+kk'=0 \ \ldots\ldots\ldots\ (7)$$

$$a'ky^2+h'ky+kk'=0 \ \ldots\ldots\ldots\ (8)$$

Subtracting (6) from (5), and also (7) from (8), we have

$$(a'h-ah')y+a'k-ak'=0 \ \ldots\ldots\ (9)$$

$$(a'k-ak')y+h'k-hk'=0 \ \ldots\ldots\ (10)$$

Multiplying (9) by $h'k-hk'$, and (10) by $a'k-ak'$, we have

$$(a'h-ah')(h'k-hk')\,y+(a'k-ak')(h'k-hk')=0 \ \ldots\ (11)$$

$$(a'k-ak')^2y+(a'k-ak')(h'k-hk')=0 \ \ldots\ (12)$$

$$\therefore\ (a'h-ah')(h'k-hk')=(a'k-ak')^2 \ \ldots\ldots\ (13)$$

Substituting the values of h, h', k, k' in equation (13), we have

$$\left\{(a'b-ab')x+a'd-ad'\right\}\,.\,\left\{(b'c-bc')x^3+(b'e-be'-c'd+cd')x^2+(b'f-bf'+d'e-de')x+d'f-df'\right\}$$
$$=\left\{(a'c-ac')x^2+(a'e-ae')x+a'f-af'\right\}^2$$

Hence, by multiplying and expanding, the final equation in x is of the fourth degree, which will, in general, be the degree of the equation obtained by eliminating between the two equations of the second degree; but the general form includes a variety of equations, according to the values of the coefficients a, b, c, &c.; when d, e, f, d', e', f' are each $=0$, the solution may be obtained by quadratics, the resulting equation in x being

$$\left\{(a'b-ab')x+a'd-ad'\right\}\,.\,\left\{(b'c-bc')x-(c'd-cd')\right\}=(a'c-ac')^2x^2.$$

Although the principles already established will not enable us to solve equations of this description *generally*, yet there are many particular cases in which they may be reduced either to pure or adfected quadratics, and the roots determined in the ordinary manner.

EXAMPLE I.

Required the values of x and y, which satisfy the equations,

$$\left\{\begin{array}{ll} x+y=p \dots\dots\dots\dots & (1) \\ xy=q^2 \dots\dots\dots\dots & (2) \end{array}\right\}$$

Squaring (1), $x^2+2xy+y^2=p^2 \dots\dots\dots (3)$

Multiply (2) by 4, $4xy=4q^2 \dots\dots\dots (4)$

Subtract (4) from (3), $x^2-2xy+y^2=p^2-4q^2$,

or $(x-y)^2=p^2-4q^2$.

Extract the root, $x-y=\pm\sqrt{p^2-4q^2} \dots\dots\dots (5)$

But by (1), $x+y=p$.

Add (1) to (5), $2x=p\pm\sqrt{p^2-4q^2}$.

Subtract (5) from (1), $2y=p\mp\sqrt{p^2-4q^2}$.

Hence the corresponding values of x and y will be

$$\left.\begin{array}{l} x=\dfrac{p+\sqrt{p^2-4q^2}}{2} \\ y=\dfrac{p-\sqrt{p^2-4q^2}}{2} \end{array}\right\} \text{ and } \left.\begin{array}{l} x=\dfrac{p-\sqrt{p^2-4q^2}}{2} \\ y=\dfrac{p+\sqrt{p^2-4q^2}}{2} \end{array}\right\}$$

EXAMPLE II.

$$\left\{\begin{array}{ll} x+y=a \dots\dots\dots\dots & (1) \\ x^2+y^2=b^2 \dots\dots\dots\dots & (2) \end{array}\right\}$$

Square (1), $x^2+2xy+y^2=a^2$.

But by (2), $x^2 \quad +y^2=b^2$.

Subtracting, $2xy \quad =a^2-b^2 \dots\dots\dots (3)$

Subtract (3) from (2), $x^2-2xy+y^2=2b^2-a^2$,

or $(x-y)^2=2b^2-a^2$.

Extracting the root, $x-y=\pm\sqrt{2b^2-a^2}$.

But by (1), $x+y=a$,

∴ adding and subtracting $2x=a\pm\sqrt{2b^2-a^2}$

$2y=a\mp\sqrt{2b^2-a^2}$.

Hence the corresponding values of x and y will be

$$\left.\begin{array}{l} x=\dfrac{a+\sqrt{2b^2-a^2}}{2} \\ y=\dfrac{a-\sqrt{2b^2-a^2}}{2} \end{array}\right\} \text{ and } \left.\begin{array}{l} x=\dfrac{a-\sqrt{2b^2-a^2}}{2} \\ y=\dfrac{a+\sqrt{2b^2-a^2}}{2} \end{array}\right\}$$

EXAMPLE III.

$$\left\{\begin{array}{ll} x+y=m \dots\dots\dots\dots & (1) \\ x^3+y^3=n^3 \dots\dots\dots\dots & (2) \end{array}\right\}$$

Cube (1), $x^3+3x^2y+3xy^2+y^3=m^3$.

But by (2), $x^3 \quad +y^3=n^3$.

Subtracting, $3x^2y+ \quad 3xy^2=m^3-n^3$,

or $3xy(x+y)=m^3-n^3$.

Substitute for $(x+y)$ its value aerived from (1),

$$3xy\,.\,m=m^3-n^3$$

$$\therefore\quad xy=\frac{m^3-n^3}{3m}$$

$$\therefore\ 4xy=\frac{4(m^3-n^3)}{3m}\quad\ldots\ldots\ldots\ldots\ (3)$$

Squaring (1), $\qquad x^2+2xy+y^2=m^2.$

But by (3) $\qquad 4xy\quad=\frac{4(m^3-n^3)}{3m}.$

Subtracting, $\qquad x^2-2xy+y^2=m^2-\frac{4(m^3-n^3)}{3m},$

or $\qquad (x-y)^2=\frac{4n^3-m^3}{3m}$

$$\therefore\ x-y\ =\pm\sqrt{\frac{4n^3-m^3}{3m}}.$$

But by (1), $\qquad x+y\ =m$

$$\therefore\ 2x=m\pm\sqrt{\frac{4n^3-m^3}{3m}}$$

$$2y=m\mp\sqrt{\frac{4n^3-m^3}{3m}}.$$

Hence the two corresponding values of x and y are

$$\left.\begin{array}{l} x=\frac{m}{2}+\sqrt{\frac{4n^3-m^3}{12m}} \\ y=\frac{m}{2}-\sqrt{\frac{4n^3-m^3}{12m}} \end{array}\right\}\ \text{and}\ \left.\begin{array}{l} x=\frac{m}{2}-\sqrt{\frac{4n^3-m^3}{12m}} \\ y=\frac{m}{2}+\sqrt{\frac{4n^3-m^3}{12m}} \end{array}\right\}$$

EXAMPLE IV.

$$\left\{\begin{array}{ll} x^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}}=a & \ldots\ldots\ldots\ldots\ (1) \\ x^3+x^{\frac{3}{2}}y^{\frac{3}{2}}+y^3=b & \ldots\ldots\ldots\ldots\ (2) \end{array}\right\}$$

Square (1), $x^3+x^{\frac{3}{2}}y^{\frac{3}{2}}+y^3+2x^{\frac{3}{2}}\,.\,x^{\frac{3}{4}}y^{\frac{3}{4}}+2x^{\frac{3}{2}}y^{\frac{3}{2}}+2y^{\frac{3}{2}}\,.\,x^{\frac{3}{4}}y^{\frac{3}{4}}=a^2.$

But by (2), $x^3+x^{\frac{3}{2}}y^{\frac{3}{2}}+y^3\qquad=b.$

Subtracting, $\qquad 2x^{\frac{3}{2}}\,.\,x^{\frac{3}{4}}y^{\frac{3}{4}}+2x^{\frac{3}{2}}y^{\frac{3}{2}}+2y^{\frac{3}{2}}x^{\frac{3}{4}}y^{\frac{3}{4}}=a^2-b,$

or $\qquad 2x^{\frac{3}{4}}y^{\frac{3}{4}}(x^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}})=a^2-b$

$$\therefore\ 2x^{\frac{3}{4}}y^{\frac{3}{4}}\,.\,a\qquad=a^2-b$$

$$\therefore\ x^{\frac{3}{4}}y^{\frac{3}{4}}\qquad=\frac{a^2-b}{2a}\ .\ .\ (3)$$

But by (1), $\qquad x^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}}=a.$

And by (3), $\qquad x^{\frac{3}{4}}y^{\frac{3}{4}}\qquad=\frac{a^2-b}{2a}.$

Adding, $\qquad x^{\frac{3}{2}}+2x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}}=a+\frac{a^2-b}{2a},$

or $\qquad (x^{\frac{3}{4}}+y^{\frac{3}{4}})^2=\frac{3a^2+b}{2a}$

$$\therefore\ x^{\frac{3}{4}}+y^{\frac{3}{4}}=\pm\sqrt{\frac{3a^2-b}{2a}}\quad\ldots\ldots\ldots\ldots\ (4)$$

Again, from (1), $x^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}}=a.$

And from (3), $3x^{\frac{3}{4}}y^{\frac{3}{4}}=\frac{3(a^2-b)}{2a}.$

Subtracting, $x^{\frac{3}{2}}-2x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}}=a-\frac{3(a^2-b)}{2a},$

or $\left(x^{\frac{3}{4}}-y^{\frac{3}{4}}\right)^2=\frac{3b-a^2}{2a}$

$$\therefore x^{\frac{3}{4}}-y^{\frac{3}{4}}=\pm\sqrt{\frac{3b-a^3}{2a}} \quad \ldots\ldots\ldots\ldots (5)$$

But by (4), $x^{\frac{3}{4}}+y^{\frac{3}{4}}=\pm\sqrt{\frac{3a^2-b}{2a}}$

$\therefore$ adding and subtracting,

$$x^{\frac{3}{4}}=\frac{\pm\sqrt{\frac{3a^2-b}{2a}}\pm\sqrt{\frac{3b-a^2}{2a}}}{2}$$

$$y^{\frac{3}{4}}=\frac{\pm\sqrt{\frac{3a^2-b}{2a}}\mp\sqrt{\frac{3b-a^2}{2a}}}{2}.$$

Hence the corresponding values of x and y are

$$x=\left\{\frac{\pm\sqrt{3a^2-b}+\sqrt{3b-a^2}}{\sqrt{8a}}\right\}^{\frac{4}{3}} \qquad x=\left\{\frac{\pm\sqrt{3a^2-b}-\sqrt{3b-a^2}}{\sqrt{8a}}\right\}^{\frac{4}{3}}$$

and

$$y=\left\{\frac{\pm\sqrt{3a^2-b}-\sqrt{3b-a^2}}{\sqrt{8a}}\right\}^{\frac{4}{3}} \qquad y=\left\{\frac{\pm\sqrt{3a^2-b}+\sqrt{3b-a^2}}{\sqrt{8a}}\right\}^{\frac{4}{3}}$$

The following require the completion of the square:

EXAMPLE V.

$$\left\{\begin{array}{ll} x+y+x^2+y^2=a \quad \ldots\ldots\ldots\ldots & (1) \\ x-y+x^2-y^2=b \quad \ldots\ldots\ldots\ldots & (2) \end{array}\right\}$$

Add (1) and (2), $2x^2+2x=a+b \quad \ldots\ldots\ldots\ldots (3)$

Subtract (2) from (1), $2y^2+2y=a-b \quad \ldots\ldots\ldots\ldots (4)$

Equations (3) and (4) are common adfected quadratics; solving these in the usual manner, we find

$$\left.\begin{array}{l} x=\frac{-1\pm\sqrt{1+2a+2b}}{2} \\ y=\frac{-1\pm\sqrt{1+2a-2b}}{2} \end{array}\right\}$$

EXAMPLE VI.

$$\left\{\begin{array}{ll} x+y=\ \ 6 \quad \ldots\ldots\ldots\ldots & (1) \\ x^4+y^4=272 \quad \ldots\ldots\ldots\ldots & (2) \end{array}\right\}$$

Raise (1) to the 4th power.

$$x^4+4x^3y+6x^2y^2+4xy^3+y^4=1296.$$

But from (2), $x^4 \qquad +y^4=\ 272.$

Subtracting $4x^3y+6x^2y^2+4xy^3=1024,$

or $2xy(2x^2+3xy+2y^2)=1024 \quad \ldots\ldots\ldots (3)$

But by (1), $2xy(2x^2+4xy+2y^2)=144xy \quad \ldots\ldots (4)$

Subtracting (3) from (4), $2x^2y^2=144xy-1024.$

Transposing and dividing by 2,

$$x^2y^2-72xy = -512.$$

Completing the square, $x^2y^2-72xy+1296=1296-512$,

or $(xy-36)^2= 784.$

$$\therefore xy-36 = \pm\sqrt{784}$$

$$xy = 36\pm 28$$

$$= 64, \text{ and } 8.$$

First, let us suppose $xy=8$.

By (1), $x^2+2xy+y^2=36$,

And $4xy = 32$.

Subtracting, $x^2-2xy+y^2= 4$

$$\therefore x-y = \pm 2,$$

But $x+y = 6$.

$\therefore$ adding and subtracting,

$$\left.\begin{matrix} x=4 \\ y=2 \end{matrix}\right\} \text{ and } \left\{\begin{matrix} x=2 \\ y=4 \end{matrix}\right\}$$

Secondly, let us take the other value of xy, or 64.

By (1), $x^2+2xy+y^2= 36$,

$4xy = 256.$

Subtracting, $x^2-2xy+y^2=-220$,

$$\therefore x-y = \pm\sqrt{-220},$$

But $x+y = 6$.

$\therefore$ adding and subtracting,

$$\left.\begin{matrix} x=\dfrac{6+\sqrt{-220}}{2} \\ y=\dfrac{6-\sqrt{-220}}{2} \end{matrix}\right\} \text{ and } \left\{\begin{matrix} x=\dfrac{6-\sqrt{-220}}{2} \\ y=\dfrac{6+\sqrt{-220}}{2} \end{matrix}\right\}.$$

Hence, in the above equations, two of the roots of x and y are possible, and two impossible.

(7)* $2x+3y = 118$ . . . . (1)
$5x^2-7y^2=4333$ . . . . (2)

$$\text{Ans. } \left.\begin{matrix} x=35 \\ y=16 \end{matrix}\right\} \text{ and } \left\{\begin{matrix} x=-229\frac{6}{17} \\ y=\ \ 192\frac{4}{17} \end{matrix}\right\}$$

(8) $8x+23y = 2x^3+2y^3$ . . . . (1)
$34y+6x^2-5y^2=13xy+24$ . . . . (2)

$$\text{Ans. } \left.\begin{matrix} x=3 \\ y=2 \end{matrix}\right\} \quad \left.\begin{matrix} x=\dfrac{-181}{133} \\ y=\dfrac{34}{133} \end{matrix}\right\} \quad \left\{\begin{matrix} x=\dfrac{55\mp\sqrt{1114}}{26} \\ y=\dfrac{-9\pm 3\sqrt{1114}}{26} \end{matrix}\right\}$$

(9) $(x-y)(x^2-y^2)=a$ . . . . (1)
$(x+y)(x^2+y^2)=b$ . . . . (2)

$$\text{Ans. } x=\frac{\sqrt{2b-a}\pm\sqrt{a}}{2\sqrt[4]{2b-a}},\ y=\frac{\sqrt{2b-c}\mp\sqrt{a}}{2\sqrt[4]{2b-a}}.$$

* The following examples, though a valuable exercise, are likely to detain the student long, and may, if necessary, be omitted.

(10) $\frac{xyz}{x+y}=a$ . . . . (1)

$\frac{xyz}{y+z}=b$ . . . . (2)

$\frac{xyz}{x+z}=c$ . . . . (3)

Ans. $x=\pm\sqrt{\frac{2abc(ab+bc-ac)}{(ab+ac-bc)(bc+ac-ab)}}$,

$y=\pm\sqrt{\frac{2abc(bc+ac-ab)}{(ab+ac-bc)(ab+bc-ac)}}$,

$z=\pm\sqrt{\frac{2abc(ab+ac-bc)}{(ab+bc-ac)(bc+ac-ab)}}$.

(11) $x+y=a$, $x^3+y^3=b$.

Ans. $x=\frac{a}{2}\pm\sqrt{\frac{4b-a^3}{12a}}$, $y=\frac{a}{2}\mp\sqrt{\frac{4b-a^3}{12a}}$.

(12) $4xy=96-x^2y^2$, $x+y=6$.

Ans. $x=4$, or 2, or $3\pm\sqrt{21}$, $y=2$, or 4, or $3\mp\sqrt{21}$.

(13) $x^n+y^n=2a^n$, $xy=c^2$.

Ans. $x=\left(a^n\pm\sqrt{a^{2n}-c^{2n}}\right)^{\frac{1}{n}}$,

$y=\frac{c^2}{x}=\frac{c^2}{(a^n\pm\sqrt{a^{2n}-c^{2n}})^{\frac{1}{n}}}$.

(14) $x^2+x+y=18-y^2$, $xy=6$.

Ans. $x=3$, or 2, or $-3\pm\sqrt{3}$, $y=2$, or 3, or $-3\mp\sqrt{3}$.

(15) $x^2+2xy+y^2+2x=120-2y$, $xy-y^2=8$.

Ans. $x=-9\mp\sqrt{5}$, $y=-3\pm\sqrt{5}$, also, $x=6$, or 9, $y=4$, or 1.

(16) $x^2+y^2-x-y=78$, $xy+x+y=39$.

Ans. $x=9$, or 3, or $\frac{-13\pm\sqrt{-39}}{2}$, $y=3$, or 9, or $\frac{-13\mp\sqrt{-39}}{2}$

(17) $x^2y^4-7xy^2-945=765$, $xy-y=12$.

Ans. $x=\frac{-19}{17\mp6\sqrt{-2}}$; also, $x=5$, or $\frac{1}{5}$, $y=-6\pm\sqrt{-2}$; also, $y=3$, or -15.

(18) $x-2\sqrt{xy}+y-\sqrt{x}+\sqrt{y}=0$, $\sqrt{x}+\sqrt{y}=5$.

Ans. $x=9$, or $\frac{25}{4}$, $y=4$, or $\frac{25}{4}$.

(19) $\frac{x^2}{y^2}+\frac{4x}{y}=\frac{85}{9}$, $x-y=2$.

Ans. $x=5$, or $\frac{17}{10}$, $y=3$, or $\frac{-3}{10}$.

(20) $\sqrt{\frac{3x}{x+y}}+\sqrt{\frac{x+y}{3x}}=2$, $xy-(x+y)=54$.

Ans. $x=6$, $y=12$.

(21) $x^4-2x^2y+y^2=49$, $x^4-2x^2y^2+y^4-x^2+y^2=20$.

Ans. $x=\pm3$, or $\pm\sqrt{6}$, or $\pm\sqrt{\frac{-13\pm\sqrt{-47}}{2}}$, or $\pm\sqrt{\frac{15\pm3\sqrt{5}}{2}}$, or $\pm\sqrt{\frac{-13\pm\sqrt{-11}}{2}}$,

$y=2$, or -1, or $\frac{1\pm\sqrt{-47}}{2}$, or $\frac{1\pm3\sqrt{5}}{2}$, or $\frac{1\pm\sqrt{-11}}{2}$.

(22) $xy+xy^2=12,$ Ans. $x=2$, or 16,

$x+xy^3=18.$ $y=2$, or $\frac{1}{2}$.

(23) $x-x^{\frac{1}{2}}=3-y,$ Ans. $x=4$, or $\frac{1}{4}$,

$4-x=y-y^{\frac{1}{2}}.$ $y=1$, or $\frac{9}{4}$.

(24) $(x^2+1)y=x\,y+126,$ Ans. $x=5$, or $\frac{1}{5}$, or $\frac{-97\pm\sqrt{6045}}{58}$,

$(x^2+1)y=x^2y^2-744.$ $y=6$, or 150, or $\frac{1682}{97\mp\sqrt{6045}}$.

(25) $x+y+\sqrt{x+y}=12,$ Ans. $x=5$, or 4,

$x^3+y^3=189.$ $y=4$, or 5.

(26) $x^2+y^2+x-y=132,$ Ans. $x=11$, or -1, or $61\pm\sqrt{-3716}$,

$(x^2+y^2)(x-y)=1220.$ $y=1$, or -11, or $61\mp\sqrt{-3716}$.

(27) $x^{\frac{2}{3}}y^{\frac{3}{2}}=2y^2.$ Ans. $x=\overline{14}|^3$, or 8,

$8x^{\frac{1}{3}}-y^{\frac{1}{2}}=14.$ $y=\overline{98}|^2$; or 4.

(28) $x^{\frac{3}{2}}+y^{\frac{2}{3}}=3x$ (see note, page 217), Ans. $x=4$, or 1,

$x^{\frac{1}{2}}+y^{\frac{1}{3}}=x.$ $y=8$.

(29) $x+x^{\frac{1}{2}}=\frac{y^2+y+2}{x^{\frac{1}{2}}}+4.$ Ans. $x=4$, or 1,

$y+xy=y^2+4y.$ $y=1$, or -2.

(30) $2x+y=26-7\sqrt{2x+y+4},$ Ans. $x=2$, or -10,

$\frac{2x+\sqrt{y}}{2x-\sqrt{y}}=\frac{16}{15}+\frac{2x-\sqrt{y}}{2x+\sqrt{y}}.$ $y=1$, or 25.

(31) $\frac{x}{y}-8\sqrt{x^2-9xy^2}=9y-16xy,$ Ans. $x=1$,

$5x=4+25y^2.$ $y=\pm\frac{1}{5}$.

(32) $16x-y^{\frac{1}{2}}=6y^{\frac{1}{4}}x^{\frac{1}{2}},$ Ans. $x=4$, or 16,

$\frac{x^4}{y}-\frac{12}{x^2}=\frac{x}{\sqrt{y}}.$ $y=256$, or $\overline{256}|^2$.

(33) $\sqrt{5\sqrt{x}+5\sqrt{y}}+\sqrt{y}=10-\sqrt{x},$ Ans. $x=9$, or 4.

$\sqrt{x^5}+\sqrt{y^5}=275.$ $y=4$, or 9.

PROBLEMS PRODUCING PURE EQUATIONS.

(1) What two numbers are those whose sum is to the greater as 10 to 7, and whose sum, multiplied by the less, produces 270?

Ans. ±21 and ±9

(2) There are two numbers in the proportion of 4 to 5, and the difference of whose squares is 81. What are the numbers?

Ans. ±12 and ±15.

(3) A detachment from an army was marching in regular column, with 5 men more in depth than in front; but upon the enemy coming in sight, the front was increased by 845 men, and by this movement the detachment was drawn up in five lines. Required the number of men?

Ans. 4550.

(4) Two workmen, A and B, were engaged to work for a certain number of days at different rates. At the end of the time, A, who had been idle 4 of

those days, had 75 shillings to receive; but B, who had been idle 7 of those days, received only 48 shillings. Now, had B been idle only 4 days and A 7, they would have received exactly alike. For how many days were they engaged, how many did each work, and what had each per day?

Ans. A worked 15 and B 12 days.
A received 5 and B 4 shillings per day.

(5) A vintner draws a certain quantity of wine out of a full vessel that holds 256 gallons, and then filling the vessel with water, draws off the same quantity of liquid as before, and so on for four draughts, when there were only 81 gallons of pure wine left. How much wine did he draw each time?

Ans. 64, 48, 36, and 27 gallons.

PROBLEMS WHICH PRODUCE ADFECTED OR COMPLETE QUADRATIC EQUATIONS.

PROBLEM 1.

190. To find a number such that twice its square, augmented by three times the number, is equal to 65.

Let x be the number required, we have for the equation of the problem,

$$2x^2+3x=65.$$

Solving the equation, $$x=-\frac{3}{4}\pm\sqrt{\frac{65}{2}+\frac{9}{16}}=-\frac{3}{4}\pm\frac{23}{4}.$$

Hence $$x=5;\ x=-\frac{13}{2}.$$

The first of these two values satisfies the conditions of the problem, as stated in the enunciation; for, in fact,

$$2(5)^2+3\times5=2\times25+15$$
$$=65.$$

In order to interpret the meaning of the second value, let us observe, that if we substitute $-x$ for $+x$ in the equation $2x^2+3x=65$, the coefficient of $3x$ alone will change its sign, for $(-x)^2=(+x)^2=x^2$. Hence the value of x will no longer be

$$x=-\frac{3}{4}\pm\frac{23}{4},$$

but will become $$x=+\frac{3}{4}\pm\frac{23}{4}.$$

Hence $$x=\frac{13}{2};\ x=-5,$$

where the values of x differ from those already found in sign alone.

Hence we may conclude that the negative solution $-\frac{13}{2}$, considered without reference to its sign, is the solution of the following problem:

To find a number such that twice its square, *diminished* by three times the number, is equal to 65.

In fact, we have

$$2\left(\frac{13}{2}\right)^2-3\times\frac{13}{2}=\frac{169}{2}-\frac{39}{2}$$
$$=65.$$

PROBLEM 2.

A tailor bought a certain number of yards of cloth for 12 pounds. If he had paid the same sum for 3 yards less of the same cloth, then the cloth would have cost 4 shillings a yard more. Required the number of yards purchased.

Let x be the number of yards purchased.

Then $\frac{240}{x}$ is the price of one yard, expressed in shillings.

If he had paid the same sum for 3 yards less, in that case the price of each would be represented by $\frac{240}{x-3}$.

But by the conditions of the problem, this last price is greater than the former by 4 shillings; hence the equation of the problem will be

$$\frac{240}{x-3}=\frac{240}{x}+4,$$

or

$$x^2-3x=180.$$

Whence

$$x=\frac{3}{2}\pm\sqrt{\frac{9}{4}+180}=\frac{3}{2}\pm\frac{27}{2}$$

$$\therefore x=15\text{ ; } x=-12.$$

The value of $x=15$ satisfies the conditions of the problem, for

$$\frac{240}{15}=16\text{ ; } \frac{240}{12}=20,$$

the price of each yard in the first case being 16 shillings, and in the last case 20, which exceeds the former by 4 shillings.

With regard to the second solution, we can form a new enunciation to which it will correspond. Resuming the original equation, and changing x into $-x$, it becomes

$$\frac{240}{-x-3}=\frac{240}{-x}+4,$$

or

$$\frac{240}{x+3}=\frac{240}{x}-4,$$

an equation which may be considered as the algebraic representation of the following problem:

A tailor bought a certain number of yards of cloth for 12 pounds. If he had paid the same sum for 3 yards *more*, then the cloth would have cost 4 shillings a yard *less*. Required the number of yards purchased.

The above equation when reduced becomes

$$x^2+3x=180,$$

instead of $x^2-3x=180$, as in the former case; solving the above, we find

$$x=12\text{ ; } x=-15.$$

The two preceding problems illustrate the principle explained with regard to problems of the first degree.

PROBLEM 3.

A merchant purchased two bills; one for \$8776, payable in 9 months, the other for \$7488, payable in 8 months. For the first he paid \$1200 more than for the second. Required the rate of interest allowed.

Let x represent the interest of \$100 for 1 month.

Then $12x$, $9x$, $8x$ severally represent the interest of \$100 for 1 year, 9 months, 8 months.

And $100+9x$, $100+8x$ represent what a capital of \$100 will become at the end of 9 and of 8 months respectively.

Hence, in order to determine the actual value of the two bills, we have the following proportions:

$$100+9x:100::8776:\frac{8776\times 100}{100+9x}$$

$$100+8x:100::7488:\frac{7488\times 100}{100+8x}.$$

The fourth terms of the above proportions express the sum paid by the merchant for each of the bills.

Hence, by the conditions of the problem,

$$\frac{877600}{100+9x}-\frac{748800}{100+8x}=1200,$$

or, dividing each member by 400,

$$\frac{2194}{100+9x}-\frac{1872}{100+8x}=3.$$

Clearing of fractions and reducing,

$$216x^2+4396x=2200.$$

Whence

$$x=-\frac{2198}{216}\pm\sqrt{\frac{2200}{216}+\left(\frac{2198}{216}\right)^2}$$

$$=\frac{-2198\pm\sqrt{5306404}}{216}$$

$$\therefore\ 12x=\frac{-2198\pm\sqrt{5306404}}{18}$$

$$=\frac{-2198\pm 2303.5\ldots\ldots}{18}$$

$$\therefore\ 12x=5.86\ldots\ldots;\ \text{and}\ 12x=-250.08\ldots\ldots$$

The positive solution, $12x=5.86\ldots\ldots$, represents the required rate of interest per cent. per annum.

With regard to the negative solution, it can only be considered as connected with the other by the same equation of the second degree. If we resume the original equation, and substitute $-x$ for $+x$, we shall find great difficulty in reconciling this new equation with an enunciation analogous to that of the proposed problem.

PROBLEM 4.

A man purchased a horse, which he afterward sold to disadvantage for 24 pounds. His loss per cent. by this bargain, upon the original price of the horse, is expressed by the number of pounds which he paid for the horse. Required the original price.

Let x be the number of pounds which he paid for the horse.

Then $x-24$ will represent his loss;

But, by the conditions of the problem, his loss per cent. is represented by the number of units in x;

His loss per cent. on one pound is $\frac{x}{100}$.

$\therefore$ his loss per cent. on x pounds must be $\frac{x^2}{100}$, or x times as great.

This gives the equation,

$$\frac{x^2}{100}=x-24$$

$$x=50\pm\sqrt{100}=50\pm10.$$

Hence $\qquad x=60\ ;\ x=40.$

Both these solutions equally fulfill the conditions of the problem.

Let us suppose, in the first place, that he paid 60 pounds for the horse; since he sold it for 24, his loss was 36. On the other hand, by the enunciation, his loss was 60 per cent. on the original price; *i. e.*, $\frac{60}{100}$ of 60, or $\frac{60\times60}{100}=36$; thus 60 satisfies the conditions.

In the second place, let us suppose that he paid 40 pounds; his loss in this case was 16. On the other hand, his loss ought to be 40 per cent. on the original price; *i. e.*, $\frac{40}{100}$ of 40, or $\frac{40\times40}{100}=16$; thus 40 also satisfies the conditions.

GENERAL DISCUSSION OF THE EQUATION OF THE SECOND DEGREE.

191. The general form of the equation, the coefficients being considered independently of their signs, is

$$x^2+px+q=0.$$

I., II. Let q be positive and $<\frac{p^2}{4}$,

I. If p be positive, $x=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q}$, and both values are negative.*

II. If p be negative, $x=+\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q}$, and both values are positive.

III., IV. Let q be positive and $>\frac{p^2}{4}$,

III. If p be positive, $x=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q}$,

IV. If p be negative, $x=+\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q}$,

and both values are imaginary.†

* In this and all the following values of x, calling the term $\frac{p}{2}$ before the radical the rational part, and $\sqrt{\frac{p^2}{4}\pm q}$ the radical part, we perceive that, when q is positive, the radical part is greater than the rational, since $\sqrt{\frac{p^2}{4}}$ alone equals $\frac{p}{2}$, the rational part; and the sign of the whole expression is that of the radical part; but when q is negative, the radical part is less than the rational, and the sign of the whole expression is that of the rational part.

† In this case, if we examine the general equation, we shall find that the conditions are absurd; for, transposing q, and completing the square, we have

V., VI. Let q be negative and $<\frac{p^2}{4}$,

V. If p be positive, $x=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}+q}$,

VI. If p be negative, $x=+\frac{p}{2}\pm\sqrt{\frac{p^2}{4}+q}$,

VII., VIII. Let q be negative and $>\frac{p^2}{4}$,

VII. If p be positive, $x=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}+q}$,

VIII. If p be negative, $x=+\frac{p}{2}\pm\sqrt{\frac{p^2}{4}+q}$,

(V.–VIII.) and one value is positive, the other negative.

IX., X. Let $q=\frac{p^2}{4}$, and be positive.

IX. If p be positive, $x=-\frac{p}{2}$.

X. If p be negative, $x=+\frac{p}{2}$.

(IX., X.) and the two values are equal.

XI., XII. Let $q=0$,

XI. If p be positive, $x=-\frac{p}{2}\pm\frac{p}{2}$, one value $=-p$, the other $=0$.

XII. If p be negative, $x=+\frac{p}{2}\pm\frac{p}{2}$, one value $=+p$, the other $=0$.

XIII. Let q be negative.

XIII. $p=0$, $x=\pm\sqrt{q}$, the two values are equal with opposite signs.

XIV. Let q be positive,

XIV. $p=0$, $x=\pm\sqrt{-q}$, both values are imaginary.

XV. Let $q=0$,

XV. $p=0$, then $x=0$, or both values are equal to 0.

$$x^2\pm px+\frac{p^2}{4}=\frac{p^2}{4}-q;$$

but since $\frac{p^2}{4}-q$ is, by hypothesis, a negative quantity, we may represent it by $-m$, where m is some positive quantity; then

$$x^2\pm px+\frac{p^2}{4}=-m$$

$$\left(x\pm\frac{p}{2}\right)^2+m=0;$$

that is, the sum of two quantities, each of which is essentially positive, is equal to 0, a manifest absurdity. Solving the equation,

$$x=\mp\frac{p}{2}\pm\sqrt{-m},$$

and the symbol $\sqrt{-m}$, which denotes absurdity, serves to distinguish this case. Hence, *when the roots are imaginary, the problem to which the equation corresponds is absurd.*

We still say, however, that the equation has two roots; for, subjecting these values of x to the same calculations as if they were real, that is, substituting them for x in the proposed equations, we shall find that they render the two members identical.

XVI. One case, attended with remarkable circumstances, still remains to be examined. Let us take the equation

$$ax^2+bx-c=0.$$

Whence $$x=\frac{-b\pm\sqrt{b^2+4ac}}{2a}.$$

Let us suppose that, in accordance with a particular hypothesis made on the given quantities in the equation, we have $a=0$; the expression for x then becomes

$$x=\frac{-b\pm b}{0};\text{ whence }\begin{cases} x=\frac{0}{0} \\ x=\frac{-2b}{0}. \end{cases}$$

The second of the above values is under the form of infinity, and may be considered as an answer, if the problem proposed be such as to admit of infinite solutions.

We must endeavor to interpret the meaning of the first, $\frac{0}{0}$.

In the first place, if we return to the equation $ax^2+bx-c=0$, we perceive that the hypothesis $a=0$ reduces it to $bx=c$, whence we derive $x=\frac{c}{b}$, a *finite* and *determinate* expression, which must be considered as representing the true value of $\frac{0}{0}$ in the case before us.

That no doubt may remain on this subject, let us assume the equation

$$ax^2+bx-c=0,$$

and put $x=\frac{1}{y}$, the expression will then become

$$\frac{a}{y^2}+\frac{b}{y}-c=0.$$

Whence $$cy^2-by-a=0.$$

Let $a=0$, this last equation will become

$$cy^2-by=0,$$

from which we have the two values $y=0$, $y=\frac{b}{c}$; substituting these values in $x=\frac{1}{y}$, we deduce

$$1^\circ.\ x=\frac{1}{0};\ 2^\circ.\ x=\frac{c}{b}.^*$$

---

* To show more distinctly how the indeterminate form arises, let us resume the general value of one of the roots.

$$x=\frac{-b+\sqrt{b^2+4ac}}{2a}.$$

If a were a factor of both the numerator and denominator, it might be suppressed, and then a, being put equal to zero, would give the true value of x. We can not, indeed, show the existence of this factor in the two terms of the fraction as it stands; but if we multiply both numerator and denominator by $-b-\sqrt{b^2+4ac}$, it becomes

$$x=\frac{(-b+\sqrt{b^2+4ac})(-b-\sqrt{b^2+4ac})}{-2a(b+\sqrt{b^2+4ac})}.$$

With respect to the value $x=\frac{-2b}{0}$, it is only to be observed that the divisor *zero*, having to be regarded as the limit of decreasing magnitudes, either positive or negative, it follows that the infinite value ought to have the ambiguous sign $\pm$.

Thus the values of x, to recapitulate, become

$$x=\frac{c}{b},\ x=\pm\infty.$$

It is remarkable that, for this particular case, we have three values of x, while in the general case there are but two.

To comprehend how these values truly belong to the equation $ax^2+bx-c=0$, put it under the form

$$\frac{-bx+c}{x^2}=a.$$

When $a=0$, the question is to find values which will render $\frac{-bx+c}{x^2}$ zero. We see that $x=\frac{c}{b}$ will do it; and as the same expression can be written under the form $-\frac{b}{x}+\frac{c}{x^2}$, we perceive that it becomes zero also, from the values $x=\pm\infty$.*

XVII. Let us consider the still more particular case still, where we have, at the same time, $a=0$, $b=0$. Then the two general values of x become $\frac{0}{0}$. We have seen above that the first may be changed into

$$x=\frac{2c}{b+\sqrt{b^2+4ac}}.$$

Transforming the second in a similar manner, it becomes

$$x=\frac{(-b-\sqrt{b^2+4ac})(-b+\sqrt{b^2+4ac})}{2a(-b+\sqrt{b^2+4ac})}=\frac{-2c}{-b+\sqrt{b^2+4ac}}.$$

In which, making $a=0$, $b=0$, the values of x, thus transformed, both give $x=\infty$; and here, also, the infinity ought to be taken with the sign $\pm$.

If we suppose $a=0$, $b=0$, $c=0$, the proposed equation will become altogether indeterminate.

The numerator, being the product of the sum and difference of two quantities, is equal to the difference of their squares, to wit: $b^2-(b^2+4ac)=-4ac$. We see, therefore, that $2a$ is a common factor to the numerator and denominator of the last expression. Suppressing it, we have

$$x=\frac{2c}{b+\sqrt{b^2+4ac}},$$

in which, if we make $a=0$, it gives $x=\frac{c}{b}$.

* In the analytic theory of curves these values answer to the intersections of the axis of abscissas with the curve of the 3° order, the equation of which is $yx+bx+c=0$. If this curve be constructed, it will be found to cut the axis of abscissas first at a finite distance from the origin, and besides has this axis for an asymptote both on the side of the positive and negative abscissas, which amounts to saying that it cuts it at infinity in either direction.

192. Let us now proceed to illustrate the principles established in this general discussion, by applying them to different problems.

PROBLEM 5.

To find in a line, A B, which joins two lights of different intensities, a point which is illuminated equally by each.

P_3 A P_1 B P_2.

(It is a principle in Optics that the intensities of the same light at different distances are inversely as the squares of the distances.)

Let a be the distance A B between the two lights.

Let b be the intensity of the light A at the distance of one foot from A.

Let c be the intensity of the light B at the distance of one foot from B.

Let P_1 be the point required.

Let $AP_1=x$; $\therefore BP_1=a-x$.

By the optical principle above enunciated, since the intensity of A at the distance of 1 foot is b, its intensity at the distance of 2, 3, 4, ... feet must be $\frac{b}{4}, \frac{b}{9}, \frac{b}{16}$; hence the intensity of A at the distance of x feet must be $\frac{b}{x^2}$. In the same manner, the intensity of B at the distance $a-x$ must be $\frac{c}{(a-x)^2}$; but according to the conditions of the question, these two intensities are equal; hence we have for the equation of the problem

$$\frac{b}{x^2}=\frac{c}{(a-x)^2}.$$

Solving this equation, and reducing the result to its most simple form,

$$x=\frac{a\sqrt{b}}{\sqrt{b}\pm\sqrt{c}}.$$

We shall now proceed to discuss these two values:

$$\left.\begin{array}{l}1°. \ldots\ldots x=\dfrac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}\\ 2°. \ldots\ldots x=\dfrac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}\end{array}\right\}\text{ whence }\left\{\begin{array}{l}a-x=\dfrac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}\\ a-x=\dfrac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}\end{array}\right.$$

I. Let $b>c$.

The first value of x, $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}$, is positive, and less than a, for $\frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}$ is a proper fraction; hence this value gives for the point equally illuminated a point P_1, situated between the points A and B. We perceive, moreover, that the point P_1 is nearer to B than to A; for, since $b>c$, we have

$$\sqrt{b}+\sqrt{b}>\sqrt{b}+\sqrt{c}, \text{ or } 2\sqrt{b}>\sqrt{b}+\sqrt{c}, \text{ and } \therefore \frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}>\frac{1}{2},$$

and, consequently, $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}>\frac{a}{2}$. This is manifestly the result at which we ought to arrive, for we here suppose the intensity of A to be greater than that of B.

The corresponding value of $a-x$, $\frac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}$, is positive, and less than $\frac{a}{2}$.

The second value of x, $\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$, is positive, and greater than a, for

$$\sqrt{b} > \sqrt{b} - \sqrt{c}, \therefore \frac{\sqrt{b}}{\sqrt{b}-\sqrt{c}} > 1, \text{ and } \therefore \frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}} > a.$$

This second value gives a point P_2, situated in the production of A B, and to the right of the two lights. In fact, we suppose that the two lights give forth rays in all directions; there may, therefore, be a point in the production of A B equally illuminated by each, but this point must be situated in the production of A B to the right, in order that it may be nearer to the less powerful of the two lights.

It is easy to perceive why the two values thus obtained are connected by the same equation. If, instead of assuming A P_1 for the unknown quantity x, we take A P_2, then B $P_2 = x - a$, thus we have the equation $\frac{b}{x^2} = \frac{c}{(x-a)^2}$; but since $(x-a)^2$ is identical with $(a-x)^2$; the new equation is the same as that already established, and which, consequently, ought to give A P_2 as well as A P_1.

The second value of $a-x$, $\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, is negative, as it ought to be, being estimated in a contrary direction from the first, on the general principle already established, that quantities estimated in a contrary sense should be represented with contrary signs; but changing the signs of the equation $a - x = \frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, we find $x - a = \frac{a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, and this value of $x-a$ represents the absolute length of B P_2.

II. Let $b < c$.

The first value of x, $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}$ is positive, and less than $\frac{a}{2}$, for $\sqrt{b}+\sqrt{c} > \sqrt{b}+\sqrt{b}$, $\therefore \sqrt{b}+\sqrt{c} > 2\sqrt{b}$, $\therefore \frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}} < \frac{1}{2}$, $\therefore \frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}} < \frac{a}{2}$.

The corresponding value of $a-x$, $\frac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}$, is positive, and greater than $\frac{a}{2}$.

Hence the point P_1 is situated between the points A and B, and is nearer to A than to B. This is manifestly the true result, for the present hypothesis supposes that the intensity of B is greater than the intensity of A.

The second value of x, $\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$, or $\frac{-a\sqrt{b}}{\sqrt{c}-\sqrt{b}}$, is essentially negative. In order to interpret the signification of this result, let us resume the original equation, and substitute $-x$ for $+x$, it thus becomes $\frac{b}{x^2} = \frac{c}{(a+x)^2}$. But since $(a-x)$ expresses in the first instance the distance of B from the point required, $a+x$ ought still to express the same distance, and, therefore, the point required must be situated to the left of A, in P_3, for example. In fact, since the intensity of the light B is, under the present hypothesis, greater than the intensity of A, the point required must be nearer to A than to B.

The corresponding value of $a-x$, $\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, or $\frac{a\sqrt{c}}{\sqrt{c}-\sqrt{b}}$, is positive, and the reason of this is, that x being negative, $a-x$ expresses, in reality, an *arithmetical sum*.

III. Let $b=c$.

The first two values of x and of $a-x$ are reduced to $\frac{a}{2}$, which gives the bisection of A B for the point equally illuminated by each light, a result which is manifestly true, upon the supposition that the intensity of the two lights is the same.

The other two values are reduced to $\frac{a\sqrt{b}}{0}$, that is, they become *infinite*, that is to say, the second point equally illuminated is situated at a distance from the points A and B greater than any which can be assigned. This result perfectly corresponds with the present hypothesis; for if we suppose the difference $b-c$, without vanishing altogether, to be exceedingly small, the second point equally illuminated, exists, but at a great distance from the two lights; this is indicated by the expression $\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$, the denominator of which is exceedingly small in comparison with the numerator if we suppose b very nearly equal to c. In the extreme case, when $b=c$, or $\sqrt{b}-\sqrt{c}=0$, the point required no longer exists, or is situated at an infinite distance.

IV. Let $b=c$ and $a=0$.

The first system of values of x and $a-x$ in this case become 0, and the second system $\frac{0}{0}$. This last result is here the symbol of indetermination; for if we recur to the equation of the problem

$$\frac{b}{x^2}=\frac{c}{(a-x)^2},$$

or

$$(b-c)x^2-2abx=-a^2b,$$

it becomes, under the present hypothesis,

$$0.x^2-0.x=0,$$

an equation which can be satisfied by the substitution of any number whatever for x. In fact, since the two lights are supposed to be equal in intensity, and to be placed at the same point, they must illuminate *every point* in the line A B *equally*.

The solution 0, given by the first system, is one of those solutions, *infinite in number*, of which the problem in this case is susceptible.

V. Let $a=0$, b not being $=c$.

Each of the two systems in this case is reduced to 0, which proves that in this case there is only one point equally illuminated, viz., *the point in which the two lights are placed.*

The above discussion affords an example of the precision with which algebra answers to all the circumstances included in the enunciation of a problem.

We shall conclude this subject by solving one or two problems which require the introduction of more than one unknown quantity.

PROBLEM 6.

To find two numbers such that, when multiplied by the numbers a and b respectively, the sum of the products may be equal to $2s$, and the product of the two numbers equal to p.

Let x and y be the two numbers sought, the equations of the problem will be

$$ax+by=2s \quad \ldots \quad (1)$$
$$xy=p \quad \ldots \quad (2)$$

From (1)

$$y=\frac{2s-ax}{b}.$$

Substituting this value in (2) and reducing, we have

$$ax^2-2sx+bp=0.$$

Whence

$$x=\frac{s}{a}\pm\frac{1}{a}\sqrt{s^2-a^2pb},$$

And $\therefore$

$$y=\frac{s}{b}\mp\frac{1}{b}\sqrt{s^2-a^2pb}.$$

The problem is, we perceive, susceptible of two direct solutions, for s is manifestly $>\sqrt{s^2-a^2bp}$; but in order that these solutions may be real we must have $s^2>$, or $=a^2bp$.

Let $a=b=1$; in this case the values of x and y are reduced to

$$x=s\pm\sqrt{s^2-p},\ y=s\mp\sqrt{s^2-p}.$$

Here we perceive that the two values of y are equal to those of x taken in an inverse order; that is to say, if $s+\sqrt{s^2-p}$ represent the value of x, then $s-\sqrt{s^2-p}$ will represent the corresponding value of y, and reciprocally.

We explain this circumstance by observing that, in this particular case, the equations of the problem are reduced to $x+y=2s$, $xy=p$, and the question then becomes, Required two numbers whose sum is $2s$, and whose product is p, or, in other words, *To divide a number* 2s *into two parts, such that their product may be equal to* p.

PROBLEM 7.

To find four numbers in proportion, the sum of the extremes being $2s$, the sum of the means $2s'$, and the sum of the squares of the four terms $4c^2$.

Let a, x, y, z represent the four terms of the proportion; by the conditions of the question, and the fundamental property of proportions, we shall have as the equations of the problem

$$a+z=2s \quad \ldots \quad (1)$$
$$x+y=2s' \quad \ldots \quad (2)$$
$$xy=az \quad \ldots \quad (3)$$
$$a^2+x^2+y^2+z^2=4c^2 \quad \ldots \quad (4)$$

Squaring (1) and (2) and adding the results,

$$a^2+x^2+y^2+z^2+2az+2xy=4(s^2+s'^2).$$

But by (4), $$a^2+x^2+y^2+z^2 \qquad =4c^2.$$

Subtracting, $$2az+2xy=4(s^2+s'^2-c^2).$$

$\therefore$ by (3), $$4az=4(s^2+s'^2-c^2)=4xy \quad . \; . \quad (5)$$

Squaring (1), $$a^2+2az+z^2=4s^2.$$

But by (5), $$4az \qquad =4(s^2+s'^2-c^2).$$

Subtracting, $$a^2-2az+z^2=4(c^2-s'^2).$$

Extracting the root, $$a-z=\pm 2\sqrt{c^2-s'^2}.$$

But by (1), $$a+z=2s.$$

∴ adding and subtracting,
$$a=s\pm\sqrt{c^2-s'^2}$$
$$z=s\mp\sqrt{c^2-s'^2}.$$

Precisely in the same manner we shall find
$$x=s'\pm\sqrt{c^2-s^2}$$
$$y=s'\mp\sqrt{c^2-s^2}.$$

The four numbers will therefore be
$$a=s+\sqrt{c^2-s'^2},\ x=s'+\sqrt{c^2-s^2}$$
$$z=s-\sqrt{c^2-s'^2},\ y=s'-\sqrt{c^2-s^2}.$$

These four numbers constitute a proportion, for we have
$$az=(s+\sqrt{c^2-s'^2})(s-\sqrt{c^2-s'^2})=s^2-c^2+s'^2$$
$$xy=(s'+\sqrt{c^2-s^2})(s'-\sqrt{c^2-s^2})=s'^2-c^2+s^2.$$

(8) What two numbers are those whose sum is 20, and their product 36?

Ans. 2 and 18.

(9) To divide the number 60 into two such parts that their product may be to the sum of their squares in the ratio of 2 to 5.

Ans. 20 and 40.

(10) The difference of two numbers is 3, and the difference of their cubes is 117. What are those numbers?

Ans. 2 and 5.

(11) A company at a tavern had £8 15*s*. to pay for their reckoning; but, before the bill was settled, two of them left the room, and then those who remained had 10*s*. apiece more to pay than before. How many were there in company?

Ans. 7.

(12) A grazier bought as many sheep as cost him £60, and after reserving 15 out of the number, he sold the remainder for £54, and gained 2*s*. a head by them. How many sheep did he buy?

Ans. 75.

(13) There are two numbers whose difference is 15, and half their product is equal to the cube of the lesser number. What are those numbers?

Ans. 3 and 18.

(14) A person bought cloth for £33 15*s*., which he sold again at £2 8*s*. per piece, and gained by the bargain as much as one piece cost him. Required the number of pieces.

Ans. 15.

(15) What number is that, which when divided by the product of its two digits, the quotient is 3; and if 18 more be added to it, the digits will be transposed?

Ans. 24.

(16) What two numbers are those whose sum, multiplied by the greater, is equal to 77, and whose difference, multiplied by the lesser, is equal to 12?

Ans. 4 and 7.

(17) To find a number such that, if you subtract it from 10, and multiply the remainder by the number itself, the product shall be 21.

Ans. 7, or 3.

(18) To divide 100 into two such parts that the sum of their square roots may be 14.

Ans. 64 and 36.

(19) It is required to divide the number 24 into two such parts that their product may be equal to 35 times their difference.

Ans. 10 and 14.

(20) The sum of two numbers is 8, and the sum of their cubes is 152. What are the numbers?

Ans. 3 and 5.

(21) The sum of two numbers is 7, and the sum of their 4th powers is 641. What are the numbers?

Ans. 2 and 5.

(22) The sum of two numbers is 6, and the sum of their 5th powers is 1056. What are the numbers?

Ans. 2 and 4.

(23) Two partners, A and B, gained £140 by trade; A's money was 3 months in trade, and his gain was £60 less than his stock; and B's money, which was £50 more than A's, was in trade 5 months. What was A's stock?

Ans. £100.

(24) To find two numbers such that the difference of their squares may be equal to a given number, q^2; and when the two numbers are multiplied by the numbers a and b respectively, the difference of the products may be equal to a given number, s^2.

Ans. $\dfrac{as^2 \pm b\sqrt{s^4-(a^2-b^2)q^2}}{a^2-b^2}$

$\dfrac{bs^2 \pm a\sqrt{s^4-(a^2-b^2)q^2}}{a^2-b^2}$

(25) There are two square buildings that are paved with stones a foot square each. The side of one building exceeds that of the other by 12 feet, and both their pavements taken together contain 2120 stones. What are the lengths of them separately?

Ans. 26 and 38 feet.

(26) A and B set out from two towns, which were at the distance of 247 miles, and traveled the direct road till they met. A went 9 miles a day, and the number of days at the end of which they met was greater by 3 than the number of miles which B went in a day. How many miles did each go?

Ans. A went 117 and B 130 miles.

(27) The joint stock of two partners was \$2080; A's money was in trade 9 months, and B's 6 months; when they shared stock and gain, A received \$1140 and B \$1260. What was each man's stock?

Ans. \$960 and \$1120.

(28) A square court-yard has a rectangular gravel walk round it. The side of the court wants 2 yards of being 6 times the breadth of the gravel walk, and the number of square yards in the walk exceeds the number of yards in the periphery of the court by 164. Required the area of the court.

Ans. 256.

(29) During the time that the shadow on a sun-dial, which shows true time, moves from 1 o'clock to 5, a clock, which is too fast a certain number of

hours and minutes, strikes a number of strokes equal to that number of hours and minutes; and it is observed that the number of minutes is less by 41 than the square of the number which the clock strikes at the last time of striking. The clock does not strike twelve during the time. How much is it too fast?

Ans. 3 hours and 23 minutes.

(30) A and B engage to reap a field for £4 10*s.*; and as A alone could reap it in 9 days, they promised to complete it in 5 days. They found, however, that they were obliged to call in C, an inferior workman, to assist them for the last two days, in consequence of which B received 3*s.* 9*d.* less than he otherwise would have done. In what time could B or C alone reap the field?

Ans. B could reap it in 15 days, C in 18.

(31) The fore wheel of a carriage makes 6 revolutions more than the hind wheel in going 120 yards; but if the periphery of each wheel be increased 1 yard, it will make only 4 revolutions more than the hind wheel in the same space. Required the circumference of each.

Ans. 4 and 5.

(32) The intensity of two lights, A and B, is as 7:17, and their distance apart 132 feet. Whereabouts between is the point of equal illumination?

(33) The loudness of a church bell is three times that of another. Now, supposing the strength of sound to be inversely as the square of the distance, at what place between the two will the bells be equally well heard.

(34) Supposing the mass of the earth to be 1 and that of the moon 0.017, their distance 240 thousand miles, and the force of attraction equal to the mass divided by the square of the distance; at what point between will a body be held in suspense, attracted toward neither?

(35) The hold of a vessel partly full of water (which is uniformly increased by a leak) is furnished with two pumps, worked by A and B, of whom A takes three strokes to two of B's; but four of B's throw out as much water as five of A's. Now B works for the time in which A alone would have emptied the hold; A then pumps out the remainder, and the hold is cleared in 13 hours and 20 minutes. Had they worked together, the hold would have been emptied in 3 hours and 45 minutes, and A would have pumped out 100 gallons more than he did. Required the quantity of water in the hold at first, and the hourly influx of the leak.

(36) To divide two numbers, a and b, each into two parts, such that the product of one part of a by one part of b may be equal to a given number, p, and the product of the remaining parts of a and b equal to another given number, p'.

$$\text{Ans. } x=\frac{ab-(p'-p)\pm\sqrt{\{ab-(p'-p)\}^2-4abp}}{2b}$$

$$+\frac{ab+(p'-p)\mp\sqrt{\{ab-(p'-p)\}^2-4abp}}{2b}$$

$$y=\frac{ab-(p'-p)\pm\sqrt{\{ab-(p'-p)\}^2-4abp}}{2a}$$

$$+\frac{ab+(p'-p)\mp\sqrt{\{ab-(p'-p)\}^2-4abp}}{2a}.$$

(37) To find a number such that its square may be to the product of the

differences of that number, and two other given numbers, a and b, in the given ratio, $p:q$.

Ans. $\frac{(a+b)p \pm \sqrt{(a-b)^2p^2+4abpq}}{2(p-q)}$.

(38) There is a number consisting of two digits, which, when divided by the sum of its digits, gives a quotient greater by 2 than the first digit; but if the digits be inverted, and the resulting number be divided by a number greater by unity than the sum of the digits, the quotient shall be greater by 2 than the former quotient. What is the number?

Ans. 24.

(39) A regiment of foot receives orders to send 216 men on garrison duty, each company sending the same number of men; but before the detachment marched, three of the companies were sent on another service, and it was then found that each company that remained would have to send 12 men additional in order to make up the complement, 216. How many companies were in the regiment, and what number of men did each of the remaining companies send on garrison duty?

Ans. There were 9 companies, and each of the remaining 6 sent 36 men.

DECOMPOSITION OF THE TRINOMIAL x^2+px-q INTO TWO FACTORS OF THE FIRST DEGREE.

193. If we add to this trinomial, in order to complete the square of the first two terms, the term $\frac{1}{4}p^2$, and afterward subtract the same, so as not to change the quantity, it becomes

$$x^2+px+\tfrac{1}{4}p^2-\tfrac{1}{4}p^2-q,$$

which may be written thus:

$$(x+\tfrac{1}{2}p)^2-(\tfrac{1}{4}p^2+q) \quad \ldots \ldots \ldots \ldots \quad (2)$$

But the difference of the squares of two quantities being equal to the product of their sum and difference, the expression (2) is equal to the following:

$$(x+\tfrac{1}{2}p+\sqrt{\tfrac{1}{4}p^2+q})(x+\tfrac{1}{2}p-\sqrt{\tfrac{1}{4}p^2+q}) \quad \ldots \quad (3)$$

We perceive from this expression that the two factors of the first degree, which compose the trinomial of the second degree, are x minus each of the roots of the equation of the second degree, formed by putting this trinomial equal to zero.

Moreover, by equating (3) to zero, we perceive that the only way of satisfying the resulting equation is by making one or other of the factors of the first degree, of which it is composed, equal to zero.

The first,

$$x+\tfrac{1}{2}p+\sqrt{\tfrac{1}{4}p^2+q}=0, \text{ gives } x=-\tfrac{1}{2}p-\sqrt{\tfrac{1}{4}p^2+q};$$

and the second,

$$x+\tfrac{1}{2}p-\sqrt{\tfrac{1}{4}p^2+q}=0, \text{ gives } x=-\tfrac{1}{2}p+\sqrt{\tfrac{1}{4}p^2+q}.$$

Hence there are but two values of x which will satisfy the general equation

$$x^2+px-q=0.$$

EXAMPLES.

1°. Decompose the trinomial $x^2-7x+10$ into two factors of the first degree.

From the equation $x^2-7x+10=0$ we find the roots $x=5$ and $x=2$. Hence

$$x^2-7x+10=(x-5)(x-2).$$

2°. $3x^2-5x-2$.

Equating this trinomial to zero, after dividing by 3, we obtain the equation $x^2-\frac{5}{3}x-\frac{2}{3}=0$, the roots of which being $x=2$ and $x=-^1$, we have

$$3x^2-5x-2=3(x-2)(x+\tfrac{1}{3})=(x-2)(3x+1).$$

3°. x^2+5x+3. Ans. $(x+\frac{5}{2}-\frac{1}{2}\sqrt{13})(x+\frac{5}{2}+\frac{1}{2}\sqrt{13})$.

4°. $4x^2-4x+1$. Ans. $(2x-1)^2$.*

5°. x^2-5x+7. Ans. $(x-\frac{5}{2})^2+\frac{3}{4}$.

194. To complete the analysis of the 2° degree, it would be necessary to consider the case where the unknown quantities exceed the equations in number. The more simple is that when there is but one equation and two unknown quantities. If it be resolved with respect to one of the unknown quantities, y, for example, an expression is found generally containing x under a radical; so that, by giving to x any rational values whatever, irrational values would be found for y. It might be proposed to find rational values for x, for which the corresponding one of y should be rational also. But the difficulty of this problem, unless it be restricted to some very simple cases, is beyond mere elements. We add one or two here. For further information upon the subject, the student is referred to the Theory of Numbers, by Legendre, a separate and very elegant treatise, in one quarto volume.

INDETERMINATE ANALYSIS OF THE SECOND DEGREE.

Resolution in whole numbers of an equation of the second degree, with two unknown quantities, which contains but the first power of one of the unknowns.

195. The questions of indeterminate analysis, which depend upon equations of a degree superior to the first, go beyond the limits which we have imposed on ourselves in the present work; but when an equation of the second degree contains the second power of but one of the unknown quantities, the solutions of this equation in whole numbers may be regarded as a question of indeterminate analysis of the first degree.

Equations of the second degree in two unknown quantities, which do not contain the second power of one of these, are represented by the equation

$$mxy+nx^2+px+qy=r \quad \ldots\ldots\ldots\ldots \quad (1)$$

Resolving this equation with respect to y, we find

$$y=\frac{-nx^2-px+r}{mx+q} \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

We deduce from it, by performing the division,

$$y=-\frac{n}{m}x+\frac{nq-mp}{m^2}+\frac{m^2r+mpq-nq^2}{m^2(mx+q)},$$

which gives

$$m^2y=-mnx+nq-mp+\frac{\mathrm{N}}{mx+q} \quad \ldots\ldots\ldots \quad (3)$$

putting to abridge $m^2r+mpq-nq^2=\mathrm{N}$.

In order that x and y should be whole numbers, it is necessary that $\frac{\mathrm{N}}{mx+q}$ should be a whole number; we must, therefore, calculate all the divisors of

* This presents a case of what are called *equal roots*.

the number N, and put $mx+q$ equal to each of these divisors successively, taken with the sign $+$ and with the sign $-$. If the equations thus obtained furnish for x a certain number of entire values, these values are to be substituted in equation (3); and it is necessary, moreover, in order that y may be a whole number, that the second number which becomes a known quantity should be divisible by m^2.

It is evident that the member of entire solutions will be very limited, and that there may not be even one.

If this method be applied to each of the following equations,

$$2xy-3x^2+y=1$$
$$5xy=2x+3y+18$$
$$xy+x^2=2x+3y+29,$$

considering only the positive solutions, we find

For the first equation . . . . . $\begin{cases} x=0, & y=1 \\ x=3, & y=4. \end{cases}$

For the second equation . . . . $\begin{cases} x=1, & y=10 \\ x=3, & y=2 \\ x=7, & y=1. \end{cases}$

For the third equation . . . . . $\begin{cases} x=4, & y=21 \\ x=5, & y=7. \end{cases}$

If the remainder, after the division of $-nx^2-px+r$ by $mx+q$, should be zero, equation (1) would be of the form $(mx+q)(ax+by+c)=0$; and we should have all the solutions of this equation by resolving separately the two equations $mx+q=0$, $ax+by+c=0$.

The method which has just been explained is applicable only in case m is not zero.

Let $m=0$; equation (1) gives

$$y=-\frac{nx^2+px-r}{q} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Suppose that one value of $x=a$ (a being a whole number) gives an entire value for y. If we place $x=a+qt$, t being any entire number whatever, we find

$$y=-\frac{na^2+pa-r}{q}-(2nat+nqt^2+pt);$$

by hypothesis, na^2+pa-r is divisible by q; the value of y, corresponding to $x=a+qt$, will be then a whole number. As this conclusion is true, whatever be the sign of t, it follows that, if the equation admits of entire solutions, they will be found to be such as answer to a value of x between 0 and q. Consequently, to obtain all the solutions in whole numbers, it will be sufficient to substitute for x in the equation the numbers 0, 1, 2, 3, . . . $q-1$, and each solution in whole numbers corresponding to one of these numbers will furnish an infinite number of others.

Equation (4), in which the object is to find values of x which render the polynomial nx^2+px-r a multiple of the given number q, M. Gauss calls *congruence* of the second degree; so, also, the equation $ax+by=c$, in which we seek to render $ax-c$ a multiple of b, is a congruence of the first degree.

Further matter on the subject of indeterminate analysis will be given in connection with the theory of numbers, for which see a subsequent part of the work.

MAXIMA AND MINIMA.

196. When a quantity which is capable of changing its value attains such a value that, after having been increasing, it begins to decrease, or, having been decreasing, it begins to increase, in the first case it is called a *maximum*, and in the second a *minimum*. The same quantity may have several maximum or minimum values.

EXAMPLE.

To find what value of x will render the fraction $\frac{x^2-2x+2}{2x-2}$ a maximum or minimum.

Equating the given function of x to z, we have

$$\frac{x^2-2x+2}{2x-2}=z \therefore x=z+1\pm\sqrt{z^2-1}.$$

We perceive at once that by making $z=+1$ we have $x=2$, and that the values of z, a little less than 1, render x imaginary; hence the given expression has a minimum value 1 corresponding to $x=2$.

In a similar manner, making $z=-1$, we have $x=0$; and a negative value of z, a little smaller than 1, would render x imaginary. But in algebra, negative quantities, which, without regard to the sign, go on increasing, ought to be regarded, when the sign is prefixed, as decreasing; we may, therefore, say that a value of z, a little greater than -1, renders x imaginary, then $z=-1$ is a maximum corresponding to $x=0$.

As the subject of maxima and minima is generally treated by the aid of the differential calculus, we shall not dwell further upon it here, though it furnishes one of the applications of equations of the second degree.

THE MODULUS OF IMAGINARY QUANTITIES.

197. We have seen (191) in the equation of the second degree

$$x^2+px+q=0,$$

that when q is positive, and greater than $\frac{p^2}{4}$, the roots are imaginary. Replace $\frac{1}{2}p$ by $-a$, to avoid fractions; and to express that $q>\frac{p^2}{4}$, put $q=a^2+b^2$; the equation will become

$$x^2-2ax+a^2+b^2=0;$$

and, by the formula for the solution of equations of the second degree,

$$x=a\pm\sqrt{-b^2},$$

or

$$x=a\pm b\sqrt{-1} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

The absolute value of the square root of the positive quantity a^2+b^2 is called the modulus of the imaginary expression (1). For example, the modulus of $3-4\sqrt{-1}$ would be $\sqrt{9+16}$, or 5.

Two quantities, such as $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$, which differ from one another only in the sign of the imaginary part, are called *conjugates* of each other. Two conjugate quantities have then the same modulus.

If we make $b=0$, the expression $a+b\sqrt{-1}$ reduces to a. Thus, the formula $x=a+b\sqrt{-1}$ may represent all quantities real or imaginary, a representing the algebraic sum of the real quantities, and b that of the coefficients of $\sqrt{-1}$ in the imaginary terms. When the quantity is real, it has for conjugate an equal quantity, and the modulus is nothing else than the quantity itself, abstraction being made of the sign.

Now I shall proceed to establish two propositions relating to moduli, which may be often useful.

Proposition I.—*The sum and difference of any two quantities whatever have a modulus comprehended between the sum and the difference of their moduli.*

Let there be two expressions $a+b\sqrt{-1}$, $a'+b'\sqrt{-1}$. Calling r and r' their moduli, we have $r^2=a^2+b^2$, $r'^2=a'^2+b'^2$. Naming R the modulus of their sum, we have evidently

$$\begin{aligned} R^2&=(a+a')^2+(b+b')^2 \\ &=a^2+a'^2+b^2+b'^2+2(aa'+bb') \\ &=r^2+r'^2+2(aa'+bb'). \end{aligned}$$

But multiplying r^2 by r'^2, it is easy to see that

$$\begin{aligned} r^2r'^2&=a^2a'^2+b^2b'^2+a^2b'^2+a'^2b^2 \\ &=(aa'+bb')^2+(ab'-ba')^2; \end{aligned}$$

then the numerical value of $aa'+bb'$ is less than, or at most equal to, rr'. Consequently, it is clear that R^2 is comprehended between the two quantities $r^2+r'^2+2rr'$ and $r^2+r'^2-2rr'$, or, what is the same thing, between $(r+r')^2$ and $(r-r')^2$. Then the modulus R is comprehended between the sum and the difference of the moduli r and r'.

The demonstration is precisely the same where, instead of the sum of the imaginary expressions, we consider their difference.

Proposition II.—*The product of two quantities has for modulus the product of the moduli of these quantities.*

In fact, multiplication gives

$$(a+b\sqrt{-1})(a'+b'\sqrt{-1})=aa'-bb'+(ab'+ba')\sqrt{-1};$$

and if we take the modulus of this product, we find, conformably to the enunciation,

$$\begin{aligned} \sqrt{(aa'-bb')^2+(ab'+ba')^2}&=\sqrt{a^2a'^2+b^2b'^2+a^2b'^2+b^2a'^2} \\ &=\sqrt{(a^2+b^2)(a'^2+b'^2)}. \end{aligned}$$

Corollary.—Then the product of any number of factors whatever must have for modulus the product q, the moduli of all the factors. Then the n^{th} power of an imaginary expression has for modulus the n^{th} power of the modulus of that expression.

The above nomenclature and propositions are from Cauchy, who exhibits in a remarkable manner the efficiency of imaginary expressions as instruments in the investigation of the properties of real quantities. The following is a specimen:

If two numbers, of which each is the sum of two squares, be multiplied together, the product must also be the sum of two squares.

Let the two numbers be

$$a^2+b^2 \text{ and } a'^2+b'^2.$$

The first of these may be considered as the product of the factors

$$a+b\sqrt{-1} \text{ and } a-b\sqrt{-1},$$

and the second as the product of the factors,

$$a'+b'\sqrt{-1} \text{ and } a'-b'\sqrt{-1};$$

so that the product of the proposed numbers will be the product of the four factors

$$a+b\sqrt{-1},\ a-b\sqrt{-1},\ a'+b'\sqrt{-1},\ a'-b'\sqrt{-1}.$$

Actually multiplying the first and third, and then the second and fourth, we have the following pair of conjugate expressions, viz.,

$$(aa'-bb')+(ab'+ba')\sqrt{-1},\ (aa'-bb')-(ab'+ba')\sqrt{-1},$$

of which the product is

$$(aa'-bb')^2+(ab'+ba')^2,$$

which is, therefore, the product of the original numbers, and proves that that product must, like each of the proposed factors, be the sum of two squares.

If we interchange the numbers a and b, or the numbers a', b', the terms of the product just deduced will be different; thus, putting a' for b', and b' for a', which produces no essential change in the proposed numbers, we have

$$(a^2+b^2)(a'^2+b'^2)=(aa'-bb')^2+(ab'+ba')^2=(ab'-ba')^2+(aa'+bb')^2.$$

Consequently there are two ways of expressing by the sum of two squares the products of two numbers, each of which is itself the sum of two squares; thus,

$$(5^2+2^2)(3^2+2^2)=11^2+16^2=4^2+19^2$$
$$(2^2+1^2)(3^2+2^2)=\ 4^2+\ 7^2=1^2+\ 8^2$$
&c., &c.

METHOD PROPOSED BY MOUREY FOR AVOIDING IMAGINARY QUANTITIES.*

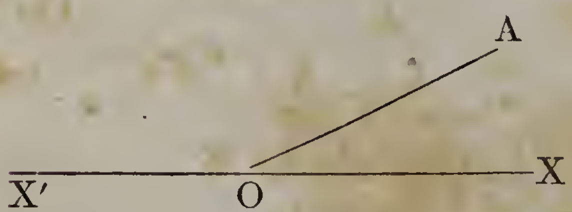
198. Objections have been made to results obtained by the calculus of imaginary expressions. The rules observed in the calculus, it is said, have only been demonstrated for real magnitudes; it is by mere analogy that they are extended to the case of imaginary quantities; we may, therefore, raise reasonable doubts as to the exactitude of the results thus deduced.

M. Mourey, who has been much occupied with these difficulties, has sought to free analysis from them entirely, in a work published in 1828, entitled the *True Theory of Negative Quantities and of the so-called Imaginary Quantities.* Without entering into long details, we shall endeavor here to give an idea of the methods proposed by this author.

Let us resume the expression $a+b\sqrt{-1}$, and give it, at first, the form

$$\sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}+\frac{b}{\sqrt{a^2+b^2}}\sqrt{-1}\right]$$

If we take the sum of the squares of the fractions, which are between the brackets, we find that this sum is equal to 1; and from thence we conclude that these two fractions can be regarded as being the sine and cosine of a same angle α. Designate also the modulus $\sqrt{a^2+b^2}$ by A; the imaginary expression can be put under the form $A(\cos\alpha+\sqrt{-1}\sin\alpha)$. Considering that this expression contains really but two quantities, the modulus A and the angle α, M. Mourey proposes to regard the modulus A as expressing the length of a right line O A, and α as being the angle A O X, which this line makes with a fixed axis O X. In other words, the modulus A represents a line of a certain length, which at first lay upon the axis O X, and which, by making a move-



* To understand this, a knowledge of the first principles of Trigonometry is necessary.

ment round the origin O upward, has departed from this axis by an angle a. M. Mourey gives the name verser to this angle, or, rather, to the arc which measures it; and then, instead of the imaginary expression, he writes simply Aa, a notation very suitable to recall at the same time the modulus A and the verser a. He proposes even to give the name *route*, or *way*, to the length O A, placed in its true position with regard to O X, so that A verser a, or Aa, is the route from O toward A.

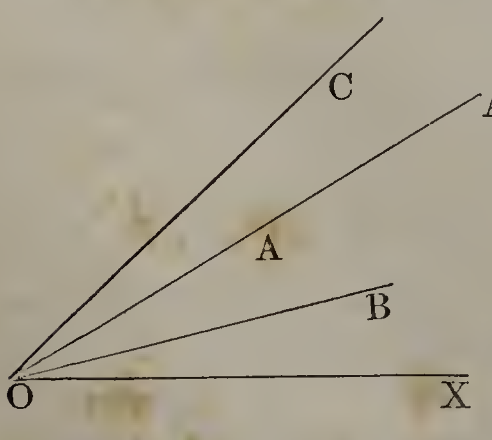
As a line can make around the origin O as many revolutions as we please, and that, also, as well by commencing its rotation below as well as above O X, it follows that the verser may pass through all states of magnitude, and be as well negative as positive. It will be positive when the movement of the line shall have commenced above; it will be negative when the movement commenced below. From this it follows that the same route can be represented with a verser which is positive, or one which is negative, provided that the sum of the versers, abstraction being made of the signs, is 360°.

From the preceding conventions it results that a *way* can be represented by giving to the length A an infinity of different versers. Suppose, to fix the ideas, that O A should be a determinate way, and that then the verser A O X should be an acute angle a; it is evident that the position of O A will undergo no change if we add or subtract from a any number whatever of entire circumferences. Thus is established this important remark, that if we designate by 2π an entire circumference, or 360°, and by n any whole number whatever, positive or negative, the expression $A2\pi n+a$ will represent the same route as Aa; this is expressed by the equality

$$A2\pi n+a=Aa.$$

When we give to A a verser equal to zero, the length A lies upon the line O X. When the verser is equal to π or 180°, this length is found in the opposite direction, O X′; then it is nothing else than the negative quantity $-A$. Thus we ought to regard as altogether equivalent the two expressions $-A$ and $A\pi$.

After these preliminaries, M. Mourey establishes the rules of algebraic calculus; then he passes to equations, and reconstructs algebra thus entirely. I shall not follow this author in all his details; I shall confine myself to the developments necessary to explain here what sense the new algebra attaches to the old imaginary expression $\sqrt{-A^2}$. I shall seek, first, the rule to be followed in the multiplication of any two quantities whatever, Aa and $B\beta$. Here the two factors are the magnitudes A and B, measured upon two lines O A and O B, which make, with a fixed axis O X, angles A O X, B O X, represented by the versers a and β. It is necessary, then, first of all, to give to the definition of multiplication the extension suitable to render it applicable to the case in question. But, considering that the multiplier $B\beta$ indicates a line B, which departs from the fixed line O X by an angle equal to β, M. Mourey regards multiplication as having for its object to take at first the length A in its actual direction as many times as there are units in B, and to turn the new line O A′ around the point O, to depart from this direc-



tion by an angle equal to β, and to give it the position O C. From this it follows that, in designating by A B the product of the two magnitudes, obstraction being made of all idea of position, the product sought will be $(AB)\alpha+\beta$.
Thus we have

$$A\alpha \times B\beta = (AB)\alpha+\beta\,;$$

that is to say, we *multiply the moduli according to the ordinary rules of arithmetic, and take the sum of the versers.*

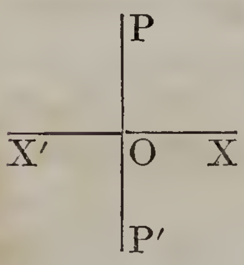
If the two versers are equal to π or 180°, we shall have $A\pi \times B\pi = (AB)2\pi$. But $A\pi$ and $B\pi$ are nothing else than $-A$ and $-B$, and $(AB)2\pi$ is the same thing as $+AB$; then $-A \times -B = +AB$. This is the known rule, $-$ by $-$ gives $+$.

According to this rule, the square of $A\alpha$ will be $(A^2)2\alpha$; that is to say, *we take the square of the modulus and double the verser.* Then, reciprocally, the square root is obtained by *extracting the square root of the modulus without regarding the verser; then take half the verser.*

Let us come now to the interpretation of the imaginary expression $\sqrt{-A^2}$. For this purpose, let us observe, first, that it is equivalent to $\sqrt{(A^2)2n\pi+\pi}$; then extracting the square root,

$$\sqrt{-A^2} = An\pi + \tfrac{1}{2}\pi.$$

If n is even, the verser $n\pi+\frac{1}{2}\pi$ places the length A in the same position as $\frac{1}{2}\pi$; that is to say, in the position O P, perpendicular to O X. If n is uneven, the verser $n\pi+\frac{1}{2}\pi$ will place the length A in a position O P′, perpendicular to O X, but below. Thus, in the system of M. Mourey, the expression $\sqrt{-A^2}$ offers no longer to the mind any idea of impossibility. It represents two routes, O P and O P′, equal and opposite, both perpendicular to the fixed axis O X.



PERMUTATIONS AND COMBINATIONS.

199. THE *Permutations* of any number of quantities are the changes which these quantities may undergo with respect to their order.

Thus, if we take the quantities a, b, c; then abc, acb, bac, bca, cab, cba are the permutations of these three quantities taken *all together*; ab, ac, ba, bc, ca, cb are the permutations of these quantities taken *two* and *two*; a, b, c are the permutations of these quantities taken singly, or *one* and *one*, &c.

The problem which we propose to resolve is,

200. *To find the number of the permutations of* n *quantities, taken* p *and* p *together.*

Let $a, b, c, d, \ldots\ldots\ldots k$, be the n quantities.

The number of the permutations of these n quantities taken singly, or one and one, is manifestly n.

The number of the permutations of these n quantities, taken two and two together, will be $n(n-1)$. For, since there are n quantities,

$$a, b, c, d, \ldots\ldots\ldots k.$$

If we remove a there will remain $(n-1)$ quantities,

$$b, c, d, \ldots\ldots\ldots k.$$

Writing a before each of these $(n-1)$ quantities, we shall have

$$ab, ac, ad, \ldots\ldots\ldots ak;$$

that is, $(n-1)$ permutations of the n quantities taken two and two, in which a stands first. Reasoning in the same manner for b, we shall have $(n-1)$ permutations of the n quantities taken two and two, in which b stands first, and so on for each of the n quantities in succession; hence the whole number of permutations will be

$$n(n-1).$$

The number of the permutations of n quantities, taken three and three together, is $n(n-1)(n-2)$. For since there are n quantities, if we remove a there will remain $(n-1)$ quantities; but, by the last case, writing $(n-1)$ for n, the number of the permutations of $(n-1)$ quantities, taken two and two, is $(n-1)(n-2)$; writing a before each of these $(n-1)(n-2)$ permutations, we shall have $(n-1)(n-2)$ permutations of the n quantities, taken three and three, in which a stands first. Reasoning in the same manner for b, we shall have $(n-1)(n-2)$ permutations of the n quantities, taken three and three, in which b stands first, and so on for each of the n quantities in succession; hence the whole number of permutations will be

$$n(n-1)(n-2).$$

In like manner, we can prove that the number of permutations of n quantities, taken four and four, will be

$$n(n-1)(n-2)(n-3).$$

Upon examining the above results, we readily perceive that a certain relation exists between the numerical part of the expressions and the class of permutations to which they correspond.

Thus the number of permutations of n quantities, taken *two and two*, is

$n(n-1)$, which may be written under the form $n(n-2+1)$.

Taken *three and three*, it is

$n(n-1)(n-2)$, which may be written under the form $n(n-1)(n-3+1)$.

Taken *four and four*, it is

$n(n-1)(n-2)(n-3)$, which may be written under the form $n(n-1)(n-2)(n-4+1)$.

Hence, from *analogy*, we may conclude that the number of permutations of n things, taken p and p together, will be

$$n(n-1)(n-2)(n-3)\ldots\ldots\ldots\ldots(n-p+1).$$

In order to *demonstrate* this, we shall employ the same species of proof already exemplified in (Arts. 23 and 78), and show that, if the above law be assumed to hold good for any one class of permutations, it must necessarily hold good for the class next superior.

Let us suppose, then, that the expression for the number of the permutations of n quantities, taken $(p-1)$ and $(p-1)$ together, is

$$n(n-1)(n-2)(n-3)\ldots\{n-(p-1)+1\}\ldots \text{(A)}$$

It is required to prove that the expression for the number of the permutations of n quantities, taken p and p together, will be

$$n(n-1)(n-2)(n-3)\ldots\ldots\ldots\ldots(n-p+1).$$

Remove a, one of the n quantities $a, b, c, d \ldots\ldots\ldots\ldots k$, then, by the expression (A), writing $(n-1)$ for n, the number of the permutations of the $(n-1)$ quantities $b, c, d \ldots\ldots\ldots\ldots k$, taken $(p-1)$ and $(p-1)$, will be

$$(n-1)(n-2)(n-3) \ldots\ldots \{(n-1)-(p-1)+1\},$$

or

$$(n-1)(n-2)(n-3) \ldots\ldots\ldots\ldots (n-p+1).$$

Writing a before each of these $(n-1)(n-2)(n-3)\ldots\ldots\ldots(n-p+1)$ permutations, we shall have $(n-1)(n-2)(n-3)\ldots\ldots\ldots(n-p+1)$ permutations of the n quantities, in which a stands first. Reasoning in the same manner for b, we shall have $(n-1)(n-2)(n-3)\ldots\ldots\ldots(n-p+1)$ permutations of the n quantities, in which b stands first; and so on for each of the n quantities in succession; hence the whole number of permutations will be

$$n(n-1)(n-2)(n-3) \ldots\ldots (n-p+1) \ldots\ldots (1)$$

Hence it appears that, if the above law of formation hold good for any one class of permutations, it must hold good for the class next superior; but it has been proved to hold good when $p=2$, or for the permutations of n quantities taken two and two; hence it must hold good when $p=3$, or for the permutation of n quantities taken three and three; $\therefore$ it must hold good when $p=4$, and so on. The law is, therefore, *general.*

EXAMPLE.

Required the number of the permutations of the eight letters a, b, c, d, e, f, g, h, taken 5 and 5 together.

Here $n=8$, $p=5$, $n-p+1=4$; hence the above formula

$$n(n-1)(n-2) \ldots\ldots (n-p+1)=8\times7\times6\times5\times4=6720,$$

the number required.

201. In formula (1) let $p=n$, it will then become

$$n(n-1)(n-2) \ldots\ldots\ldots\ldots 2.1,$$

or

$$1.2.3 \ldots\ldots\ldots (n-1)n \ldots\ldots\ldots (2)$$

which expresses the number of the permutations of n quantities taken all together.*

EXAMPLE.

Required the number of the permutations of the eight letters a, b, c, d, e, f, g, h.

Here $n=8$; hence the above formula (2) in this case becomes

$$1.2.3.4.5.6.7.8=40320,$$

the number required.

202. The number of the permutations of n quantities, supposing them all different from each other, we have found to be

$$1.2.3 \ldots\ldots\ldots\ldots\ldots\ldots (n-1)n.$$

But if the same quantity be repeated a certain number of times, then it is manifest that a certain number of the above permutations will become identical.

Thus, if one of the quantities be repeated a times, the number of identical permutations will be represented by $1.2.3\ldots\ldots\ldots a$; and hence, in order to

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* Many writers on algebra confine the term *permutations* to this class where the quantities are taken *all together*, and give the title of *arrangements* or *variations* to the groups of the n quantities when taken *two and two, three and three, four and four*, &c. The introduction of these additional designations appears unnecessary; but, in using the word *permutations* absolutely, we must always be understood to mean those represented by formula (2), unless the contrary be specified.

obtain the number of permutations different from each other, we must divide (2) by $1.2.3 \ldots\ldots a$, and it will then become

$$\frac{1.2.3\ldots\ldots(n-1)n}{1.2.3\ldots\ldots a}.$$

If one of the quantities be repeated a times, and another of the quantities be repeated β times, then we must divide by $1.2\ldots\ldots a \times 1.2\ldots\ldots \beta$; and, in general, if among the n quantities there be a of one kind, β of another kind, γ of another kind, and so on, the expression for the number of the permutations different from each other of these n quantities will be

$$\frac{1.2.3\ldots\ldots\ldots\ldots n}{1.2\ldots a \times 1.2\ldots\ldots \beta \times 1.2\ldots\ldots\gamma, \text{ \&c.}} \ldots\ldots (3)$$

EXAMPLE I.

Required the numbers of the permutations of the letters in the word *algebra.*

Here $n=7$, and the letter a is repeated twice; hence formula (3) becomes

$$\frac{1.2.3.4.5.6.7}{1.2}=2520, \text{ the number required.}$$

EXAMPLE II.

Required the number of the permutations of the letters in the word *caifacarataddarada.*

Here $n=18$, a is repeated eight times, c twice, d thrice, r twice; hence the number sought will be

$$\frac{1.2.3.4.5.6.7.8.9.10.11.12.13.14.15.16.17.18}{1.2.3.4.5.6.7.8 \times 1.2 \times 1.2.3 \times 1.2}=6616209600.$$

EXAMPLE III.

Required the number of the permutations of the product $a^x\, b^y\, c^z$, written at full length.

Here $n=x+y+z$, the letter a is repeated x times, the letter b, y times, and the letter c, z times; the expression sought will, therefore, be

$$\frac{1.2.3\ldots\ldots\ldots\ldots(x+y+z)}{1.2.3\ldots\ldots x \times 1.2.3\ldots\ldots y \times 1.2.3\ldots\ldots z}.$$

203. The *Combinations** of any number of quantities signify the different collections which may be formed of these quantities, without regard to the *order* in which they are arranged in each collection. Each combination must, therefore, have one letter different from any other of the combinations.

Thus the quantities a, b, c, when taken *all together*, will form only one combination, abc; but will form six different permutations, abc, acb, bac, bca, cab, cba; taken *two and two*, they will form the three combinations ab, ac, bc, and the six permutations ab, ba, ac, ca, bc, cb.

The problem which we propose to resolve is,

To find the number of the combinations of n *quantities, taken* p *and* p *together.*

Each of these combinations of p quantities being separately permutated, will furnish $1.2.3\ldots p$ permutations, which, multiplied by the whole number of combinations, will give the whole number of permutations of n quantities, taken

* Where numerical or literal factors are combined, the term combination may be considered as signifying the same as product.

p and p. Therefore the latter, namely, the *whole number of permutations, or* n(n—1)(n—2)....(n—p+1), *divided by the number of permutations of each combination, or* 1.2.3...p, *will give the number of combinations of* n *quantities, taken* p *and* p. Denoting it by C, we have

$$C=\frac{n(n-1)(n-2)\ldots\ldots\ldots\ldots(n-p+1)}{1.2.3\ldots\ldots\ldots\ldots\ldots(p-1)p}\ldots\ldots(4)$$

204. There is a species of notation employed to denote permutations and combinations, which is sometimes used with advantage from its conciseness.

The number of the permutations of n quantities, taken p and p, are represented by . . . . (nPp)

The number of the permutations of n quantities, taken *all together*, are represented by . . . . (nPn)

The number of the combinations of n quantities, taken p and p, are represented by . . . . (nCp)

and so on. It is manifest that the above proposition may be expressed according to this notation by

$$(n\mathrm{C}p)=\frac{(n\mathrm{P}p)}{(p\mathrm{P}p)}.$$

M. Cauchy employs the notation $(m)_n$ to express the number of combinations of m letters, taken n at a time. The German notation for the same is $\overset{n}{\underset{m}{\mathrm{C}}}$.

When the series of natural numbers, or the letters of the alphabet up to any required number, are to be permuted or combined, an abbreviated notation has been employed as follows:

P(1, 2, 3) stands for 123, 132, 213, 231, 312, 321.

$\overset{2}{\mathrm{P}}$(1..4) stands for 12, 13, 14, 21, 23, 24, 31, 32, 34, 41, 42, 43.

$\overset{3}{\mathrm{C}}$(a...e) stands for *abc, abd, abe, acd, ace, ade, bcd, bce, bde, cde.*

If one or more of the numbers or letters may be repeated, this can also be expressed in the notation. Thus,

P(1, 1, 2)=112, 121, 211.

$\overset{2}{\mathrm{P}}$(1, 1, 2, 3)=11, 12, 13, 21, 23, 31, 32.

$\overset{3}{\mathrm{C}}$(1, 1, 2, 2, 3)=112, 113, 122, 123, 223.

If all the letters, numbers, or single things may be repeated an equal number of times, this can be expressed with the aid of an exponent; thus,

$$\overset{6}{\mathrm{C}}(1,\ 2,\ 3)^5,\ \overset{3}{\mathrm{P}}(0,\ 1,\ 2)^2,\ \overset{4}{\mathrm{C}}(1..7)^n.$$

205. If n single things be arranged in combinations of k, or of $n-k=r$, the number of combinations in either case will be the same, *i. e.*,

$$\overset{k}{\underset{n}{\mathrm{C}}}=\frac{n(n-1)\ldots(n-k+1)}{1.2.3\ldots k}=\overset{r}{\underset{n}{\mathrm{C}}}=\frac{n(n-1)\ldots(n-r+1)}{1.2.3\ldots r},$$

for every new combination of k letters must leave a new one of r letters.

By a similar reasoning, if n be divided into three parts, the first k, the second r, and the third s, it may be shown that

$$\overset{k}{\underset{n}{\mathrm{C}}}\times\overset{r}{\underset{n-k}{\mathrm{C}}}=\overset{k}{\underset{n}{\mathrm{C}}}\times\overset{s}{\underset{n-k}{\mathrm{C}}}=\overset{r}{\underset{n}{\mathrm{C}}}\times\overset{k}{\underset{n-r}{\mathrm{C}}},\ \&c.$$

206. Cases may occur in which not all possible combinations, but only such

as fulfill certain conditions, are required. Many such may be imagined. For instance, where the numbers to be combined increase by a common difference, or by a common ratio, as 1357, 2468, or 124, or 248. The most useful case is where the number in each combination must amount to the same sum. The method of proceeding in this case is to fill up all the places except the last with the lowest numbers, the last place being occupied by the supplementary number necessary to produce the given sum; then diminishing the last number and increasing one of the preceding by the same amount, taking care not to allow a lower ever to follow a higher number. We give examples of such combinations, the general formula for which is ${}^{r}\overset{k}{C}(1\ldots.n)$.

(1) ${}^{10}\overset{3}{C}(1\ldots7)=127,\ 136,\ 145,\ 235.$

(2) ${}^{14}\overset{4}{C}(1\ldots8)=1238,\ 1247,\ 1256,\ 1346,\ 2345.$

(3) ${}^{5}\overset{4}{C}(0..5)n=0005,\ 0014,\ 0023,\ 0113,\ 0122,\ 1112.$

(4) ${}^{20}\overset{5}{C}(3\ldots.)n=33338,\ 33347,\ 33356,\ 33446,\ 33455,\ 34445,\ 44444.$

It is easy to be perceived that in two cases this kind of combination is impossible. 1°. When the highest form does not amount to the required sum; and, 2°. When the lowest form exceeds it, as in

$${}^{10}\overset{3}{C}(123)n, \text{ or } {}^{10}\overset{3}{C}(4\ldots)n.$$

207. Similar conditions may be imposed upon permutations. In order that the permutations of a given series of numbers, taken a certain number at a time, should amount always to a given sum, the same rule will apply, with this difference, that lower numbers may follow higher; in other words, the combinations formed by the previous rule may each be permuted.

The following examples will render this more intelligible:

(1) ${}^{9}\overset{2}{P}(1..8)=18,\ 27,\ 36,\ 45,\ 54,\ 63,\ 72,\ 81.$

(2) ${}^{7}\overset{3}{P}(1\ldots)=124,\ 142,\ 214,\ 241,\ 412,\ 421.$

(3) ${}^{6}\overset{4}{P}(1\ldots)n=1113,\ 1122,\ 1131,\ 1212,\ 1221,\ 1311,\ 2112,\ 2121,\ 2211,\ 3111.$

(4) ${}^{4}\overset{3}{P}(0..)n=013,\ 022,\ 031,\ 103,\ 112,\ 121,\ 130,\ 202,\ 211,\ 220,\ 301,\ 310.$

Under this head, also, two contradictory cases occur: 1°. When the highest form amounts to too little; and, 2°. When the lowest form amounts to too much. As, for instance, in

$${}^{9}\overset{2}{P}(1..4)n, \text{ or } {}^{9}\overset{2}{P}(5\ldots)n.$$

208. The applications of the theory of permutations and combinations are numerous. One of the most useful is the determination of the coefficients of a series of the form

$$a+bx+cx^2+dx^3+ex^4+\ldots+kx^n\ldots,^*$$

especially the coefficients of the binomial formula, the method of determining which, by the theory of permutations and combinations, will be given hereafter.

Another extensive application of the theory of permutations and combina-

* These coefficients are supposed to depend upon some given law. A common case is when the number of factors combined in each coefficient is indicated by the exponent of the letter of arrangement, x.

tions is to be found in geometric relations, such as where the combinations of a certain number of points, lines, angles, &c., from among a given number of these, are required.

Not less useful is this theory in natural science: as in crystalography, when the manifold forms of crystals are required; in chemistry, when the various combinations of chemical elements; and in music, of consonant tones, &c.

But perhaps its most important use is in the doctrine of chances, or, as it is mathematically named, the

CALCULUS OF PROBABILITIES.

The outlines of this extensive subject we shall here briefly indicate, referring the student for further information to the admirable treatises of La Place and Lacroix, and to the practical work of De Morgan.

I. Let there be among m possible cases g, which, as fulfilling certain requisitions, are considered as favorable, $(m-g)=u$ unfavorable. Then the ratio of the favorable to all possible cases is called the *mathematical probability* for the occurrence of a favorable case. The ratio of the unfavorable to all possible cases is the mathematical improbability of the occurrence. If the first be expressed by w, the second by v, then

$$w=\frac{g}{m} \text{ and } v=\frac{u}{m} \quad \ldots\ldots\ldots \quad \text{(I.)}$$

The probability is, therefore, the less, the smaller the number of the favorable in comparison with that of all possible cases, and *vice versâ*. Should all possible cases be favorable, then $w=1$, which is, therefore, the expression for certainty. Thus the mathematical probability and improbability of a pictured card, of which there are 12, being drawn from 52, are expressed by

$$w=\frac{12}{52}=\frac{3}{13},\ v=\frac{40}{52}=\frac{10}{13};$$

that of drawing one card from 52,

$$w=\frac{52}{52}=1.$$

II. Let there be among m possible cases g favorable, of different (first, second, third, &c.) kinds, expressed by g_1, g_2, g_3, &c., the partial probabilities by w_1, w_2, w_3, &c.; then

$$w=w_1+w_2+w_3+, \&c., =\frac{g_1+g_2+g_3+, \&c.}{m} \quad \ldots\ldots \quad \text{(II.)}$$

that is, the probability of one of several different kinds is equal to the sum of their partial probabilities. Thus, for the probability of one of the six faces of a die, marked 1, 2, or 3, being thrown, we have

$$w_1=\frac{1}{6},\ w_2=\frac{1}{6},\ w_3=\frac{1}{6};$$

$$\therefore w=\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{3}{6}=\frac{1}{2}.$$

III. Let the occurrence be favorable only on the supposition that two or more of the single favorable cases concur, then the formula for the compound probability is

$$w=w_1\times w_2\times w_3\cdot\cdot=\frac{g_1\times g_2\times g_3\cdot\cdot}{m_1\times m\ \ \times m_3\cdot\cdot} \quad \ldots\ldots\ldots \quad \text{(III.)}$$

in which m_1, m_2, m_3, &c., express the possible cases of the partial occurrences; that is, the probability of the compound occurrence is equal to the products of the partial probabilities. For as each of the m_1 may concur with each of the m_2 cases, there will be $m_1 \times m_2$ possible cases, which, by the supervening of m_3 new cases, increase to $m_1 \times m_2 \times m_3$, and so on. The same reasoning applies to the favorable cases g_1, g_2, g_3, &c., from whence, by the principles already established, results formula (III.). Let it be required, for example, to draw out of a vase which contains the numbers 1, 2, 3, 4, 5, and 6, first 1, then either 2 or 3, and, finally, 4, 5, or 6, in three drawings; the probability is expressed by

$$w=\frac{1}{6}\times\frac{2}{5}\times\frac{3}{4}=\frac{1}{20}.$$

If the partial occurrences are equal (that is, repetitions of the same), then $w=\left(\frac{g}{m}\right)^n$. Thus, if with each of three dice, 6 shall be thrown,

$$w=\left(\frac{1}{6}\right)^3=\frac{1}{216}.$$

IV. Should there be m possible cases, of which g are favorable and u unfavorable, and of these $k+r$ are to occur, so that k of the favorable, with r of the unfavorable, must come in juxtaposition, then the expression for the probability of the occurrence of every such order is

$$w=\left(\frac{g}{m}\right)\left(\frac{g-1}{m-1}\right)\cdots\left(\frac{g-k+1}{m-k+1}\right)\times\left(\frac{u}{m-k}\right)\left(\frac{u-1}{m-k-1}\right)\cdots\left(\frac{u-r+1}{m-k-r+1}\right) \text{(IV.)}$$

This depends on (III.), each of the factors in the above value of w expressing the partial probability of the single occurrence of a 1st, 2d, ...kth favorable case, also of a 1st, 2d, ...rth unfavorable case, and the product expressing the probability of these occurring in a certain order.

EXAMPLE.

If from 20 tickets, 8 of which are prizes and 12 blanks, 6 are to be drawn; then, in favor of the requisition that exactly two prizes shall be first drawn, or shall occupy any given place in the order,

$$w=\left(\frac{8}{20}\right)\left(\frac{7}{19}\right)\times\left(\frac{12}{18}\right)\left(\frac{11}{17}\right)\left(\frac{10}{16}\right)\left(\frac{9}{15}\right)=\frac{77}{3230}.$$

V. Should there be required in the supposition of the last case no particular order for the single cases which occur, the expression becomes

$$w=\overset{k}{\underset{k+r}{C}}\cdot\left(\frac{g}{m}\right)\cdots\left(\frac{g-k-1}{m-k-1}\right)\cdot\left(\frac{u}{m-k}\right)\cdots\left(\frac{u-r+1}{m-k-r+1}\right).\ \ldots\ \text{(V.)}$$

Thus it will be found that, if from 30 appointed numbers out of 90, 5 of the whole 90 are to be drawn, so that just 3 of the 30 shall be among those drawn, it being immaterial at which three of the five drawings, the expression for the probability in this case is

$$w=\left(\frac{5.4.3}{1.2.3}\right)\cdot\left(\frac{30}{90}\right)\left(\frac{29}{89}\right)\left(\frac{28}{88}\right)\cdot\left(\frac{60}{87}\right)\left(\frac{59}{86}\right)=\frac{20650}{126291}.$$

VI. Should the number of possible cases continue to remain the same, while the other circumstances are as in (V.), the formula would be

$$\overset{k}{\underset{k+r}{C}}\left(\frac{g}{m}\right)^k\cdot\left(\frac{u}{m}\right)^r \ldots\ldots\ldots\ldots \text{(VI.)}$$

EXAMPLE.

The probability of throwing the same face three times in 7 casts of a die, or one cast of 7 dice, would be expressed by

$$\frac{7.6.5}{1.2.3}\left(\frac{1}{6}\right)^3.\left(\frac{5}{6}\right)^4=\frac{21875}{279936}.$$

VII. Let the probability be required that of two different occurrences the first, or, if this does not, the second, shall happen; if the single probability of the first happening be expressed by w, the probability of its failing will be expressed by $1-w$; this must be combined with the probability of the second happening, according to (III.), giving

$$(1-w_1)w_2$$

for the probability of the second happening, if the first fails: then the compound probability required is expressed (II.) by

$$w=w_1+w_2(1-w_1)=w_1+w_2-w_1.w_2.$$

EXAMPLE.

Required the probability of throwing with two dice, at the first cast 8, and, if this does not happen, 9 at the second cast.

$$w=\frac{5}{36}*+\frac{4}{36}\left(1-\frac{5}{36}\right)=\frac{5}{36}+\frac{4}{36}.\frac{31}{36}=\frac{19}{81}.$$

VIII. Above we have considered the *absolute* probability of the happening of an event; the relative probability of the happening of two events is expressed by the formula

$$\frac{w_1}{w_1+w_2}, \text{ or } \frac{w_2}{w_1+w_2}.$$

EXAMPLE.

The relative probability of throwing with two dice rather 7 than 10, is expressed by $\frac{w_1}{w_1+w_2}=\frac{6}{6+3}=\frac{2}{3}$.

IX. When money depends on the happening of an event, the product of the sum risked, multiplied by the expression for the probability of the event on which it depends, is called the *mathematical expectation.* If there be among m_1+m_2 cases, m_1 favorable for one party, and m_2 for the other, the sum risked by the first a_1, and by the second a_2, then for the mathematical expectation of each we have

$$e_1=\frac{m_1}{m_1+m_2}.a_2=w_1a_2 \quad . . . \ (1) \qquad e_2=\frac{m_2}{m_1+m_2}.a_1=w_2a_1 \quad . . . \ (2)$$

Therefore, when $e_1=e_2$, it is necessary that $a_1:a_2=w_1:w_2$. This principle is important in the subject of annuities and life insurance. For its application, and that of all the foregoing theory to which, see De Morgan on Probabilities.

EXAMPLES.

(1) How many binary combinations of oxygen, hydrogen, nitrogen, carbon, sulphur, and phosphorus? How many ternary combinations of the same?

(2) How many combinations of 5 colors among those of the prism, viz., red, orange, yellow, green, blue, indigo, and violet?

* 12 and 2 can each be thrown with two dice but in one way, 11 and 3 each in two ways, 10 and 4 in three ways, 5 and 9 in four ways, 6 and 8 in five ways, 7 in six ways.

(3) What is the probability of throwing with three dice two equal numbers? with five dice, three equal?

(4) What of throwing with two dice the faces 2, 4, and 6?

(5) What the probability that a dollar tossed twice will fall head up once?

(6) Of which is the probability greater, the drawing at three trials from 52 cards three cards of different colors, of which there are four, or three face cards, of which there are 12?

(7) What of drawing out of a vase containing 5 white, 6 red, and 7 black balls, in two drawings, 2 red, or else a white and a black ball?

(8) What of drawing out of the same vase, in three drawings, 3 of different colors, or else 2 black and 1 white?

(9) What of throwing with four dice 15, or with three dice 12?

METHOD OF UNDETERMINED COEFFICIENTS.

209. The method of undetermined coefficients is a method for the expansion or development of algebraic functions into infinite series, arranged according to the ascending powers of one of the quantities considered as a variable.* The principle employed in this method may be stated in the following

THEOREM.

If $Ax^{\alpha}+Bx^{\beta}+Cx^{\gamma}+$, &c., $=A'x^{\alpha'}+B'x^{\beta'}+C'x^{\gamma'}+$, &c. (1), for all values of x, then must the exponents of x in the two members be the same, and the coefficients of the same powers of x the same. For, dividing (1) by x^{α}, we have

$$A+Bx^{\beta-\alpha}+Cx^{\gamma-\alpha}+, \text{ \&c.}, =A'x^{\alpha'-\alpha}+B'x^{\beta'-\alpha}+C'x^{\gamma'-\alpha}+, \text{ \&c.} \quad (2)$$

Since x may have any value, make it zero; the first member thus reduces to A, while the second becomes zero, unless we suppose α equal to some one of the exponents α', β', γ', ... Suppose it to be α'. Then we have $\alpha=\alpha'$, and $\therefore A=A'$. Suppressing the equal terms A and $A'x^{\alpha'-\alpha}$ from the two members of (2), and dividing it by $x^{\beta-\alpha}$, it becomes

$$B+Cx^{\gamma-\beta}+, \text{ \&c.}, =B'x^{\beta'-\beta}+Cx^{\gamma'-\beta}+, \text{ \&c.}$$

Making, again, $x=0$, the first member reduces to B, and the second to zero, which is absurd, unless we make β equal to some one of the exponents of x, say β', in the second member, and then $B=B'$. Proceeding in this way, the exponents of x, and the coefficients of the same powers of x in the one member, may be proved equal to those in the other.

The above theorem may be expressed in a modified form; thus, if all the terms of (1) be transposed to the first member, it becomes, collecting the equal powers of x, α and α', β and β', &c.,

$$(A-A')x^{\alpha}+(B-B')x^{\beta}+(C-C')x^{\gamma}+, \text{ \&c.}, =0;$$

from which, since $A=A'$, $B=B'$, &c., we perceive that when a function of x is equal to zero for all values of x, the coefficients of the different powers of x are equal to zero separately.

EXAMPLES.

(1). Expand the fraction $\dfrac{1}{1-2x+x^2}$ into an infinite series.

Assume $$\frac{1}{1-2x+x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\ldots,$$

* A variable quantity is one which is either entirely indeterminate, so that it may have any value at pleasure, or one which varies in conformity with certain conditions imposed.

in which some of the coefficients A, B, C, &c., may be zero, and thus certain powers of x be wanting; then, multiplying by $1-2x+x^2$, we have

$$\begin{aligned}1=A+Bx+Cx^2+Dx^3+Ex^4+\ldots\\ -2Ax-2Bx^2-2Cx^3-2Dx^4-\ldots\\ +Ax^2+Bx^3+Cx^4+\ldots\end{aligned}$$

Hence, by the preceding theorem, we have

$$\begin{array}{rl} A=1 \quad \therefore A= \;\ldots & =1 \\ B-2A=0 \quad B=2A & =2 \\ C-2B+A=0 \quad C=2B-A & =3 \\ D-2C+B=0 \quad D=2C-B & =4 \\ E-2D+C=0 \quad E=2D-C & =5 \\ \&c. \qquad \&c. & \end{array}$$

Therefore $\quad \dfrac{1}{1-2x+x^2}=1+2x+3x^2+4x^3+5x^4+6x^5+\ldots\ldots$

The equality of a function to a series is hypothetical; and after A, B, C, ... have been found, the result must be carefully examined. If we put the function $\dfrac{1}{3x-x^2}=A+Bx+$, &c., it gives the absurdity $-1=0$. We must put $\dfrac{1}{3x-x^2}=Ax^{-1}+Bx^0+Cx+Dx^2+$, &c. The method of indeterminate coefficients is to be avoided where other methods will apply.

(2) Extract the square root of $1+x$.

Assume $\sqrt{1+x}=A+Bx+Cx^2+Dx^3+\ldots$, and square both sides;

$$\begin{aligned}\therefore 1+x=A^2+ABx+ACx^2+ADx^3+AEx^4+\ldots\\ +ABx+B^2x^2+BCx^3+BDx^4+\ldots\\ +ACx^2+BCx^3+C^2x^4+\ldots\\ +ADx^3+BDx^4+\ldots\\ +AEx^4+\ldots\end{aligned}$$

Hence, equating the coefficients of the like powers of x, we have

$$\begin{array}{rl} A^2=1 & \therefore A= 1 \\ 2AB=1 & B=\dfrac{1}{2A}=\dfrac{1}{1\cdot 2}=\dfrac{1}{2} \\ 2AC+B^2=0 & C=-\dfrac{B^2}{2A}=-\dfrac{1}{2\cdot 4}=-\dfrac{1}{8} \\ 2AD+2BC=0 & D=-\dfrac{BC}{A}=\dfrac{1}{2\cdot 8}=\dfrac{1}{16} \\ 2AE+2BD+C^2=0 & E=-\dfrac{2BD+C^2}{2A}=-\dfrac{1}{2}\left\{\dfrac{1}{16}+\dfrac{1}{64}\right\}=-\dfrac{5}{128}, \\ \&c. & \qquad \&c. \end{array}$$

Therefore $\quad \sqrt{1+x}=\pm(1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4+\ldots)$.

(3) Decompose $\dfrac{3x-5}{x^2-13x+40}$ into two fractions having simple binomial denominators.

By quadratics we find $x^2-13x+40=(x-5)(x-8)$; hence we may assume

$$\frac{3x-5}{x^2-13x+40}=\frac{A}{x-5}+\frac{B}{x-8}=\frac{A(x-8)+B(x-5)}{(x-5)(x-8)}=\frac{(A+B)x-8A-5B}{x^2-13x+40};$$

$$\therefore 3x-5=(A+B)x-(8A+5B);$$

and by the principle of undetermined coefficients we have

$$A+B=3, \text{ and } 8A+5B=5.$$

Whence $A=-\frac{10}{3}$ and $B=\frac{19}{3}$; and therefore we get

$$\frac{3x-5}{x^2-13x+40}=\frac{6\frac{1}{3}}{x-8}-\frac{3\frac{1}{3}}{x-5}=\frac{19}{3}\cdot\frac{1}{x-8}-\frac{10}{3}\cdot\frac{1}{x-5}.^*$$

Note.—The values of A and B might have been determined in the following manner:

Since $$\frac{3x-5}{x^2-13x+40}=\frac{A}{x-5}+\frac{B}{x-8}=\frac{A(x-8)+B(x-5)}{x^2-13x+40};$$

$$\therefore\ 3x-5=A(x-8)+B(x-5).$$

Now this equation must subsist for every value of x; and therefore,

if $x=5$, we have $15-5=A(5-8)$; $\therefore\ A=\frac{15-5}{5-8}=-\frac{10}{3}$;

if $x=8$, we have $24-5=B(8-5)$; $\therefore\ B=\frac{24-5}{8-5}=\frac{19}{3}$.

This method may frequently be employed with advantage, and will be found useful in the integration of rational fractions, in the Integral Calculus.

EXAMPLES FOR EXERCISE.

(1) Expand $\frac{1-x}{1+x}$ into an infinite series.

Ans. $1-2x+2x^2-2x^3+2x^4-2x^5+\ldots\ldots$

(2) Expand $\sqrt{a^2-x^2}$ in a series.

Ans. $a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}-\ldots\ldots$

(3) Find the development of $\frac{1-x}{1+x+x^2}$.

Ans. $1-2x+x^2+x^3-2x^4+x^5+x^6-2x^7+\ldots\ldots$

(4) Decompose the fraction $\frac{2x+3}{x^3+x^2-2x}$.

Ans. $-\frac{3}{2x}-\frac{1}{6(x+2)}+\frac{5}{3(x-1)}$.

(5) Expand the fraction $\frac{1+2x}{1-3x}$ in a series.

Ans. $1+5x+15x^2+45x^3+135x^4+\ldots\ldots$

(6) Resolve $\frac{x^2}{(x+1)(x+2)(x+3)}$ into partial fractions.

Ans. $\frac{1}{2(x+1)}-\frac{4}{x+2}+\frac{9}{2(x+3)}$.

(7) Resolve $\frac{13+21x+2x^2}{1-5x^2+4x^4}$ into partial fractions.

Ans. $\frac{1}{1+x}-\frac{6}{1-x}+\frac{2}{1+2x}+\frac{16}{1-2x}$.

* When the denominator is composed of equal factors, such as $(x+a)^3$, $(x-b)^2$, it will be necessary to assume the given function equal to

$$\frac{A}{(x+a)^3}+\frac{B}{(x+a)^2}+\frac{C}{x+a}+\frac{D}{(x+b)^2}+\frac{E}{(x-b)^2}.$$

(8) Expand $\frac{a-bx}{a+cx}$ to four terms.

Ans. $1-(b+c)\frac{x}{a}+c(b+c)\frac{x^2}{a^2}-c^2(b+c)\frac{x^3}{a^3}+\ldots\ldots$

(9) Resolve $\frac{x+2}{x^3-x}$ into partial fractions.

Ans. $\frac{1}{2(x+1)}+\frac{3}{2(x-1)}-\frac{2}{x}$.

(10) Resolve $\frac{1}{x^3(1-x^2)(1+x)}$ into partial fractions.

Ans. $\frac{1}{x^3}+\frac{1}{x^2}+\frac{2}{x}+\frac{1}{2(1-x)^2}+\frac{7}{4(1-x)}-\frac{1}{4(1+x)}$.

(11) Expand $\frac{x^2}{x^2+2ax+a^2}$ to four terms.

Ans. $1-\frac{2a}{x}+\frac{3a^2}{x^2}-\frac{4a^3}{x^3}+\ldots\ldots$

(12) Resolve $\frac{1}{x^4-1}$ into partial fractions.

Ans. $\frac{1}{4(x-1)}-\frac{1}{4(x+1)}-\frac{1}{2(x^2+1)}$.

LOGARITHMS.

210. LOGARITHMS are artificial numbers adapted to natural numbers, in order to facilitate numerical calculations; and we shall now proceed to explain the theory of these numbers, and illustrate the principles upon which their properties depend.

DEFINITION.—*In a system of logarithms, all numbers are considered as the powers of some one number, arbitrarily assumed, which is called the* BASE *of the system, and the exponent of that power of the base which is equal to any given number is called the* LOGARITHM *of that number.*

Thus, if a be the base of a system of logarithms, N any number, and x such that

$$N=a^x,$$

then x is called the logarithm of N, in the system whose base is a.

The base of the common system of logarithms (called, from their inventor, "Briggs's Logarithms") is the number 10. Hence, since

$(10)^0=1$,	0 is the logarithm of 1	in this system,
$(10)^1=10$,	1 is the logarithm of 10	in this system,
$(10)^2=100$,	2 is the logarithm of 100	in this system,
$(10)^3=1000$,	3 is the logarithm of 1000	in this system,
$(10)^4=10000$,	4 is the logarithm of 10000	in this system,
&c. = &c.	&c.	

211. In order to have the numbers corresponding to the logarithms 1, $\frac{1}{2}$ or 0.5, $\frac{1}{4}$ or 0.25, &c., it is necessary to extract the square, 4th, and so on, root of 10, or to extract the square root successively, as exhibited in the following table :

Number of times that the square root is extracted successively.	Numbers.	Exponents or Logarithms.
0	10,000 0000	1,000 0000
1	3,162 2777	0,500 0000
2	1,778 2794	0,250 0000
3	1,333 5214	0,125 0000
4	1,154 7819	0,062 5000
5	1,074 6078	0,031 2500
6	1,036 6329	0,015 6250
7	1,018 1517	0,007 8125
8	1,009 0350	0,003 9062
9	1,004 5073	0,001 9531
10	1,002 2511	0,000 9765
11	1,001 1249	0,000 4882
12	1,000 5623	0,000 2441
13	1,000 2811	0,000 1220
14	1,000 1405	0,000 0610
15	1,000 0702	0,000 0305
16	1,000 0351	0,000 0152
17	1,000 0175	0,000 0076
18	1,000 0087	0,000 0038
19	1,000 0043	0,000 0019
20	1,000 0021	0,000 0009
21	1,000 0010	0,000 0004
22	1,000 0005	0,000 0002
23	1,000 0002	0,000 0001
24	1,000 0001	0,000 0000

By means of the above table, to calculate the logarithm of any number (A) between 1 and 10 accurately to 5 places of decimals, take out from the second column the nearest number to A, but less, and divide A by this. Take out, again, the next less number than the quotient B, as a divisor for B, and so on until the last quotient contains only millionths; the logarithm sought is the sum of all the exponents or logarithms in the third column corresponding to the divisors used from the second. For, calling these exponents α, β, γ, δ we have

$$\frac{A}{10^{\alpha}}=B; \quad \frac{B}{10^{\beta}}=C; \quad \frac{C}{10^{\gamma}}=D; \quad \frac{D}{10^{\delta}}=E;$$

$$\therefore A=10^{\alpha}B=10^{\alpha}\times 10^{\beta}C=10^{\alpha}\times 10^{\beta}\times 10^{\gamma}D=10^{\alpha}.10^{\beta}.10^{\gamma}.10^{\delta}\ldots$$

$$\therefore A=10^{\alpha+\beta+\gamma+\delta\ldots}.$$

Any exponent beyond δ being added to the others would not affect the millionth place, or fifth decimal. Q. E. D.

Now, inasmuch as all numbers lying between the 1st, 2d, 3d, &c., powers of 10 must have broken numbers for logarithms, these numbers will be of the form $10^{a+\frac{k}{m}}=10^{a}.10^{\frac{k}{m}}$; hence the calculation of their logarithms will in every case depend on the calculation of a fractional logarithm such as has been just exhibited.

A table of logarithms is a table containing all numbers from 1 up to 10000 or 100000, or some high number, with their corresponding logarithms.

These tables are made with certain abbreviations and conveniences, which we shall presently explain.

From the scheme of numbers in (210) it appears, that in the common system the logarithm of every number between 1 and 10 is some number between

0 and 1, *i. e.*, is a fraction. The logarithm of every number between 10 and 100 is some number between 1 and 2, *i. e.*, is 1 plus a fraction. The logarithm of every number between 100 and 1000 is some number between 2 and 3, *i. e.*, is 2 plus a fraction, and so on. The whole number, or integral part of the logarithm, is called the *index*, or, more commonly, the *characteristic*.

212. In the common tables of logarithms the fractional part alone of the logarithm is registered, and from what has been said above, the rule usually given for finding the *characteristic*, or *index*, will be readily understood, viz.: *The index of the logarithm of any number greater than unity is equal to one less than the number of integral figures in the given number;* for if the number be between 10 and 100, it will contain two integral figures; if between 100 and 1000, it will contain three, and so on. Thus, in searching for the logarithm of such a number as 2970, we find in the tables opposite to 2970 the number 4727564; but since 2970 is a number between 1000 and 10000, its logarithm must be some number between 3 and 4, *i. e.*, must be 3 plus a fraction; the fractional part is the number 4727564, which we have found in the tables; prefixing to this the index 3, and interposing a decimal point, we have 3.4727564, the logarithm of 2970.

We must not, however, suppose that the number 3.4727564 is the exact logarithm of 2970, or that

$$2970=(10)^{3.4727564}$$

accurately. The above is only an approximate value of the logarithm of 2970; we can obtain the exact logarithms of very few numbers; but, taking a sufficient number of decimals, we can approach as nearly as we please to the true logarithms.

213. It has been shown that in Briggs's system the logarithm of 1 is 0; consequently, if we wish to extend the application of logarithms to fractions, we must establish a convention by which the logarithms of numbers less than 1 may be represented by numbers less than zero, *i. e.*, by *negative numbers*.

Extending, therefore, the above principles to negative exponents, since

$\frac{1}{10}$ or $(10)^{-1}=0.1$, -1 is the logarithm of .1 in this system,

$\frac{1}{100}$ or $(10)^{-2}=0.01$, -2 is the logarithm of .01 in this system,

$\frac{1}{1000}$ or $(10)^{-3}=0.001$, -3 is the logarithm of .001 in this system,

$\frac{1}{10000}$ or $(10)^{-4}=0.0001$, -4 is the logarithm of .0001 in this system,

&c. &c.

It appears, then, from this *convention*, that the logarithm of every number between 1 and .1 is some number between 0 and -1; the logarithm of every number between .1 and .01 is some number between -1 and -2; the logarithm of every number between .01 and .001 is some number between -2 and -3, and so on.

From this will be understood the rule given in books of tables for finding the *characteristic*, or *index*, of the logarithm of a decimal fraction, viz.: *The index of any decimal fraction is a negative number, equal to unity, added to the number of zeros immediately following the decimal point.* Thus, in searching for a logarithm of the number such as .00462, we find in the tables opposite to

462 the number 6646420; but since .00462 is a number between .001 and .01, its logarithm must be some number between -3 and -2, *i. e.*, must be -3 plus a fraction; the fractional part is the number 6646420, which we have found in the tables; therefore $-3+.6646420$ is the logarithm of .00462. It is customary to write the sign $-$ over the characteristic to show that it affects that alone, and not the decimal part of the logarithm, which is positive; thus, $\bar{3}.6646420$.

GENERAL PROPERTIES OF LOGARITHMS.

214. Let N and N′ be any two numbers, x and x' their respective logarithms, a the base of the system. Then, by definition,

$$N = a^x \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$
$$N' = a^{x'} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

I. Multiply equations (1) and (2) together,

$$NN' = a^x a^{x'}$$
$$= a^{x+x'}$$

$\therefore$ by definition, $x+x'$ is the logarithm of NN′; that is to say,

The logarithm of the product of two or more factors is equal to the sum of the logarithms of those factors.

II. Divide equation (1) by (2).

$$\frac{N}{N'} = \frac{a^x}{a^{x'}}$$
$$= a^{x-x'}.$$

$\therefore$ by definition, $x-x'$ is the logarithm of $\frac{N}{N'}$; that is to say,

The logarithm of a fraction, or of the quotient of two numbers, is equal to the logarithm of the numerator minus the logarithm of the denominator.

III. Raise both members of equation (1) to the nth power.

$$N^n = a^{nx}.$$

$\therefore$ by definition, nx is the logarithm of N^n; that is to say,

The logarithm of any power of a given number is equal to the logarithm of the number multiplied by the exponent of the power.

IV. Extract the n^{th} root of both members of equation (1).

$$N^{\frac{1}{n}} = a^{\frac{x}{n}}.$$

$\therefore$ by definition, $\frac{x}{n}$ is the logarithm of $N^{\frac{1}{n}}$; that is to say,

The logarithm of any root of a given number is equal to the logarithm of the number divided by the index of the root.

Combining the last two cases, we shall find

$$N^{\frac{m}{n}} = a^{\frac{mx}{n}},$$

whence $\frac{mx}{n}$ is the logarithm of $N^{\frac{m}{n}}$.

It is of the highest importance to the student to make himself familiar with the application of the above principles to algebraic calculations. The following examples will afford a useful exercise:

(1) Log. $(a, b, c, d \ldots\ldots) =$ log. $a +$ log. $b +$ log. $c +$ log. $d \ldots\ldots$

(2) Log. $\left(\frac{abc}{de}\right) =$ log. $a +$ log. $b +$ log. $c -$ log. $d -$ log. e.

(3) $\text{Log.}\ (a^m b^n c^p \ldots) = m \log.\ a + n \log.\ b + p \log.\ c \ldots$

(4) $\text{Log.}\ \left(\frac{a^m b^n}{c^p}\right) = m \log.\ a + n \log.\ b - p.\ \log.\ c.$

(5) $\text{Log.}\ (a^2 - x^2) = \log.\ (a+x) \times (a-x) = \log.\ (a+x) + \log.\ (a-x).$

(6) $\text{Log.}\ \sqrt{a^2 - x^2} = \frac{1}{2} \log.\ (a+x) + \frac{1}{2} \log.\ (a-x).$

(7) $\text{Log.}\ (a^3 \sqrt[4]{a^3}) = \log.\ a^3 + \frac{1}{4} \log.\ a^3 = 3 \log.\ a + \frac{3}{4} \log.\ a = \frac{15}{4} \log.\ a.$

(8) $\text{Log.}\ \sqrt[n]{(a^3 - x^3)^m} = \frac{m}{n} \log.\ (a-x) + \frac{m}{n} \log.\ (a^2 + ax + x^2)$

$$= \frac{m}{n} \{\log.\ (a-x) + \log.\ (a+x+z) + \log.\ (a+x-z)\}$$

where $z^2 = ax$.

(9) $\text{Log.}\ \sqrt{a^2 + x^2} = \frac{1}{2} \{\log.\ (a+x+z) + \log.\ (a+x-z)\}$, where $z^2 = 2ax$.

(10) $\text{Log.}\ \frac{\sqrt{a^2 - x^2}}{(a+x)^2} = \frac{1}{2} \{\log.\ (a-x) - 3 \log.\ (a+x)\}.$

TABLES OF LOGARITHMS.

The principal French tables are those of M. Callet, an American edition of which has been made by the late Mr. Hasler. The first of these tables, marked Chiliade I., occupying only five pages, contains the series of numbers from 1 up to 1200, with their logarithms expressed to eight places of decimals, the numbers being in the column marked N, and their logarithms in the column marked Log.* The second table, which is of far greater bulk, exhibits the logarithms of all entire numbers from 1020 up to 10800. The numbers are in the column entitled N, and their logarithms in the following column, marked 0. The characteristics of the logarithms are not written in the tables, since they may be known without, being always one less than the number of digits of which the number to which the logarithm belongs is composed. The logarithms of numbers containing one figure more than those in the column N, are found by means of the columns marked at top 1, 2, 3, ... 9. Thus, to find the logarithm of 27796, seek in the column N the number 2779; run along the horizontal line which contains this number to the column marked 6; you find there the last four figures of the logarithm sought: the first three figures of it are found in the column marked 0, to the left of the period, on the same horizontal line, or a little above. You obtain thus, after prefixing the proper characteristic,

$$\log.\ 27796 = 4.4439823.$$

It will be seen, by inspecting the tables, that the differences of the consecutive logarithms is constantly the same for a considerable number of them, and as the differences of the consecutive numbers is also constant, it follows that

* This table also contains an arrangement for reducing minutes and seconds to seconds without the trouble of multiplying by 60. Thus, on the fourth page, we find 12′ in the first of the columns marked log., and against 20, in the first column marked ″, we find 740, which is the number of seconds in 12′ 20″. By this arrangement we find readily the logarithm of the seconds in any given number of minutes and seconds, which is often convenient in astronomical calculations. It is evident that these numbers might be considered as degrees and minutes, or hours and minutes, as well as minutes and seconds.

the differences of the logarithms are proportional to the differences of the numbers. Suppose, then, that the logarithm of 14518469 were required.

From the tables we find, as before, neglecting for the present the characteristic (see a page of the tables of Callet at the end of this volume),

$$\text{log. } 14518 = 1619068.$$

This is also the logarithm of 14518000, which differs from the logarithm of the next number 14519, or 14519000, viz., 1619367 by 299, while the numbers themselves differ by 1000. But the number 14518000 differs from the given number 14518469 by 469, the last three figures not yet used; hence the proportion

Dif. Nos. Dif. Logs. Dif. Nos. Dif. of Logs.

$$1000 : 299 :: 469 : x = 141,$$

which result, added to 1619068, gives 7.1619209 for the logarithm required, 7 being the proper characteristic for the logarithm of a number consisting of eight figures.

The proportion is solved by multiplying the difference 469 by $\frac{299}{1000}$, or by $\frac{2}{10}+\frac{9}{100}+\frac{9}{1000}$. Now, by inspecting the last column of the page, this difference, 299, will be found ready calculated, and its product as nearly as it can be expressed in two or three figures by $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, &c., or .1, .2, .3, &c., the multiplier being in the left hand and the product in the right hand of the two small columns of figures under the difference, 299. These multipliers may be regarded as hundredths or thousandths, only giving the products their proper place. With this explanation, the following calculation will be understood:

Log. 14518	. . . .	1619068
0.4	. . . .	120
0.06	. . . .	18*
0.009	. . . .	3
Log. 14518469	. . . .	7.1619209

215. To find the number corresponding to a given logarithm, say 1619209, look in the column marked 0 for the nearest less logarithm, and take the corresponding number, which is 1451. Run the eye along the horizontal line till the number most nearly approaching 9209, forming the last four figures of the given logarithm, is found. This is 9068, which is found in column 8. Subtract this from 9209, and the difference is 141. Find in the right hand of the two columns of small figures marked dif. et p., or simply dif., at the top of the page, the nearest less number than 141; this is 120, which answers to 4 in the left hand. The difference between 120 and 141 is 21. Multiply 21 by 10, and seek, as before, in the small column, the number nearest 210; this is 209, which answers to 7. The calculation is below.

Log. $x = 1619209$

For	1619068	14518
First remainder,	141	04
Second remainder,	21	007
		$x = 1451847.$

The numbers 4 and 7 thus found may be simply annexed to 14518.

* The number in the table is 179; but, as the 9 is rejected, the 7 is increased by 1, since 179 is nearer 180 than 170.

If the characteristic of the logarithm had been

6, the number would have been 1451847;
5, the number would have been 145184.7;
4, the number would have been 14518.47;
1, the number would have been 14.51847;
0, the number would have been 1.451847;
$\bar{1}$, the number would have been .1451847;
$\bar{2}$, the number would have been .01451847.

This table contains in the first three columns an arrangement for reducing any number of degrees, minutes, and seconds, or hours, minutes, and seconds, to seconds, which is particularly useful in astronomical calculations, where the logarithm of the number of seconds in a given number of degrees, minutes, and seconds is frequently required.

EXAMPLE I.

Reduce $0°$ or $0^h\ 24'\ 57''$ to seconds. In the table (see last page), at the head of the first column, find $0°$, and immediately under it $24'$; descending this column to $55''$, near the bottom, and opposite $57''$, which is understood to be two numbers below, is found 1497, the number of seconds required.

If the degrees or hours exceed 3, the proceeding is different.

EXAMPLE II.

To reduce $4°$ or $4^h\ 2'\ 39''$ to seconds. Find $4°\ 0'$ at the head of the *second* column, and below, in this same column, $2'\ 30''$, to which corresponds, in the third column, 1455. Thus, $4°\ 2'\ 30''=14550''$ $\therefore 4°\ 2'\ 39''=14559''$.

EXAMPLES OF THE APPLICATION OF LOGARITHMS.

(1) To find the value to within 0.01 of the expression

$$x=\frac{7340\times 3549}{681.8\times 593.1}.$$

By the properties of logarithms,

$$\log. x=\log. 7340+\log. 3549-\log. 681.8-\log. 593.1.$$

The following is the calculation:

log. 7340 = 3.8656961	log. 681.8 = 2.8336570
log. 3549 = 3.5501060	log. 593.1 = 2.7731279
sum = 7.4158021	sum = 5.6067849

First sum, = 7.4158021
Second sum, = 5.6067849
Diff. or log. x = 1.8090172

216. The *arithmetical complement* of a logarithm is what remains after the logarithm is subtracted from 10. Thus, the arithmetical complement of the logarithm 2.7190826 is $10-2.7190826=7.2809174$, which is obtained by beginning on the right and subtracting each figure (carrying 1 to all except the first) from 10, or beginning on the left and subtracting each figure of the logarithm from 9, except the last, which is subtracted from 10.

217. The operation of subtraction of logarithms can be replaced by addition, if we use the arithmetic complement; for if, to a given logarithm, log. a, we add the arithmetical complement of another logarithm, such as $10-\log. b$ we have

$$\log. a + 10 - \log. b,$$

from which, rejecting 10, the result is

$$\log. a - \log. b,$$

the same as would be obtained by simply subtracting the second logarithm from the first.

We have then the following rule for operating with arithmetical complements: *Add the arithmetical complements of the logarithms of the divisors and the logarithms of the multipliers of a formula together, rejecting* 10 *from the sum for every arithmetical complement employed.*

The above example would be wrought by this rule as follows:

$$\begin{aligned} \log.\ 7340 &= 3.8656961 \\ \log.\ 3549 &= 3.5501060 \\ \text{ar. comp. } \log.\ 681.8 &= 7.1663430 \\ \text{ar. comp. } \log.\ 593.1 &= 7.2268721 \\ \text{sum rejecting } 20 &= \overline{1.8090172} = \log. x, \therefore x = 64.42. \end{aligned}$$

We thus obtain the same result as by the other method. The number corresponding need be taken from the tables only to four figures, because, the characteristic being 1, the entire part of the number will contain but two places, which will leave two places for the decimal part, as required, since the value of x was to be obtained to within 0.01.

(2) To find the value within 0.00001 of the quotient.

$$x = \frac{(\sqrt[5]{146298})^4}{(\sqrt[6]{988789})^5}.$$

By the rules,

$$\log. x = \tfrac{4}{5} \log. 146298 - \tfrac{5}{6} \log. 988789,$$

and the calculation will be as follows:

$\frac{4}{5}$ log. 146298.		$\frac{5}{6}$ log. 988789.	
log. 14629	0.1652146	log. 98878	0.9950997
for 0.8	238	for 0.9	40
log. 146298	5.1652384	log. 988789	5.9951037
product by 4	20.6609536	product by 5	29.9755185
quotient by 5	4.1321907	quotient by 6	4.9959197

$$\begin{aligned} \tfrac{4}{5} \log.\ 146298 &= 4.1321907 \\ \text{ar. comp. } \tfrac{5}{6} \log.\ 988789 &= 5.0040803 \\ \text{sum } -10, \text{ or } \log. x &= \bar{1}.1362710 \\ \therefore x &= 0.13686. \end{aligned}$$

(3) Required $\sqrt[11]{\frac{13}{27}}$ by means of logarithms.

$$\begin{aligned} 13 \log.\ & 1.1139434 \\ 27 \log.\ & 1.4313638 \\ 11)\ & \bar{1}.6825796 \\ \sqrt[11]{\tfrac{13}{27}} = .9357149 \log.\ & \bar{1}.9711436 \end{aligned}$$

The division by 11 is performed by adding -10 to the negative part of the logarithm and $+10$ to the positive.

The logarithm to be divided is viewed as if written thus:

$$-11 + 10.6825796.$$

EXERCISES IN LOGARITHMS.

(4) Calculate the logarithm of 8 from the table on page 259.

(5) Also of 7, 70, 700, 7000, 70000.

(6) Also of 356, 35600, 3560000.

(7) From the tables find the logarithms of 314, 3.721, 41.2.

(8) Also of 7315, 8416, 91.75, 34760, 1708000.

(9) Find the numbers the logarithms of which are 0.13130, 4.56502.

(10) Also those the logarithms of which are 3.6520528, 7.4891144.

(11) Those the logarithms of which are 4.49010, 0.66200, 5.72403.

(12) Find by proportional parts the logarithms of 314761, 440736, 37025400, 2111768.

(13) Also of 22.3345, 137.2014, 46.27835.

(14) Of .75, .341, .7391, .0347, .000536, .0000083.

(15) Of $\frac{5}{7}, \frac{3}{8}, \frac{6}{11}, \frac{4}{13}, \frac{7}{40}$.

(16) Find the logarithm of the product of 9.734 and 5.639.

(17) Also of $35.98 \times 7.433 \times 6.543 \times 29.78$.

(18) Also of $22.74 \times 31.201 \times 0.0067 \times 0.9298$.

(19) Divide 3758000 by 4986 by means of logarithms.

(20) Also 16.87 : 0.07658 and 1.687 : 7658.

(21) Also 14.307 : 30415, 761.23 : 0.01871, 3.16 : 0.942.

(22) Find the logarithm of $\frac{7}{133}, \frac{125}{114}, \frac{31}{4566}, \frac{734}{1000}, \frac{1}{3946}$.

(23) Find the power $(5486)^4$ by means of logarithms.

(24) Also the powers $(37.49)^9$, $(106.4)^5$, $(0.032)^7$, $(7.0034)^8$.

(25) Also $\left(\frac{1}{2}\right)^{32}, \left(\frac{3}{5}\right)^4, \left(\frac{1}{32}\right)^5, \left(\frac{3}{8}\right)^4, \left(\frac{127}{100}\right)^{13}$.

(26) Also $\left(3+\frac{1}{5}\right)^6, \left(4-\frac{1}{3}\right)^8, \left(7+\frac{20}{21}\right)^9, \left(100-\frac{1}{100}\right)^3$.

(27) Find the cube root by logarithms of 1728000.

(28) Also $\sqrt[3]{34{\cdot}782}$, $\sqrt[4]{23990}$, $\sqrt[5]{628.73}$.

(29) Also $\sqrt[7]{\frac{1337}{2239}}, \sqrt[10]{\frac{9466}{8871}}, \sqrt[12]{\frac{120300}{7098}}, \sqrt[6]{0.1563}, \sqrt[5]{0.0082}$.

(30) Also $\sqrt[34]{7368}$, $\sqrt[100]{45390000}$, $\sqrt[77]{800.9}$.

(31) Also $\sqrt[11]{(1347)^8}$, $\sqrt[9]{(70.44)^{11}}$, $\sqrt[20]{(8.664)^{19}}$.

(32) Also $\sqrt[3]{\left(\frac{1722}{3347}\right)^5}, \sqrt[7]{\left(\frac{0.006}{8.48}\right)^{25}}, \sqrt[4]{\left(\frac{72.93}{0.086}\right)^7}$.

(33) Find by means of logarithms, using the arithmetical complement, the value of $\frac{27630 \times 2678 \times 5428}{36940 \times 5302 \times 7013}$.

(34) Also of $\frac{207.3 \times 50.66 \times 38.09 \times 2713 \times 0.098}{344 \times 0.763 \times 0.4 \times 6984 \times 7034.2}$.

(35) Also of $\sqrt[3]{\frac{0.85762 \times 0.00853}{7.58913 \times 86.24}}$.

GAUSS LOGARITHMS.

218. The common logarithms, or logarithms of Briggs, are applicable only to the operations of multiplication, division, formation of powers, or extraction of roots, and do not apply when the required operation is that of addition or sub-

traction, indicated in formulas by the quantities to be operated upon being connected by the signs $+$ and $-$.

A system of logarithms has, however, been invented by Gauss,* designed exclusively for sums and differences. The arrangement of these tables, which contain three columns, marked A, B, C, is founded upon the following simple considerations.

We have for the form of a sum $p+q$, and of a difference $p-q$, the following identities:

$$p+q=p\left(\frac{p+q}{p}\right) \ldots\ldots\ldots\ldots (1)$$

$$p-q=p:\left(\frac{p}{p-q}\right) \ldots\ldots\ldots\ldots (2)$$

$$\therefore \log.\ (p+q)=\log.\ p+\log.\ \left(\frac{p+q}{p}\right) \qquad (3)$$

$$\text{and } \log.\ (p-q)=\log.\ p-\log.\ \left(\frac{p}{p-q}\right)$$

The logarithms of the sum $p+q$ and the difference $p-q$ appear, therefore, in these formulas, equal to the sum or difference of two logarithms, the first of which is to be considered as directly given, but the second of which must be found by the Gauss tables. They contain,

I. In the column A logarithms of numbers of the form $\left(\frac{p}{q}\right)$, increasing from 0.000 to 5.000.

II. In column B logarithms of numbers of the form $\left(\frac{p+q}{p}\right)$, decreasing from 0.30103 to 0.00000.

III. In column C logarithms of numbers of the form $\frac{p+q}{q}$, increasing from 0.30103 to 5.00000.

Now, therefore, inasmuch as $\log.\ \left(\frac{p}{q}\right)=\log.\ p-\log.\ q$, by the tables of common logarithms, the first thing to be done is to take the difference of the common logarithms of p and q, enter with this column A in the Gauss logarithms, and take out the corresponding number from column B. The addition of this number to logarithm p will give, according to (3), the logarithm sought of $p+q$.

In order to find the logarithm of the difference $p-q$, by means of the logarithms of p and q, two cases must be considered:

1°. Where $\frac{p}{q}<2 \therefore \log.\ p-\log.\ q<0.30103$, it is only necessary to enter with this difference column B, and to subtract the adjoining logarithm of column C from logarithm p. For, corresponding to the logarithms of numbers of the form $\left(\frac{p}{q}\right)$ in B, C contains the logarithms of those of the form $\left(\frac{p}{p-q}\right)$.

2°. If $\frac{p}{q}>2 \therefore \log.\ p-\log.\ q>0.30103$, and, therefore, is contained in the column C; subtract the corresponding logarithm in column B from loga-

* They are found in the latest edition of the tables of Vega, and those edited by Köhler.

rithm p: because, if the numbers in C are considered $=\frac{p}{q}$, the corresponding numbers in B are $=\frac{p}{p-q}$.

The existence of the foregoing relations between B and C is easily perceived if we substitute in II. and III. the value $p-q$ for p, and afterward q for $p-q$.

EXAMPLES.

(1) Let log. $p=3.24502$ and log. $q=2.74194$, to find log. $(p+q)$. We enter column A with the log. $p-$ log. $q=0.50308$, and the corresponding log. in column $B=0.11861$, $\therefore$

$$\text{log. } p+\text{B}=3.24502+0.11861=3.36363=\text{ log. } 2310.$$

(2) From log. $p=3.32675$ and log. $q=2.09482$, to determine log. $(p-q)$. Find by means of proportional parts for the value of log. $p-$ log. q in column B the corresponding log. in $C=0.38325$; consequently,

$$\text{log. } p-\text{C}=3.32675-0.38325=2.94350=\text{ log. } 878.$$

(3) From log. $p=2.64207$ and log. $q=1.87640$ the log. of $(p-q)$ is found by subtracting from the nearest value of log. $p-$ log. $q=0.76567$, in column C, the corresponding log. from $B=0.08171$. Thus,

$$\text{log. } p-\text{B}=2.64207-0.08171=2.56036=\text{ log. } 363.4.$$

The Gauss logarithms would be applicable in the solution of the exponentials on page 269.

(4) Find by the Gauss logarithms the log. of $\sqrt[3]{200}+\sqrt[3]{100}$.

(5) Also the log. of $[(0.7345)^3+(0.2349)^3]$.

(6) Also the log. of the difference $(\sqrt[5]{36}-\sqrt[5]{27})$.

(7) Also of $\{(1.237)^{14}-(0.9864)^{15}\}$.

219. Let us resume the equation

$$N=a^x.$$

1°. If $a>1$, making $x=0$, we have $N=1$; the hypothesis $x=1$ gives $N=a$. As x increases from 0 up to 1, and from 1 up to infinity, N will increase from 1 up to a, and from a up to infinity; so that x being supposed to pass through all intermediate values, according to the law of continuity, N increases also, but with much greater rapidity. If we attribute negative values to x, we have $N=a^{-x}$, or $N=\frac{1}{a^x}$. Here, as x increases, N diminishes, so that x being supposed to increase negatively, N will decrease from 1 toward 0, the hypothesis $x=\infty$ gives $N=0$; *i. e., the logarithm of zero is an infinite negative quantity.*

2°. If $a<1$, put $a=\frac{1}{b}$, where $b>1$, and we shall then have $N=\frac{1}{b^x}$, or $N=b^x$, according as we attribute positive or negative values to x. We here arrive at the same conclusion as in the former case, with this difference, that when x is positive $N<1$, and when x is negative $N>1$.

3°. If $a=1$, then $N=1$, whatever may be the value of x.

From this it appears that,

I. *In every system of logarithms the logarithm of* 1 *is* 0, *and the logarithm of the base is* 1.

II. *If the base be* >1, *the logarithms of numbers* >1 *are positive, and the logarithms of numbers* <1 *are negative. The contrary takes place if the base be* <1.

III. *The base being fixed, any number has only one real logarithm; but the same number has manifestly a different logarithm for each value of the base, so that every number has an infinite number of real logarithms.* Thus, since $9^2=81$ and $3^4=81$, 2 and 4 are the logarithms of the same number 81, according as the base is 9 or 3.

IV. *Negative numbers have no real logarithms; for, attributing to* x *all values from* $-\infty$ *up to* $+\infty$, *we find that the corresponding values of* N *are positive numbers only, from* 0 *up to* $+\infty$.

220. In order to solve the equation

$$c=a^x,$$

where c and a are given, and where x is unknown, we equate the logarithms of the two members, which gives us

$$\log. c=x \log. a.$$

Whence

$$x=\frac{\log. c}{\log. a}.$$

To determine the value of x in the equation

$$Aa^x+Ba^{x-b}+Ca^{x-c}+\ldots\ldots\ldots =P,$$

we have

$$a^x\left(A+\frac{B}{a^b}+\frac{C}{a^c}+\ldots\ldots\ldots\right)=P,$$

or

$$Qa^x =P,$$

substituting Q for the term in the parenthesis.

$$\therefore x=\frac{\log. P-\log. Q}{\log. a}.$$

If we have an equation $a^z=b$, where z depends upon an unknown quantity, x, and we have

$$z=Ax^n+Bx^{n-1}+\ldots\ldots\ldots$$

Since $z=\dfrac{\log. b}{\log. a}=K$ some known number, the problem depends upon the solution of the equation of the n^{th} degree

$$K=Ax^n+Bx^{n-1}+\ldots\ldots\ldots\ldots$$

For example, let

$$4\left(\frac{2}{3}\right)^{x^2-5x+4}=9.$$

Hence

$$(x^2-5x+4)\log.\left(\frac{2}{3}\right)=\log.\frac{9}{4}$$

$$\therefore x^2-5x+4 =-2;*$$

an equation of the second degree, from which we find $x=2$, $x=3$.

---

* This result may be readily seen by observing that $\left(\frac{3}{2}\right)^2=\frac{9}{4}\ \therefore\ 2\log.\frac{3}{2}=\log.\frac{9}{4}$, and $\log.\frac{3}{2}=-\log.\frac{2}{3}$.

To find the value of x from the equation

$$b^{n-\frac{a}{x}}=c^{mx}f^{x-p}.$$

Taking the logarithms of each member,

$$\left(n-\frac{a}{x}\right)\text{ log. }b=mx\text{ log. }c+(x-p)\text{ log. }f,$$

or

$$(m\text{ log. }c+\text{ log. }f)x^2-(n\text{ log. }b+p\text{ log. }f)x+a\text{ log. }b=0,$$

a quadratic equation, from which the value of x may be determined.

In like manner, from the equation

$$c^{mx}=ab^{nx-1},$$

we find

$$x=\frac{\text{log. }a-\text{ log. }b}{m\text{ log. }c-n\text{ log. }b}.$$

Equations of this nature are called *Exponential Equations.*

To resolve the exponential equation

$$\left(\frac{117}{337}\right)^x=\frac{8493}{73}.$$

By the rule,

$$x(\text{log. }117-\text{ log. }337)=\text{ log. }8493-\text{ log. }73$$

$$\therefore\ x=-\frac{\text{log. }8493-\text{ log. }73}{\text{log. }337-\text{ log. }117}=.$$

Calculation,

8493 log. 3.9290611	337 log. 2.5276299	
73 log. 1.8633229	117 log. 2.0681859	
diff. 2.0657382	. . . .	log. 0.3150752
	diff. .0.4594440	log. $\overline{1}$.6622326
	$x=-$ 4.49616 log.=diff.	0.6528426

This example admits the use of the Gauss logarithms.

Let $10^x=-100$ $\therefore x$ log. $10=$ log. (-100) ; log. (-100) here must be regarded, like an imaginary quantity, as a symbol of absurdity. It is evident that there is no power of 10 equal to -100.

221. Let N and N+1 be two consecutive numbers, the difference of their logarithms, taken in any system, will be

$$\text{log. }(N+1)-\text{ log. }N=\text{ log. }\left(\frac{N+1}{N}\right)=\text{ log. }\left(1+\frac{1}{N}\right),$$

a quantity which approaches to the logarithm of 1, or zero, in proportion as $\frac{1}{N}$ decreases, that is, as N increases. Hence it appears that

The difference of the logarithms of two consecutive numbers is less in proportion as the numbers themselves are greater.

Let $a^x=N$ and $b^y=N$; then we have

$$x=\text{ log. N to the base }a,\text{ or }x=\text{ log. }_aN^*$$

$$y=\text{ log. N to the base }b,\text{ or }y=\text{ log. }_bN.$$

Hence $\text{log. }_aN=\text{ log. }_ab^y=y\text{ log. }_ab$ (Art. 214, III.) ;

$$\therefore\ x=y\text{ log. }_ab,$$

* Understanding by the notation log. $_a$N the logarithm of N in the system whose base is a.

and

$$y=\frac{1}{\log._a b}.x; \quad . \; . \; . \; . \; . \; . \; . \; (A)$$

and by means of this equation we can pass from one system of logs. to another, by multiplying x, the log. of any number in the system whose base is a, by the reciprocal of log. b in the same system; and thus we shall obtain the log. of the same number in the system whose base is b.

The factor $\frac{1}{\log._a b}$ is constant for all numbers, and is called the *Modulus*, that is to say, if we divide the logs. of the same number c, taken in two systems, the quotient will be invariable for these systems, whatever may be the value of c, and will be the modulus, the constant multiplier which reduces the first system of logs. to the second.*

If we find it inconvenient to make use of a log. calculated to the base 10, we can in this manner, by aid of a set of tables calculated to the base 10, discover the logarithm of the given number in any required system.

For example, let it be required, by aid of Briggs's tables, to find the log. of $\frac{2}{3}$ in a system whose base is $\frac{5}{7}$.

Let x be the log. sought, then by (A)

$$x=\frac{\log.\frac{2}{3}}{\log.\frac{5}{7}}$$

$$=\frac{\log. 2-\log. 3}{\log. 5-\log. 7}.$$

Taking these logs. in Briggs's system, and reducing, we find

$$x=\frac{-0.17609125}{-0.14612804}$$

$$=1.2050476=\log. \frac{2}{3} \text{ to base } \frac{5}{7}.$$

Similarly, the log. of $\frac{2}{3}$, in the system whose base is $\frac{3}{2}$, is

$$x=\frac{\log. 2-\log. 3}{\log. 3-\log. 2}=-1,$$

which is manifestly the true result; for in this case the general equation $N=a^x$ becomes $\frac{2}{3}=\left(\frac{3}{2}\right)^x=\left(\frac{2}{3}\right)^{-x}$, and x is evidently $=-1$.

In a system whose base is a, we have

$$n=a^{\log. n};$$

for, by the definition of a logarithm in the equation $n=a^x$, x is the log. n.

In like manner,

$$n^h=a^{\log. (n^h)}=a^{h \log. n}.$$

* The term *Modulus*, of a system of logarithms, is generally understood to be the number by which it is necessary to multiply Napierian logarithms of numbers, in order to obtain the logarithms of the system in question. The peculiar character of Napierian logarithms will be presently explained.

EXAMPLES FOR EXERCISE.

(1) Given $2^{2x}+2^x=12$ to find the value of x.
(2) Given $x+y=a$, and $m^{(x-y)}=n$ to find x and y.
(3) Given $m^x n^x=a$, and $hx=ky$ to find x and y.

ANSWERS.

(1) $x=1\cdot584962$, or $x=\log.\ (-4)\div\log.\ 2$.
(2) $x=\frac{1}{2}\{a+\log.\ n\div\log.\ m\}$ and $y=\frac{1}{2}\{a-\log. n\div\log.\ m\}$.
(3) $x=\log.\ a\div(\log.\ m+\log.\ n)$ and $y=\frac{h}{k}\log.\ a\div(\log.\ m+\log.\ n)$.

THE EXPONENTIAL THEOREM.

222. *It is required to expand* a^x *in a series ascending by the powers of* x.
Since $a=1+a-1$, therefore $a^x=\{1+(a-1)\}^x$; and by the binomial theorem we have

$$\{1+(a-1)\}^x=1+x(a-1)+\frac{x(x-1)}{1\cdot2}(a-1)^2+\frac{x(x-1)(x-2)}{1\cdot2\cdot3}(a-1)^3+\ldots.$$
$$=1+\{(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\tfrac{1}{4}(a-1)^4+\ldots.\}x+Bx^2+Cx^3\ldots$$

where B, C.... denote the coefficients of x^2, x^3.....; and if we put

$$A=(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3+\tfrac{1}{4}(a-1)^4+\ldots..$$

Then $a^x=1+Ax+Bx^2+Cx^3+Dx^4+Ex^5+\ldots..$

For x write $x+h$; then we have

$$\begin{aligned} a^{x+h}&=1+A(x+h)+B(x+h)^2+C(x+h)^3+\ldots.\\ &=1+Ax+Bx^2+Cx^3+Dx^4+\ldots.\\ &\quad+Ah+2Bxh+3Cx^2h+4Dx^3h+\ldots.\\ &\quad+Bh^2+3Cxh^2+6Dx^2h^2+\ldots.\\ &\quad+Ch^3+4Dxh^3+\ldots.\\ &\quad+Dh^4+\ldots. \end{aligned}$$

But $a^{x+h}=a^x\times a^h=(1+Ax+Bx^2+Cx^3+\ldots.)(1+Ah+Bh^2+Ch^3+\ldots$

$$\begin{aligned} &=1+Ax+Bx^2+Cx^3+Dx^4+\ldots.\\ &\quad+Ah+A^2xh+ABx^2h+ACx^3h+\ldots.\\ &\quad+Bh^2+ABxh^2+B^2x^2h^2+\ldots.\\ &\quad+Ch^3+ACxh^3+\ldots.\\ &\quad+Dh^4+\ldots. \end{aligned}$$

Now these two expansions must be identical; and we must, therefore, have the coefficients of like powers of x and h equal; hence

$$2B=A^2 \qquad \therefore\ B=\frac{A^2}{2}$$
$$3C=AB \qquad C=\frac{A\cdot B}{3}=\frac{A^3}{2\cdot3}$$
$$4D=AC \qquad D=\frac{AC}{4}=\frac{A^4}{2\cdot3\cdot4}$$

&c. &c. &c. &c.

Hence $$a^x=1+Ax+\frac{A^2x^2}{1\cdot2}+\frac{A^3x^3}{1\cdot2\cdot3}+\frac{A^4x^4}{1\cdot2\cdot3\cdot4}+\ldots.$$

which is the *exponential theorem*; where

$$A=(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\tfrac{1}{4}(a-1)^4+\ldots...$$

Let ε be the value of a, which renders $A=1$, then

$$(\varepsilon-1)-\tfrac{1}{2}(\varepsilon-1)^2+\tfrac{1}{3}(\varepsilon-1)^3-\tfrac{1}{4}(\varepsilon-1)^4+\ldots=1$$

$$\varepsilon^x=1+x+\frac{x^2}{1\cdot2}+\frac{x^3}{1\cdot2\cdot3}+\frac{x^4}{1\cdot2\cdot3\cdot4}+\ldots.$$

Now, since this equation is true for every value of x, let $x=1$; then

$$\varepsilon=1+1+\frac{1}{1\cdot2}+\frac{1}{1\cdot2\cdot3}+\frac{1}{1\cdot2\cdot3\cdot4}+\ldots\ldots$$

$$=1+1+\tfrac{1}{2}(1)+\tfrac{1}{3}\left(\frac{1}{1\cdot2}\right)+\tfrac{1}{4}\left(\frac{1}{1\cdot2\cdot3}\right)+\ldots\ldots$$

$$=2\cdot718281828459\ldots\ldots$$

223. *We add another method of calculating the logarithm of any given number.*

Let N be any given number whose logarithm is x, in a system whose base is a; then

$$a^x=N \text{ and } a^{xz}=N^z.$$

Hence, by the exponential theorem, we have from the last equation

$$1+Axz+A^2\frac{x^2z^2}{1.2}+\ldots=1+A_1z+A_1^2\frac{z^2}{1\cdot2}+\ldots.;$$

and equating the coefficients of z, we get $Ax=A_1$; hence

$$x=\frac{A_1}{A}=\frac{(N-1)-\frac{1}{2}(N-1)^2+\frac{1}{3}(N-1)^3-\ldots.}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\ldots.};$$

because $A=(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\ldots$ in the expansion of a^{xz}

and $A_1=(N-1)-\frac{1}{2}(N-1)^2+\frac{1}{3}(N-1)^3-\ldots$ in the expansion of N^z

224. *To find the logarithm of a number in a converging series.*

We have seen that if $a^x=N$, then

$$x=\frac{(N-1)-\frac{1}{2}(N-1)^2+\frac{1}{3}(N-1)^3-\frac{1}{4}(N-1)^4+\ldots}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4+\ldots}$$

Now the reciprocal of the denominator is the *modulus* of the system;*and, representing the modulus by M, we have

$$x=\log. N=M\{(N-1)-\tfrac{1}{2}(N-1)^2+\tfrac{1}{3}(N-1)^3-\tfrac{1}{4}(N-1)^4+\ldots\}$$

Put $N=1+n$; then $N-1=n$, and we have

$$\log.(1+n)=M(+n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4+\tfrac{1}{5}n^5-\ldots)\quad\ldots[A]$$

Similarly, $$\log.(1-n)=M(-n-\tfrac{1}{2}n^2-\tfrac{1}{3}n^3-\tfrac{1}{4}n^4-\tfrac{1}{5}n^5+\ldots)$$

$$\therefore \log.(1+n)-\log.(1-n)=2M(n+\tfrac{1}{3}n^3+\tfrac{1}{5}n^5+\tfrac{1}{7}n^7+\ldots)$$

$$\text{or } \log.\frac{1+n}{1-n}=2M(n+\tfrac{1}{3}n^3+\tfrac{1}{5}n^5+\tfrac{1}{7}n^7+\ldots)$$

---

* If, in the expression for a^x deduced in (Art. 222), we make $x=\frac{1}{A}$, we obtain

$$a^{\frac{1}{A}}=1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+,\ \&c.,$$

which is the value of ε, given at the end of the same art.:

$$\therefore a^{\frac{1}{A}}=\varepsilon \therefore a=\varepsilon^A \therefore A\log.\varepsilon=\log. a \therefore \frac{1}{A}=\frac{\log.\varepsilon}{\log. a}=\frac{1}{\log. a},$$

if ε be the base of the system of logarithms expressed by log. Therefore $\frac{1}{A}=\frac{1}{\log. a}$ is, by a previous definition (Art. 221), the modulus for passing from the system whose base is to that whose base is a. If log. a refers to the base a, $\frac{1}{A}$ becomes equal to log. ε.

Put $n=\frac{1}{2P+1}$; then $1+n=\frac{2P+2}{2P+1}$, $1-n=\frac{2P}{2P+1}$, and $\frac{1+n}{1-n}=\frac{P+1}{P}$;

consequently,

$$\log.(P+1)-\log.P=2M\left\{\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}+\dots\right\}$$

$$\therefore \log.(P+1)=\log.P+2M\left\{\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}+\dots\right\}$$

Hence, if log. P be known, the log. of the next greater number can be found by this rapidly converging series.

By substituting the series of natural numbers for N in this formula, the corresponding values of x will be their logarithms.

224. *To find the Napierian logarithms of numbers.*

In the preceding series, which we have deduced for log. (P+1), we find a number M, called the *modulus* of the system; and we must assign some value to this number before we can compute the value of the series. Now, as the value of M is arbitrary, we may follow the steps of the celebrated Lord Napier, the inventor of logarithms, and assign to M the simplest possible value. This value will therefore be unity, and we have

$$\log.(P+1)=\log.P+2\left\{\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}+\dots\right\}$$

Expounding P successively by 1, 2, 3, 4, &c., we find

$$\log.\ 2=\qquad 2\left(\frac{1}{3}+\frac{1}{3\cdot3^3}+\frac{1}{5\cdot3^5}+\frac{1}{7\cdot3^7}+\dots\right)=\cdot6931472$$

$$\log.\ 3=\log.2+2\left(\frac{1}{5}+\frac{1}{3\cdot5^3}+\frac{1}{5\cdot5^5}+\frac{1}{7\cdot5^7}+\dots\right)=1\cdot0986123$$

$$\log.\ 4=2\log.2\ \dots\dots\dots\dots =1\cdot3862944$$

$$\log.\ 5=\log.4+2\left(\frac{1}{9}+\frac{1}{3\cdot9^3}+\frac{1}{5\cdot9^5}+\frac{1}{7\cdot9^7}+\dots\right)=1\cdot6094379$$

$$\log.\ 6=\log.2+\log.3\ \dots\dots\dots\dots =1.7917595$$

$$\log.\ 7=\log.6+2\left(\frac{1}{13}+\frac{1}{3\cdot13^3}+\frac{1}{5\cdot13^5}+\dots\dots\right)=1\cdot9459101$$

$$\log.\ 8=\log.2+\log.4,\text{ or }3\log.2\ \dots\dots =2\cdot0794415$$

$$\log.\ 9=2\log.3\ \dots\dots\dots\dots =2\cdot1972246$$

$$\log.\ 10=\log.2+\log.5\ \dots\dots\dots\dots =2\cdot3025851$$

In this manner the Napierian logarithms of all numbers may be computed.

225. *To find the common logarithms of numbers.*

The base of the Napierian system is $\varepsilon=2\cdot718281828\dots$, and the base of the common system is $b=10$, the base of our common system of arithmetic; then we have $b=10$, and $a=\varepsilon=2\cdot718281828\dots$, and consequently, if N denote any number, we shall have

$$\log._{10}N=\frac{1}{\log._{\varepsilon}10}\cdot\log._{\varepsilon}N;\ \text{that is,}$$

$$\text{com. log. }N=\frac{1}{2\cdot3025851}\times\text{Nap. log. }N=\cdot43429448\times\text{Nap. log. }N;*$$

* To find the value of the Napierian base, observe that, since com. log. N=·43429448× Nap. log. N., if we make in this expression N=ε, the Napierian base, we have

com. log. ε=·43429448.

From a table of common logs., therefore, we find the number corresponding to the logaand the modulus of the common system is, therefore,

$$M=\frac{1}{2\cdot3025851}=\cdot43429448 \therefore 2M=\cdot86858896$$

Hence, to construct a table of common logarithms, we have

$$\text{log.}\ (P+1)=\text{log.}\ P+\cdot86858896\left\{\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}+\ldots\right\}$$

Expounding P successively by 1, 2, 3, &c., we get

$$\text{log.}\ 2=\cdot86858896\left(\frac{1}{3}+\frac{1}{3^4}+\frac{1}{5\cdot3^5}+\ldots\right)$$

$$=\cdot86858896\times\cdot6931472 \ldots\ldots = \cdot3010300$$

$$\text{log.}\ 3=\text{log.}\ 2+\cdot86858896\left(\frac{1}{5}+\frac{1}{3\cdot5^2}+\frac{1}{5^3}+\ldots\right)\ \ldots = \cdot4771213$$

$$\text{log.}\ 4=2\ \text{log.}\ 2 \ldots\ldots = \cdot6020600$$

$$\text{log.}\ 5=\text{log.}\ \tfrac{10}{2}=\text{log.}\ 10-\text{log.}\ 2=1-\text{log.}\ 2 \ \ldots = \cdot6989700$$

$$\text{log.}\ 6=\text{log.}\ 2+\text{log}\ 3 \ldots\ldots = \cdot7781513$$

$$\text{log.}\ 7=\text{log.}\ 6+\cdot86858896\left(\frac{1}{13}+\frac{1}{3\cdot13^3}+\frac{1}{5\cdot13^5}+\ldots\right)= \cdot8450980$$

$$\text{log.}\ 8=\text{log.}\ 2^3=3\ \text{log.}\ 2 \ldots\ldots = \cdot9030900$$

$$\text{log.}\ 9=\text{log.}\ 3^2=2\ \text{log.}\ 3 \ldots\ldots = \cdot9542426$$

$$\text{log.}\ 10= \ldots\ldots =1\cdot0000000$$

&c. &c.

226. Since $\text{log.}\ \frac{1+n}{1-n}=2M(n+\frac{1}{3}n^3+\frac{1}{5}n^5+\frac{1}{7}n^7+\ldots)$

let $\frac{1+n}{1-n}=P$; then $1+n=P(1-n)$, or $n=\frac{P-1}{P+1}$

$$\therefore \text{log.}\ P=2M\left\{\frac{P-1}{P+1}+\frac{1}{3}\cdot\left(\frac{P-1}{P+1}\right)^3+\frac{1}{5}\cdot\left(\frac{P-1}{P+1}\right)^5+\ldots\right\}$$

and thus we have a series for computing the logs. of all numbers, without knowing the log. of the previous number.

EXAMPLES.

(1) Given the log. of $2=0\cdot3010300$, to find the logs. of 25 and $\cdot0125$.

Here $25=\frac{100}{4}=\frac{10^2}{2^2}$; therefore log. $25=2$ log. $10-2$ log. $2=1\cdot3979406$.

Again, $\cdot0125=\frac{125}{10000}=\frac{1}{80}=\frac{1}{10\times2^3}$

$\therefore$ log. $\cdot0125=$ log. $1-$ log. $10-3$ log. $2=-1-3$ log. $2=\bar{2}\cdot0969100$

(2) Calculate the common logarithm of 17.

Ans. 1.2304489.

(3) Given the logs. of 2 and 3 to find the logarithm of $22\cdot5$.

Ans. $1+2$ log. $3-2$ log. 2.

(4) Having given the logs. of 3 and $\cdot21$, to find the logarithm of 83349.

Ans. $6+2$ log. $3+3$ log. $\cdot21$.

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rithm $\cdot43429448$, which is $2\cdot7182818$, the Napierian base. This also furnishes us with another definition of the modulus of the common (or any other) system of logarithms; it is *the common* (or, &c.) *logarithm of the Napierian base.* See further note at the end of Progressions.

PROGRESSIONS.

ARITHMETICAL PROGRESSION.

227. WHEN a series of quantities continually increase or decrease by the addition or subtraction of the same quantity, the quantities are said to be in *Arithmetical Progression.* A more appropriate name is *Progression by Differences.*

Thus the numbers 1, 3, 5, 7, ... which differ from each other by the addition of 2 to each successive term, form what is called *an increasing arithmetical progression*, or *progression by differences*, and the numbers 100, 97, 94, 91, ... which differ from each other by the subtraction of 3 from each successive term, form what is called *a decreasing progression by differences.*

Generally, if a be the first term of an arithmetical progression, and δ the common difference, the successive terms of the series will be

$$a,\ a\pm\delta,\ a\pm2\delta,\ a\pm3\delta,\ \ldots\ldots$$

in which the positive or negative sign will be employed, according as the series is an increasing or decreasing progression.

Since the coefficient of δ in the *second* term is 1, in the *third* term 2, in the *fourth* term 3, and so on, in the n^{th} term it will be $n-1$, and the n^{th} term of the series will be of the form

$$a\pm(n-1)\delta \quad \ldots\ldots\ldots \quad (1)$$

In what follows we shall consider the progression as an increasing one, since all the results which we obtain can be immediately applied to a decreasing series by changing the sign of δ.

228. *To find the sum of* n *terms of a series in arithmetical progression.*

Let $a=$ first term.
$l=$ last term.
$\delta=$ common difference.
$n=$ number of terms.
$S=$ sum of the series.

Then $\quad S=a+(a+\delta)+(a+2\delta)+\ldots\ldots+l.$

Write the same series in a reverse order, and we have

$$S=\quad l+(l-\delta)+(l-2\delta)+\ldots\ldots+a$$

Adding, $2S=(a+l)+(a+l)+(a+l)\ +\ldots\ldots+(a+l)$

$=n(a+l)$, since the series consists of n terms.

$$\therefore\ S=\frac{n(a+l)}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Or, since $l=a+(n-1)\delta$ (Art. 227),

$$S=\frac{2na+n(n-1)\delta}{2} \quad \ldots\ldots\ldots\ldots \quad (3)$$

Hence, if any three of the five quantities a, l, δ, n, S be given, the remaining two may be found by eliminating between equations (1) and (2).

It is manifest from the above process that

The sum of any two terms which are equally distant from the extreme terms is equal to the sum of the extreme terms, and if the number of terms in the series be uneven, the middle term will be equal to one half the sum of the extreme terms, or of any two terms equally distant from the extreme terms.

EXAMPLE I.

Required the sum of 60 terms of an arithmetical series, whose first term is 5 and common difference 10.

Here $a=5,\ \delta=10,\ n=60$

$$\therefore\ l=a+(n-1)\delta=5+59\times 10=595$$

$$\therefore\ S=\frac{(5+595)\times 60}{2}$$

$$=600\times 30=18000=\text{ sum required.}$$

EXAMPLE II.

A body descends in vacuo through a space of $16\frac{1}{12}$ feet during the first second of its fall, but in each succeeding second $32\frac{1}{6}$ feet more than in the one immediately preceding. If a body fall during the space of 20 seconds, how many feet will it fall in the last second, and how many in the whole time?

Here $a=\frac{193}{12},\ \delta=\frac{386}{12},\ n=20$

$$\therefore\ l=\frac{193}{12}+19\times\frac{386}{12}$$

$$=\frac{7527}{12}=627\tfrac{1}{4}\text{ feet}$$

$$S=\frac{(193+7527)\times 20}{2\times 12}$$

$$=\frac{77200}{12}$$

$$=6433\tfrac{1}{3}\text{ feet.}$$

EXAMPLE III.

To insert m arithmetical means between a and b.

Here we are required to form an arithmetical series of which the first and last terms, a and b, are given, and the number of terms $=m+2$; in order, then, to determine the series, we must find the common difference.

Eliminating S by equations (1) and (2), we have

$$2a+(n-1)\delta=l+a$$

$$\delta=\frac{l-a}{n-1}$$

But here $l=b,\ a=a,\ n=m+2$

∴ the required series will be

$$a+\left(a+\frac{b-a}{m+1}\right)+\left(a+\frac{2(b-a)}{m+1}\right)+\dots\dots+\left(a+\frac{m(b-a)}{m+1}\right)+\left(a+\frac{(m+1)(b-a)}{m+1}\right)$$

or

$$a+\frac{b+ma}{m+1}+\frac{2b+(m-1)a}{m+1}+\dots\dots+\frac{mb+a}{m+1}+b.$$

(4) Required the sum of the odd numbers 1, 3, 5, 7, 9, &c., continued to 101 terms?

Ans. 10201.

(5) How many strokes do the clocks of Venice, which go on to 24 o'clock, strike in the compass of a day?

Ans. 300.

(6) The first term of a decreasing arithmetical series is 10, the common difference $\frac{1}{3}$, and the number of terms 21; required the sum of the series.

Ans. 140.

(7) One hundred stones being placed on the ground in a straight line, at the distance of 2 yards from each other; how far will a person travel who shall bring them one by one to a basket which is placed 2 yards from the first stone?

Ans. 11 miles and 840 yards.

The relations (1) and (2), in which five quantities, a, δ, n, l, S, enter, will serve to determine any two of these when the other three are given. Thus they furnish the solution of as many distinct problems as there are ways of taking two quantities from among five; and, consequently, the number of problems will be $\frac{5\cdot 4}{2}$ or 10. In order that they may be possible, it is necessary that the value of n should be not only real, but entire and positive. Without entering into the details of the calculation, we place below the solutions of these ten problems.

I. Given a, δ, n. Required l, S. $\quad l=a+(n-1)\delta,\ S=\frac{1}{2}n[2a+(n-1)\delta]$

II. Given l, δ, n. Required a, S. $\quad a=l-(n-1)\delta,\ S=\frac{1}{2}n[2l-(n-1)\delta]$.

III. Given a, n, l. Required δ, S. $\quad \delta=\frac{l-a}{n-1},\ S=\frac{1}{2}n(a+l)$.

IV. Given δ, n, S. Required a, l. $\quad a=\frac{2S-n(n-1)\delta}{2n},\ l=\frac{2S+n(n-1)\delta}{2n}$.

V. Given a, n, S. Required δ, l. $\quad l=\frac{2S}{n}-a,\ \delta=\frac{2(S-an)}{n(n-1)}$.

VI. Given l, n, S. Required a, δ. $\quad a=\frac{2S}{n}-l,\ \delta=\frac{2(nl-S)}{n(n-1)}$.

VII. Given a, δ, l. Required n, S. $\quad n=\frac{l-a}{\delta}+1,\ S=\frac{(l+a)(l-a+\delta)}{2\delta}$.

VIII. Given a, l. S. Required n, δ. $\quad n=\frac{2S}{a+l},\ \delta=\frac{(l+a)(l-a)}{2S-(l+a)}$.

IX. Given a, δ, S. Required l, n. $\quad n=\frac{\delta-2a\pm\sqrt{(\delta-2a)^2+8\delta S}}{2\delta}$, $\quad l=a+(n-1)\delta$.

X. Given l, δ, S. Required a, n. $\quad n=\frac{\delta+2l\pm\sqrt{(\delta+2l)^2-8\delta S}}{2\delta}$, $\quad a=l-(n-1)\delta$.

GEOMETRICAL PROGRESSION.

229. A series of quantities, in which each is derived from that which immediately precedes it, by multiplication by a constant quantity, is called a *Geometrical Progression*, or *Progression by Quotients*.

Thus, the numbers 2, 4, 8, 16, 32, ... in which each is derived from the preceding by multiplying it by 2, form what is called an *increasing geometrical* *progression*; and the numbers 243, 81, 27, 9, 3, ... in which each is derived from the preceding by multiplying it by the number $\frac{1}{3}$, form what is called a *decreasing geometrical progression.*

The common multiplier in a geometrical progression is called the *common ratio.*

Generally, if a be the first term and ρ the common ratio, the successive terms of the series will be of the form

$$a,\ a\rho,\ a\rho^2,\ a\rho^3 \ldots\ldots$$

The exponent of ρ in the *second* term is 1, in the *third* term is 2, in the *fourth* term 3, and so on; hence the n^{th} term of a series will be of the form,

$$a\rho^{n-1}.$$

230. *To find the sum of* n *terms of a series in geometrical progression.*

Let $a=$ first term,
$l=$ last term,
$\rho=$ common ratio,
$n=$ number of terms,
$S=$ sum of the series.

Then

$$S = a + a\rho + a\rho^2 + a\rho^3 + \ldots\ldots\ldots\ldots\ldots\ldots + a\rho^{n-1}.$$

Multiply both sides of the equation by ρ,

$$S\rho = \quad a\rho + a\rho^2 + a\rho^3 + \ldots\ldots\ldots\ldots\ldots\ldots + a\rho^{n-1} + a\rho^n.$$

Subtract the first from the second,

$$S(\rho-1) = a\rho^n - a$$

$$\therefore\ S = \frac{a(\rho^n-1)}{\rho-1} \ldots\ldots\ldots\ldots (1)$$

Or, since

$$l = a\rho^{n-1}$$

$$S = \frac{\rho l - a}{\rho-1} \ldots\ldots\ldots\ldots (2)$$

If the series be a decreasing one, and consequently ρ fractional, it will be convenient to change the signs of both numerator and denominator in the above expressions, which then become

$$S = \frac{a(1-\rho^n)}{1-\rho}$$

$$S = \frac{a-\rho l}{1-\rho}.$$

231. If two progressions have different first terms, but the same ratio, the ratio of the sums of the two is equal to the ratio of their first terms. For

$$(a + a\rho + a\rho^2 + a\rho^3 +, \text{\&c.}) : (b + b\rho + b\rho^2 + b\rho^3 +, \text{\&c.})$$
$$= a(1 + \rho + \rho^2 + \rho^3 +, \text{\&c.}) : b(1 + \rho + \rho^2 + \rho^3 +, \text{\&c.}) = a : b$$

232. It appears that if any three of the five quantities, a, l, ρ, n, S, be given, the remaining two may be found by eliminating between equations (1) and (2). It must be remarked, however, that when it is required to find ρ from a, n, S given, or from n, l, S given, we shall obtain ρ in an equation of the n^{th} degree, a general solution of which can not be given. If n be required, it will be convenient to apply logarithms, as the equation to be resolved will be an exponential.

EXAMPLE I.

Required the sum of 10 terms of the series 1, 2, 4, 8, ...

Here $$a=1,\ \rho=2,\ n=10$$

$$\therefore S=\frac{a(\rho^n-1)}{\rho-1}$$
$$=2^{10}-1$$
$$=1023.$$

EXAMPLE II.

Required the sum of 10 terms of the series $1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \ldots.$

Here $$a=1,\ \rho=\frac{2}{3},\ n=10$$

$$\therefore S=\frac{a(1-\rho^n)}{1-\rho}$$
$$=\frac{1-\left(\frac{2}{3}\right)^{10}}{1-\frac{2}{3}}$$
$$=\frac{174075}{59049}.$$

EXAMPLE III.

To insert m geometric means between a and b.

Here we are required to form a geometric series, of which the first and last terms, a and b, are given, and the number of terms $=m+2$; in order, then, to determine the series, we must find the common ratio.

Eliminating S by equations (1) and (2),

$$a\rho^n-a=\rho l-a$$
$$\rho=\sqrt[n-1]{\frac{l}{a}}.$$

But here

$$l=b,\ n=m+2$$
$$\therefore \rho=\sqrt[m+1]{\frac{b}{a}}.$$

Hence the series required will be

$$a+a\,.\sqrt[m+1]{\frac{b}{a}}+a\,.\sqrt[m+1]{\frac{b^2}{a^2}}+\ldots+a\,.\sqrt[m+1]{\frac{b^{m-1}}{a^{m-1}}}+a\,.\sqrt[m+1]{\frac{b^m}{a^m}}+a\,.\sqrt[m+1]{\frac{b^{m+1}}{a^{m+1}}},$$

or

$$a+\sqrt[m+1]{a^mb}+\sqrt[m+1]{a^{m-1}b^2}+\ldots+\sqrt[m+1]{a^2b^{m-1}}+\sqrt[m+1]{ab^m}+b,$$

or

$$a+a^{\frac{m}{m+1}}b^{\frac{1}{m+1}}+a^{\frac{m-1}{m+1}}b^{\frac{2}{m+1}}+\ldots+a^{\frac{2}{m+1}}b^{\frac{m-1}{m+1}}+a^{\frac{1}{m+1}}b^{\frac{m}{m+1}}+b.$$

233. *To find the sum of an infinite series decreasing in geometrical progression.*

We have already found that the sum of n terms of a decreasing geometrical series is

$$S=\frac{a-a\rho^n}{1-\rho},$$

which may be put under the form

$$S=\frac{a}{1-\rho}-\frac{a}{1-\rho}\cdot\rho^n.$$

Since ρ is a fraction, ρ^n is less than unity, and the greater the number n, the smaller will be the quantity ρ^n; if, therefore, we take a very great number of terms of a decreasing series, the quantity ρ^n, and, consequently, the term $\frac{a\rho^n}{1-\rho}$, will be very small in comparison with $\frac{a}{1-\rho}$; and if we take n greater than any assignable number, or make $n=\infty$, then ρ^n will be smaller than any assignable number, and therefore may be considered $=0$, and the second term in the above expression will vanish.

Hence we may conclude that the sum of an infinite series, decreasing in geometrical progression, is

$$S=\frac{a}{1-\rho}$$

Strictly speaking, $\frac{a}{1-\rho}$ is the *limit* to which the sum of any number of terms approaches, and the above expression will approach more or less nearly to perfect accuracy, according as the number of terms is greater or smaller.

Thus, let it be required to find the sum of the infinite series

$$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+, \text{\&c.}$$

Here $$a=1,\ \rho=\frac{1}{3},\ n=\infty$$

$$\therefore S=\frac{a}{1-\rho}$$

$$=\frac{1}{1-\frac{1}{3}}$$

$$=\frac{3}{2}.$$

The error which we should commit in taking $\frac{3}{2}$ for the sum of the first n terms of the above series is determined by the quantity

$$\frac{a\rho^n}{1-\rho}=\frac{3}{2}\left(\frac{1}{3}\right)^n.$$

Thus, if $n=5$, then $\frac{3}{2}\left(\frac{1}{3}\right)^5=\frac{1}{2.3^4}=\frac{1}{162}$;

$n=6$, then $\frac{3}{2}\left(\frac{1}{3}\right)^6=\frac{1}{2.3^5}=\frac{1}{486}$.

Hence, if we take $\frac{3}{2}$ as the sum of 5 terms of the above series, the amount would be too great by $\frac{1}{162}$.

If we take $\frac{3}{2}$ as the sum of 6 terms, the amount will be too great by $\frac{1}{486}$, and so on.*

* I. The theory of progressions involves that of logarithms. Let there be two progressions, the one geometric, beginning with 1, the other arithmetical, beginning with 0.

$$\div 1:2:4:8:16:32:64:128, \&c.$$
$$\div 0.3.6.9.12.15.18.\ 21, \&c.,$$

which exhibit a notation sometimes employed.

If we compare these with each other, we perceive that, multiplying together any two terms of the first, and adding the corresponding terms of the second, we obtain two corresponding terms, again, of these same progressions. Thus, $4\times16=64$, $6+12=18$; and we perceive that 18 corresponds to 64. Thus a multiplication is effected by addition. This simple observation is, no doubt, very ancient; but it was the genius of Napier, a Scottish baronet, which derived from it the theory of logarithms, one of the most useful of modern discoveries. It was published in 1644, under the title of *Mirifici Logarithmorum Descriptio.*

Logarithms, then, according to Napier, were regarded as a series of numbers in arithmetical progression, while the numbers themselves corresponding, formed a geometrical progression. I proceed to explain his method of constructing them.

In order that the geometrical progression should embrace all numbers greater than 1, it is necessary to conceive it formed of terms which increase in an insensible manner, setting out from 1; and, to have their logarithms, it is necessary to conceive the arithmetical progression as composed of terms which vary by insensible degrees, setting out from zero.

At their origin, the simultaneous increments which the terms 1 and 0 receive are inappreciably small; but, however small they may be, we may conceive that there is a certain relation established between them, which is entirely arbitrary. Thus, when these increments begin to arise, we can suppose that that of the logarithm 0 is double, triple, &c., of that of the number 1. This relation is called the modulus of the logarithms, which designate by M.

Suppose, now, that to the term 1 of the geometric progression an increment ω, very small, but yet appreciable in numbers, is given. The corresponding increment of the term zero of the arithmetical progression will be very nearly equal to $M\omega$; and we can take for the two progressions these:

$$\div 1:1+\omega:(1+\omega)^2:(1+\omega)^3:(1+\omega)^4:\&c.$$
$$\div 0.\ \ M\omega.\ \ 2M\omega\ .\ 3M\omega\ .\ 4M\omega\ .\&c.$$

We have said that the relation or modulus M can be taken at pleasure; consequently, according to the values attributed to it, will be obtained different *systems* of logarithms. The logarithms which Napier published were derived from the progressions

$$\div 1:1+\omega:(1+\omega)^2:(1+\omega)^3:\&c.$$
$$\div 0.\quad \omega.\quad 2\omega\ .\quad 3\omega\ .\&c.,$$

which supposes $M=1$.

This avoids the multiplications by M. The logarithms of numbers in Napier's table serve to find those of any other system, by simply multiplying each by the modulus of that system.

The terms of these two series vary slowly, so that, in prolonging both as far as we please, we are sure of finding in the first, terms equal to the entire numbers 2, 3, &c., or so near them that the difference may be neglected. The corresponding terms of the second may then be taken for the logarithms of these numbers, and are those written in the tables.

By this we perceive that these logarithms are not exactly those of the numbers beside which they are written. But there is another cause of inaccuracy, viz., that ω represents only approximately the increment, which the logarithm 0 takes when ω is that taken by 1. The smaller ω is, however, the greater the exactness.

II. Let it be proposed to determine the error produced by assuming that the difference of the numbers is proportional to the difference of their logarithms, when the number of places in the numbers is 5, and their difference not greater than 1.

If in the series [A], Art. 224, we make $n=\frac{1}{x}$, we have

$$l\left(\frac{1+x}{x}\right)=l'1+x)-lx=M\left\{\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{3x^3}-\frac{1}{4x^4}+, \&c.\right\},$$

As in arithmetical progressions, all the questions which can be proposed for solution in geometric progressions reduce to 10, the solutions of which are deduced from

$$l = a\rho^{n-1} \quad \dots\dots\dots\dots\dots \quad (1)$$

$$S = \frac{\rho l - a}{\rho - 1} \quad \dots\dots\dots\dots\dots \quad (2)$$

---

from which it appears generally that as the number x increases, the difference of the logarithms of x and $1+x$ diminishes. Also, since $\frac{1}{x}$ is greater than the whole series, $\frac{1}{x}$ being diminished by more than it is increased, we have

$$l(1+x) - lx < \frac{M}{x}.$$

If the base be 10, we have seen that $M = 0.4342\ldots < \frac{1}{2}$. Hence, in this case,

$$l(1+x) - lx < \frac{1}{2x}.$$

If x consist of five places, its least value is 10000. Therefore the greatest value of $l(1+x) - lx$ is less than $\frac{1}{20000} = 0.00005$.

Hence we may infer that the logarithms of every two consecutive whole numbers consisting of five places must agree in the first four decimal places at least.

Now let

$$\Delta = l(1+x) - lx = l\frac{1+x}{x}.$$

$$\Delta' = l(2+x) - l(1+x) = l\frac{2+x}{1+x}.$$

$$\Delta - \Delta' = l\frac{1+x}{x} - l\frac{2+x}{1+x}.$$

$$= l\frac{(1+x)^2}{x(2+x)} = l\left(1 + \frac{1}{x(2+x)}\right).$$

But by [A], Art. 224,

$$l\left(1 + \frac{1}{x(2+x)}\right) = M\left\{\frac{1}{x(2+x)} - \frac{1}{2x^2(2+x)^2} + \frac{1}{3x^3(2+x)^3} - \&c.\right\}$$

$$\therefore \Delta - \Delta' < \frac{1}{2x(2+x)}.$$

If x consist of five places, its least value is 10000, and, therefore, the greatest value of $\Delta - \Delta'$ is less than $\frac{1}{20000 \times 10002} = \frac{1}{200040000}$, which, when reduced to a decimal, has no significant figure within the first eight places. Hence, in tables which extend only to seven places, we may assume that $\Delta - \Delta' = 0$, or $\Delta = \Delta'$.

Thus we infer that, under the circumstances which have been supposed, the logarithms of numbers in arithmetical progression will themselves be in arithmetical progression

Let now n and $n+1$ be two consecutive whole numbers, and $n + \frac{p}{q}$ an intermediate fraction. These may be looked upon as three terms of an arithmetical progression, whose first term is n, whose common difference is $\frac{1}{q}$, whose $(p+1)^{\text{th}}$ term is $n + \frac{p}{q}$, and whose $(q+1)^{\text{th}}$ term is $n+1$. By what has been already shown, the logarithms of the several terms of this series will also be in arithmetical progression.

Let δ be their common difference. The $(p+1)^{\text{th}}$ term of this series will be

$$ln + p\delta,$$

which will be the logarithm of the $(p+1)^{\text{th}}$ term of the former series;

$$\therefore ln + p\delta = l\left(n + \frac{p}{q}\right) \quad \dots\dots \quad [B]$$

These solutions are contained in the following table:

I.	Given $a, \rho, n.$ Required l, S.		$l=a\rho^{n-1}$, $S=\frac{\rho l-a}{\rho-1}=\frac{a(\rho^n-1)}{\rho-1}$.
II.	Given $l, \rho, n.$ Required a, S.		$a=\frac{l}{\rho^{n-1}}$, $S=\frac{l(\rho^n-1)}{\rho^{n-1}(\rho-1)}$.
III.	Given $a, n, l.$ Required ρ, S.		$\rho=\sqrt[n-1]{\frac{l}{a}}$, $S=\frac{\sqrt[n-1]{l^n}-\sqrt[n-1]{a^n}}{\sqrt[n-1]{l}-\sqrt[n-1]{a}}$.
IV.	Given ρ, n, S. Required $a, l.$		$a=\frac{S(\rho-1)}{\rho^n-1}$, $l=\frac{S\rho^{n-1}(\rho-1)}{\rho^n-1}$.
V.	Given a, n, S. Required $\rho, l.$		$\rho^{n-1}+\rho^{n-2}\ldots+1=\frac{S}{a}$, $l=a\rho^{n-1}$.
VI.	Given l, n, S. Required $\rho, a.$		$\left(\frac{1}{\rho}\right)^{n-1}+\left(\frac{1}{\rho}\right)^{n-2}\ldots+1=\frac{S}{l}$ $a=l\left(\frac{1}{\rho}\right)^{n-1}$.
VII.	Given $a, \rho, l.$ Required n, S.		$S=\frac{\rho l-a}{\rho-1}$, $n=1+\frac{\log. l-\log. a}{\log. \rho}$.
VIII.	Given a, l, S. Required $\rho, n.$		$\rho=\frac{S-a}{S-l}$, $n=1+\frac{\log. l-\log. a}{\log. \rho}$.
IX.	Given a, ρ, S. Required $l, n.$		$l=\frac{a+S(\rho-1)}{\rho}$, $n=1+\frac{\log. l-\log. a}{\log. \rho}$.
X.	Given l, ρ, S. Required $a, n.$		$a=l\rho-S(\rho-1)$, $n=1+\frac{\log. l-\log. a}{\log. \rho}$.

HARMONICAL PROGRESSION.

234. A series of quantities is called a *harmonical progression* when, if any three consecutive terms be taken, the first is to the third as the difference of the first and second to the difference of the second and third.

Thus, if $a, b, c, d \ldots$ be a series of quantities in harmonical progression, we shall have

$$a:c::a-b:b-c;\ b:d::b-c:c-d, \text{ \&c.}$$

235. *The reciprocals of a series of terms in harmonical progression are in arithmetical progression.*

Let $a, b, c, d, e, f \ldots$ be a series in harmonical progression.

Then, by definition,

Also, the last term of the latter series, which will be

$$ln+q\delta,$$

will be the logarithm of the last term of the former series;

$$\therefore l(n+1)=ln+q\delta,\ \therefore l(n+1)-ln=q\delta.$$

But by [B],

$$l\left(n+\frac{p}{q}\right)-ln=p\delta, \therefore \frac{l\left(n+\frac{p}{q}\right)-ln}{l(n+1)-ln}=\frac{p}{q}.$$

But, also,

$$\frac{\left(n+\frac{p}{q}\right)-n}{(n+1)-n}=\frac{p}{q}.$$

Hence the differences of the logarithms are as the differences of the numbers.

$$a:c::a-b:b-c;\ b:d::b-c:c-d;\ c:e::c-d:d-e, \text{ \&c.}$$

$$\therefore ab-ac=ac-bc,\ bc-bd=bd-dc,\ cd-ce=ce-ed, \text{ \&c.}$$

$$\therefore \frac{ab}{abc}-\frac{ac}{abc}=\frac{ac}{abc}-\frac{bc}{abc},\ \frac{bc}{bcd}-\frac{bd}{bcd}=\frac{bd}{bcd}-\frac{dc}{bcd},\ \frac{cd}{cde}-\frac{ce}{cde}=\frac{ce}{cde}-\frac{ed}{cde},$$

or

$$\frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a},\ \frac{1}{d}-\frac{1}{c}=\frac{1}{c}-\frac{1}{b},\ \frac{1}{e}-\frac{1}{d}=\frac{1}{d}-\frac{1}{c};$$

from which it appears that the quantities $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e}$, &c., are in arithmetical progression.

To insert m harmonic means between a and b.

Since the reciprocals of quantities in harmonical progression are in arithmetical progression, let us insert m arithmetic means between $\frac{1}{a}$ and $\frac{1}{b}$.

Generally, in arithmetical progression,

$$l=a+(n-1)\delta$$

$$\therefore \delta=\frac{l-a}{n-1}.$$

In this case, $l=\frac{1}{b}$, $a=\frac{1}{a}$, $n=m+2$, and $\therefore\ \delta=\frac{a-b}{(m+1)ab}$.

The arithmetic series will be

$$\frac{1}{a}+\frac{a+mb}{(m+1)ab}+\frac{2a+(m-1)b}{(m+1)ab}+\dots\dots\frac{(m-1)a+2b}{(m+1)ab}+\frac{ma+b}{(m+1)ab}+\frac{1}{b}.$$

Therefore the harmonical series will be

$$a+\frac{(m+1)ab}{a+mb}+\frac{(m+1)ab}{2a+(m-1)b}+\dots\dots+\frac{(m+1)ab}{(m-1)a+2b}+\frac{(m+1)ab}{ma+b}+b.$$

INTEREST AND ANNUITIES.

236. The solution of all questions connected with interest and annuities may be greatly facilitated by the employment of the algebraical formulæ.

In treating of this subject we may employ the following notation:

Let p dollars denote the principal.

r the interest of $1 for one year.

i the interest of p dollars for t years.

s the amount of p dollars for t years at the rate of interest denoted by r.

t the number of years that p is put out at interest.

SIMPLE INTEREST.

Problem I.—*To find the interest of a sum* p *for* t *years at the rate* r.

Since the interest of one dollar for one year is r, the interest of p dollars for one year must be p times as much, or pr; and for t years t times as much as for one year; consequently,

$$i=ptr \quad \dots\dots\dots\dots\dots\dots \quad (1)$$

PROBLEM II.—*To find the amount of a sum* p *laid out for* t *years at simple interest at the rate* r.

The amount must evidently be equal to the principal, together with the interest upon that principal for the given time.

Hence $s = p + ptr$

$= p(1+tr) \dots\dots\dots\dots (2)$

EXAMPLE I.

Required the interest of \$873.75 for $2\frac{1}{2}$ years at $4\frac{3}{4}$ per cent. per annum.

It will be convenient to reduce broken periods of time to decimals of a year.

By the formula (1) we have

$$i = ptr.$$

In the example before us,

$p \dots\dots\dots = \$873.75$

$r \dots\dots\dots = \$.0475$*

$t = 2\frac{1}{2}$ years $\dots\dots\dots = 2.5$ years.

$\therefore i = 873.75 \times 2.5 \times .0475$ dollars.

$= \$103.7578125.$

The *amount* of the above sum at the end of the given time will be

$$s = d + dtr$$

$$= \$873.75 + \$103.757.$$

PRESENT VALUE AND DISCOUNT AT SIMPLE INTEREST.

The present value of any sum s *due* t *years hence is the principal which in the time* t *will amount to* s.

The discount upon any sum due t *years hence is the difference between that sum and its present value.*

PROBLEM III.—*To find the present value of* s *dollars due* t *years hence, simple interest being calculated at the rate* r.

By formula (2) we find the amount of a sum p at the end of t years to be

$$s = p + ptr.$$

Consequently, p will represent the present value of the sum s due t years hence, and we shall have

$$p = \frac{s}{1+tr} \dots\dots\dots\dots (3)$$

for the expression required.

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* r is the interest of \$1 for one year. To find the value of r when interest is calculated at the rate of $\$4\frac{3}{4}$ or \$4.75 *per cent.* per annum, we have the following proportion:

$$\$100 : \$1 :: \$4.75 : r$$

$$\therefore r = \$\frac{4.75}{100} = \$0.0475.$$

In like manner,

When the rate of interest *per cent.* is \$7, then $r = \$0.07$.
When the rate of interest *per cent.* is 6, then $r =$ 0.06.
When the rate of interest *per cent.* is 5, then $r =$ 0.05.
When the rate of interest *per cent.* is $4\frac{3}{4}$, then $r =$ 0.0475.
When the rate of interest *per cent.* is $4\frac{1}{2}$, then $r =$ 0.045.
When the rate of interest *per cent.* is $4\frac{1}{4}$, then $r =$ 0.0425
When the rate of interest *per cent.* is 4, then $r =$ 0.04.
When the rate of interest *per cent.* is $3\frac{3}{4}$, then $r =$ 0.0375.
&c. &c. &c.

PROBLEM IV.—*To find the discount on* s *dollars due* t *years hence, at the rate* r, *simple interest.*

Since the discount on s is the difference between s and its present value, we shall have

$$d = s - \frac{s}{1+tr}$$

$$= \frac{str}{1+tr} \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

EXAMPLE.

Required the discount on $100, due 3 months hence, interest being calculated at the rate of 5 per cent. per annum.

Here $s = \$100$

$t =$ 3 months $=$ .25 years.

$r = \$ .05$.

Here the present value of p is

$$p = \frac{s}{1+tr}$$

$$= \frac{100}{1+.25 \times .05}$$

$$= \frac{100}{1.0125}$$

$$= 98.76543 \text{ dollars.}$$

But $s = \$100$

$p = \$98.76543$

$\therefore s - p$ or $dis = \$1.235$.

ANNUITIES AT SIMPLE INTEREST.

PROBLEM V.—*To find the amount which must be paid at the end of* t *years, for the enjoyment of an annuity* a, *simple interest being allowed at the rate* r.

At the end of the first year the annuity a will be due; at the end of the second year a second payment a will become due, together with ar the interest for one year upon the first payment; at the end of the third year a third payment a becomes due, together with $2ar$ the interest for one year upon the former two payments, and so on; the sum of all these will be the amount required.

Thus:

At the end of the first year, the sum due is a.

At the end of the second year, the sum due is $a + ar$.

At the end of the third year, the sum due is $a + 2ar$.

At the end of the fourth year, the sum due is $a + 3ar$.

&c. &c. &c.

At the end of the t^{th} year, the sum due is $a + (t-1)ar$.

Hence, adding these all together for the whole amount,

$$s = ta + ar(1+2+3+\ldots\ldots(t-1)).$$

Or, taking the expression for the sum of the arithmetical series, $1+2+3+\ldots\ldots\ldots(t-1)$

$$s = ta + ra.\frac{t(t-1)}{1.2} \quad \ldots\ldots\ldots \quad (5)$$

Problem VI.—*To find the present value of an annuity* a *payable for* t *years, simple interest being allowed at the rate* r.

It is manifest that the present value of the annuity must be a sum such that, if put out at interest for t years at the rate r, its amount at the end of that period will be the same with the amount of the annuity.

Hence, if we call this present value p, we shall have, by Problems I. and V.,

$$p+ptr = \text{amount of annuity.}$$
$$= ta+ra.\frac{t(t-1)}{1.2}$$
$$\therefore p=\frac{ta+ra.\frac{t(t-1)}{1.2}}{1+tr}$$
$$=\frac{ta}{2}.\frac{2+(t-1)r}{1+tr} \quad \ldots\ldots\ldots\ldots (6)^*$$

COMPOUND INTEREST.

Problem VII.—*To find the amount of a sum* p *laid out for* t *years, compound interest being allowed at the rate* r.

At the end of the first year the amount will be, by Problem II.,

$$p+pr, \text{ or } p(1+r).$$

Since compound interest is allowed, this sum $p(1+r)$ now becomes the principal, and hence, at the end of the second year, the amount will be $p(1+r)$, together with the interest on $p(1+r)$ for one year; that is, it will be

$$p(1+r)+pr(1+r), \text{ or } p(1+r)^2.$$

The sum $p(1+r)^2$ must now be considered as the principal, and hence the whole amount, at the end of the third year, will be

$$p(1+r)^2+pr(1+r)^2, \text{ or } p(1+r)^3.$$

And, in like manner, at the end of the t^{th} year, we shall have

$$s=p(1+r)^t \quad \ldots\ldots\ldots\ldots (7)$$

Any three of the four quantities, s, p, r, t, being given, the fourth may always be found from the above equation.

EXAMPLE I.

Find the amount of $15.50 for 9 years, compound interest being allowed at the rate of $3\frac{1}{2}$ per cent. per annum, the interest payable at the end of each year.

By equation (7),

$$s=p(1+r)^t$$
$$\therefore \log. s = \log. p + t \log. (1+r).$$

Hence

$$p=\$15.50$$
$$t=9 \text{ years}$$
$$r=\$.035$$
$$\therefore \log p = 1.1903317$$
$$t \log. (1+r) = 0.1344627$$
$$\therefore \log. s = 1.3247944 = \log. \text{ of } 21.12481$$
$$\therefore s = \$21.12481.$$

* It is unnecessary to give any examples under this rule, as the purchase of annuities at simple interest can never be of practical utility.

EXAMPLE II.

Find the amount of £182 12*s.* 6*d.* for 18 years, 6 months, and 10 days, at the rate of $3\frac{1}{2}$ per cent. per annum, compound interest, the interest being payable at the end of each year.

In this case, it will be convenient, first, to find the amount at compound interest of the above sum for 18 years, and then calculate the interest on the result for the remaining period.

By formula (7),

$$s=p(1+r)^t$$
$$\log. s=\log. p+t \log. (1+r)$$

Here $p=£182.\ 12s.\ 6d.=£182.625$
$r= \quad =£.035$
$t= \quad =18$ years

$$\therefore \log. p=2.2615602$$
$$t \log. (1+r)=0.2689254$$
$$\therefore \log. s=2.5304856 = \log. \text{ of } 339.224.$$

Again, to find the interest on this sum for the short period, we have

$$i=s\,t'\,r$$
$$\therefore \log. i= \log. s+ \log. t'+ \log r.$$

Here $s=£339.224$
$r=£.035$
$t'=6$ months, 10 days$=$ .527402 years

$$\therefore \log. s=2.5304856$$
$$\log. r=\bar{2}.5440680$$
$$\log. t'=\bar{1}.7221401$$
$$\therefore \log. s\,t'\,r=.07966937= \log. \text{ of } 6.2617200$$
$$\therefore s\,t'\,r=£6.26172.$$

The whole amount required will, therefore, be

$$s+s\,t'\,r=£339.224+£6.26172$$
$$=£345\ 9s.\ 8\tfrac{1}{2}d.$$

EXAMPLE III.

Required the compound interest upon \$410 for $2\frac{1}{2}$ years at $4\frac{1}{2}$ per cent. per annum, the interest being payable half yearly.

In this case the time t must be calculated in *half years;* and, since we have supposed r to be the interest of \$1 for one year, we must substitute $\frac{r}{2}$, which will be the interest of \$1 for half a year; the formula (7) will thus become

$$s=p\left(1+\frac{r}{2}\right)^{2t}$$
$$\therefore \log. s= \log. p+2t \log. \left(1+\frac{r}{2}\right).$$

Here $p=\$410$
$r=\$.045$
$2t=5$ half years

$$\therefore \log. p=2.6127839$$
$$5 \log. 1.0225=0.0483165$$
$$\therefore \log. s=2.6611004=\log. \text{ of } 458.2471$$
$$\therefore s=\$458.2471.$$

The *interest* must be the difference between this amount and the original principal;

$$\therefore i = s - p$$
$$= \$458.247 - \$410$$
$$= \$48.247.$$

EXAMPLE IV.

\$400 was put out at compound interest, and at the end of 9 years amounted to \$569.333 ; required the rate of interest per cent.

Here s, p, t are given, and r is sought.

From formula

$$s = p(1+r)^t$$

we have

$$\log.(1+r) = \frac{1}{t}(\log. s - \log. p).$$

Here

$$s = \$569.3333$$
$$p = \$400$$
$$t = 9 \text{ years}$$
$$\therefore \log. s = 2.7553666$$
$$\log. p = 2.6020600$$
$$\therefore \log. s - \log. p = .1533066$$
$$\log.(1+r) = \frac{.1533066}{9}$$
$$= .0170340$$
$$= \log. \text{ of } 1.04$$
$$\therefore r = .04 = 4 \text{ per cent.}$$

EXAMPLE V.

In what time will a sum of money double itself, allowing 4 per cent. compound interest?

Here s, p, r are given, and t is sought.

From the formula (7) we have

$$s = p(1+r)^t.$$

But here

$$s = 2p$$
$$\therefore 2p = p(1+r)^t$$
$$\therefore 2 = (1+r)^t$$
$$t = \frac{\log. 2}{\log.(1+r)}$$
$$= \frac{.3010300}{.0170333}$$
$$= 17.673 \text{ years}$$
$$= 17 \text{ years, 8 months, 2 days.}$$

In like manner, if it be required to find in what time a sum will triple itself at the same rate, we have

$$t = \frac{\log. 3}{\log. 1.04}$$
$$= \frac{.4771213}{.0170333}$$
$$= 28.011 \text{ years}$$
$$= 28 \text{ years, 0 months, 3 days.}$$

PRESENT VALUE AND DISCOUNT AT COMPOUND INTEREST.

If we call p the present value of a sum s due t years hence, and d its discount, reasoning precisely in the same manner as in the case of simple interest, we shall find

$$p=\frac{s}{(1+r)^t} \quad \dots\dots\dots (8)$$

$$d=s\left(1-\frac{1}{(1+r)^t}\right) \quad \dots\dots\dots (9)$$

ANNUITIES AT COMPOUND INTEREST.

PROBLEM VIII.—*To find the amount of an annuity* a *continued for* t *years, compound interest being allowed at the rate* r.

At the end of the first year the annuity a will become due; at the end of the second year a second payment a will become due, together with the interest of the first payment a for one year, that is, ar; the whole sum upon which interest must now be computed is thus, $2a+ar$.

At the end of the third year a further payment a becomes due, together with the interest on $2a+ar$, i. e., $2ar+ar^2$; the whole sum upon which interest must now be computed is $3a+3ar+ar^2$. The result will appear evident when exhibited under the following form:

Whole amount at the end of first year, $=a$.

Whole amount at the end of second year, $=a+a+ar$
$=a+a(1+r)$.

Whole amount at the end of third year, $=a+a+a(1+r)+ar+ar(1+r)$
$=a+a(1+r)+a(1+r)^2$.

Whole amount at the end of fourth year, $=a+a+a(1+r)+a(1+r)^2+ar$
$+ar(1+r)+ar(1+r)^2$.
$=a+a(1+r)+a(1+r)^2+a(1+r)^3$

&c. &c. &c.

Whole amount at the end of t^{h} year, $=a+a(1+r)+a(1+r)^2+a(1+r)^3$
$+\dots\dots a(1+r)^{t-1}$.

Hence the whole amount is, in terms of the sum of a geometric progression,

$$s=a\{1+(1+r)+(1+r)^2+\dots\dots+(1+r)^{t-1}\}$$

$$=a\cdot\frac{(1+r)^t-1}{r} \quad \dots\dots\dots (10)$$

PROBLEM IX.—*To find the present value of an annuity* a *payable for* t *years, compound interest being allowed at the rate* r.

It is manifest that the present value of this annuity must be a sum such, that if put out at interest for t years at the rate r, its amount at the end of that period will be the same as the amount of the annuity.

Hence, if we call this present value p, we shall have, by Probs. VII. and VIII.,

$$p(1+r)^t=\text{amount of annuity}$$

$$=a\cdot\frac{(1+r)^t-1}{r}$$

$$\therefore p=\frac{(1+r)^t-1}{r(1+r)^t}\cdot a$$

$$=\frac{a}{r}\cdot\frac{(1+r)^t-1}{(1+r)^t} \quad \dots\dots\dots (11)$$

EXAMPLE.

What is the present value of an annuity of $500, to last for 40 years, compound interest being allowed at the rate of $2\frac{1}{2}$ per cent. per annum.

By formula (11),

$$p=\frac{a}{r}\cdot\frac{(1+r)^t-1}{(1+r)^t}.$$

Here

$$a=\$500$$
$$r=\$.025$$
$$t=40 \text{ years};$$
$$\therefore (1+r)^t=(1.025)^{40}.$$

Now

$$\begin{aligned}\log. (1.025)^{40}&=40 \log. 1.025\\ &=40\times .0107239\\ &=.4289560\\ &=\log. 2.685072\end{aligned}$$
$$\therefore (1.025)^{40}=2.685072=(1+r)^t.$$

Also,

$$\frac{a}{r}=\frac{500}{.025}=20000$$
$$\begin{aligned}\therefore p&=20000\times\frac{1.685072}{2.685072}\\ &=20000\times .62757\ldots\\ &=12551.40 \text{ dollars.}\end{aligned}$$

REVERSION OF ANNUITIES.

PROBLEM X.—*To find the present value* (P) *of an annuity* a *which is to commence after* T *years, and to continue for* t *years.*

The present value required is manifestly the present value of a for $T+t$ years, minus the present value of a for T years.

By Problem IX., the present value of a for $T+t$ years $=\frac{a}{r}\cdot\frac{(1+r)^{T+t}-1}{(1+r)^{T+t}}$.

By Problem IX., the present value of a for T years $=\frac{a}{r}\cdot\frac{(1+r)^T-1}{(1+r)^T}$

$$P=\frac{a}{r}\cdot\left\{(1+r)^{-T}-(1+r)^{-(T+t)}\right\} \quad \ldots\ldots (12)$$

PURCHASE OF ESTATES.

PROBLEM XI.—*To find the present value* p *of an estate, or perpetuity, whose annual rental is* a, *compound interest being calculated at the rate* r.

The present value of an annuity a, to continue for t years, by Prob. IX., is

$$p=\frac{a}{r}\left\{1-(1+r)^{-t}\right\};$$

but if the annuity last *forever*, as in the case of an estate, then $t=\infty$, and $\therefore \frac{1}{(1+r)^t}=\frac{1}{\infty}=0$; hence, in the present case,

$$p=\frac{a}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

EXAMPLE.

What is the value of an estate whose rental is \$1000, allowing the purchaser 5 per cent. for his money ?

Here

$$a = \$1000$$
$$r = \$.05$$

$$\therefore p = \frac{1000}{.05}$$
$$= 20000, \text{ or 20 years' purchase.}$$

REVERSION OF PERPETUITIES.

PROBLEM XII.—*To find the present value of an estate, or perpetuity, whose annual rental is* a *dollars, to a person to whom it will revert after* T *years, compound interest being allowed at the rate* r.

By Problem X., the present value of an annuity, to commence after T years, and to continue for t years, is

$$p = \frac{a}{r}\left\{(1+r)^{-T} - (1+r)^{-(T+t)}\right\}$$

In the present case, $t = \infty$, and $\therefore (1+r)^{-(T+t)} = 0$; hence we shall have

$$p = \frac{a}{r} \cdot \frac{1}{(1+r)^T} \quad \ldots\ldots\ldots\ldots \quad (14)$$

EXAMPLES FOR PRACTICE.

(1) Find the interest of \$555 for $2\frac{1}{2}$ years at $4\frac{3}{4}$ per cent. simple interest.

Ans. \$65.906.

(2) In what time will the interest of \$1 amount to 75 cents, allowing $4\frac{1}{2}$ per cent. simple interest?

Ans. 16 years, 8 months.

(3) What is the amount of \$120.50 for $2\frac{1}{2}$ years at $4\frac{3}{4}$ per cent. simple interest?

Ans. \$134.809.

(4) The interest of £25 for $3\frac{1}{2}$ years, at simple interest, was found to be £3 18*s.* 9*d.* ; required the rate per cent. per annum.

Ans. $4\frac{1}{2}$.

(5) Find the discount on £100 due at the end of 3 months, interest being calculated at the rate of 5 per cent. per annum.

Ans. £1 4*s.* $8\frac{1}{4}d$

(6) What is the present value of the compound interest of £100 to be received five years hence at 5 per cent. per annum ?

Ans. £78 7*s.* $0\frac{1}{2}d$.

(7) What is the amount of £721 for 21 years at 4 per cent. per annum compound interest?

Ans. £1642 19*s.* $9\frac{1}{4}d$

(8) The rate of interest being 5 per cent., in what number of years, at compound interest, will \$1 amount to \$100 ?

Ans. 94 years, 141.4 days.

(9) Find the present value of £430, due nine months hence, discount being allowed at $4\frac{1}{2}$ per cent. per annum.

Ans. £415 19*s.* $2\frac{1}{2}d$.

(10) Find the amount of \$1000 for 1 year at 5 per cent. per annum, compound interest, the interest being payable daily.

Ans. \$1051.288 nearly.

(11) What sum ought to be given for the lease of an estate for 20 years, of the clear annual rental of £100, in order that the purchaser may make 8 per cent. of his money?

Ans. £981 16*s*. $3\frac{3}{4}d$.

(12) Find the present value of £20, to be paid at the end of every five years, forever, interest being calculated at 5 per cent.

Ans. £72 7*s*. $9\frac{1}{2}d$.

(13) What is the present value of an annuity of £20, to continue forever, and to commence after two years, interest being calculated at 5 per cent.?

Ans. £362 16*s*. $2\frac{3}{4}d$.

(14) The present value of a freehold estate of £100 per annum, subject to the payment of a certain sum (A) at the end of every two years, is £1000, allowing 5 per cent. compound interest. Find the sum (A).

Ans. A = £102 10*s*.

(15) What is the present value of an annuity of £79 4*s*., to commence 7 years hence and continue forever, interest being calculated at the rate of $4\frac{1}{2}$ per cent.?

Ans. £1293 5*s*. $11\frac{1}{2}d$.

INTERPOLATION.

234. This name is applied to the process of finding intermediate numbers between those given in tables.

Tables are generally calculated from an algebraic formula in which there are two variable quantities, the one of which is called a *function* of the other, the latter being usually called the *argument* of the function.

Thus, logarithms are functions of the numbers to which they belong, the numbers being the arguments. Several formulas expressing the relation between a number and its logarithm have been seen by the student, and will serve to exemplify the formulas in general of which we are now speaking.

The substitution of successive numbers for the argument, the calculating of the corresponding values of the function, and writing the results in a table, is called *tabulating* the formula.

If the formulas which have been derived under our articles upon interest and annuities should be tabulated, they would furnish what are called interest tables.

The function frequently depends upon two arguments, as in the formula for simple interest,

$$i = ptr \ldots\ldots\ldots\ldots (1)$$

Here the function is i, the interest, and the arguments are, p the principal, and r the rate. This requires a table of double entry, the usual form of which is a table in several columns occupying the whole width of the page, the arguments being placed, the successive values of the one in a horizontal line at the heads of the columns, and of the other in a vertical line at the side of the page, the corresponding values of the function being placed in the column under one of its arguments, and on the horizontal line of the other. The formula (1) above may employ a table of triple entry, the three arguments being the principal, the rate, and the time. Such a table is formed by giving a whole page to the argument of rate, the side and top being occupied by the arguments of principal and time.

235. Where the differences of the functions are proportional to the differences of their arguments, then the interpolation is made by simply solving a proportion, the first two terms of which are the difference of the tabulated functions and the difference of their arguments; the third term being the difference between one of the tabulated arguments and that whose function is to be interpolated; the fourth, or unknown, term of this proportion will be the interpolated function required. This is called the method by first differences, and has been exemplified in taking out logarithms of large numbers not found exactly in the tables.

When the differences of the functions are not nearly proportional to the differences of the arguments, as in the case of the logarithms of small numbers, the method of interpolation above described would not be sufficiently accurate. The nature of the variation of the function, as the argument varies in value, is made sensible by taking the difference between each two of three consecutive functions in the table, and comparing the difference between the first and second with the difference between the second and third. If these differences are the same, we have seen, in the note to (Art. 233), that the method of first differences already explained applies; but if they are not, their difference, which is called a second difference, will, by its magnitude, indicate the degree of inaccuracy of the method of first differences. This exposition will serve to exhibit, in a general way, the nature and office of second differences. We proceed to give a more analytic development of the use of second, third, &c., differences, the latter holding the same relation to the second differences that these do to the first.

236. Let f and $f+\delta_1$ represent two consecutive functions in the table, δ_1 being their first difference. The next consecutive function, if the first differences were constant, would be expressed by $f+2\delta_1$; but as they are supposed not to be, it must be expressed by the form $f+2\delta_1+\delta_2$, δ_2 being the second difference, or difference between the two first differences, δ_1 and $\delta_1+\delta_2$. The scheme below will show the form of the successive functions:

Functions.	1st Differences.	2d Differences.	3d Differences.	4th Differences.
f				
$f+\delta_1$	δ_1			
$f+2\delta_1+\delta_2$	$\delta_1+\delta_2$	δ_2		
$f+3\delta_1+3\delta_2+\delta_3$	$\delta_1+2\delta_2+\delta_3$	$\delta_2+\delta_3$	δ_3	
$f+4\delta_1+6\delta_2+4\delta_3+\delta_4$	$\delta_1+3\delta_2+3\delta_3+\delta_4$	$\delta_2+2\delta_3+\delta_4$	$\delta_3+\delta_4$	δ_4

and so on; from which we perceive that the coefficients are the same as in the expansion of a binomial, that of the second term being the number of the consecutive function after the first function. Denoting this number by n, we have for the general form of the nth function after the first,

$$f+n\delta_1+\frac{n(n-1)}{1.2}\delta_2+\frac{n(n-1)(n-2)}{1.2.3}\delta_3+\ldots+\delta_n \quad . \quad . \quad . \quad . \quad [C]$$

Suppose, now, that a value of the function intermediate between the first and second of the series in the table be required, n here, instead of being an entire number, is a fraction. If the value of the function be required, corresponding to a value of the argument midway between its consecutive values in the table, n becomes equal to $\frac{1}{2}$. If the arguments of the tables differ by 24 hours, and the function be required for 3 hours, n becomes equal to $\frac{3}{24}$, or $\frac{1}{8}$. If the tabu lar arguments differ by 1 hour, or 60 minutes, and the function be required for an argument 15 minutes beyond an even hour, $n=\frac{15}{60}=\frac{1}{4}$.

EXAMPLE.

Given the logs. of 15, 16, 17, 18, 19, to find that of 17.25.

Arg. or No.	Func. or Log.	1st Difs. δ_1.	2d Difs. δ_2.	3d Difs. δ_3.	δ_4.
15	1.17609126	2802872			
16	1.20411998	2632894	—169978		
17	1.23044892	2482359	—150535	+19443	—3158
18	1.25527251	2348109	—134250	+16285	
19	1.27875360				

The numbers in the third column are obtained by taking the differences of the consecutive numbers in the second. The numbers in the fourth column from the second in the same way.

As 2.25 is $\frac{9}{4}$ the interval between 15 and 18, we make $n=\frac{9}{4}$, and have for formula (C), taking $\delta_1=2802872$, $\delta_2=-69978$, $\delta_3=19443$, $\delta_4=-3158$.

The result would be nearly the same by neglecting δ_4 and using the mean of the two third differences.*

$$f= \quad 1.176126$$

$$n\delta_1=\frac{9}{4}\delta_1= \quad 306462$$

$$\frac{n(n-1)}{1.2}\delta_2=\frac{45}{32}\delta_2= \quad -239031$$

$$\frac{n(n-1)(n-2)}{1.2.3}\delta_3=\frac{45}{384}\delta_3= \quad 2278$$

$$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\delta_4=-\frac{135}{6144}\delta_4= \quad 69$$

Value of func. required, viz., log. 17.25 = 1.23678904

The formula for interpolation may be derived very elegantly by the method of indeterminate coefficients. Thus, let y represent the value of the interpolated function to be found, A the argument in the table, m the number of parts (4ths in the example above) between A and the consecutive argument of the table, and n the whole number of parts (4 in the above example) between these consecutive arguments. It is evident that y, depending on A and m, may be expressed in terms of these. Assume, therefore,

$$y=A+Bm+Cm^2+Dm^3+, \&c.,$$

in which B, C, D, &c., are undetermined coefficients, whose values are to be found.

Now let m have successive values, represented by 0, n, $2n$, $3n$, &c., then the corresponding values of y will be

* As *means* are much used in calculations with tables, it may be well to advertise the student that a mean of three numbers is obtained by adding them together and dividing by 3; of five numbers, by adding them together and dividing the sum by 5, and so on.

$$A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$
$$A+Bn+Cn^2+Dn^3+, \&c. \ldots\ldots\ldots (2)$$
$$A+B.2n+C(2n)^2+D(2n)^3+, \&c. \ldots\ldots (3)$$
$$A+B.3n+C(3n)^2+D(3n)^3+, \&c. \ldots\ldots (4)$$
&c.

Subtracting successively (1) from (2), (2) form (3), &c., and representing the remainders by P', Q', R', &c., and dividing by n, we have

$$\frac{P'}{n}=B+C.n+Dn^2+, \&c. \ldots\ldots\ldots (5)$$
$$\frac{Q'}{n}=B+C.3n+D7n^2+, \&c. \ldots\ldots\ldots (6)$$
$$\frac{R'}{n}=B+C.5n+D19n^2+, \&c. \ldots\ldots\ldots (7)$$
&c. &c.

Again, subtracting successively (5) from (6), (6) from (7), &c., and representing the remainders by P'', Q'', &c., and dividing by $2n$, we get

$$\frac{P''}{2n}=C+D.3n+, \&c. \ldots\ldots\ldots (8)$$
$$\frac{Q''}{2n}=C+D.6n+, \&c. \ldots\ldots\ldots (9)$$
&c. &c.

Next, subtracting (8) from (9), &c., and representing the remainders by P''', &c., and dividing by $3n$, we have

$$\frac{P'''}{3n}=D+, \&c. \ldots\ldots\ldots (10)$$

But $$P'''=\frac{Q''-P''}{2n}; \text{ also } Q''=\frac{R'-Q'}{n} \text{ and } P''=\frac{Q'-P'}{n};$$

$$\therefore P'''=\frac{(R'-Q')-(Q'-P')}{2n^2}.$$

Putting δ_3 for the numerator of this fraction, we have by (10),

$$D=\frac{P'''}{3n}=\frac{\delta_3}{6n^3}.$$

Substituting this value of D in (8), and transposing, there results

$$C=\frac{P''}{2n}-\frac{\delta_3}{2n^2}.$$

But $P''=\frac{Q'-P'}{n}$, and putting δ_2 for $Q'-P'$, we obtain

$$C=\frac{\delta_2}{2n^2}-\frac{\delta_3}{2n^2}.$$

Again, substituting these values of D and C in (5), and transposing, we have

$$B=\frac{P'}{n}-\frac{\delta_2}{2n}+\frac{\delta_3}{2n}-\frac{\delta_3}{6n};$$

or, putting δ_1 for P', and simplifying,

$$B=\frac{\delta_1}{n}-\frac{\delta_2}{2n}+\frac{\delta_3}{3n}.$$

Finally, substituting these values of the coefficients B, C, D ... in the as sumed equation, we obtain

$$y=A+\frac{m}{n}\delta_1+\frac{1}{2}\frac{m}{n}\left(\frac{m}{n}-1\right)\delta_2+\frac{1}{6}\frac{m}{n}\left(\frac{m^2}{n^2}-\frac{3m}{n}+2\right)\delta_3+, \&c.,$$

as the formula for interpolation, which coincides with the one obtained before, δ_1, δ_2, δ_3 .. being the first, second, and third differences of the functions, as is evident from the manner in which they have been assumed above.

Let us apply it to a table in the Nautical Almanac, which gives the moon's latitude at noon and midnight for every day in the year.

EXAMPLE.

Let it be required to find the moon's latitude for August 4, 1842, at $16^h\ 18^m$ mean time at Greenwich, that is, at 4.3 hours after midnight.

Moon's Latitude.	δ_1.	δ_2.	Mean Second Difference.
Aug. 4. Noon, $+0°\ 45'\ 48''.1$ Midnight, $+0\ \ 5\ 54.6$ Aug. 5. Noon,* $-0\ 34\ 33.1$ Midnight, $-1\ 14\ 49.4$	$-39'\ 53''.5$ $-40\ 27.7$ $-40\ 16.3$	$+34.2$ -11.4	$+11''.4$

Now, to apply the formula, we have

$$A=0°\ 5'\ 54''.6,\ \delta_1=-40'\ 27''.7,\text{ or }-40.463\text{ minutes};$$

$$\frac{m}{n}=\frac{4.3}{12}=0.358,\ \frac{m}{n}\delta_1=-14'\ 29''.16;$$

$$\delta_2=+11''.4,\ \frac{m}{n}-1=-0.642,\ \tfrac{1}{2}\frac{m}{n}\left(\frac{m}{n}-1\right)\delta_2=-1''.31.$$

Therefore, $y=-0°\ 8'\ 35''.87$, which, without the sign $-$, is the moon's correct latitude south at the time for which it was required.

Second differences will ordinarily insure sufficient accuracy. Third and fourth differences are rarely used.

INEQUATIONS.

237. In discussing algebraical problems, it is frequently necessary to introduce *inequations*, that is, expressions connected by the sign $>$. Generally speaking, the principles already detailed for the transformation of equations are applicable to inequations also. There are, however, some important exceptions which it is necessary to notice, in order that the student may guard against falling into error in employing the sign of inequality. These exceptions will be readily understood by considering the different transformations in succession.

I. *If we add the same quantity to, or subtract it from, the two members of any inequation, the resulting inequation will always hold good, in the same sense as the original inequation*; that is, if

$$a>b,\text{ then } a+a'>b+a',\text{ and } a-a'>b-a'.$$

Thus, if

$$8>3,\quad\text{we have still}\quad 8+5>3+5,\quad\text{and}\quad 8-5>3-5.$$

So, also, if

$$-3<-2,\text{ we have still } -3+6<-2+6,\text{ and } -3-6<-2-6.†$$

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* The moon's latitude is marked $+$ when north, $-$ when south.

† The negative quantity of greater numerical value is always considered less than the negative quantity of less numerical value.

The truth of this proposition is evident from what has been said with reference to equations.

This principle enables us, as in equations, to transpose any term from one member of an inequation to the other by changing its sign.

Thus, from the inequation

$$a^2+b^2>3b^2-2a^2,$$

we deduce

$$a^2+2a^2>3b^2-b^2,$$

or

$$3a^2>2b^2.$$

II. *If we add together the corresponding members of two or more inequations which hold good in the same sense, the resulting inequation will always hold good in the same sense as the original individual inequations*; that is, if

$$a>b,\ c>d,\ e>f,$$

then

$$a+c+e>b+d+f.$$

III. *But if we subtract the corresponding members of two or more inequations which hold good in the same sense, the resulting inequation* WILL NOT ALWAYS *hold good in the same sense as the original inequations.*

Take the inequations $4<7$, $2<3$, we have still $4-2<7-3$, or $2<4$.

But take $9<10$ and $6<8$, the result is $9-6>$ (not $<$) $10-8$, or $3>2$.

We must, therefore, avoid as much as possible making use of a transformation of this nature, unless we can assure ourselves of the sense in which the resulting inequality will subsist.

IV. *If we multiply or divide the two members of an inequation by a positive quantity, the resulting inequation will hold good in the same sense as the original inequation.* Thus, if

$$a<b,\quad \text{then}\quad ma<mb,\quad \frac{a}{m}<\frac{b}{m}$$

$$-a>-b,\ \text{then}\ -na>-nb,\ -\frac{a}{n}>-\frac{b}{n}.$$

This principle will enable us to clear an inequation of fractions.

Thus, if we have

$$\frac{a^2-b^2}{2d}>\frac{c^2-d^2}{3a},$$

multiplying both members by $6ad$, it becomes

$$3a(a^2-b^2)>2d(c^2-d^2).$$

But,

V. *If we multiply or divide the two members of an inequation by a negative quantity, the resulting inequation will hold in a sense opposite to that of the original inequation.*

Thus, if we take the inequation $8>7$, multiplying both members by -3, we have the opposite inequation, $-24<-21$.

Similarly, $8>7$, but $\frac{8}{-3}<\frac{7}{-3}$, or $-\frac{8}{3}<-\frac{7}{3}$.

VI. *We can not change the signs of both members of an inequation unless we reverse the sense of the inequation, for this transformation is manifestly the same thing as multiplying both members by* -1.

VII. *If both members of an inequation be positive numbers, we can raise them to any power without altering the sense of the inequation*; that is, if

$$a>b, \text{ then } a^n>b^n.$$

Thus, from $5>3$ we have $(5)^2>(3)^2$, or $25>9$.

So, also, from $(a+b)>c$, we have $(a+b)^2>c^2$.

But,

VIII. *If both members of an inequation be not positive numbers, we can not determine,* a priori, *the sense in which the resulting inequation will hold good, unless the power to which they are raised be of an uneven degree.*

Thus, $-2<3$ gives $(-2)^2<(3)^2$, or $4<9$;

But, $-3>-5$ gives $(-3)^2<(-5)^2$, or $9<25$;

Again, $-3>-5$ gives $(-3)^3>(-5)^3$, or $-27>-125$.

In like manner,

IX. *We can extract any root of both members of an inequation without altering the sense of the inequation*; that is, if

$$a>b, \text{ then } \sqrt[n]{a}>\sqrt[n]{b}.$$

If the root be of an even degree, both members of the inequation must necessarily be positive, otherwise we should be obliged to introduce imaginary quantities, which can not be compared with each other.

EXAMPLES IN INEQUATIONS.

(1) The double of a number, diminished by 6, is greater than 24; and triple the number, diminished by 6, is less than double the number increased by 10. Required a number which will fulfill the conditions.

Let x represent a number fulfilling the conditions of the question; then, in the language of inequations, we have

$$2x-6>24, \text{ and } 3x-6<2x+10.$$

From the former of these inequations we have

$$2x>30, \text{ or } x>15;$$

and from the latter we get

$$3x-2x<10+6, \text{ or } x<16;$$

therefore 15 and 16 are the limits, and any number between these limits will satisfy the conditions of the question. Thus, if we take the number 15·9, we have

$$15{\cdot}9\times2-6>24 \text{ by } 1{\cdot}8,$$

while $15{\cdot}9\times3-6<15{\cdot}9\times2+10$ by 0·1.

(2) $$3x-2>\frac{5}{2}x-\frac{4}{5}$$

$$\therefore 30x-20>25x-8$$

$$30x-25x>20-8$$

$$5x>12$$

$$x>\frac{12}{5}.$$

(3) $$43-5x<10-8x.$$

Ans. $x<-11$.

(4) $$\frac{7}{6}-\frac{5}{4}x<8-2x.$$

Ans. $x<\frac{82}{9}$.

In the second example, $\frac{12}{5}$, or $2\frac{2}{5}$, is *an inferior limit* of the values of x.

In the second, -11, and, in the third, $\frac{82}{9}$, or $9\frac{1}{9}$, are superior limits of the value of x. If the second and fourth of the above inequalities must be verified simultaneously by the values of x, these values must be comprised between $2\frac{2}{5}$ and $9\frac{1}{9}$. If the third and fourth, it is sufficient that it be less than -11. Finally, there is no value which will verify at the same time the 2° and 3°.

(5) $$3x-2y>5,\ 5x+3y>16;$$
$$\therefore x>\frac{5+2y}{3} \text{ and } x>\frac{16-3y}{5}.$$

We can attribute to y any value whatever, and for each arbitrary value of y we can give to x all the values greater than the greatest of the two quantities

$$\frac{5+2y}{3},\ \frac{16-3y}{5}.$$

We determine, also, from the proposed inequalities,

$$y<\frac{3x-5}{2},\ y>\frac{16-5x}{3}.$$

In order that these last two may be fulfilled,

$$\frac{3x-5}{2}>\frac{16-5x}{3};$$
$$\therefore x>\frac{47}{19}.$$

Thus x can receive only values superior to $\frac{47}{19}$, or $2\frac{9}{19}$, and for each value of x there should be admitted for y but values comprised between the two limits above.

(6) $$x^2+4x>12$$
$$\therefore x^2+4x+4>16$$
$$x+2>\pm 4$$
$$x>2, \text{ or } -2.$$

The inferior limit of x is $+2$.

(7) $$x^2+7x<30.$$

Ans. $x<3$ or -10.

The superior limit of x is -10.

GENERAL THEORY OF EQUATIONS.

THE NATURE AND COMPOSITION OF EQUATIONS.

238. The valuable improvements recently made in the process for the determination of the roots of equations of all degrees, render it indispensably necessary to present to the student a view of the present state of this interesting department of analytical investigation. The beautiful *theorem* of M. Sturm for the complete separation of the real and imaginary roots, and for discovering their initial figures, combined with the admirable method of continuous approximation as improved by Horner, has given a fresh impulse to this branch of scientific research, entirely changed the state of the subject, and completed the theory and numerical solution of equations of all degrees.

We recapitulate here two or three

DEFINITIONS.

1. An *equation* is an algebraical expression of equality between two quantities.

2. A *root of an equation* is that number, or quantity, which, when substituted for the unknown quantity in the equation, verifies that equation.

3. A *function* of a quantity is any expression involving that quantity; thus, ax^2+b, ax^3+cx+d, $\frac{ax^2+b}{cx+d}$, a^x are all functions of x; and also ax^2-by^2, $\sqrt{4x-5y}$, $\frac{2x+3y}{3x-2y}$, $y^2+yx+x^2+a^2+b+2$, are all functions of x and y. These functions are usually written $f(x)$, and $f(x, y)$.

4. To express that two members of an equation are identical or true for every value of x, the sign $\equiv$ is sometimes used.

PROPOSITION I.

Any function of x, *of the form*

$$x^n+px^{n-1}+qx^{n-2}+rx^{n-3}+\ldots\ldots$$

when divided by x—a, *will leave a remainder, which is the same function of* a *that the given polynomial is of* x.

Let $f(x)=x^n+px^{n-1}+qx^{n-2}+\ldots\ldots$; and, dividing $f(x)$ by $x-a$, let Q denote the quotient thus obtained, and R the remainder which does not involve x; hence, by the nature of division, we have

$$f(x)\equiv Q(x-a)+R.$$

Now this equation must be true for every value of x, because its truth depends upon a principle of division which is independent of the particular values of the letters; hence, if $x=a$, we have

$$^*f(a)=0+R;$$

and, therefore, the remainder R is the same function of a that the proposed polynomial is of x.

EXAMPLES.

(1) What is the remainder of x^2-6x+7, divided by $x-2$, without actually performing the operation?

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* The student will recollect that $f(x)$ stands for $x^n+px^{n-1}+$, &c., and that, therefore, $f(a)$ will stand for $a^n+pa^{n-1}+qa^{n-2}+$, &c.

(2) What is the remainder of $x^3-6x^2+8x-19$, divided by $x+3$?
(3) What is the remainder of $x^4+6x^3+7x^2+5x-4$, divided by $x-5$?
(4) What is the remainder of x^3+px^2+qx+r, divided by $x-a$?

ANSWERS.

(1) $R=2^2-6\times2+7=-1$.
(2) $R=(-3)^3-6(-3)^2+8(-3)-19=-124$.
(3) 1571.
(4) a^3+pa^2+qa+r.

PROPOSITION II.

If a *is the root of the equation,*

$$x^n+A_1x^{n-1}+A_2x^{n-2}+\ldots\ldots A_{n-2}x^2+A_{n-1}x+A_n=0,$$

the first member of the equation is divisible by $x-a$.

If the division be performed, the remainder, according to the preceding proposition, must be of the form

$$a^n+A_1a^{n-1}+A_2a^{n-2}\ldots\ldots+A_{n-2}a^2+A_{n-1}a+A_n;$$

i. e., the same function of a that the first member of the proposed equation is of x; and, therefore, since a is a root of the equation, the remainder vanishes, and the polynomial, or first member of the equation, is divisible exactly by $x-a$.

Conversely, if the first member of an equation $f(x)=0$ *be divisible by* $x-a$, *then* a *is a root of the equation.*

For, by the foregoing demonstration, the final remainder is $f(a)$; but since $f(x)$, or the first member of the equation, is divisible by $x-a$, the remainder must vanish; hence $f(a)=0$; and therefore, a being substituted for x in the equation $f(x)=0$, verifies the equation, and, consequently, a is a root of the equation.

PROPOSITION III.

239. The proposition that every equation has a root, has in most treatises on Algebra been taken for granted. It has, however, of late years been thought to require a demonstration, and we add one which is as brief and clear as any of the best modifications of that by Cauchy.

As it will prove a little tedious, the student may, if he please to admit the proposition, pass on to Prop. IV.

It will be necessary to premise a few lemmas relating to the properties of moduli, some of which have been already demonstrated (Art. 197), but we repeat them here for convenience of reference.

LEMMA I.—*The sum or difference of any two quantities whatever has a modulus comprehended between the sum and difference of the moduli of the two quantities.*

LEMMA II.—*The modulus of a product of two factors is equal to the product of their moduli.*

Corollary.—Hence the product of the moduli of any number of factors is the modulus of their product, and the modulus of the n^{th} power of a quantity is the n^{th} power of its modulus.

LEMMA III.—*In order that a quantity of the form* $a+b\sqrt{-1}$ *may be zero, it is necessary, and it is sufficient, that its modulus should be zero;* for a and b being real quantities, let

$$a+b\sqrt{-1}=0.$$

As the real part a can not destroy the imaginary part $b\sqrt{-1}$, we must have separately $a=0$ and $b=0$ ∴ $\sqrt{a^2+b^2}=0$.

Lemma IV.—Let there be a polynomial of the form

$$X=x^m-px^{m-1}-qx^{m-2}\ldots-u,$$

in which the coefficients of all the terms after the first are essentially negative. A value of x can always be found sufficiently great to render the first term x^m greater than all the others together, and, consequently, the expression X essentially positive, and as great as we please.

For we can write X thus,

$$X=x^m\left(1-\frac{p}{x}-\frac{q}{x^2}\ldots-\frac{u}{x^m}\right),$$

in which, if x be supposed to increase indefinitely, the negative terms in the parenthesis will decrease indefinitely. As soon as x has attained a value λ sufficiently great to make these negative terms together equal to 1, the value of the expression X will go on increasing indefinitely, and be always positive.

If λ be taken negatively instead of positively, X will still be positive, provided m be *even;* but if m be *odd*, then, when $-\lambda$ is put for x, the leading term will be negative, and, consequently, X negative.

Corollary.—If the first term p of a series $p+qx+rx^2+$, &c., be constant, x may be taken a sufficiently small fraction to make the sign of the whole depend on that of the first term.*

* From the above it may be shown, that in every equation of an odd degree two values can always be found, which, when separately substituted for the unknown quantity, will furnish two results with opposite signs, and that in every equation of an even degree two such values can also be assigned, whenever the final term or absolute number is *negative;* for, in this case, the substitution of zero for x will give a negative result, viz., the absolute number itself, and the substitution of $+\lambda$ or $-\lambda$ will give a *positive* result.

From these inferences it may be proved, without difficulty, that every equation of an odd degree, without exception, has a real root, and every equation of an even degree, provided its final term be negative, has two real roots, the one positive, the other negative. This conclusion might be deduced immediately from what has just been established, if it be conceded that every polynomial $f(x)$, which gives results of opposite signs when two values a, b are successively given to x, passes from $f(a)$ to $f(b)$ continuously through all intermediate values, as x passes continuously from a to b. But this is a principle that requires demonstration. We proceed to establish it with the necessary rigor.

PROPOSITION.

If in the polynomial

$$f(x)=x^n+A_{n-1}x^{n-1}\ldots+A_2x^2+A_1x+N$$

x be supposed to vary continuously from $x=a$ to $x=b$, then the function $f(x)$ will vary continuously from $f(a)$ to $f(b)$.

DEMONSTRATION.

Let a' be any value intermediate between a and b. Substitute $a'+h$ for x in the polynomial, and it will become

$$f(a'+h)=(a'+h)^n+A_{n-1}(a'+h)^{n-1}\ldots A_2(a'+h)^2+A_1(a'+h)+N;$$

that is, actually developing, in the second member, by the binomial theorem, and arranging the results according to the ascending powers of h,

PRELIMINARY DEMONSTRATION.

240. Each of the equations

$$x^m = \pm 1,\ x^m = \pm \sqrt{-1}$$

has a root of the form $a+b\sqrt{-1}$. This is true of the equation $x^m=+1$, whether m be even or odd, since $x=1$ always satisfies it. It is also true of the equation $x^m=-1$ when m is odd, for then $x=-1$ satisfies it.

When m is even, it must either be some power of 2, or else some power of 2 multiplied by an odd number; if it be a power of 2, then the value of x will be obtained after the extraction of the square root repeated as many times in succession as there are units in the said power. Now the square root of the form $a+b\sqrt{-1}$ is always of the same form (Art. 118). Hence, when m is a power of 2, each of the equations $x^m=-1$, $x^m=\pm\sqrt{-1}$ has a root of the form announced. When m is a power of 2 multiplied by an odd number, then, if we extract the root of this odd degree first, there will remain to be extracted only a succession of square roots.

We have, therefore, merely to show that, when m is odd, a root of $\pm\sqrt{-1}$ is of the predicted form.

Now the odd *powers*, 1, 3, 5, &c., of $+\sqrt{-1}$, are (Art. 66)

$$+\sqrt{-1},\ -\sqrt{-1},\ +\sqrt{-1}\ldots.$$

and the same powers of $-\sqrt{-1}$ are

$$-\sqrt{-1},\ +\sqrt{-1},\ -\sqrt{-1}\ldots.$$

consequently, when m is odd, a *root* of $\pm\sqrt{-1}$ is either $+\sqrt{-1}$ or $-\sqrt{-1}$. Hence the predicted form occurs, whether m be odd or even.

It follows from this proposition that, whatever positive whole number m may be, $(-1)^{\frac{1}{m}}$ and $(\sqrt{-1})^{\frac{1}{m}}$ will always be of the form $a+b\sqrt{-1}$; or, more generally, $(-1)^{\frac{n}{m}}$ and $(\sqrt{-1})^{\frac{n}{m}}$ will always be of this form, n and m being any integers positive or negative (Cor. to Lemma II.).

THEOREM.

241. Every algebraical equation, of whatever degree, has a root of the form

$$\begin{array}{lll|ll|l}
f(a'+h)=a'^n & & +na'^{n-1} & h & +n(n-1)a'^{n-2} & \frac{h^2}{2}\cdots+h^n \\
& +A_{n-1}a'^{n-1} & +(n-1)A_{n-1}a'^{n-2} & & +(n-1)(n-2)A_{n-1}a'^{n-3} & \\
& \cdot \quad \cdot & \cdot & & \cdot \quad \cdot \quad \cdot & \\
& +A_2a'^2 & & & & \\
& +A_1a' & & & & \\
& +N & & & &
\end{array}$$

which may be written

$$f(a'+h)=f(a')+f_1(a')h+f_2(a')\frac{h^2}{2}+f_3(a')\frac{h^3}{2.3}\cdots h^n.$$

Now, by what has been above shown, a value so small may be given to h that the sum of the terms after $f(a')$ shall be less than any assignable quantity, however small. Hence, whatever intermediate value a' between a and b be fixed upon for x in $f(x)$, in proceeding to a neighboring value, by the addition to a' of a quantity h ever so minute, we obtain for $f(a'+h)$ a like minute increase of the preceding value $f(a')$. In other words, in proceeding continuously from a to b in our substitutions for x, the results of those substitutions must be, in like manner, continuous, or all connected together without any unoccupied interval.

$a+b\sqrt{-1}$, whether the coefficients of the equation be all real, or any of them imaginary and of the same form.

Let $f(x)=x^n+A_{n-1}x^{n-1}+\ldots A_3x^3+A_2x^2+A_1x+N=0$ . . . . . . (1)
represent any equation the coefficients of which are either real or imaginary.

If in this equation we substitute $p+q\sqrt{-1}$ for x, p and q being real, the first member will furnish a result of the form $P+Q\sqrt{-1}$, P and Q being real (Lemma II.). Should $p+q\sqrt{-1}$ be a root of the equation, this result must be zero; or, which is the same thing, the modulus of $P+Q\sqrt{-1}$, viz., $\sqrt{P^2+Q^2}$, must be zero (Lemma III.). And we have now to prove that values of p and q always exist that will fulfill this latter condition.

In order to this, it will be sufficient to show that whatever value of $\sqrt{P^2+Q^2}$, greater than zero, arises from any proposed values of p and q, other values of p and q necessarily exist, for which $\sqrt{P^2+Q^2}$ becomes still smaller, so that the *smallest* value of which $\sqrt{P^2+Q^2}$ is capable must be zero, and the particular expression $p+q\sqrt{-1}$, whence this value has arisen, must be a root of the equation.

For the purpose of examining the effect upon any function, $f(x)$, of changes introduced into the value of x, the development exhibited at Art. 239, Note, is very convenient. By changing x into $x+h$, the altered value of the function is thus expressed by

$$f(x+h)=f(x)+f_1(x)h+f_2(x)\frac{h^2}{1\,.\,2}+f_3(x)\frac{h^3}{1\,.\,2\,.\,3}\cdots h^n \quad . \ . \ . \ . \ . \ . \ . \ . \ (2)$$

where $f(x)$ is the original polynomial, and $f_1(x), f_2(x)$, &c., contain none but integral and positive powers of x (Art. 239, Note).

The first of these functions, $f(x)$, becomes $P+Q\sqrt{-1}$ when $p+q\sqrt{-1}$ is substituted for x; the other functions may some of them vanish for the same substitution, for aught we know to the contrary; but *all* the terms after $f(x)$ can not vanish; the last h^n, which does not contain x, must necessarily remain.

Without assuming any hypothesis as to what terms of $f(x+h)$ vanish for the value $x=p+q\sqrt{-1}$, which causes the first of those terms, $f(x)$, to become $P+Q\sqrt{-1}$, let us represent by h^m the *least* power of h for which the coefficient does not vanish when $p+q\sqrt{-1}$ is put for x. This coefficient will be of the form $R+S\sqrt{-1}$, in which R and S can not *both* be zero.

When $p+q\sqrt{-1}$ is put for x, we have represented $f(x)$ by $P+Q\sqrt{-1}$. In like manner, when $p+q\sqrt{-1}+h$ is put for x, we may represent the function by $P'+Q'\sqrt{-1}$. The development (2) will then be

$$P'+Q'\sqrt{-1}=(P+Q\sqrt{-1})+(R+S\sqrt{-1})h^m+\text{ terms } h^{m+1}, h^{m+2}, \ldots\ldots h^n.$$

Now h is quite arbitrary; we may give to it any sign and any value we please, provided only it come under the general form $a+b\sqrt{-1}$. Leaving the *absolute value* still arbitrary, we may therefore replace it by either $+k$ or $-k$, or $\pm(-1)^{\frac{1}{m}}k$; and thus render h^m either positive or negative, whichever we please, whatever be the value of m; and we have seen that $(-1)^{\frac{1}{m}}$ comes within the stipulated form (Art. 240). Hence we may write the foregoing development thus, the sign of k^m being under our own control:

$$P'+Q'\sqrt{-1}=(P+Q\sqrt{-1})+(R+S\sqrt{-1})k^m+ \text{ terms in } k^{m+1}, k^{m+2}, \ldots k^n.$$

But in any equation of this kind the real terms in one member are together equal to those in the other, and the imaginary terms in one to the imaginary terms in the other. Consequently,

$$P'=P+Rk^m+ \text{ the } real \text{ terms in } k^{m+1}, k^{m+2}, \ldots k^n;$$
$$Q'=Q+Sk^m+ real \text{ terms involving powers above } k^m.$$

Hence the square of the modulus of $P'+Q'\sqrt{-1}$ is

$$P'^2+Q'^2=P^2+Q^2+2(PR+QS)k^m+ \text{ real terms in } k^{m+1}, k^{m+2}, \ldots k^{2n}.$$

Now k may be taken so small that the sum of all the terms after P^2+Q^2 may take the same sign as $2(PR+QS)k^m$ by (239), which sign we can always render negative whatever $PR+QS$ may be, because, as observed above, k^m may be made either positive or negative, as we please.

Hence we can always render

$$P'^2+Q'^2<P^2+Q^2, \text{ or } \sqrt{P'^2+Q'^2}<\sqrt{P^2+Q^2}.$$

In other words, whatever values of p and q, in the expression $p+q\sqrt{-1}$, cause the modulus $\sqrt{P^2+Q^2}$ to exceed zero, other values exist for which the modulus will become smaller; and, consequently, one case at least must exist for which the modulus, and, consequently, the expression $P+Q\sqrt{-1}$, must become zero.

This conclusion presumes, however, that $PR+QS$ is not zero. If such should be the case, then our having chosen the form of h, so as to secure a command over the *sign* of $2(PR+QS)$, will have been unnecessary. The form must then be so chosen that a command may be secured over the sign of the first term *after* $2(PR+QS)k^m$, in the above series, for $P'^2+Q'^2$, which does not vanish, when the preceding conclusion will follow.

242. The values of a and b in the expression $a+b\sqrt{-1}$, which, when put for x in $f(x)$, cause that polynomial to vanish, can never be infinite.

We may write $f(x)$ as follows, viz.,

$$f(x)=x^n\left(1+\frac{A_{n-1}}{x}+\frac{A_{n-2}}{x^2}+\ldots\frac{N}{x^n}\right);$$

or, putting $P+Q\sqrt{-1}$ for what $f(x)$ becomes, when $p+q\sqrt{-1}$ is substituted for x, we have

$$P+Q\sqrt{-1}=$$
$$(p+q\sqrt{-1})^n\left(1+\frac{A_{n-1}}{p+q\sqrt{-1}}+\frac{A_{n-2}}{(p+q\sqrt{-1})^2}+\ldots\frac{N}{(p+q\sqrt{-1})^n}\right).$$

Now the modulus of a quotient is the quotient of the modulus of the dividend by the modulus of the divisor (Lemma II.). In each of the dividends A_{n-1}, A_{n-2}, &c., above, the modulus is finite by hypothesis. Hence, if either p or q be infinite, and, consequently, the modulus of every denominator or divisor also infinite, the modulus of each quotient must be *zero*. Hence, in this case, each of the above fractions must itself be zero (Lemma III.), and therefore the modulus of the entire quantity within the parenthesis simply 1; and the modulus of a product is the product of the moduli of the factors, so that the modulus of the preceding product, viz., $\sqrt{P^2+Q^2}$, is the modulus of $(p+q\sqrt{-1})^n$. But the n^{th} power of $p+q\sqrt{-1}$ has for modulus the n^{th} power of the modulus of $p+q\sqrt{-1}$, that is, the n^{th} power of $\sqrt{p^2+q^2}$ (Lemma

II., Cor.), which is infinite; consequently, $\sqrt{P^2+Q^2}$ must be infinite. But when $p+q\sqrt{-1}$ is a root of the equation $f(x)=0$, $\sqrt{P^2+Q^2}$ is zero. Hence, in this case, neither p nor q can be infinite.

243. An objection may be brought against the preceding reasoning that ought not to be concealed. It may be denied that the modulus of the product above referred to is simply the modulus of $(p+q\sqrt{-1})^n$ in the case of p or q infinite; for it may be maintained that although in this case all the quantities within the parenthesis after the 1 become *zero*, yet the combination of these with $(p+q\sqrt{-1})^n$, which involves *infinite* quantities, *may* produce quantities also infinite; and thus the modulus of the product may differ from the modulus of $(p+q\sqrt{-1})^n$ by a quantity infinitely great. It is not to be denied that there is weight in this objection. But it is not difficult to see that although the true modulus may thus differ from the modulus of $(p+q\sqrt{-1})^n$ by an infinite quantity, yet the modulus of $(p+q\sqrt{-1})^n$, involving higher powers than enter into the part neglected, is infinitely greater than that part. This part, therefore, is justly regarded as nothing in comparison to the part preserved, the former standing in relation to the latter as a finite quantity to infinity.

But the proposition may be established somewhat differently, as follows:

Substituting $(p+q\sqrt{-1})$ for x in $f(x)$, we have

$$P+Q\sqrt{-1}=$$
$$(p+q\sqrt{-1})^n+A_{n-1}(p+q\sqrt{-1})^{n-1}+\ldots A_1(p+q\sqrt{-1})+N.$$

Call the aggregate of all these terms after the first $P'+Q'\sqrt{-1}$; then it is plain that the modulus of the first term, that is, $(\sqrt{p^2+q^2})^n$, must infinitely exceed the modulus $\sqrt{P'^2+Q'^2}$ of the remaining terms whenever p or q is infinite, because in this latter modulus so high a power of the infinite quantity p or q can not enter as enters into the former. Now the modulus of the whole expression, that is, of the sum of $(p+q\sqrt{-1})^n$ and $P'+Q'\sqrt{-1}$, is not less than the difference of the moduli of these quantities themselves (Lemma I.), which difference is infinite. Hence, as before, $\sqrt{P^2+Q^2}$ must be infinite when p or q is infinite.

PROPOSITION IV.

244. *Every equation containing but one unknown quantity has as many roots as there are units in the highest power of the unknown quantity.*

Let $f(x)=0$ be an equation of the n^{th} degree; then if a_1 be a root of this equation, we have, by last proposition,

$$(x-a_1)f_1(x)=f(x)=0,$$

where $f_1(x)$ represents the quotient arising from the division of $f(x)$ by $x-a_1$, and will be a polynomial, arranged according to the powers of x, one degree lower than the given polynomial $f(x)$. Now, if a_2 is also a root of the equation $f(x)=0$, it is obvious that $f_1(x)$ must be divisible by $x-a_2$, for $x-a_1$ is not divisible by $x-a_2$ (see Art. 84, Note); hence, if $f_2(x)$, a polynomial of a degree one lower than $f_1(x)$, or of a degree two lower than $f(x)$, represent the quotient of $f_1(x)$ divided by $x-a_2$, we have

$$(x-a_1)(x-a_2)f_2(x)=f(x)=0.$$

Proceeding in this manner, if $a_3, a_4, a_5, \ldots\ldots a_n$ are roots of the equation, the degree of the quotient reducing by one each time, the equation will assume the form

$$(x-a_1)(x-a_2)(x-a_3)\ldots\ldots(x-a_n)=0;$$

and, consequently, there are as many roots as factors, that is, as units in the highest power of x, the unknown quantity; for the last equation will be verified by any one of the n conditions,

$$x=a_1,\ x=a_2,\ x=a_3,\ x=a_4,\ \ldots\ x=a_n;$$

and since the equation, being of the n^{th} degree, contains n of these factors of the 1st degree, $(x-a_1)$, &c., there are n roots.

Corollary 1. When one root of an equation is known, the depressed equation containing the remaining roots is readily found by synthetic division.

Corollary 2. The number of factors of the 2° degree in an equation is $n(n-1)\div 1.2$; of the 3°, $n(n-1)(n-2)\div 1.2.3$, and so on (see Art. 203).

EXAMPLES.

(1) One root of the equation $x^4-25x^2+60x-36=0$ is 3; find the equation containing the remaining roots.

$$\begin{array}{llll} 1+0 & -25 & +60-36 & (3 \\ \quad\ 3 & +\ 9 & -48+36 & \\ \hline 1+3 & -16 & +12. & \end{array}$$

Hence $\quad x^3+3x^2-16x+12=0$

is the equation containing the remaining roots.

(2) Two roots of the equation $x^4-12x^3+48x^2-68x+15=0$ are 3 and 5; find the quadratic containing the remaining roots.

$$\begin{array}{llll} 1 & -12 & +48-68+15 & (3 \\ & \quad 3 & -27+63-15 & \\ \hline 1 & -\ 9 & +21-\ 5 & (5 \\ & \quad 5 & -20 & \\ \hline 1 & -\ 4 & +1 & \end{array}$$

$$\therefore x^2-\ 4x+1=0$$

is the equation containing the two remaining roots.

(3) One root of the cubic equation $x^3-6x^2+11x-6=0$ is 1; find the quadratic containing the other roots.

Ans. $x^2-5x+6=0$

(4) Two roots of the biquadratic equation $4x^4-14x^3-5x^2+31x+6=0$ are 2 and 3; find the reduced equation.

Ans. $4x^2+6x+1=0$.

(5) One root of the cubic equation $x^3+3x^2-16x+12=0$ is 1; find the remaining roots.

Ans. 2 and -6.

(6) Two roots of the biquadratic equation $x^4-6x^3+24x-16=0$ are 2 and -2; find the other two roots.

Ans. $3\pm\sqrt{5}$.

PROPOSITION V.

245. *To form the equation whose roots are* $a_1,\ a_2,\ a_3,\ a_4,\ \ldots\ldots\ a_n$.

The polynomial, $f(x)$, which constitutes the first member of the equation required, being equal to the continued product of $x-a_1$, $x-a_2$, $x-a_3$, ... $x-a_n$, by the last proposition, we have

$$(x-a_1)(x-a_2)(x-a_3)\ldots\ldots(x-a_n)=0;$$

and by performing the multiplication here indicated, we have, when

$$n=2,\ \begin{array}{r|l} x^2-a_1 & x+a_1a_2=0 \\ -a_2 & \end{array}$$

$$n=3,\ \begin{array}{r|l|l} x^3-a_1 & x^2+a_1a_2 & x-a_1a_2a_3=0 \\ -a_2 & \quad +a_1a_3 & \\ -a_3 & \quad +a_2a_3 & \end{array}$$

$$n=4,\ \begin{array}{r|l|l|l} x^4-a_1 & x^3+a_1a_2 & x^2-a_1a_2a_3 & x+a_1a_2a_3a_4=0,\text{ and so on.} \\ -a_2 & \quad +a_1a_3 & \quad -a_1a_2a_4 & \\ -a_3 & \quad +a_2a_3 & \quad -a_1a_3a_4 & \\ -a_4 & \quad +a_1a_4 & \quad -a_2a_3a_4 & \\ & \quad +a_2a_4 & & \\ & \quad +a_3a_4 & & \end{array}$$

By continuing the multiplication to the last, the equation will be found whose roots are those proposed; and from what has been done we learn that

(1) The coefficient of the *second* term in the resulting polynomial will be the sum of all the roots with their signs changed.

(2) The coefficient of the *third* term will be the sum of the products of every two roots with their signs changed.

(3) The coefficient of the *fourth* term will be the sum of the products of every three roots with their signs changed.

(4) The coefficient of the *fifth* term will be the sum of the products of every four roots with their signs changed, and so on; the *last* or *absolute* term being the product of all the roots with their signs changed.*

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* I. The generality of this law may be proved as follows: Let us suppose it to hold good for the product of n binomial factors, we shall prove that it will for the product of $n+1$ of these. Let

$$x^n-A_1x^{n-1}+A_2x^{n-2}—,\ \&c.,\ \pm A_n$$

represent the product of n binomial factors, in which A_1 represents the sum $a_1+a_2+a_3+$, &c., $+a_n$ of the n second terms of the binomials, A_2 the sum of their products two and two, A_3 the sum of their products three and three, and so on, and A_n the product of all the n second terms $a_1a_2a_3$, &c., a_n.

Introduce now a new factor $(x-a_{n+1})$. Performing the multiplication of the above polynomial by this new factor,

$$\begin{array}{l} x^n-A_1x^{n-1}+A_2x^{n-2}—,\ \&c.,\ \pm A_n \\ x-a_{n+1} \\ \hline x^{n+1}-A_1x^n+A_2x^{n-1}—,\ \&c.,\ \pm A_nx \\ \qquad -a_{n+1}x^n+A_1a_{n+1}x^{n-1}—,\ \&c. \qquad \mp A_na_{n+1} \\ \hline \end{array}$$

$$\begin{array}{r|l|l} x^{n+1}-A_1 & x^n+A_2 & x^{n-1}—,\ \&c.,\ \mp A_na_{n+1} \\ -a_{n+1} & \quad +A_1a_{n+1} & \end{array}$$

Here the coefficient of the second term $\begin{matrix}-A_1\\-a_{n+1}\end{matrix}$ is composed of A_1, the sum of all the second terms of the n binomials $(x-a_1)$, $(x-a_2)$, &c., and a_{n+1}, the second term of the $(n+1)^{\text{th}}$ binomial, and is, therefore, equal to the sum of the second terms of the $n+1$ binomials. The coefficient of the third term $\begin{matrix}+A_2\\+A_1a_{n+1}\end{matrix}$ is composed of A_2, the sum of the products of the n second terms two and two, and A_1a_{n+1}, the sum of the n second terms, each

Corollary 1.—If the coefficient of the second term in any equation be 0, that is, if the second term be absent, the sum of the positive roots is equal to the sum of the negative roots.

Corollary 2.—If the signs of the terms of the equation be all positive, the roots will be all negative, and if the signs be alternately positive and negative, the roots will be all positive.

Corollary 3.—Every root of an equation is a divisor of the last or absolute term.

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multiplied by the new second term a_{n+1}; hence $\begin{matrix}+A_2 \\ +A_1 a_{n+1}\end{matrix}$ will be the sum of the products of the $n+1$ second terms two and two.

The last term $A_n a_{n+1}$ is the product of A_n, which is the product of all the n second terms multiplied by the new second term a_{n+1}, so that $A_n a_{n+1}$ is the product of all the $n+1$ second terms.

We have thus proved that if the law for the formation of the coefficients above stated hold good for a certain number of binomial factors n, it will hold good for one more, or $n+1$. We have seen, by experiment, that it holds good for four, it therefore holds good for five; if for five, it must for six, and so on *ad infinitum.*

II. One might imagine, at first view, that the above relations would make known the roots. They give at once equations into which these roots enter, and which are equal in number to the coefficients of the equation (excepting the coefficient of the first term, which is unity). The number of these coefficients is equal to the number of the roots of the equation. Unfortunately, when we seek to resolve these secondary equations, we are led to the very equation proposed, so that no progress is made.

For simplicity, I will take the equation of the 3° degree.

$$x^3+Px^2+Qx+R=0 \quad \ldots\ldots\ldots\ldots\ (1)$$

Designating the three roots by a, b, c, we have, to determine the roots, the three relations

$$\begin{aligned} P&=-a-b-c \\ Q&=ab+ac+bc \quad \ldots\ldots\ldots\ldots\ldots\ (2) \\ R&=-abc \end{aligned}$$

To deduce from them an equation which contains but the unknown a, the most simple mode of proceeding is, to multiply the 1° by a^2, the 2° by a, and add them to the 3°. There results

$$\begin{aligned} Pa^2+Qa+R=-a^3&-a^2b-a^2c \\ &+a^2b+a^2c+abc \\ &\qquad\qquad -abc. \end{aligned}$$

Reducing, and transposing the term $-a^3$, we have

$$a^3+Pa^2+Qa+R=0.$$

The unknown quantities b and c are thus eliminated, but the equation resulting is of the same degree with the proposed. From the symmetrical form of the relations (2) we perceive that the elimination of a and b, or a and c, would have been attended with similar consequences.

III. To find the sum of the squares of the roots of any equation.

$$-A_1=a+b+c\ldots+l;$$

$$\therefore A_1^2=a^2+b^2+c^2\ldots+l^2+2(ab+ac+bc+\ldots.)$$

$$=\text{sum of the squares}+2A_2;$$

$$\therefore \text{sum of squares}=A_1^2-2A_2.$$

To find the sum of the reciprocals of the roots.

$$(-1)^{n-1}A_{n-1}=bc\ldots l+ac\ldots l+ab..l+..$$

$$(-1)^n A_n=abc\ldots l;$$

$$\therefore \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ldots.+\frac{1}{l}=-\frac{A_{n-1}}{A_n}.$$

Corollary 4.—In any equation, when the roots are all real, and the last or absolute term very small compared with the coefficients of the other terms, then will the roots of such an equation be also very small.

EXAMPLES.

(1) Form the equation whose roots are 2, 3, 5, and -6

Here we have simply to perform the multiplication indicated in the equation

$$(x-2)(x-3)(x-5)(x+6)=0,$$

and this is best done by detached coefficients in the following manner:

$$\begin{array}{l} 1-\ 2\ (-3 \\ \underline{\ \ -\ 3+\ 6} \\ 1-\ 5+\ 6\ (-5 \\ \underline{\ \ -\ 5+25-30} \\ 1-10+31-30\ (6 \\ \underline{\qquad 6-60+186-180} \\ 1-\ 4-29+156-180 \end{array}$$

$\therefore x^4-4x^3-29x^2+156x-180=0$ is the equation sought.

(2) Form the equation whose roots are 1, 2, and -3.

(3) Form the equation whose roots are 3, -4, $2+\sqrt{3}$, and $2-\sqrt{3}$.

(4) Form the equation whose roots are $3+\sqrt{5}$, $3-\sqrt{5}$, and -6.

ANSWERS.

(2) $x^3-7x+6=0$.

(3) $x^4-3x^3-15x^2+49x-12=0$.

(4) $x^3-32x+24=0$.

PROPOSITION VI.

246. *No equation whose coefficients are all integers, and that of the highest power of the unknown quantity unity, can have a fractional root.*

If possible, let the equation

$$x^n+A_{n-1}x^{n-1}+\ldots+A_3x^3+A_2x^2+A_1x+N=0,$$

whose coefficients are all integral, have a fractional root, expressed in its lowest terms by $\frac{a}{b}$. If we substitute this for x, and multiply the resulting equation by b^{n-1}, we shall have

$$\frac{a^n}{b}+A_{n-1}a^{n-1}+\ldots+A_3a^3b^{n-3}+Aab^{n-2}+Nb^{n-1}=0.$$

In this polynomial, every term after the first is integral; hence the first term must be integral also. But $\frac{a}{b}$ being a fraction in its lowest terms, $\frac{a^n}{b}$ must also be a fraction in its lowest terms, and can not be an integral. (See Note to Art. 84.) Therefore the proposed equation can not have a fractional root.

PROPOSITION VII.

247. *If the signs of the alternate terms in an equation be changed, the signs of all the roots will be changed.*

Let $\quad x^n+A_1x^{n-1}+A_2x^{n-2}+\ldots\ldots A_{n-1}x+A_n=0 \quad \ldots\ldots (1)$

be an equation; then, changing the signs of the alternate terms, we have

$$x^n-A_1x^{n-1}+A_2x^{n-2}-\ldots\ldots\pm A_{n-1}x\mp A_n=0 \quad \ldots (2)$$

or $\quad -x^n+A_1x^{n-1}-A_2x^{n-2}+\ldots\ldots\mp A_{n-1}x\pm A_n=0 \quad \ldots (3)$

But equations (2) and (3) are identical, for the sum of the positive terms in each is equal to the sum of the negative terms, and therefore they are identical. Now if a be a root of equation (1), and if a be substituted for x in equation (1) and $-a$ in equation (2), if n be an even number, or in equation (3) if n be an odd number, the results will be the very same; and since the former is verified by such substitution, a being a root, the latter, viz., equation (2) or (3), as the case may be, is also verified, and therefore $-a$ is a root of the identical equations (2) and (3).

Corollary.—If the signs of all the terms are changed, the signs of the roots remain unchanged.

EXAMPLES.

(1) The roots of the equation $x^3-6x^2+11x-6=0$ are 1, 2, 3. What are the roots of the equation $x^3+6x^2+11x+6=0$?

Ans. -1, -2, -3.

(2) The roots of the equation $x^4-6x^3+24x-16=0$ are 2, -2, $3\pm\sqrt{5}$. Express the equation whose roots are 2, -2, $-3+\sqrt{5}$, and $-3-\sqrt{5}$.

Ans. $x^4+6x^3-24x-16=0$.

PROPOSITION VIII.

248. *Surds and impossible roots enter equations by pairs.*

Let $x^n+A_1x^{n-1}+A_2x^{n-2}+\ldots\ldots A_{n-1}x+A_n=0$ be an equation having a root of the form $a+b\sqrt{-1}$, then will $a-b\sqrt{-1}$ be also a root of the equation; for, let $a+b\sqrt{-1}$ be substituted for x in the equation, and we have

$$(a+b\sqrt{-1})^n+A_1(a+b\sqrt{-1})^{n-1}+\ldots\ldots A_{n-1}(a+b\sqrt{-1})+A_n=0.$$

Now, by expanding the several terms of this equation, we shall have a series of monomials, all of which will be real except the odd powers of $b\sqrt{-1}$, which will be imaginary. Let P represent the real and $Q\sqrt{-1}$ the imaginary terms of the expanded equation; then

$$P+Q\sqrt{-1}=0,$$

an equation which can exist only when $P=0$ and $Q=0$, for the imaginary quantities can not cancel the real ones, but the real must cancel one another, and the imaginary one another separately.

Again, let $a-b\sqrt{-1}$ be substituted for x in the proposed equation; then the only difference in the expanded result will be in the signs of the odd powers of $b\sqrt{-1}$, and the collected monomials, by the previous notation, will assume the form $P-Q\sqrt{-1}$ but we have seen that $P=0$ and $Q=0$;

$$\therefore\ P-Q\sqrt{-1}=0,$$

and hence $a-b\sqrt{-1}$ also verifies the equation, and is therefore a root.

Such roots are called conjugate.

In a similar manner, it is proved that if $a+\sqrt{b}$ be one root of an equation, $a-\sqrt{b}$ will also be a root of that equation.

Corollary 1.—An equation which has impossible roots is divisible by

$$\{x-(a+b\sqrt{-1})\}\{x-(a-b\sqrt{-1})\},\ \text{or}\ x^2-2ax+a^2+b^2,$$

and, therefore, every equation may be resolved into rational factors, simple or quadratic.

Corollary 2.—All the roots of an equation of an even degree may be impossible, but if they are not all impossible, the equation must have at least two real roots.

Corollary 3.—The product of every pair of impossible roots being of the form a^2+b^2 is positive; and, therefore, the absolute term of an equation whose roots are all impossible must be positive.

Corollary 4.—Every equation of an odd degree has at least one real root, and if there be but one, that root must necessarily have a contrary sign to that of the last term.

Corollary 5.—Every equation of an even degree whose last term is negative has at least two real roots, and if there be but two, the one is positive, and the other negative.

PROPOSITION IX.

249. *The* m *roots of the equation* $X=0$, or

$$x^m+Px^{m-1}+Qx^{m-2}+, \&c, =0 \; . \; . \; . \; . \; . \; . \; [A]$$

must be of the form $a+b\sqrt{-1}$, of which form we have already shown (Art. 241) that it must have one.

For, let $a+b\sqrt{-1}$ be the root whose existence is demonstrated. We know (Prop. II.) that the polynomial x^m+, &c., is divisible by $x-(a+b\sqrt{-1})$; but when we effect this division, the quantities $a+b\sqrt{-1}$, P, Q, &c., can combine only by addition, by subtraction, and by multiplication; then the coefficients of the quotient $x^{m-1}+$, &c., will still be of the form $a+b\sqrt{-1}$. Consequently, the equation $x^{m-1}+$, &c., will also have at least one root of the form $a'+b'\sqrt{-1}$; dividing $x^{m-1}+$, &c., by $x-(a'+b'\sqrt{-1})$, the coefficients of the quotient $x^{m-2}+$, &c., will be still of the same form. Continuing to reason thus, it is evident that the primitive polynomial X will be divided into m factors of the form $x-(a+b\sqrt{-1})$, and, consequently, the roots of the equation will all be of the form $a+b\sqrt{-1}$.

PROPOSITION X.

250. *The roots of the two conjugate equations,*

$$Y+Z\sqrt{-1}=0 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (1)$$

$$Y-Z\sqrt{-1}=0 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (2)$$

will be conjugates of each other.

Let $x=a+b\sqrt{-1}$ be a root of equation (1), and $Y'+Z'\sqrt{-1}$ the quotient of its first member, by $x-a-b\sqrt{-1}$, we have the identity

$$(Y'+Z'\sqrt{-1})(x-a-b\sqrt{-1})=Y+Z\sqrt{-1} \; . \; . \; . \; . \; . \; . \; (3)$$

Effecting the multiplication in the 1° member, we find

$$(x-a)Y'+bZ'+[(x-a)Z'-bY']\sqrt{-1}$$

Changing now in the two factors Z' into $-Z'$, and b into $-b$, we see that in the product the part which does not contain $\sqrt{-1}$ remains the same, and that that which does contain $\sqrt{-1}$ only changes its sign; by virtue of (3), therefore, we have

$$(Y'-Z'\sqrt{-1})(x-a+b\sqrt{-1})=Y-Z\sqrt{-1} \; . \; . \; . \; . \; (4)$$

From whence we conclude that $a-b\sqrt{-1}$ is a root of (2); that is, all the roots of (2) are obtained by changing in those of (1) the sign of $\sqrt{-1}$. The real roots, according to this, must be the same in the two equations.

We may now consider the following beautiful proposition as demonstrated from the foregoing.

PROPOSITION XI.

An algebraic equation which has real coefficients is always composed of as many real factors of the 1° *degree as it has real roots, and of as many real factors of the* 2° *degree as it has pairs of imaginary roots.*

DEPRESSION OR ELEVATION OF ROOTS OF EQUATIONS.

PROPOSITION.

251. *To transform an equation into another whose roots shall be the roots of the proposed equation increased or diminished by any given quantity.*

Let $ax^n+A_1x^{n-1}+A_2x^{n-2}+\ldots\ldots A_{n-1}x+A_n=0$, be an equation, and let it be required to transform it into an equation whose roots shall be the roots of this equation diminished by r.

This transformation might be effected by substituting $y+r$ for x in the proposed equation, and the resulting equation in y would be that required; but this operation is generally very tedious, and we must therefore have recourse to some more simple mode of forming the transformed equation. If we write $y+r$ for x in the proposed equation, it will obviously be an equation of the very same dimensions, and its form will evidently be

$$ay^n+B_1y^{n-1}+B_2y^{n-2}+\ldots\ldots B_{n-1}y+B_n=0 \quad \ldots\ldots \quad (1)^*$$

in which B_1, B_2, &c., will be polynomials involving r. But $y=x-r$, and therefore (1) becomes

$$a(x-r)^n+B_1(x-r)^{n-1}+\ldots\ldots B_{n-1}(x-r)+B_n=0 \quad \ldots \quad (2)$$

which, when developed, must be identical with the proposed equation; for, since $y+r$ was substituted for x in the proposed, and then $x-r$ for y in (2), the transformed equation, we must necessarily have reverted to the original equation; hence we have

$$a(x-r)^n+B_1(x-r)^{n-1}+\ldots B_{n-1}(x-r)+B_n=ax^n+A_1x^{n-1}+\ldots A_{n-1}x+A_n.$$

* It will be of the same form with the development in the note to (Art. 239). We give it again below, arranged according to the powers of r instead of y. After substituting $y+r$ for x, we write the development of each term of the proposed equation in a horizontal line; the first horizontal line is the development of ax^n, the second of A_1x^{n-1}, and so on.

$$\begin{array}{llll} ay^n+any^{n-1}r & +\frac{an(n-1)}{1.2}y^{n-2}r^2 & +\ldots \\ +A_1y^{n-1}+A_1(n-1)y^{n-2}r & +\frac{A_1(n-1)(n-2)}{1.2}y^{n-3}r^2 & +\ldots \\ +A_2y^{n-2}+A_2(n-2)y^{n-3}r & +\frac{A_2(n-2)(n-3)}{1.2}y^{n-4}r^2 & +\ldots \\ \ldots\ldots \quad \ldots\ldots & \ldots\ldots & \ldots \\ +A_n. & & \end{array}$$

In which the first column is of the same form as the proposed equation; the second column, or coefficient of r, is derived from the first by multiplying the coefficient of each term by its exponent, and diminishing the exponent by unity; the third column, or coefficient of $\frac{r^2}{1.2}$, is derived from the second in a similar manner, and so on.

If we designate by $f(x)$ the first member of the given equation, and by $f'(x)$ the first derived function, by $f''(x)$ the second derived, and so on, we shall have

$$f(x+r)=f(x)+f'(x)r+\frac{f''(x)}{1.2}r^2+\frac{f'''(x)}{1.2.3}r^3+, \&c.$$

Now, if we divide the first member by $x-r$, every term will evidently be divisible, except the last, B_n, which will be the remainder, and the quotient will be

$$a(x-r)^{n-1}+B_1(x-r)^{n-2}+\ldots\ldots B_{n-2}(x-r)+B_{n-1};$$

and since the second member is identical with the first, the very same quotient and remainder would arise by dividing this second member also by $x-r$; hence it appears that if the first member of the original equation be divided by $x-r$, the remainder will be the last or absolute term of the sought transformed equation.

Again, if we divide the quotient thus obtained, viz.,

$$a(x-r)^{n-1}+B_1(x-r)^{n-2}+\ldots\ldots B_{n-2}(x-r)+B_{n-1}$$

by $x-r$, the remainder will obviously be B_{n-1}, the coefficient of the term last but one in the transformed equation; and thus, by successive divisions of the polynomial in the first member of the proposed equation by $x-r$, we shall obtain the whole of the coefficients of the required equation.

RULE.

Let the polynomial in the first member of the proposed equation be a function of x, and r the quantity by which the roots of the equation are to be diminished or increased; then divide the proposed polynomial by $x-r$, or $x+r$, according as the roots of the proposed equation are to be diminished or increased, and the quotient thus obtained by the same divisor, giving a second quotient, which divide by the same divisor, and so on till the division terminates; then will the coefficients of the transformed equation, beginning with the highest power of the unknown quantity, be the coefficient of the highest power of the unknown quantity in the proposed equation, and the several remainders arising from the successive divisions taken in a reverse order, the first remainder being the last or absolute term in the required transformed equation.

Note.—When there is an absent term in the equation, its place must be supplied with a cipher.

EXAMPLES.

(1) Transform the equation $5x^4-12x^3+3x^2+4x-5=0$ into another whose roots shall be less than those of the proposed equation by 2.

$$\begin{array}{l}
x-2)\ 5x^4-12x^3+3x^2+4x-5\ (5x^3-2x^2-x+2 \\
\qquad 5x^4-10x^3 \\
\qquad\quad -2x^3+3x^2 \\
\qquad\quad -2x^3+4x^2 \\
\qquad\qquad\quad -x^2+4x \\
\qquad\qquad\quad -x^2+2x \\
\qquad\qquad\qquad\quad 2x-5 \\
\qquad\qquad\qquad\quad 2x-4 \\
\qquad\qquad\qquad\qquad -1.\ \text{First remainder.}
\end{array}$$

$$\begin{array}{l}
x-2)\ 5x^3-2x^2-x+2\ (5x^2+8x+15 \\
\qquad 5x^3-10x^2 \\
\qquad\qquad 8x^2-x \\
\qquad\qquad 8x^2-16x \\
\qquad\qquad\quad 15x+2 \\
\qquad\qquad\quad 15x-30 \\
\qquad\qquad\qquad\quad 32.\ \text{Second remainder.}
\end{array}$$

$$\begin{array}{l} x-2)\ 5x^2+8x+15\ (5x+18 \\ \qquad\ \ 5x^2-10x \\ \qquad\qquad\ \ 18x+15 \\ \qquad\qquad\ \ 18x-36 \\ \qquad\qquad\qquad\quad 51.\quad \text{Third remainder.} \end{array}$$

$$\begin{array}{l} x-2)\ 5x+18\ (5 \\ \qquad\ \ 5x-10 \\ \qquad\qquad\ \ 28.\quad \text{Fourth remainder.} \end{array}$$

Therefore the transformed equation is

$$5y^4+28y^3+51y^2+32y-1=0.$$

This laborious operation can be avoided by *Horner's Synthetic Method* of division, and its great superiority over the usual method will be at once apparent by comparing the subsequent elegant process with the work above. Taking the same example, and writing the modified or changed term of the divisor $x-2$ on the right hand instead of the left, the whole of the work will be thus arranged:

$$\begin{array}{rrrrrl} 5 & -12 & +\ 3 & +\ 4 & -5 & (2 \\ & 10 & -\ 4 & -\ 2 & 4 & \\ \hline & -\ 2 & -\ 1 & 2 & -1 & \therefore B_4=-1 \\ & 10 & 16 & 30 & & \\ \hline & 8 & 15 & 32 & & \therefore B_3=32 \\ & 10 & 36 & & & \\ \hline & 18 & 51\ldots & & & \therefore B_2=51 \\ & 10 & & & & \\ \hline & 28 & & & & \therefore B_1=28 \end{array}$$

$\therefore 5y^4+28y^3+51y^2+32y-1=0$ is the required equation, as before.

(2) Transform the equation $5y^4+28y^3+51y^2+32y-1=0$ into another having its roots greater by 2 than those of the proposed equation.

$$\begin{array}{rrrrrl} 5 & +28 & +\ 51 & +32 & -1 & (-2 \\ & -10 & -36 & -30 & -4 & \\ \hline & 18 & 15 & 2 & -5 & \\ & -10 & -16 & 2 & & \\ \hline & 8 & -\ 1 & 4 & & \\ & -10 & 4 & & & \\ \hline & -\ 2 & 3 & & & \\ & -10 & & & & \\ \hline & -12 & & & & \end{array}$$

$\therefore 5x^4-12x^3+3x^2+4x-5=0$ is the sought equation, which, from the transformations we have made, must be the original equation in Example 1.

(3) Find the equation whose roots are less by 1·7 than those of the equation

$$z^3-2z^2+3z-4=0.$$

$$\begin{array}{rrrrl} 1 & -2 & +3 & -4 & (1 \\ & 1 & -1 & 2 & \\ \hline & -1 & 2 & -2 & \\ & 1 & 0 & & \\ \hline & 0 & 2 & & \\ & 1 & & & \\ \hline & 1 & & & \end{array}$$

Now we know the equation whose roots are less by 1 than those of the given equation: it is $x^3+x^2+2x-2=0$; and by a similar process for ·7, remembering the localities of the decimals, we have the required equation; thus:

$$\begin{array}{rrrl}
1+1 & +2 & -2 & (\cdot 7 \\
.7 & 1\cdot 19 & 2\cdot 233 & \\ \hline
1\cdot 7 & 3\cdot 19 & \cdot 233 & \\
7 & 1\cdot 68 & & \\ \hline
2\cdot 4 & 4\cdot 87 & & \\
7 & & & \\ \hline
3\cdot 1 & & &
\end{array}$$

$\therefore\ y^3+3\cdot 1y^2+4\cdot 87y+\cdot 233=0$ is the required equation.

This latter operation can be continued from the former without arranging the coefficients anew in a horizontal line, recourse being had to this second operation merely to show the several steps in the transformation, and to point out the equations at each step of the successive diminutions of the roots. Combining these two operations, then, we have the subsequent arrangement.

$$\begin{array}{rrrl}
1-2 & +3 & -4 & (1\cdot 7 \\
1 & -1 & 2 & \\ \hline
-1 & 2 & -2 & \\
1 & 0 & 2\cdot 233 & \\ \hline
0 & 2 & \cdot 233 & \\
1 & 1\cdot 19 & & \\ \hline
1\cdot 7 & 3\cdot 19 & & \\
\cdot 7 & 1\cdot 68 & & \\ \hline
2\cdot 4 & 4\cdot 87 & & \\
\cdot 7 & & & \\ \hline
3\cdot 1 & & &
\end{array}$$

or

$$\begin{array}{rrrl}
1-2 & +3 & -4 & (1\cdot 7 \\
1\cdot 7 & -\ \cdot 51 & 4\cdot 233 & \\ \hline
-\ \cdot 3 & 2\cdot 49 & \cdot 233 & \\
1\cdot 7 & 2\cdot 38 & & \\ \hline
1\cdot 4 & 4\cdot 87 & & \\
1\cdot 7 & & & \\ \hline
3\cdot 1 & & &
\end{array}$$

We have then the same resulting equation as before, and in the latter of these we have used 1·7 at once. It is always better, however, to reduce continuously as in the former, to avoid mistakes incident to the multiplier 1·7.

(4) Find the equation whose roots shall be less by 1 than those of the equation

$$x^3-7x+7=0.$$

(5) Find the equation whose roots shall be less by 3 than the roots of the equation

$$x^4-3x^3-15x^2+49x-12=0,$$

and transform the resulting equation into another whose roots shall be greater by 4.

(6) Give the equation whose roots shall be less by 10 than the roots of the equation

$$x^4+2x^3+3x^2+4x-12340=0.$$

(7) Give the equation whose roots shall be less by 2 than those of the equation

$$x^5+2x^3-6x^2-10x+8=0.$$

(8) Give the equation whose roots shall each be less by $\frac{1}{2}$ than the roots of the equation

$$2x^4-6x^3+4x^2-2x+1=0.$$

ANSWERS.

(4) $y^3+3y^2-4y+1=0$ . . . . whence $x=y+1$
(5) $y^4+9y^3+12y^2-14y=0$ . . . . whence $x=y+3$
and $z^4-7z^3+66z-72=0$ . . . . whence $x=z-1$
(6) $y^4+42y^3+663y^2+4664y=0$ . . . . whence $x=y+10$
(7) $y^5+10y^4+42y^3+86y^2+70y+12=0$ . . . . . . whence $x=y+2$
(8) $2y^4-2y^3-2y^2-\frac{3}{2}y+\frac{3}{8}=0$ . . . . whence $x=y+\frac{1}{2}$

PROPOSITION

252. *If the real roots of an equation, taken in the order of their magnitudes, be*

$$a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ \ldots\ldots$$

where a_1 *is the greatest,* a_2 *the next, and so on; then if a series of numbers,*

$$b_1,\ b_2,\ b_3,\ b_4,\ b_5,\ \ldots\ldots$$

in which b_1 *is greater than* a_1, b_2 *a number between* a_1 *and* a_2, b_3 *a number between* a_2 *and* a_3, *and so on, be substituted for* x *in the proposed equation, the results will be alternately positive and negative.*

The polynomial in the first member of the proposed equation is the product of the simple factors

$$(x-a_1)(x-a_2)(x-a_3)(x-a_4)\ \ldots\ldots$$

and quadratic factors, involving the imaginary roots; but the quadratic factors have always a positive value for every real value of x (Art. 248, Cor. 3); therefore we may omit these positive factors; and substituting for x the proposed series of values, b_1, b_2, b_3, &c., we have these results:

$$(b_1-a_1)(b_1-a_2)(b_1-a_3)(b_1-a_4)\ldots=+\cdot+\cdot+\cdot+\ldots=+$$
$$(b_2-a_1)(b_2-a_2)(b_2-a_3)(b_2-a_4)\ldots=-\cdot+\cdot+\cdot+\ldots=-$$
$$(b_3-a_1)(b_3-a_2)(b_3-a_3)(b_3-a_4)\ldots=-\cdot-\cdot+\cdot+\ldots=+$$
$$(b_4-a_1)(b_4-a_2)(b_4-a_3)(b_4-a_4)\ldots=-\cdot-\cdot-\cdot+\ldots=-$$

&c. &c. &c.

Corollary 1.—If two numbers be successively substituted for x in any equation, and give results with different signs, then between these numbers there must be *one*, *three*, *five*, or some *odd* number of roots.

Corollary 2.—If the results of the substitution in corollary 1 are affected with like signs, then between these numbers there must be *two*, *four*, or some *even* number of roots, or *no* root between these numbers.

Corollary 3.—If any quantity q, and every quantity greater than q, renders the result positive, then q is greater than the greatest root of the equation.

Corollary 4.—Hence, if the signs of the alternate terms be changed, and if p, and every quantity greater than p, renders the result positive, then $-p$ is less than the least root.

EXAMPLE.

Find the initial figure in one of the roots of the equation

$$x^3-4x^2-6x+8=0.$$

Here one value of x does not differ greatly from unity, for the value of the given polynomial, when $x=1$, is -1, and when $x=\cdot 9$, it is found thus:

$$\begin{array}{rrrl} 1- & 4- & 6 & +8\ (\cdot 9 \\ & \cdot 9- & 2\cdot 79- & 7911 \\ \hline & -3\cdot 1- & 8\cdot 79+ & \cdot 089. \end{array}$$

The value, therefore, when $x=\cdot 9$ is (Art. 251) $\cdot 089$. Hence the former value being negative, and the latter positive, the initial figure of one root is $\cdot 9$.

PROPOSITION.

253. *Given an equation of the* n^{th} *degree to determine another of the* $(n-1)^{th}$ *degree, such that the real roots of the former shall separate those of the latter.*

Let $a_1, a_2, a_3, a_4, \ldots . a_n$ be the roots taken in order of the equation

$$x^n+A_1x^{n-1}+A_2x^{n-2}+\ldots . A_{n-1}x+A_n=0;$$

then diminishing the roots of this equation by r (Art. 251), we have the following process, viz.:

$$\begin{array}{cccccc} 1+A_1+ & A_2+\ldots\ldots & A_{n-2}+ & A_{n-1}+ & A_n\ (r \\ r & rB_1 & rB_{n-3} & rB_{n-2} & rB_{n-1} \\ \hline B_1 & B_2 & B_{n-2} & B_{n-1} & B_n \\ r & rC_1 & rC_{n-3} & rC_{n-2} & \\ \hline C_1 & C_2 & C_{n-2} & C_{n-1} & \end{array}$$

Whence

$$\begin{aligned} C_{n-1} &= A_{n-1}+r\,B_{n-2}+r\,C_{n-2} \\ &= A_{n-1}+r(A_{n-2}+r\,B_{n-3})+r\,(A_{n-2}+r\,B_{n-3}+rC_{n-3}) \\ &= A_{n-1}+2r\,A_{n-2}+2r^2\,B_{n-3}+r^2\,C_{n-3} \\ &= A_{n-1}+2r\,A_{n-2}+2r^2(A_{n-3}+r\,B_{n-4})+r^2(A_{n-3}+rB_{n-4}+rC_{n-4}) \\ &= A_{n-1}+2r\,A_{n-2}+3r^2\,A_{n-3}+3r^3\,B_{n-4}+r^3\,C_{n-4} \\ &\ \vdots \\ &= A_{n-1}+2r\,A_{n-2}+3r^2\,A_{n-3}+\ldots\ldots\ldots (n-1)r^{n-2}A_1+nr^{n-1}, \end{aligned}$$

or

$$C_{n-1}=nr^{n-1}+(n-1)A_1r^{n-2}+(n-2)A_2r^{n-3}+\ldots 2A_{n-2}r+A_{n-1}\ldots \quad (1)$$

Again, the roots of the transformed equation will evidently be

$$a_1-r,\ a_2-r,\ a_3-r,\ a_4-r,\ \ldots .\ a_n-r,$$

and as we have found the coefficient, C_{n-1}, of the last term but one in the transformed equation, by one process, we shall now find the same coefficient, C_{n-1}, by another process (Prop. V., p. 309); it is the product of every $(n-1)$ roots of the equation (1) with their signs changed; hence we have

$$\left.\begin{aligned} C_{n-1} = &(r-a_1)(r-a_2)(r-a_3)\ldots\ldots\ldots \text{ to } (n-1) \text{ factors} \\ &+(r-a_1)(r-a_2)(r-a_4)\ldots\ldots\ldots \text{ to } (n-1) \text{ factors} \\ &+(r-a_1)(r-a_3)(r-a_4)\ldots\ldots\ldots \text{ to } (n-1) \text{ factors} \\ &\ \vdots \\ &+(r-a_2)(r-a_3)(r-a_4)\ldots\ldots\ldots \text{ to } (n-1) \text{ factors} \end{aligned}\right\} \ldots\ldots (2)$$

Now these two expressions which we have obtained for C_{n-1} are equal to one another, and, therefore, whatever changes arise by substitution in the one, the same changes will be produced, by a like substitution, in the other; hence, substituting a_1, a_2, a_3, &c., successively for r in the second member of equation (2), we have these results:

$$(a_1-a_2)(a_1-a_3)(a_1-a_4)\ldots\ldots=+.+.+\ldots\ldots=+$$
$$(a_2-a_1)(a_2-a_3)(a_2-a_4)\ldots\ldots=-.+.+\ldots\ldots=-$$
$$(a_3-a_1)(a_3-a_2)(a_3-a_4)\ldots\ldots=-.-.+\ldots\ldots=+$$
&c. &c. &c.

But when a series of quantities, a_1, a_2, a_3, a_4, &c., are substituted for the unknown quantity in any equation, and give results which are alternately $+$ and $-$, then, by Art. 352, these quantities, taken in order, are situated in the successive intervals of the real roots of the proposed equation; hence, making $C_{n-1}=0$, and changing r into x, we have from equation (1)

$$nx^{n-1}+(n-1)A_1x^{n-2}+(n-2)A_2x^{n-3}+\ldots.2A_{n-2}x+A_{n-1}=0 \;\ldots\; (3)$$

an equation whose roots, therefore, separate those of the original equation

$$x^n+A_1x^{n-1}+A_2x^{n-2}+\ldots.A_{n-1}x+A_n=0,$$

and the manner of deriving it from the proposed equation is to multiply each term of the proposed equation by the exponent of x, and to diminish the exponent one. It is identical with the second column of the development in the note to Article 251. It is known by the name of the derived equation.

Let a_1, a_2, a_3, a_4, &c., be the roots of the proposed equation, and b_1, b_2, b_3, &c., those of the derived equation (3), ranged in the order of magnitude; then the roots of both the given, and the derived equation will be represented in order of magnitude by the following arrangement, viz.:

$$a_1,\ b_1,\ a_2,\ b_2,\ a_3,\ b_3,\ a_4,\ b_4,\ a_5,\ b_5,\ \&c.\ldots$$

Corollary 1.—If $a_2=a_1$, then $r-a_1$ will be found as a factor in each of the groups of factors in equation (2), which has been shown to be the separating equation (3), and, therefore, the separating equation and the original equation will obviously have a common measure of the form $x-a_1$.

Corollary 2.—If $a_3=a_2=a_1$, then $(r-a_1)(r-a_1)$ will occur as a common factor in each group of factors in (2); that is, the separating equation (3) is divisible by $(x-a_1)^2$; and, therefore, the proposed equation and the separating equation have a common measure of the form $(x-a_1)^2$.

Corollary 3.—If the proposed equation have also $a_4=a_5$, then it will have a common measure with the separating equation of the form $(x-a_1)^2\,(x-a_4)$, and so on.

Scholium.—When, therefore, we wish to ascertain whether a proposed equation has *equal roots*, we must first find the separating equation, and then find the greatest common measure of the polynomials constituting the first members of these two equations. If the greatest common measure be of the form

$$(x-a_1)^p\,(x-a_2)^q\,(x-a_3)^r\ldots\ldots$$

then the proposed equation will have $(p+1)$ roots $=a_1$, $(q+1)$ roots $=a_2$, $(r+1)$ roots $=a_3$, &c. The equation may then be depressed to another of lower dimensions, by dividing it by the difference between x and the repeated root raised to a power of the degree expressed by the number of times it is repeated.

EXAMPLES.

Find the equal roots of the equation

$$x^7+5x^6+6x^5-6x^4-15x^3-3x^2+8x+4=0 \dots\dots (1)$$

The derived polynomial is

$$7x^6+30x^5+30x^4-24x^3-45x^2-6x+8 \dots\dots (2)$$

and the common divisor of (1) and (2)

$$x^4+3x^3+x^2-3x-2 \dots\dots (3)$$

The values of x, found by putting this equal to zero, would be the repeated roots of the proposed equation. This itself will be found to have equal roots, for its derived is

$$4x^3+9x^2+2x-3,$$

and their common divisor

$$x+1.$$

Hence, by the rule,

$$(x+1)^2 \dots\dots (4)$$

is a factor of (3), and

$$(x+1)^3$$

a factor of the proposed.

Dividing (3) by (4), the quotient is

$$x^2+x-2,$$

which, put equal to zero, gives

$$x=1, \text{ or } -2.$$

Hence (3) may be put under the form

$$(x+1)^2 (x-1) (x+2),$$

and by the rule in the above scholium the given equation may be put under the form

$$(x+1)^3 (x-1)^2 (x+2)^2,$$

so that in the proposed equation there are three roots equal to -1, two to $+1$, and two to -2.

(2) $$x^3-3a^2x-2a^3=0.$$

By the process above it may be transformed into

$$(x+a)^2 (x-2a)=0,$$

so that the three roots are two equal to $-a$, and the third $2a$.

(3) $$x^8-12x^7+53x^6-92x^5-9x^4+212x^3-153x^2-108x+108=0$$

decomposes into

$$(x-1) (x-2)^2 (x+1)^2 (x-3)^3=0.$$

254. The most satisfactory and unfailing criterion for the determination of the number of imaginary roots in any equation is furnished by the admirable theorem of Sturm, which gives the precise number of real roots, and, consequently, the exact number of imaginary ones, since both the real and imaginary roots are together equal to the number denoted by the degree of the proposed equation.

PROPOSITION.

To find the number of real and imaginary roots in any proposed equation.

The acknowledged difficulty which has hitherto been experienced in the important problem of the separation of the real and imaginary roots of any proposed equation is now completely removed by the recent valuable researches of the celebrated Sturm; and we shall now give the demonstration of the theorem by which this desirable object has been so fully accomplished, nearly as given by the author himself, deeming it far more satisfactory than any other version which we have seen.

THEOREM OF STURM.

I. Let $Nx^m+Px^{m-1}+Qx^{m-2}+\ldots\ldots+Tx+U=0$
be a numerical equation of any degree whatever, of which it is proposed to determine all the real roots.

We begin by performing upon this equation the operation which serves to determine whether or not it has equal roots (Art. 253, Sch.), in a manner which we proceed to point out. If V designate the entire function $Nx^m+Px^{m-1}+$, &c., and V_1 its derived function (which is formed by multiplying each term of V by the exponent of x in this term, and diminishing that exponent by unity), we must seek for the greatest common divisor of the two polynomials V and V_1. Divide, at first, V by V_1, and when a remainder is obtained of a degree inferior to that of the divisor V_1, change the signs of all the terms of this remainder (the signs + into — and — into +). Designate by V_2 what this remainder becomes after the change of signs. Divide in the same manner V_1 by V_2, and, after having changed the signs of the remainder, it becomes a new polynomial V_3, of a degree inferior to that of V_2. The division of V_2 by V_3 conducts, in the same manner, to a function V_4, which will be the remainder resulting from this division after having changed the signs. This series of divisions is to be continued, taking care to change the signs of the terms of each remainder. This change of signs, which would be useless if our object was to find the greatest common divisor of the polynomials V and V_1, is necessary in the theory about to be explained. As the degrees of the successive remainders go on diminishing, we arrive finally either at a numerical remainder independent of x, and differing from zero, or at a remainder a function of x, which exactly divides the preceding remainder.

We shall examine these two cases separately.

II. Suppose, in the first place, that, after a certain number of divisions, we arrive at a numerical remainder, which may be represented by V_r.

In this case we know that the equation $V=0$ has no equal roots, since the polynomials V and V_1 have no common divisor function of x. Representing by Q_1, $Q_2\ldots Q_{r-1}$, the quotients given by the successive divisions, which leave for remainders $-V_2$, $-V_3\ldots-V_r$, we have this series of equalities:

$$\begin{aligned} V&=V_1Q_1-V_2\\ V_1&=V_2Q_2-V_3\\ V_2&=V_3Q_3-V_4\ \ldots\ldots\ldots\ (1)\\ &\vdots\\ V_{r-2}&=V_{r-1}Q_{r-1}-V_r. \end{aligned}$$

Thus much being premised, the consideration of this system of functions $V, V_1, V_2\ldots V_r$ furnishes a sure and easy means of knowing *how many real roots the equation* $V=0$ *has comprehended between two numbers* A *and* B *of any magnitude or signs whatever*, B *being greater than* A. The following is the rule which attains this object:

Substitute in place of x the number A in all the functions $V, V_1, V_2\ldots V_{r-1}, V_r$, then write in order, in one line, the signs of the results, and count the number of variations which are found in this succession of signs. Write, in the same manner, the succession of signs which these same functions take by the substitution of the other member B, and count the number of variations which are found in this second succession. *The number of variations which it has less than the first will be the number of real roots of the equation* $V=0$ *comprehended between the numbers* A *and* B. *If the second succession has as many variations as the first, the equation* $V=0$ *has no real root between* A *and* B.

III. We shall demonstrate this theorem by examining how the number of variations formed by the signs of the functions V, V_1, $V_2 \ldots V_r$, for any one value whatever of x, can change, when x passes through different states of magnitude.

Whatever may be the signs of these functions for one determinate value of x, when x increases by insensible degrees to beyond this value, there can take place no change of signs in this succession of signs, unless one of the functions, V, $V_1 \ldots$, changes sign, and, consequently (153, note), becomes zero. There are then two cases to examine, according as the function which vanishes is the first, V, or some one of the other functions, V_1, $V_2 \ldots V_{r-1}$, *intermediate* between V and V_r: the last, V_r, can not change sign, since it is a number positive or negative.

IV. Let us see first what alteration the succession of signs experiences when x, in increasing in a continuous manner, attains and passes by a value which destroys the first function V. Designate this value by c. The function V_1, derived from V, can not be zero at the same time with V for $x=c$, because by the hypothesis the equation $V=0$ has not equal roots. We see, besides, by the equations (1), without falling back upon the theory of equal roots, that if the functions V and V_1 were zero for $x=c$, all the other functions, V_2, $V_3 \ldots$, and, finally, V_r, would be zero at the same time; but, on the contrary, V_r is by hypothesis a number different from zero. V_1 has then for $x=c$ a value different from zero, positive or negative.

Let us consider values of x very little different from c. If in designating by u a positive quantity as small as we please, we make by turns $x=c-u$ and $x=c+u$, the function V_1 will have for these two values of x the same sign that it has for $x=c$; because we can take u sufficiently small, to insure that V_1 shall have for these two values of x the same sign that it has for $x=c$; since we can take u so small that V_1 will not vanish, and not change sign, while x increases from the value $c-u$ to $c+u$.*

We must now determine the sign of V for $x=c+u$. Designate for a moment V by $f(x)$, V_1 by $f'(x)$, and the other derived functions of V by $f''(x)$, $f'''(x) \ldots, f^{m}(x)$, which are not to be confounded with V_2, V_3, &c., these latter not being derived functions. When we make $x=c+u$, V becomes $f(c+u)$, and we have (see note to Prop. III., Art. 239)

$$f(c+u)=f(c)+f'(c)u+\frac{f''(c)}{1.2}u^2+\frac{f'''(c)}{1.2.3}u^3+, \&c.;$$

or, rather, observing that $f(c)$ is zero, and that $f'(c)$ is not,

$$f(c+u)=u\left[f'(c)+\frac{f''(c)}{1.2}u+\frac{f'''(c)}{1.2.3}u^2+\ldots\right].$$

We see from this expression of $f(c+u)$, that in attributing to u very small

* The delicate point on which the theorem hinges is the one stated here. Let it be distinctly seen that since V_1 can not be zero at the same time with V when $x=c$, therefore, however little c may differ from a value which reduces V_1 to zero, u may be taken smaller than this difference.

positive values, $f(c+u)$ will have the same sign as $f'(c)$,* and, consequently, $f(c+u)$ will have also the same sign as $f'(c+u)$, since $f'(c+u)$ has the same sign as $f'(c)$. Thus, V has the same sign as V_1 for $x=c+u$.

By changing u into $-u$ in the preceding formula, we have

$$f(c-u)=-u\left[f'(c)-\frac{f''(c)}{1.2}u+, \&c.\right]$$

And we perceive, in the same manner, that $f(c-u)$ has a sign contrary to that of $f'(c)$, from whence it follows that for $x=c-u$ the sign of V is contrary to that of V_1.

Then, if the sign of $f'(c)$ or of V_1, for $x=c$, is $+$, the sign of V will be $+$ for $x=c+u$, and $-$ for $x=c-u$. If, on the contrary, the sign of V_1 is $-$ for $x=c$, that of V will be $-$ for $x=c+u$, and $+$ for $x=c-u$. Besides, V_1 has for $x=c+u$ and for $x=c-u$ the same sign as it has for $x=c$.

These results are indicated in the following table:

		V	V_1		V	V_1
For	$x=c-u$,	$-$	$+$,		$+$	$-$,
	$x=c$.	0	$+$,	or else	0	$-$,
	$x=c+u$,	$+$	$+$,		$-$	$-$.

Thus, when the function V vanishes, the sign of V forms with the sign of V_1 a variation, before x attains the value c, which reduces V to zero, and this variation is changed into a permanence after x passes this value.

As to the other functions, V_2, V_3, &c., each will have, as V_1, either for $x=c+u$ or for $x=c-u$, the same sign that it has for $x=c$, that is, if none of them vanish for $x=c$ at the same time with V.

The succession of the signs of the functions V, V_1, $V_2 \ldots V_r$, loses then a variation, when x, going on increasing, passes over a value c, which reduces the first function V to zero without destroying any of the other functions, V_1, V_2, &c. It is necessary now to examine what happens when one of these functions vanishes.

V. Let there be a function intermediate between V and V_r, which is destroyed when x becomes equal to b. This value of x can not reduce to zero either the function V_{n-1}, which precedes immediately V_n, or the function V_{n+1}, which follows V_n. Indeed, we have between the three functions V_{n-1}, V_n, V_{n+1}, the following equation, which is one of the equations (1).

$$V_{n-1}=V_nQ_n-V_{n+1}.$$

It proves that if the two consecutive functions, V_{n-1}, V_n, were zero for the same value of x, V_{n+1} would be zero at the same time; and as we have also

$$V_n=V_{n+1}Q_{n+1}-V_{n+2},$$

we should have, again, $V_{n+2}=0$, and so on, so that we should have finally $V_r=0$, which is contrary to the hypothesis.

The two functions, V_{n-1} and V_{n+1}, have then for $x=b$ values different from zero; moreover, these values are of contrary signs, because the same equation,

$$V_{n-1}=V_nQ_n-V_{n+1},$$

gives $V_{n-1}=-V_{n+1}$, when we have $V_n=0$.

* This depends upon a principle demonstrated at Art. 239, Cor., that if a function of u be arranged according to the ascending powers of u, u may be taken so small that the sign of the whole function shall depend upon that of its first term.

This being established, substitute in place of x two numbers, $b-u$ and $b+u$, very little different from b; the two functions, V_{n-1} and V_{n+1}, will have for these two values of x the same signs as they have for $x=b$, since we can always take u sufficiently small, to insure that neither V_{n-1} nor V_{n+1} shall change sign when x enlarges in the interval from $b-u$ to $b+u$. Whatever may be the sign of V_n for $x=b-u$, as it is placed in the succession of signs between those of V_{n-1} and V_{n+1}, which are contrary, the signs of these three consecutive functions, V_{n-1}, V_n, V_{n+1}, for $x=b-u$, will form always either a permanence followed by a variation, or a variation followed by a permanence, as is here seen.

$$\text{For} \quad x=b-u \quad \begin{matrix} V_{n-1} & V_n & V_{n+1} \\ + & \pm & - \end{matrix}, \text{ or else, } \begin{matrix} V_{n-1} & V_n & V_{n+1} \\ - & \pm & + \end{matrix}.$$

Similarly, the signs of the three functions, V_{n-1}, V_n, V_{n+1}, for $x=b+u$, whatever may be that of V_n, will form one variation, and will form but one.

Besides, each of the other functions will have the same sign for $x=b-u$ and $x=b+u$, provided no one of them is found to be zero for $x=b$ at the same time as V_n.

Consequently, the succession of the signs of all the functions, V, $V_1 \dots V_r$, for $x=b+u$, will contain precisely as many variations as the succession of their signs for $x=b-u$. Thus, the number of variations in the succession of signs is not changed when any intermediate function whatever passes through zero.

One arrives evidently at the same conclusion, if many intermediate functions, not consecutive, vanish for the same value of x. But if this value should destroy also the first function, V, the change of sign of this one would then make one variation disappear at the left of the succession of signs, as has been shown in IV.

VI. It is then demonstrated that each time that the variable x, in increasing by insensible degrees, attains and passes a value which renders V equal to zero, the series of the signs of the functions V, V_1, $V_2 \dots V_r$ loses a variation formed on its left by the signs of V and V_1, which is replaced by a permanence, while the changes of signs of the intermediate functions, V_1, $V_2 \dots V_{r-1}$, can never either augment or diminish the number of variations which existed already. Consequently, if we take any number whatever, A, positive or negative, and any other number whatever, B, greater than A, and if we make x increase from A to B, as many values of x as are comprised between A and B, which render V equal to zero, so many variations will the succession of signs of the functions V, $V_1 \dots V_r$ for $x=B$ contain less than the succession of their signs for $x=A$. This was the theorem to be demonstrated.

Remark.—In the successive divisions which serve to form the functions V_2, V_3, &c., we can, before taking a polynomial for a dividend or divisor, multiply or divide it by any positive number at pleasure. The functions V, V_1, $V_2 \dots V_r$, obtained by this operation, will differ only by positive numerical factors from those which we have previously considered, and which appear in equations (1), so that they will have respectively the same signs as these for each value of x.

With this modification we can, when the coefficients of the equation $V=0$ are whole numbers, form polynomials V_2, V_3, &c., the coefficients of which shall be also entire. But it is necessary to take good care that the numerical factors thus introduced or suppressed be all positive.

VII. This theorem gives the means of knowing the whole number of real roots of the equation $V=0$.

In fact, an entire polynomial function of x being given, we can always assign to x such a positive value as that for this and every greater value the polynomial will have constantly the sign of its first term (see Art. 239). It is the same with all negative values of x below a certain limit. All the real roots of the equation $V=0$ being comprised between $-\infty$ and $+\infty$, it will be sufficient, in order to know their number, to substitute $-\infty$ and $+\infty$ instead of A and B, in the functions V, V_1, $V_2 \ldots V_r$, and to note the two successions of signs for $-\infty$ and $+\infty$. When we make $x=+\infty$, each function is of the same sign as its first term. For $x=-\infty$, each function of an even degree, including V_r, has the same sign that it has for $x=+\infty$; but each function of an uneven degree takes for $x=-\infty$ a contrary sign to that which it has for $x=+\infty$. The excess of the number of variations formed by the signs of the functions V, $V_1 \ldots V_r$, for $x=-\infty$, over the number of variations for $x=+\infty$, will express the whole number of real roots of the equation $V=0$.*

To determine the initial figures of the roots, we may substitute the successive numbers of the series

$$0,\ -1,\ -2,\ -3,\ -4,\ \ldots\ldots$$

till we have as many variations as $-\infty$ produced; and if we substitute the numbers of the series

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* One might be curious to know how the succession of signs of the functions V, V_1, $V_2 \ldots V_r$ must undergo change so as that a variation is lost every time that V vanishes.

We have seen (IV.) that if c is a root of the equation $V=0$, the two functions V and V_1 must have contrary signs for $x=c-u$, and the same sign for $x=c+u$. So that if we designate by c' the root of the equation $V=0$, which is next greater than c, so that between c and c' there is no other root, V_1 will have for $x=c'-u$ a sign contrary to that of V. But V has constantly the same sign for all values of x comprised between c and c'; and as V_1 has the same sign as V for $x=c+u$, and a contrary sign to that of V for $x=c'-u$, we see that V_1 has two values with contrary signs for $x=c+u$ and for $x=c'-u$; then, while x increases from $c+u$ to $c'-u$, V_1 must change sign once, or an uneven number of times (I., or Prop. of Art. 252, Cor. 1).

Let γ be the only value of x, or the least value of x between c and c', for which V_1 changes sign. V and V_1 will have for $x=\gamma-u$ the same common sign that they have for $x=c+u$. For $x=\gamma+u$ V will have this same sign; but V_1 will have the contrary sign. V_2 will have a sign contrary to that of V for the three values for $\gamma-u$, γ, and $\gamma+u$ (V.). If, for example, V is positive for $x=c+u$, we have the following table:

		V	V_1	V_2
For	$x=\gamma-u$	$+$	$+$	$-$
	$x=\gamma$	$+$	0	$-$
	$x=\gamma+u$	$+$	$-$	$-$

Thus, before x attained the value c, which destroys V, the signs of V and V_1 formed a variation which is changed into a permanence after x has overpassed this value c; this permanence subsists until V_1 changes sign, then it is anew replaced by a variation after the change of sign of V_1; but, at the same time, there is a variation formed by the signs of V_1 and V_2 which changes into a permanence, so that the number of variations in the total succession of signs is neither increased nor diminished.

If V_1 changes sign a second time for a new value of x comprehended between c and c', the variation which the signs of V and V_1 form before x attains this value will be again replaced by a permanence; and still, on account of V_2, the number of variations will remain the same in the succession of signs. As V_1 can thus change sign only an uneven number of times, after its last change the signs of V and V_1 will form a variation which will subsist until x attains the value c', which destroys V. We have not to consider here the case where V vanishes without changing sign.

$$0, 1, 2, 3, 4, \ldots\ldots$$

till we arrive at a number which produces as many variations as $+\infty$, then the numbers thus obtained will be the limits of the roots of the equation, and the situation of the roots will be indicated by the signs arising from the substitution of the intermediate numbers.

We shall now apply the theorem to a few

EXAMPLES.

(1) Find the number and situation of the roots of the equation

$$x^3-4x^2-6x+8=0.*$$

Here we have

$$V = x^3-4x^2-6x+8$$
$$V_1=3x^2-8x-6;$$

then, multiplying the polynomial V by 3, in order to avoid fractions,

$$\begin{array}{rl} 3x^2-8x-6) & 3x^3-12x^2-18x+24\ (x-1 \\ & 3x^3-\ 8x^2-\ 6x \\ \hline & -\ 4x^2-12x+24, \text{ multiply by } \frac{3}{4}; \\ \text{or} & -\ 3x^2-\ 9x+18 \\ & -\ 3x^2+\ 8x+\ 6 \\ \hline & -17x+12 \ \therefore\ V_2=17x-12 \end{array}$$

$$\begin{array}{rl} & 3x^2-\ 8x-6 \\ & 17 \\ \hline 17x-12) & 51x^2-136x-102\ (3x \\ & 51x^2-\ 36x \\ \hline & -100x-102. \end{array}$$

It is now unnecessary to continue the division further, since it is very obvious that the sign of the remainder, which is independent of x, is $-$; and, therefore, the series of functions are

$$V = x^3-4x^2-6x+8$$
$$V_1= 3x^2-8x-6$$
$$V_2=17x-12$$
$$V_3=+.$$

Put $+\infty$ and $-\infty$ for x in the leading terms of these functions, and the signs of the results are

* The process applied to the general cubic equation $x^3+ax^2+bx+c=0$, gives the following functions, viz.:

With the second term.		*Without the second term, or $a=0$.*	
$V = x^3+ax^2+bx+c$	(1)	$V = x^3+bx+c$	(2)
$V_1=3x^2+2ax+b$		$V_1=3x^2+b$	
$V_2=2(a^2-3b)x+ab-9c$		$V_2=-2bx-3c$	
$V_3=-4a^3c+a^2b^2-18abc-4b^3-27c^2$		$V_3=-4b^3-27c^2$	

These functions in (1) and (2) will frequently be found useful in the application of Sturm's theorem to equations of the third degree, since the derived functions in any particular example may be found by substitution only. In order that all the roots of the equation $x^3+bx+c=0$ may be real, the first terms of the functions must be positive; hence $-2bx$ and $-4b^3-27c^2$ must be positive; and as $-27c^2$ is always negative, b must be negative, in order that $-4b^3$ and $-2b$ may be positive; therefore, when all the roots are real, $4b^3$ must be greater than $27c^2$, or $\left(\frac{b}{3}\right)^3$ greater than $\left(\frac{c}{2}\right)^2$. When, therefore, b is negative and $\left(\frac{b}{3}\right)^3>\left(\frac{c}{2}\right)^2$, all the roots are real, a criterion which has been long known, and as simple as can be given.

For $x=+\infty$, $++++$ no variation,

$x=-\infty$, $-+-+$ three variations,

$\therefore 3-0=3$, the number of real roots in the proposed cubic equation.

Next, to find the situation of the roots we must employ narrower limits than $+\infty$ and $-\infty$. Commencing at zero, let us extend the limits both ways, and, since the proposed equation has only one permanence of sign, one of the roots is negative, and the remaining roots are positive.

	V	V_1	V_2	V_3	*Var.*
For $x=0$ signs	+	−	−	+	2
$x=1$	−	−	+	+	1
$x=2$	−	−	+	+	1
$x=3$	−	−	+	+	1
$x=4$	−	+	+	+	1
$x=5$	+	+	+	+	1
$x=6$	+	+	+	+	0

	V	V_1	V_2	V_3	*Var.*
For $x=\ \ 0$ signs	+	−	−	+	2
$x=-1$	+	+	−	+	2
$x=-2$	−	+	−	+	3

We perceive, then, by the columns of variations, that the roots are between 0 and 1, 4 and 5, -1 and -2; hence the initial figures of the roots are -1, 0, and 4; and, in order to narrow still further the limits of the root between 0 and 1, we shall resume the substitutions for x in the series of functions as before. But as the substitution of 1 for x, in the function V, gives a value nearly zero, we shall commence with 1, and descend in the scale of tenths, until we arrive at the first decimal figure of the root.

Let $x=1$ signs $--++$ one variation,

$x=\cdot 9$ . . . . $+-++$ two variations;

hence the initial figures are -1, $\cdot 9$, and 4.

(2) Find the number and situation of the real roots of the equation

$$x^4+x^3-x^2-2x+4=0.$$

Here the several functions are

$$V = x^4+x^3-x^2-2x+4$$
$$V_1= 4x^3+3x^2-2x-2$$
$$V_2= x^2+2x-6$$
$$V_3=-x+1$$
$$V_4=+.$$

Let $x=+\infty$, signs of leading terms $+++-+$ two variations

$x=-\infty$ . . . . $+-+++$ two variations;

and all the roots of the equation are imaginary.

When, in seeking for the greatest common divisor of V and V_1, we arrive at a polynomial V_n (for example, at that of the second degree), which, put equal to zero, will only give imaginary values of x, it is not necessary to carry the divisions further, because this polynomial V_n will be constantly of the same sign as its first term for all real values of x; for if it gave a plus sign for one value, and a minus for another, there must be a real root between.*

(3) Required the number and situation of the real roots of the equation

$$2x^4-11x^2+8x-16=0.$$

The first three functions are

$$V = 2x^4-11x^2+8x-16$$
$$V_1= 4x^3-11x+4$$
$$V_2=11x^2-12x+32;$$

* This consideration is of importance, as the calculations for determining the functions V_2, V_3 are long, especially toward the last, on account of the magnitude of their numerical coefficients.

and the roots of the quadratic $11x^2-12x+32=0$ are imaginary, for $11\times32\times4$ is greater than 12^2; hence V_2 must preserve the same sign for every value of x, and the subsequent functions can not change the number of variations, for a variation is only lost by the change of the sign of V. Hence,

For $x=+\infty$ signs $+++$ no variation,
$x=-\infty$ . . . $+-+$ two variations;

and the proposed equation has two real roots, the one positive and the other negative, since the last term is negative. (Prop. VIII., Cor. 5, p. 314.)

When	$x=0$ signs $-++$	$x=\ \ 0$ signs $-++$
	$x=1$ . . . . $--+$	$x=-1$ . . . . $-++$
	$x=2$ . . . . $-++$	$x=-2$ . . . . $--+$
	$x=3$ . . . . $+++$	$x=-3$ . . . . $+-+$.

Hence the initial figures of the real roots are 2 and -2.

When two roots are nearly equal to each other.

(4) Find the roots of the equation

$$x^3+11x^2-102x+181=0.$$

The functions are

$$V=x^3+11x^2-102x+181$$
$$V_1=3x^2+22x-102$$
$$V_2=122x-393$$
$$V_3=+;$$

and the signs of the leading terms are all $+$; hence the substitution of $-\infty$ and $+\infty$ must give three real roots.

To discover the situation of the roots, we make the substitutions

$x=0$ which gives $+--+$ two variations,
$x=1$ . . . . $+--+$
$x=2$ . . . . $+--+$
$x=3$ . . . . $+--+$ two variations,
$x=4$ . . . . $++++$ no variation;

hence the two positive roots are between 3 and 4, and we must, therefore, transform the several functions into others, in which x shall be diminished by 3. This is effected by Art. 251, p. 315; and we get

$$V'=y^3+20y^2-9y+1$$
$$V'_1=3y^2+40y-9$$
$$V'_2=122y-27$$
$$V'_3=+.$$

Make the following substitutions in these functions, viz.:

$y=0$ signs $+--+$ two variations,
$y=\cdot1$ . . . $+--+$
$y=\cdot2$ . . . $+--+$ two variations,
$y=\cdot3$ . . . $++++$ no variation;

hence the two positive roots are between 3·2 and 3·3, and we must, again, transform the last functions into others, in which y shall be diminished by ·2. Effecting this transformation, we have

$$V''=z^3+20\cdot6z^2-\cdot88z+\cdot008$$
$$V''_1=3z^2+41\cdot2z-\cdot88$$
$$V''_2=122z-2\cdot6$$
$$V''_3=+.$$

Let	$z=0$ then signs are	$+--+$	two variations,
	$z=.01$	$+--+$	two variations,
	$z=\cdot02$	$---+$	one variation,
	$z=\cdot03$	$++++$	no variation;

hence we have 3·21 and 3·22 for the positive roots, and the sum of the roots is —11; therefore, $-11-3\cdot21-3\cdot22=-17\cdot4$ is the negative root.

When the equation has equal roots.

255. When the equation has equal roots, one of the divisors will divide the preceding without a remainder, and the process will thus terminate without a remainder, independent of x. In this case, the last divisor is a common measure of V and V_1; and it has been shown (Art. 253, Scholium 3, p. 321) that if $(x-a_1)(x-a_2)^2$ be the greatest common measure of V and V_1, then V is divisible by $(x-a_1)^2(x-a_2)^3$, and the depressed equation furnishes the distinct and separate roots of the equation, for Sturm's theorem takes no notice of the repetition of a root. The several functions may be divided by the greatest common measure so found, and the depressed functions employed for the determination of the distinct roots; but it is obvious that the original functions will furnish the separate roots just as well as the depressed ones, for the former differ only from the latter in being multiplied by a common factor (29); and whether the sign of this factor be + or —, the number of variations of sign must obviously remain unchanged, since multiplying or dividing by a positive quantity does not affect the signs of the functions; and if the factor or divisor be negative, *all* the signs of the functions will be changed, and the number of variations of sign will remain precisely as before.

Find the number and situation of the real roots of the equation

$$x^5-7x^4+13x^3+x^2-16x+4=0.$$

By the usual process, we find

$$\begin{aligned}
V &= x^5-7x^4+13x^3+x^2-16x+4\\
V_1 &= 5x^4-28x^3+39x^2+2x-16\\
V_2 &= 11x^3-48x^2-51x+2\\
V_3 &= 3x^2-8x+4\\
V_4 &= x-2\\
V_5 &= 0.
\end{aligned}$$

Hence $x-2$ is a common measure of V and V_1; and if

$x=-\infty$ the signs are	$-+-+-$	four variations,
$x=-2$	$-+-+-$	four variations,
$x=-1$	$0+-+-$	
$x=\ \ 0$	$+-++-$	three variations,
$x=\ \ 1$	$-++--$	two variations,
$x=\ \ 2$	0 0 0 0 0	
$x=\ \ 3$	$--+++$	one variation,
$x=\ \ 4$	$+++++$	no variation.

Therefore we infer that there are four distinct and separate roots; one is —1, for V vanishes for this value of x; another between 0 and 1; a third is 2, and a fourth is between 3 and 4. The common measure $x-2$ indicates that the polynomial V is divisible by $(x-2)^2$; and hence there are two roots equal to 2 (Art. 253, Cor. 1).

It may happen that one of the functions, V_1, $V_2 \ldots V_{r-1}$, should be found zero either for $x=A$ or $x=B$. In this case it is sufficient to count the variations which are found in the succession of signs of the functions V, V_1, $V_2 \ldots V_r$, omitting the function which is zero. This results from the demonstration in Art. 254, V, for the case where an intermediate function vanishes.

When the number of the auxiliary functions, V_1, V_2, &c., is equal to the degree of the equation, as is ordinarily the case, in consequence of each remainder in seeking for the common divisor being one degree less than the preceding, the number of imaginary roots in the equation may be found by the following rule: *The equation* $V=0$ *will have as many pairs of imaginary roots as there are variations of sign in the succession of the signs of the first terms of the functions* V_1, V_2, &c., *to the sign of the constant* V_m *inclusive.*

This follows from the fact that two consecutive functions, V_{n-1}, V_n, are the one of an even, the other of an odd degree. Then, if the two functions have the same sign for $x=+\infty$, they will have contrary for $x=-\infty$, and *vice versa.* So that if we write the succession of signs of V, V_1, $V_2 \ldots V_m$, for $x=-\infty$ and for $x=+\infty$, each variation in the one succession will correspond to a permanence in the other. Thus, the number of permanences for $x=-\infty$ is equal to the number of variations for $x=+\infty$.

But for $x=+\infty$ the number of variations will be that of the *first terms* of the functions V, $V_1 \ldots V_m$, which denote by i. Then there will be i permanences for $x=-\infty$ and $m-i$ variations. The excess of the number of variations $m-i$ for $x=-\infty$ over the number i for $x=+\infty$, is $m-2i$, which is therefore the number of real roots of the equation, and therefore $2i$ the number of imaginary roots, the whole number of roots being m.

HORNER'S METHOD OF RESOLVING NUMERICAL EQUATIONS OF ALL ORDERS.

256. The method of approximating to the roots of numerical equations of all orders, discovered by W. G. Horner, Esq., of Bath, England, is a process of very remarkable simplicity and elegance, consisting simply in a succession of transformations of one equation to another, each transformed equation as it arises having its roots less or greater than those of the preceding by the corresponding figure in the root of the proposed equation. We have shown how to discover the initial figures of the roots by the theorem of STURM; and by making the penultimate coefficient in each transformation available as a trial divisor of the absolute term, we are enabled to discover the succeeding figure of the root; and thus proceeding from one transformation to another, we are enabled to evolve, one by one, the figures of the root of the given equation, and push it to any degree of accuracy required.

GENERAL RULES.

1. Find the number and situation of the roots by *Sturm's* theorem, and let the root required to be found be positive.

2. Transform the equation into another whose roots shall be less than those of the proposed equation by the initial figure of the root.

3. Divide the absolute term of the transformed equation by the *trial divisor*, or penultimate coefficient, and the next figure of the root will be obtained, by which diminish the root of the transformed equation as before, and proceed in this manner till the root be found to the required accuracy.

Note 1.—When a negative root is to be found, change the signs of the alternate terms of the equation, and proceed as for a positive root.

Note 2.—When three or four decimal places in the root are obtained, the operation may be contracted, and much labor saved, as will be seen in the following examples:

EXAMPLES.

(1) Find all the roots of the cubic equation

$$x^3-7x+7=0.$$

By Sturm's theorem, the several functions are (Note, p. 328),

$$V = x^3-7x+7$$
$$V_1=3x^2-7$$
$$V_2=2x-3$$
$$V_3=+$$

Hence, for $x=+\infty$ the signs are $++++$ no variation,

$x=-\infty$ . . . . $-+-+$ three variations;

therefore the equation has *three* real roots, one negative, and two positive.

To determine the initial figures of these roots, we have

for $x=0$ signs	$+--+$	for $x=0$ signs	$+--+$
$x=1$ . . .	$+--+$	$x=-1$ . . .	$+--+$
$x=2$ . . .	$++++$	$x=-2$ . . .	$++-+$
		$x=-3$ . . .	$++-+$
		$x=-4$ . . .	$-+-+$

hence there are two roots between 1 and 2, and one between -3 and -4.

But in order to ascertain the first figures in the decimal parts of the two roots situated between 1 and 2, we shall transform the preceding functions into others, in which the value of x is diminished by unity. Thus, for the function V we have this operation:

$$\begin{array}{rrrl}
1+0 & -7 & +7 & (1 \\
1 & 1 & -6 & \\
\hline
1 & -6 & 1 & \\
1 & 2 & & \\
\hline
2 & -4 & & \\
1 & & & \\
\hline
3 & & &
\end{array}$$

And transforming the others in the same way, we obtain the functions

$$V'=y^3+3y^2-4y+1;\ V'_1=3y^2+6y-4;\ V'_2=2y-1;\ V'_3=+.$$

Let $y=\cdot1$ then the signs are $+--+$ two variations,

$y=\cdot2$ . . . . $+--+$ do.

$y=\cdot3$ . . . . $+--+$ do.

$y=\cdot4$ . . . . $---+$ one variation,

$y=\cdot5$ . . . . $--\mp+$ do.

$y=\cdot6$ . . . . $-+++$ do.

$y=\cdot7$ . . . . $++++$ no variation.

Therefore, the initial figures of the three roots are 1·3, 1·6, and -3.

The rest of the process, with a repetition of the above, is exhibited and afterward explained below.

<table>
<tr><td>1+0</td><td>— 7</td><td>+7 (1·356895867</td></tr>
<tr><td>1</td><td>1</td><td>—6</td></tr>
<tr><td>1</td><td>— 6</td><td>*1 . . .</td></tr>
<tr><td>1</td><td>2</td><td>—903</td></tr>
<tr><td>2</td><td>—*4 . .</td><td>*97 . . .</td></tr>
<tr><td>1</td><td>9 9</td><td>—86625</td></tr>
<tr><td>* 33</td><td>— 3 0 1</td><td>*10375 . .</td></tr>
<tr><td>3</td><td>1 0 8</td><td>—9048984</td></tr>
<tr><td>36</td><td>—*1 9 3</td><td>*1326016</td></tr>
<tr><td>3</td><td>1 9 7 5</td><td>—1184430</td></tr>
<tr><td>* 39 5</td><td>— 1 7 3 2 5</td><td>141586</td></tr>
<tr><td>5</td><td>2 0 0 0</td><td>—132923</td></tr>
<tr><td>40 0</td><td>—*1 5 3 2 5 . .</td><td>8663</td></tr>
<tr><td>5</td><td>2 4 3 3 6</td><td>—7382</td></tr>
<tr><td>* 40 56</td><td>— 1 5 0 8 1 6 4</td><td>1281</td></tr>
<tr><td>6</td><td>2 4 3 7 2</td><td>—1181</td></tr>
<tr><td>40 62</td><td>—*1 4 8 3 7 9|2</td><td>100</td></tr>
<tr><td>6</td><td>3 2 5|4</td><td>—89</td></tr>
<tr><td>*|40|68</td><td>— 1 4 8 0 5 3|8</td><td>11</td></tr>
<tr><td></td><td>3 2 5|4</td><td>—10</td></tr>
<tr><td></td><td>— 1 4 7 7 2|8</td><td>1</td></tr>
<tr><td></td><td>3|6</td><td></td></tr>
<tr><td></td><td>— 1 4 7 6 9|2</td><td></td></tr>
<tr><td></td><td>3|6</td><td></td></tr>
<tr><td></td><td>— 1|4|7|6|5</td><td></td></tr>
</table>

The process here is similar to that on p. 318. The numbers marked with stars are the coefficients of the equation having the reduced roots. Thus, *3, *4, and *1 are the coefficients of the equation whose roots are 1 less than those of the proposed equation. The right-hand 3 of *33 is the 3 tenths added in the next step of the process, which has for its object to reduce the roots by ·3. The coefficients of the resulting equation are *39, —*193, and *97. Now, instead of going on in this manner to obtain the following figures, 568, &c., of the root, the method of proceeding changes; the 193, which is the penultimate coefficient, becomes a trial divisor, by which dividing the absolute term 97, which is .0097, the divisor being 1·93, the quotient is 5, the next figure of the root, which is .05. This 5 is annexed to the *39, and we proceed as before; that is, multiply the *395 in the first column by this 5, producing 1975 in the second column, and by addition, 1·7325, and so on. To show that the quotient figure 5 is obtained by means of the trial divisor, observe that the 1·7325 is nearly equal to the *1·93 above, and that the ·088625 in the third column, which is the product of 1·7325 by the ·05, is nearly equal to the *·097 above; hence the quotient of *·097 by 1·93 is nearly this same ·05.

The further we proceed, the more accurate this process becomes, for the first figure of each number in the first column being units, this, multiplied by the decimal figure found in the root, which is thousandths, tens of thousandths, and so on; that is, soon a very small fraction gives thousandths, ten of thousandths, and so on, or a very small fraction, for the product; and, the first figure in the numbers of the second column being also units, these numbers are not much affected by the addition of the above-named products.*

When the number of decimal places in the numbers of the third column becomes equal to the number of decimal places required in the root, it will not be necessary to obtain any more in the third column; and as each new decimal figure in the root, multiplied by the number in the second column, would make one more place in the third, it will be necessary to cut off one figure in the second column, and, for a similar reason, two figures in the first column. As soon as the figures are all cut off in the first column, the process becomes simply one of division, the divisor and dividend rapidly diminishing.

We have thus found one root $x=1·356895867$......., and the coefficients of the successive transformed equations are indicated by the asterisks in each column. To find another, we have the following:

1+0	−7	+7 (1·692021471
1	1	−6
1	−6	1...
1	2	−1104
2	−4 . .	− 104...
1	2 1 6	100809
36	−1 8 4	−3191...
6	2 5 2	3156888
42	6 8 . .	−34112
6	4 4 0 1	31774
48 9	1 1 2 0 1	−2338
9	4 4 8 2	1589
49 8	1 5 6 8 3 . .	−749
9	1 0 1 4 4	635
50 72	1 5 7 8 4 4 4	−114
2	1 0 1 4 8	111
50 74	1 5 8 8 5|9|2	3
2	1|	
|50|76	1|5|8|8|7|	

Another root is $x=1·692021471$ . . . .

For the negative root, change the signs of the second and fourth terms.

---

* To show this in a more general way, let

$$ax^n+Bx^{n-1}+Bx^{n-2}....+B_{n-1}x+B_n=0$$

be one of the depressed equations which is to furnish the next decimal place of the root of the proposed equation; the value of x in this depressed equation will of course be a very small fraction; hence the higher powers of it may, without much error, be neglected. The depressed equation thus reduces to

$$B_{n-1}x+B_n=0.$$

Hence the value of x, without regard to its sign, is

$$x=\frac{B_n}{B_{n-1}}$$

nearly; that is, it may be obtained by dividing the ultimate by the penultimate coefficient.

<table>
<tr><td>1—0</td><td>—7</td><td>—7 (3·0489173396</td></tr>
<tr><td>3</td><td>9</td><td>+6</td></tr>
<tr><td>3</td><td>2</td><td>—1......</td></tr>
<tr><td>3</td><td>18</td><td>814464</td></tr>
<tr><td>6</td><td>20....</td><td>—185536...</td></tr>
<tr><td>3</td><td>3616</td><td>166382592</td></tr>
<tr><td>904</td><td>203616</td><td>—19153408</td></tr>
<tr><td>4</td><td>3632</td><td>18791228</td></tr>
<tr><td>908</td><td>207248..</td><td>—362180</td></tr>
<tr><td>4</td><td>73024</td><td>208875</td></tr>
<tr><td>9128</td><td>20797824</td><td>—153305</td></tr>
<tr><td>8</td><td>73088</td><td>146212</td></tr>
<tr><td>9136</td><td>2087091|2</td><td>—7093</td></tr>
<tr><td>8</td><td>823|0</td><td>6266</td></tr>
<tr><td>|..|91|44</td><td>2087914|2</td><td>—827</td></tr>
<tr><td></td><td>823|0</td><td>626</td></tr>
<tr><td></td><td>208873|7</td><td>—201</td></tr>
<tr><td></td><td>|9</td><td>188</td></tr>
<tr><td></td><td>208874|6</td><td>—13</td></tr>
<tr><td></td><td>|9</td><td>12</td></tr>
<tr><td></td><td>2|0|8|8|7|5</td><td>1</td></tr>
</table>

Hence the three roots of the proposed cubic equation are

$$x = 1·356895867 \ldots\ldots$$
$$x = 1·692021471 \ldots\ldots$$
$$x = -3·048917339 \ldots\ldots$$

(2) Find the roots of the equation $x^3+11x^3-102x+181=0$.

We have already found the roots to be nearly 3·21, 3·22, and —17. (See Example 4, p. 330.)

<table>
<tr><td>1+11</td><td>—102</td><td>+181 (3·21312775</td></tr>
<tr><td>3</td><td>42</td><td>—180</td></tr>
<tr><td>14</td><td>— 60</td><td>1...</td></tr>
<tr><td>3</td><td>51</td><td>—992</td></tr>
<tr><td>17</td><td>— 9..</td><td>8...</td></tr>
<tr><td>3</td><td>404</td><td>—6739</td></tr>
<tr><td>202</td><td>— 496</td><td>1261...</td></tr>
<tr><td>2</td><td>408</td><td>—1217403</td></tr>
<tr><td>204</td><td>— 88..</td><td>43597</td></tr>
<tr><td>2</td><td>2061</td><td>—34183</td></tr>
<tr><td>2061</td><td>— 6739</td><td>9414</td></tr>
<tr><td>1</td><td>2062</td><td>—6787</td></tr>
<tr><td>2062</td><td>— 4677..</td><td>2627</td></tr>
</table>

Carried to next page.

<table>
<tr><td>2 06 2</td><td>— 4 6 7 7 . .</td><td>2627</td></tr>
<tr><td>1</td><td>6 1 8 9 9</td><td>—2372</td></tr>
<tr><td>2 06 33</td><td>— 4 0 5 8 0 1</td><td>255</td></tr>
<tr><td>3</td><td>6 1 9 0 8</td><td>—237</td></tr>
<tr><td>2 06 36</td><td>— 3 4 3 8 9|3</td><td>18</td></tr>
<tr><td>3</td><td>2 0 6|4</td><td>—16</td></tr>
<tr><td>|·2|06|39</td><td>— 3 4 1 8 2|9</td><td>2</td></tr>
<tr><td></td><td>2 0 6|4</td><td></td></tr>
<tr><td></td><td>— 3 3 9 7|6</td><td></td></tr>
<tr><td></td><td>4|1</td><td></td></tr>
<tr><td></td><td>— 3 3 9 3|5</td><td></td></tr>
<tr><td></td><td>4|1</td><td></td></tr>
<tr><td></td><td>— 3|3|8|9</td><td></td></tr>
</table>

In a similar manner, the two remaining roots will be found to be

$$x=3{\cdot}22952121$$

and

$$x=-17{\cdot}44264896.$$

(3) Given $x^4+x^3+x^2+3x-100=0$, to find the number and situation of the real roots.

Here we have

$$V = x^4+ x^3+ x^2+3x-100$$
$$V_1=4x^3+3x^2+2x+3$$
$$V_2=-5x^2-34x+1603$$
$$V_3=-1132x+6059$$
$$V_4=-.$$

Let $x=-\infty$ then signs are $+ - - + -$ three variations,
$x=+\infty$ . . . . $+ + - - -$ one variation ;

hence two roots are real and two imaginary ; and the real roots must have contrary signs, for the last term of the equation is negative. To find the situation of the roots

	in $V\ V_1 V_2 V_3 V_4$
Let $x=0$	signs $- + + + -$
$x=1$	. . . $- + + + -$
$x=2$	. . . $- + + + -$
$x=3$	. . . $+ + + + -$

	in $V\ V_1 V_2 V_3 V_4$
Also, $x=\ 0$	signs $- + + + -$
$x=-1$	. . . $-\ 0 + + -$
$x=-2$	. . . $- - + + -$
$x=-3$	. . . $- - + + -$
$x=-4$	. . . $+ - + + -$

In this example the function V_1 vanishes for $x=-1$, and for the same value of x the functions V and V_2 have contrary signs, agreeably to Lemma 2, and writing + or — for 0 gives the same number of variations. The initial figures of the root are, therefore, 2 and —3.

To find the negative root, we have the following operation:

1—1	+1	— 3	—100 (3·43357786336599
3	6	2 1	54
2	7	1 8	—46....
3	15	6 6	416896
5	22	8 4 . . .	—43104...
3	24	2 0 2 2 4	384456501
8	46 . .	1 0 4 2 2 4	—46583499..
3	4 56	2 2 1 1 2	390491222121
11 4	50 56	1 2 6 3 3 6 . . .	—75343767879
4	4 72	1 8 1 6 1 6 7	65189289046
11 8	55 28	1 2 8 1 5 2 1 6 7	—10154478833
4	4 88	1 8 2 7 5 6 1	9128951421
12 2	60 16 . .	1 2 9 9 7 9 7 2 8 . . .	—1025527412
4	37 89	1 8 4 0 1 2 7 0 7	912928254
12 63	60 53 89	1 3 0 1 6 3 7 4 0 7 0 7	—112599158
3	37 98	1 8 4 1 2 7 2 4 1	104335040
12 66	60 91 87	1 3 0 3 4 7 8 6 7 9 4 8	—8264118
3	38 07	3 0 7 1 0 1 4 5	7825130
12 69	61 29 94 . .	1 3 0 3 7 8 5 7 8 0 9 3	—438988
3	3 81 69	3 0 7 1 3 3 2 5	391256
12 723	61 33 75 69	1 3 0 4 0 9 2 9 1 4 2	—47732
3	3 81 78	4 3 0 0 3 1	39126
12 726	61 37 57 47	1 3 0 4 1 3 5 9 1 7 3	—8606
3	3 81 87	4 3 0 0 3 1	7825
12 729	61 41 39 34	1 3 0 4 1 7 8 9 2 0	—781
3	63 6	4 3 0 0	652
·12 732	61 42 02 9	1 3 0 4 1 8 3 2 2	—129
	63 6	4 3 0	117
	61 42 66 5	1 3 0 4 1 8 7 5 2	—12
	63 6	4 8	11
	61 43 30	1 3 0 4 1 8 8 0 4	1

For the positive root we have a similar operation,

$$1 \; +1 \; +1 \; +3 \; -100 \; (2{\cdot}8028512181582;$$

but this we shall leave for the student to perform, and the two roots will be found to be

$$x = 2{\cdot}8028512181582\ldots$$
$$x = -3{\cdot}4335778633659\ldots$$

(4) Find the roots of the equation $x^5+2x^4+3x^3+4x^2+5x-20=0$

Here we have $V = x^5 + 2x^4 + 3x^3 + 4x^2 + 5x - 20$

$V_1 = 5x^4 + 8x^3 + 9x^2 + 8x + 5$

$V_2 = -7x^3 - 21x^2 - 42x + 255$

$V_3 = -13x + 14$

$V_4 = -.$

For $x = -\infty$ we have signs $- + + + -$ two variations;

$x = +\infty \ldots\ldots\ldots\; + + - - -$ one variation.

Hence the difference of variations of sign indicates the existence of one real and four imaginary roots, the real root being situated between 1 and 2.

1+2	+ 3	+ 4	+ 5	−20 (1·125790..
1	3	6	10	15
3	6	10	15	−5....
1	4	10	20	387171
4	10	20	35....	−112829
1	5	15	37171	87005
5	15	35...	387171	−25824
1	6	2171	39414	22285
6	21..	37171	42658|5	−3539
1	71	2243	844|	3136
71	2171	39414	43502|5	−403
1	72	2316	853|4	403
72	2243	417|30	4435|6	
1	73	4|7	21|5	
73	2316	422|0	4457|1	
1	74	4|7	21|5	
74	|..2|390	426|7	447|8	
1		4|7	|2	
|..75		|.4|31	44|8	

Hence the real root is nearly 1·125790; and by using another period of ciphers we should have the root correct to ten places of decimals, with very little additional labor.

ADDITIONAL EXAMPLES FOR PRACTICE.

(1) Find all the roots of the equation $x^3-3x-1=0$.
(2) Find all the roots of the equation $x^3-22x-24=0$.
(3) Find the roots of the equation $x^3+x^2-500=0$.
(4) Find the roots of the equation $x^3+x^2+x-100=0$.
(5) Find the roots of the equation $2x^3+3x^2-4x-10=0$.
(6) Find the roots of the equation $x^4-12x^2+12x-3=0$.
(7) Find the roots of the equation $x^4-8x^3+14x^2+4x-8=0$.
(8) Find the roots of the equation $x^4-x^3+2x^2+x-4=0$.
(9) Find the roots of the equation $x^5-10x^3+6x+1=0$.
(10) Find the roots of the equation $x^5+3x^4+2x^3-3x^2-2x-2=0$
(11) Find all the roots of the equation

$$x^6+4x^5-3x^4-16x^3+11x^2+12x-9=0.$$

ANSWERS.

(1) $x=+1·879385242$, $x=-1·532088886$, $x=-·347296355$

(2) $x=+5·1622776601$, $x=-1·1622776601$, $x=-4$

(3) $x=7·61727975596$

(4) $x=4·2644299731$

(5) $x=1·6248190836$

(6) $x=+2·858083308163$, $x=+·606018306917$, $x=+·443276939605$, $x=-3·907378554685$

(7) $x=+5·2360679775$, $x=+·7639320225$, $x=+2·7320508075$, $x=-·7320508075$

(8) $x=+1·146994592$, $x=-1·090593586$

$$(9)\ \begin{cases} x=-3\cdot0653157912983 \\ x=-\ \cdot6915762804900 \\ x=-\ \cdot1756747992883 \\ x=+\ \cdot8795087084144 \\ x=+3\cdot0530581626622 \end{cases}$$

$$(10)\quad x=1\cdot059109003461882$$

$$(11)\ \begin{cases} x=-1;\ x=-3;\ x=1 \\ \qquad x=-3;\ x=1 \\ \qquad\qquad x=1 \end{cases}$$

257. The theorem of Sturm gives a simple means of establishing the conditions of the reality of the roots. As the real roots are comprised between two limits, $-L'$ and $+L$, the one negative and the other positive, which may be chosen as large as we please, the question reduces to seeking the conditions necessary, in order that from $x=-L'$ to $x=+L$ the series V, V_1, V_2, &c., should lose a number of variations equal to the degree of the equation.

Supposing this degree to be m, it must then lose m variations. But in order that it may have m variations, it is necessary that it should have at least $m+1$ terms; and as it can not have more, we are sure that the quantities V, V_1, V_2, &c., exist to the number $m+1$, and that they are respectively of the degree m, $m-1$, $m-2$, &c. The last, which does not contain x, will then be represented by V_m.

When in the polynomial functions of x we substitute very large numbers, positive or negative, for x, we know that the results are of the same sign as if each polynomial were reduced to its first term; therefore, in the present investigation, we need occupy ourselves only with the first term. Let us take the equation $V=0$ under the ordinary form

$$x^m+px^{m-1}+qx^{m-2}+, \text{\&c.}, =0.$$

The first term of V is x^m; that of the derived polynomial, V_1, will be mx^{m-1}. With regard to those of the polynomials V_2, V_3, &c., they are functions composed of the coefficients p, q, &c., determined by the successive divisions in accordance with the rule. Let us represent these functions by G_2, $G_3 \ldots G_m$ and write in order the $m+1$ quantities,

$$x^m,\ mx^{m-1},\ G_2x^{m-2},\ G_3x^{m-3} \ldots G_m.$$

The question will be reduced to finding the conditions which will cause the loss of m variations from this series when we pass from $x=-L'$ to $x=+L$. In order that this may be the case, it must have m variations upon the substitution of $-L'$, and m permanences upon the substitution of $+L$. But in this series the powers of x go on decreasing by unity; consequently, if it has nothing but permanences when $x=+L$, it will have nothing but variations when $x=-L'$. Thus, the conditions are reduced simply to such as require this series to have only positive coefficients, that is to say, to the following,

$$G_2>0,\ G_3>0 \ldots\ldots G_m>0.$$

These conditions will never be greater in number than $m-1$, but they may be less in number, inasmuch as some of the above inequalities may involve the others.

EXAMPLE.

258. Find the conditions necessary for the reality of the roots of the equation $x^3+qx+r=0$.

Here we have $m=3$, and the conditions are only two in number, $G_2>0$ and $G_3>0$.

To find G_2 and G_3, we calculate V_2 and V_3 by successive divisions, as follows:

First Division.

$$\begin{array}{r|l} x^3+\ qx+\ r & 3x^2+q \\ \hline 3x^3+3qx+3r & x \\ 3x^3+\ qx & \\ \hline 2qx+3r & \end{array}$$

$$\therefore V_2=-2qx-3r.$$

Second Division.

$$\begin{array}{r|l} 3x^2+\quad q & -\ 2qx-\ 3r \\ \hline 12q^2x^2+\ 4q^3 & -\ 6qx+\ 9r \\ 12q^2x^2+18qrx & \\ \hline -18qrx+\ 4q^3 & \\ -18qrx\ -27r^2 & \\ \hline 4q^3+27r^2 & \end{array}$$

$$\therefore V_2=-\ 4q^3-27r^2.$$

Consequently, the inequalities $G_2>0$, $G_3>0$, become

$$-2q>0,\quad -4q^3-27r^2>0;$$

observing, however, that the first inequality is embraced in the second, since r^2 is always positive; and changing the signs of the second, we have for the sole condition of the roots of an equation of the third degree, being real,

$$4q^3+27r^2<0.$$

We have now given so much of the general properties of equations of all degrees, and such modes of proceeding, as will insure their numerical solution in a manner the most certain and infallible, and ordinarily the best.

There are, however, many transformations of equations, which, by reducing their degree, or by giving them a particular form, serve to facilitate their solution in certain cases. There are also many general principles applicable to the resolution of equations of the higher orders by the methods in use previous to the discovery of Sturm, which, with these methods themselves, it is desirable to know for many purposes in the application of algebraic analysis to the higher branches of both pure and mixed mathematics, for ulterior improvements in the general theory of equations itself, and even for use in the solution of equations, in some cases, to which they are more conveniently adapted than the method of Sturm. A treatise on algebra could scarcely be regarded as complete without some notice of these. We shall therefore give as extensive an exhibition of them as can in any way be useful in an elementary work like the present, commencing with the well known

RULE OF DES CARTES.

259. *An equation can not have a greater number of positive roots than there are variations of sign in the successive terms from + to —, or from — to +, nor can it have a greater number of negative roots than there are permanences, or successive repetitions of the same sign in the successive terms.*

Let an equation have the following signs in the successive terms, viz. :

$$+-+---+++-,\ \text{or}\ +----+-+++.$$

Now, if we introduce another positive root, we must multiply the equation by $x-a$, and the signs in the partial and final products will be

$$\begin{array}{l} +-+---+++- \\ \quad -+-+++---+ \\ \hline +-+-\pm\pm+\pm\pm-+ \end{array} \qquad \begin{array}{l} +----+-+++ \\ \quad -++++-+--- \\ \hline +-\pm\pm+-+\pm\pm- \end{array}$$

where the ambiguous sign $\pm$ indicates that the sign may be + or — according to the relative magnitudes of the terms with contrary signs in the partial products, and where it will be observed the permanences in the proposed equation are changed into signs of ambiguity; hence the permanences, take the ambiguous sign as you will, are not increased in the final product by the introduction of the positive root $+a$; but the number of signs is increased by *one*, and, therefore, the number of variations must be increased by *one*. Hence it is obvious that the introduction of every positive root also introduces one additional variation of sign, and, therefore, the whole number of positive roots can not exceed the number of variations of signs in the successive terms of the proposed equation.

Again, by changing the signs of the alternate terms, the roots will be changed from positive to negative, and *vice versa* (see Prop. VII.). Moreover, by this change the permanences in the proposed equation will be replaced by variations in the changed equation, and the variations in the former by permanences in the latter; and since the changed equation can not have a greater number of positive roots than there are variations of sign, the proposed equation can not have a greater number of negative roots than there are permanences of sign.

Let v be the number of variations, v' the number of variations of the transformed equation obtained by changing x into $-x$. The number of real roots of the equation can not surpass $v+v'$. Then, if this sum is less than the degree m, the equation will have imaginary roots.

The sum $v+v'$ is never greater than the degree, and when it is less the difference is an even number. (See Art. 248.)

EXAMPLES.

(1) The equation $x^6+3x^5-41x^4-87x^3+400x^2+444x-720=0$ has six real roots. How many are positive?

(2) The equation $x^4-3x^3-15x^2+49x-12=0$ has four real roots. How many of these are negative?

260. We give next the repetition of a principle already presented, but which may be derived as a direct consequence of the theorem of Sturm.

THEOREM OF ROLLE.

Let $F(x)=0$ be an equation which has no equal roots, $F'(x)$ its derived polynomial. We have seen that as x increases, the series of Sturm loses a variation every time that x passes over a root of the equation $F(x)=0$, and that it can not lose one in any other way. Moreover, we have seen that this variation is lost at the commencement of the series of functions, in consequence of $F(x)$ changing sign, while $F'(x)$ does not; so that $F(x)$ is always of a sign contrary to that of $F'(x)$ for a value of x a little less than the root, and always of the same sign for a value a little greater.

Thus, when we ascend from a root r to a root r', which is immediately above r, $F(x)$ must be of the same sign as $F'(x)$ for a value of x a little greater than r, and of a sign contrary to $F(x)$ for a value of x a little less than r'. But in the interval $F(x)$ does not change sign; then $F'(x)$ must change sign at least once; therefore the equation $F'(x)=0$ has at least one root between r and r'.

Let $a, b, c, d \ldots g$ be the real roots of $F(x)=0$, arranged in order of magnitude, beginning with the largest; and let $a_1, b_1, c_1 \ldots g_1$ be the real roots of $F'(x)=0$, disposed in the same manner. We have just seen that these last are comprised, some between a and b, some between b and c, &c.; but as the degree of $F'(x)$, and, consequently, the number of its roots, is one less than the degree and number of roots of $F(x)=0$, it follows that the equation $F(x)=0$ can have but one root above a_1, but one between a_1 and $b_1 \ldots$, and, finally, but one below g_1. This property, which has been long known, and of which we have given an independent demonstration at (Art. 253), is identical with the theorem of Rolle.

261. The considerations which lead to the theorem of Rolle furnish also the means of determining whether the m roots of the equation $F(x)=0$ are real and unequal.

Since a_1 is between a and b, b_1 between b and c, &c., it is easy to see (Art. 252) that if we substitute successively a_1, b_1, &c., in place of x in $F(x)$, the results will be alternately negative and positive ; so that

For . . . . $F(a_1)$, $F(b_1)$, $F(c_1)$, &c.,
we have . . . . $-$, $+$, $-$, &c.

But we may apply to the function $F'(x)$ and its derived function $F''(x)$ all that has been said in the preceding article of $F(x)$ and $F'(x)$; then,

For . . . . $F''(a_1)$, $F''(b_1)$, $F''(c_1)$, &c.,
we have. . $+$, $-$, $+$, &c.

Then the products $F(a_1)\times F''(a_1)$, $F(b_1)\times F''(b_1)$, &c., of which there are $m-1$, will be all negative.

But if we make $F(x)\times F''(x)=y$, and eliminate (as at p. 157) x between the two equations,

$$F'(x)=0,\ F(x)\times F''(x)=y \ . \ . \ . \ . \ . \ . \ (2)$$

the $m-1$ roots of the final equation in y will be precisely the products above; but since all these products are negative, the equation in y will have only negative roots, and, consequently, all its terms will have the sign $+$. Thus, when the equation $F(x)=0$ has none but real and unequal roots, the theorem of Rolle shows that the roots of $F'(x)=0$ must be real and unequal also; and from what has just been said above, it appears that besides this, the signs are all plus in the equation in y, resulting from the elimination of x between the equations (2).

262. Conversely, these conditions being fulfilled, we can demonstrate that all the roots of $F(x)=0$ will be real and unequal. And first, the $m-1$ roots of $F'(x)=0$ being real, from what has just been said, those of $F''(x)=0$ must be real, and the $m-1$ values of y, or $F(x)\times F''(x)$ real also; and the roots of $F'(x)=0$ being by hypothesis unequal, the theorem of Rolle proves that the quantities $F''(a_1)$, $F''(b_1)$, &c., have their signs alternately $+$ and $-$. Again, since the equation in y has its signs all $+$, we conclude that it has no positive roots; and since all its roots are real, they can only be negative; then the $m-1$ products

$$F(a_1)\times F''(a_1),\ F(b_1)\times F''(b_1),\ \&c.,$$

are negative. But the second factors have their signs alternately $+$ and $-$; then the quantities $F(a_1)$, $F(b_1)$, &c., must have their signs alternately $-$ and $+$. Then there exists above a_1 a root of the equation $F(x)=0$, another between a_1 and b_1, another between b_1 and c_1, &c., therefore the m roots of this equation are real and unequal.

The conditions drawn from the equation in y may be regarded as actually known, because this equation is obtained by simple elimination. As to the other condition which requires that the roots of $F'(x)=0$ be real, let it be observed that this equation is of the degree $m-1$, and, applying to it the same reasoning as to $F(x)=0$, we reduce the question to determining the reality of the roots of $F''(x)=0$, which is only of the degree $m-2$. Continuing thus, we descend to an equation of the second degree, the derived function of which being of the first degree, can not have an imaginary root. Then the only condition to fulfill will be that the equation y, which is also of the first degree, have its two terms of the same sign.

REMARK.—By recurring to the reasoning which led to the use of the equation $y=F(x)\times F''(x)$, it is easily perceived that this may be replaced by $M\times F(x)\times F''(x)$, M being any positive quantity whatever. We can then introduce or suppress in the polynomials $F(x)$, $F'(x)$, $F''(x)$, &c., such positive factors as may be judged suitable to simplify the calculation.

263. The equation in y, resulting from the elimination of x in the equations (2), being of the degree $m-1$, will have $m-1$ coefficients, thus presenting $m-1$ conditions to be fulfilled; the second equation in y, obtained by eliminating x between the two, $F''(x)=0$, $y=F'(x)\times F'''(x)$, will be of the degree $m-2$, and present $m-2$ conditions to be fulfilled, and so on, till we arrive at an equation of the first degree in y, which will give but a single condition; then, taking all the conditions in an inverse order, their number will be expressed (Art. 228) by

$$1+2+3\ldots+m-1=\frac{m(m-1)}{2}.$$

264. For an application of the above, let us take the general equation of the second degree,

$$x^2+px+q=0.$$

Here we have $F(x)=x^2+px+q$, $F'(x)=2x+p$, $F''(x)=2$, and we perceive at once that $F'(x)$ has no imaginary root, since it is of the first degree.

In order to have the equation in y, the two equations between which we must eliminate x are

$$2x+p=0,\ y=(x^2+px+q)\times 2.$$

The elimination gives

$$y+2\left(\frac{1}{4}p^2-q\right)=0.$$

Then, in order that the terms of this equation may have the same sign, we must have $\frac{1}{4}p^2-q>0$; and this is the only condition necessary to insure the reality of the roots of the equation of the second degree. It accords with what we have seen at (Art. 191).

265. Let us consider next the general equation of the third degree. The second term, it will be seen hereafter, may be made to disappear without changing the number of the real roots; we may therefore take it under the form

$$x^3+qx+r=0.$$

In this case $F(x)=x^3+qx+r$, $F'(x)=3x^2+q$, $F''(x)=6x$. It is necessary, first, that the derived equation, $3x^2+q=0$, should have only real and unequal roots; and for this the condition is evidently $q<0$.

Secondly, it is necessary to eliminate x between the two equations

$$3x^2+q=0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$y=(x^3+qx+r)\times 6x,$$

or

$$y=6x^4+6qx^2+6rx \quad . \quad . \quad . \quad . \quad (2)$$

The first gives

$$x^2=-\frac{1}{3}q \ \therefore \ x^4=\frac{1}{9}q^2,$$

and (2) becomes

$$y=-\frac{4}{3}q^2+6rx$$

$$\therefore x=\frac{3y+4q^2}{18r}.$$

Substituting this in (1), we have, after reducing,

$$y^2+\frac{8}{3}q^2y+\frac{4}{9}q(4q^3+27r^2)=0.$$

In order that the three terms of this equation may have the same sign, it is necessary, and it is sufficient, that the known term should be positive. We have already seen that q must be negative, but q^2 in the second term is positive; then the new condition is $4q^3+27r^2<0$. Finally, as this new condition can be fulfilled only when q is negative, it is the only one necessary, in order that the roots of the equation of the third degree should be real and unequal.

FOURIER'S METHOD OF SEPARATING THE ROOTS.

266. We shall now give another method of separating the roots, proposed by Fourier, which has the recommendation that the auxiliary functions employed in it are $f(x)$ and its successive derived functions, which can be formed by inspection;* so that the method can be applied nearly with equal ease to an equation of any degree; in particular, the intervals in which no real root can be situated are, by Fourier's method, immediately assigned. The objection to this method is, that by its immediate application we only find a limit which the number of real roots in a given interval can not exceed, and not the absolute number; and that the subsidiary propositions by which this defect is supplied are not of the same simple character as the original theorem. The enunciation and proof are as follows.

THEOREM.

The number of real roots of f(x)=0 *which lie between two numbers* a *and* b, *can not exceed the difference between the number of variations of signs in the results of the substitutions of* a *and* b *for* x, *in the series formed by* f(x) *and its derived functions:* viz., $f(x), f'(x), f''(x), \ldots f^n(x)$.

If none of the equations

$$f(x)=0, f'(x)=0, \text{\&c.},$$

have a root between a and b, it is manifest that the substitution of a and b, and of any intermediate quantity, in $f(x)$, $f'(x)$, &c., will always produce exactly

* The method of Sturm employs only the given and first derived function $f(x)$ and $f'(x)$, which are the same as V and V_1, the other functions in his method, viz., V_2, V_3, &c., being obtained by the method of the common divisor, which, in practice, is tedious for functions of the higher degrees, especially if they have large coefficients. For methods of simplifying these laborious operations, see Young's Theory and Solution of the higher Equations

the same series of signs; but if any of these equations have roots between a and b, then changes in the series of signs will occur in substituting gradually ascending quantities from a to b; our object is to show that by such substitutions the number of variations of signs can never increase, and that one variation will be lost every time the substituted quantity passes through a real root $f(x)=0$; this we shall do by examining separately each of the cases in which the series of signs can be affected; namely, 1, when $f(x)$ alone vanishes; 2, when some derived function, $f^m(x)$, alone vanishes; 3 and 4, when some group of derived functions, of which $f(x)$ either is not or is a part, alone vanishes; and lastly, when several or all of these cases of vanishing happen at the same time.

First, suppose that $x=c$ (c being some quantity between a and b) makes $f(x)$ vanish, without making any of the derived functions vanish; then the result of substituting $c+h$ for x in $f(x)$ and $f'(x)$ is (supposing h so small that the signs of the whole of the two series which express $f(c+h)$ and $f'(c+h)$ depend upon those of their first terms, and writing down only the first terms)

$$h \cdot f'(c) \text{ and } f'(c),$$

which have different or the same signs according as h is $-$ or $+$; therefore, in passing from $c-h$ to $c+h$ through a root of the equation, a variation of signs is lost, but none gained.*

Secondly, suppose that $x=c$ makes one of the derived functions, $f^m(x)$, vanish, without making any other of the derived functions, or $f(x)$, vanish; then the result of substituting $c+h$ for x in the three consecutive functions

$$f^{m-1}(x),\ f^m(x),\ f^{m+1}(x),$$

(these being the only terms which it is necessary to examine)* is

$$f^{m-1}(c),\ h \cdot f^{m+1}(c),\ f^{m+1}(c).$$

If, then, the first and third terms have the same sign, there will be two variations when h is negative, and two permanences when h is positive; if the extreme terms have contrary signs, there will be one variation, and one only, whether h be negative or positive; therefore, in passing from $c-h$ to $c+h$ through a value which makes one of the derived functions vanish, either two variations or none will be lost, but none ever gained.

Thirdly, suppose that $x=c$ makes r consecutive derived functions vanish, without making any other derived function, or $f(x)$, vanish; then the result of the substitution of $c+h$ for x in the series

$$f^{m-r}(x),\ f^{m-r+1}(x),\ \ldots f^{m-1}(x),\ f^m(x),\ f^{m+1}(x),$$

(these being the only terms necessary to be examined) is

$$f^{m-r}(c),\ \frac{h^r}{\lfloor r}f^{m+1}(c),\ \ldots,\ \frac{h^2}{\lfloor 2}f^{m+1}(c),\ \frac{h}{1}f^{m+1}(c),\ f^{m+1}(c),$$

where $\lfloor r$ denotes $1 \cdot 2 \cdot 3 \ldots . r$.

If, then, the extremes of this series have the same sign, there will be r or $r+1$ changes (according as r is even or odd) when h is negative, and no change when h is positive; if the extreme terms have contrary signs, there

* It is unnecessary to attend to the other functions of the series of derived functions, because h is supposed so small that not one of them vanishes by the substitution of any quantity between $c-h$ and $c+h$, and therefore each has the same sign for $c-h$ as for $c+h$.

will be r or $r+1$ variations (according as r is odd or even) when h is negative, and one change when h is positive; therefore, in passing from $c-h$ to $c+h$ through a value which makes r consecutive derived functions vanish, r or $r\pm1$ changes are lost (according as r is even or odd) but none ever gained.

Fourthly, suppose the vanishing group to consist of $f(x)$ and the first $r-1$ derived functions (which corresponds to r roots $=c$ in $f(x)=0$);* then the result of the substitution of $c+h$ for x in $f(x), f'(x), \ldots f^{r-1}(x), f^{r}(x)$, is

$$\frac{h^{r}}{\underline{|r}}f^{r}(c),\ \frac{h^{r-1}}{\underline{|r-1}}f^{r}(c), \ldots \frac{h}{1}f^{r}(c), f^{r}(c),$$

in which there are r variations when h is negative, and none when h is positive; therefore, in passing through a root which occurs r times in the equation, r changes are lost, but none gained.

Lastly, suppose the substitution of $x=c$ to produce several, or all of the above cases at the same time; then, because the conclusions respecting the effect of the passage through c upon the series of signs in one part of the series of derived functions are not at all influenced by what happens, in consequence of the same passage, at another distinct part of the series, by what has been proved, several variations will be lost, but none ever gained.

Since then, in substituting gradually ascending values from a to b, variations of signs are generally lost for every passage through a quantity which makes one or more of the derived functions vanish, and invariably one for every passage through a root of $f(x)=0$, but none under any circumstances gained, it follows that the number of roots of $f(x)=0$, which lie between a and b, can not be greater than the excess of the number of variations given by $x=a$, above that given by $x=b$.

267. Hence, if the limits, a and b, be $-\infty$ and $+\infty$, or any two numbers the first of which gives only variations, and the second only permanences; and if, in the series formed by $f(x)$ and its derived functions,

$$f(x),\ f'(x),\ f''(x),\ \ldots\ f^{n}(x),$$

c be substituted for x and be then made to assume all values between these limits, the series of signs of the results will have the following properties; there will at first be n variations of sign, and at last no variation, but n permanences; these variations disappear gradually as c increases, and when once lost, can never be recovered; one variation disappears every time c passes through a real unequal root of $f(x)=0$; r variations disappear every time c passes through a root which occurs r times in $f(x)=0$; either two or none of the variations disappear every time one only of the derived functions vanishes, without $f(x)$ vanishing at the same time; an even number p of variations disappears every time an even group of p functions (not including the first $f(x)$) vanishes; and an even number $q\pm1$ of variations disappears every time an odd group of q functions (not including the first $f(x)$) vanishes. Also, if a value causes $f(x)$ and the first $r-1$ derived functions to vanish, and an even group of p functions in one part of the series, and an odd group of q functions in another part, to vanish at the same time, the number of variations lost in passing through that value will be $r+p+q\pm1$.

268. Hence, if $f(x)=0$ have all its roots real, no value of x can make any of the derived functions vanish, and thereby exterminate variations of signs,

* See (Art. 253, Schol).

without at the same time making $f(x)$ vanish; for if it could, since those variations can never be restored, and since a variation must disappear for every passage through a real root, the total number of variations lost would surpass n, the degree of the equation, which is absurd, since there are but n derived functions in all. Whenever, therefore, variations disappear between values of x which do not include a root of $f(x)=0$, there is, corresponding to that occurrence, an equal number of imaginary roots of $f(x)=0$. Hence, if $x=c$ produces a zero between two similar signs, or if it produces an even number p of consecutive zeros either between similar or contrary signs, there will be respectively two, or p, imaginary roots corresponding; or if it produces an odd number q of consecutive zeros, there will be $q\pm1$ imaginary roots corresponding, according as they stand between similar or contrary signs; c, of course, not being a root of $f(x)=0$.

Observation.—Since the derivatives which follow any one $f^{r}(x)$ may be supposed to arise originally from it, it is manifest that the same conclusions respecting the roots of $f^{r}(x)=0$ may be drawn from observing the part of the series of derived functions

$$f^{r}(x), f^{r+1}(x), \dots f^{n}(x)$$

as were drawn respecting the root of $f(x)=0$ from the whole series.

269. Des Cartes's rule of signs is included in Fourier's theorem as a particular case.

For when, in the series formed by $f(x)$ and its derived functions, we put $x=-\infty$, there are n variations; and when we put $x=0$, the signs of the series of functions become the same as those of the coefficients of the proposed equation

$$p_n, p_{n-1}, \dots p_1, 1.$$

Let the number of variations in this series of coefficients $=k$, and therefore the number of permanences (supposing the equation complete) $=n-k$: if we make $x=+\infty$, the signs of the functions are all positive, and the number of variations $=0$. Hence, between $x=-\infty$ and $x=0$, the number of variations lost is $n-k$; therefore in a complete equation there can not be more than $n-k$ negative roots, *i. e.*, than the number of permanences in the series of coefficients; also, between $x=0$ and $x=\infty$, the number of variations lost is k, whether the equation be complete or incomplete; hence in any equation there can not be more positive roots than k, *i. e.*, than the number of variations in the series of coefficients, which is Des Cartes's rule of signs.

270. Fourier's theorem may also be presented under the following form:

If an equation have m real roots between a and b, then the equation whose roots are those of the proposed, each diminished by a, has at least m more variations of signs than the equation whose roots are those of the proposed, each diminished by b.

The transformed equations would be

$$f(y+a)=0, f(y+b)=0;$$

and if these were arranged according to ascending powers of y, the coefficients would be the values assumed by $f(x), f'(x)$, &c., when a and b are respectively written for x. Therefore, whatever number of variations of signs is lost in the series $f(x), f'(x)$, &c., in passing from a to b, the same is lost in passing from one transformed equation to the other; but the series for a has at least m

more variations than that for b; therefore $f(y+a)=0$ has at least m more variations than $f(y+b)=0$.

271. To apply this method to find the intervals in which the roots of $f(x)=0$ are to be sought, we must substitute successively for x, in the series formed by $f(x)$ and its derived functions, the numbers

$$-a, \ldots -10, \ -1, \ 0, \ 1, \ 10, \ldots, \ +\beta \ (1),$$

(a and $+\beta$ being the least negative and least positive number, which give respectively only variations and permanences), and observe the number of variations of sign in each result.

Let h and k be the numbers of variations of sign when any two consecutive terms in series (1), a and b, are respectively written for x; therefore $h-k$ is the number of real roots that may lie between a and b: if this equals zero, $f(x)=0$ has no real root between a and b, and the interval is excluded; if $h-k=1$, or any odd number, there is at least one real root between a and b; if $h-k=2$, or any even number, there may be two, or some even number, or none; the latter case will happen when, as explained above (Art. 268), some number between a and b makes two or some even number of variations vanish, without satisfying $f(x)=0$. Similarly, we must examine all the other partial intervals; and when two or more roots are indicated as lying in any interval, their nature must be determined by a succeeding proposition.

The two former of the following examples are extracted from Fourier's work.

EXAMPLE I.

$$f\ (x)= \quad x^5-\ 3x^4-\ 24x^3+\ 95x^2-46x-101=0$$
$$f'\ (x)= \ 5x^4-12x^3-\ 72x^2+190x\ -46$$
$$f''\ (x)=\ 20x^3-36x^2-144x\ +190$$
$$f'''(x)=\ 60x^2-72x\ -144$$
$$f^4\ (x)=120x\ -72$$
$$f^5\ (x)=120.$$

Hence we have the following series of signs resulting from the substitutions of -10, -1, 0, &c., for x, in the series of quantities

	f	f'	f''	f'''	f^4	f^5
(-10)	$-$	$+$	$-$	$+$	$-$	$+$
(-1)	$+$	$-$	$+$	$-$	$-$	$+$
(0)	$-$	$-$	$+$	$-$	$-$	$+$
(1)	$-$	$+$	$+$	$-$	$+$	$+$
(10)	$+$	$+$	$+$	$+$	$+$	$+$

Hence all the roots lie between -10 and $+10$, because five variations have disappeared; one root lies in each of the intervals -10 to -1, and -1 to 0, because in each of them a single variation is lost; no root lies between 0 and 1, because no variation is lost between those limits; and three roots may be sought between 1 and 10 (because three variations have disappeared), one of which is certainly real; it is doubtful whether the other two are real or imaginary.

Observation.—When any value c of x makes one of the derived functions, $f^m(x)$, vanish, we may substitute $c\pm h$ instead of c, h being indefinitely small; then all the other functions will have the same sign as when $x=c$, and the sign of $f^m(c\pm h)$ will depend upon that of $\pm hf^{m+1}(c)$; *i. e.*, it will be the same or contrary to that of the following derivative, $f^{m+1}(c)$, according as h is positive or negative, or according as we substitute a quantity a little less or a little greater than the value which makes $f^{m}(x)$ vanish. The use of this remark will be seen in the following example.

EXAMPLE II.

$$f\ (x)=\ x^4-\ 4x^3-3x+23=0$$
$$f'\ (x)=\ 4x^3-12x^2-3$$
$$f''\ (x)=12x^2-24x$$
$$f'''(x)=24x\ -24$$
$$f^4\ (x)=24.$$

	f	f'	f''	f'''	f^4
$x=0$	$+$	$-$	0	$-$	$+$
$x=0\mp h,$	$+$	$-$	$\pm$	$-$	$+$
$x=1$	$+$	$-$	$-$	0	$+$
$x=1\mp h,$	$+$	$-$	$-$	$\mp$	$+$
$x=10$	$+$	$+$	$+$	$+$	$+$

Every value less than 0 gives results alternately $+$ and $-$, therefore there is no real negative root; for $x=0$, we have a result zero placed between two similar signs, and therefore corresponding to it there is a pair of imaginary roots. There is no root between 0 and 1, but there may be two roots between 1 and 10.

EXAMPLE III.

$$f(x)=x^6-6x^5+40x^3+60x^2-x-1=0.$$

Here there is no root <-1; there is one, and there may be three, between -1 and 0; there is one root between 0 and 1, and there may be two roots between 2 and 3.

272. The above process will determine the intervals in which the roots are to be sought, but not always their nature; when an even number of roots is indicated, they may all turn out to be impossible. The series of magnitudes between $-\infty$ and $+\infty$, to be substituted for x in the derived functions, has been divided into intervals of two sorts, each contained by assigned limits, a and b. The first sort of interval is one within which no root is comprehended, *i. e.*, the limits of which give the same number of variations of signs in the series of derived functions. The second sort is one within which roots may lie, *i. e.*, where the number of variations resulting from the substitution of b is less than the number resulting from the substitution of a, in the series of derived functions. This second sort of interval has two subdivisions, viz., cases where the indicated roots do really exist, and others where they are imaginary. When we have ascertained that a certain number of roots may lie between a and b, we may substitute c (a quantity between a and b) in the series of derived functions, and if any variations disappear, our interval is broken into two others; if no variations disappear, we may increase or diminish c, and make a second substitution, and it may still happen that no variation is lost, and so on continually; and we may be left, after all, in a state of uncertainty, whether the separation of the roots is impossible because they are imaginary, or only retarded because their difference is extremely small. This uncertainty is relieved by taking the interval so small as to be sure to include the real roots, if they exist.

One method of arriving at the proper interval is by means of the so-called equation of the squares of the differences of the roots of the given equation, which we shall hereafter have occasion to deduce. This process is tedious in practice; and as our object in unfolding the method of Fourier was to pursue it only so far as it threw light upon the general theory of equations, we shall here leave it.

We should now introduce the theorem of Budan, but it requires a transformation which we have not yet exhibited, and we therefore take this opportunity to complete a subject, one proposition of which (Art. 251) we have already had occasion to anticipate.

TRANSFORMATION OF EQUATIONS.

PROPOSITION I.

273. *To transform an equation into another whose second term shall be removed.*

Let the proposed equation be

$$x^n+A_1x^{n-1}+A_2x^{n-2}+\ldots\ldots A_{n-1}x+A_n=0;$$

and by Art. 245 we know that the sum of the roots of this equation is $-A_1$; therefore, the sum of all the roots must be increased by A_1 in order that the transformed equation may want its second term; but there are n roots, and hence each root must be increased by $\frac{A_1}{n}$, and then the changed equation will have its second term absent. If the sign of the second term of the proposed equation be negative, then the sum of all the roots is $+A_1$; and in this case we must evidently diminish each root by $\frac{A_1}{n}$, and the changed equation will then have its second term removed. Hence this

RULE.

Find the quotient of the coefficient of the second term of the equation divided by the highest power of the unknown quantity, and decrease or increase the roots of the equation by this quotient, according as the sign of the second term is negative or positive.

EXAMPLES.

(1) Transform the equation $x^3-6x^2+8x-2=0$ into another whose second term shall be absent.

Here $A_1=-6$, and $n=3$; $\therefore$ we must diminish each root by $\frac{6}{3}$ or 2

$$\begin{array}{rrrl} 1 & -6 & +8 & -2\ (2 \\ & 2 & -8 & 0 \\ \hline & -4 & 0 & -2 \\ & 2 & -4 & \\ \hline & -2 & -4 & \\ & 2 & & \\ \hline & 0 & & \end{array}$$

$\therefore$ $y^3-4y-2=0$ is the changed equation.

And since the roots are diminished, we must have the relation $x=y+2$.

(2) Transform the equation $x^4-16x^3-6x+15=0$ into another whose second term shall be removed.

(3) Transform the equation $x^5+15x^4+12x^3-20x^2+14x-25=0$ into another whose second term shall be absent.

(4) Change the equation $x^2+ax+b=0$ into another deficient of the second term.

(5) Change the equation $x^3+ax^2+bx+c=0$ into another wanting the second term.

ANSWERS.

(2) $y^4-96y^2-518y-777=0$.

(3) $y^5-78y^3+412y^2-757y+401=0$.

(4) $z^2-\frac{a^2}{4}+b=0$.

(5) $z^3-\left(\frac{a^2}{3}-b\right)z+\frac{2a^3}{27}-\frac{ab}{3}+c=0$.

PROPOSITION II.

274. *To transform an equation into another whose roots shall be the reciprocals of the roots of the proposed equation.*

Let $ax^n+A_1x^{n-1}+A_2x^{n-2}+\ldots\ldots A_{n-1}x+A_n=0$ be the proposed equation, and put $y=\frac{1}{x}$; then $x=\frac{1}{y}$, and by writing $\frac{1}{y}$ for x in the proposed equation, multiplying by y^n, and reversing the order of the terms, we have the equation

$$A_ny^n+A_{n-1}y^{n-1}+A_{n-2}y^{n-2}+\ldots\ldots A_2y^2+A_1y+a=0,$$

whose roots are the reciprocals of the roots of the proposed equation.

The transformation is then effected by simply changing the order of the coefficients of the given equation.

Corollary 1.—Hence an equation may be transformed into another whose roots shall be greater or less than the *reciprocals* of the roots of the proposed equation, simply by reversing the order of the coefficients, and then proceeding as in the Proposition to Art. 251.

Corollary 2.—If the coefficients of the proposed equation be the same, whether taken in reverse or direct order, then it is evident that the transformed equation will be the same as the original one; and, therefore, the roots of such equations must be of the form

$$r_1, \frac{1}{r_1};\ r_2, \frac{1}{r_2};\ r_3, \frac{1}{r_3};\ r_4, \frac{1}{r_4}, \text{\&c.}$$

Corollary 3.—If the coefficients of an equation of an odd degree be the same whether taken in direct or inverse order, but have contrary signs, then, also, the roots of the transformed equation will be the same as the roots of the proposed equation; for, changing the signs of all the terms, the original and transformed equations will be identical, and the roots remain unchanged when the signs of *all* the terms are changed. And this will likewise be the case in an equation of an even degree, provided only the middle term be absent, in order that the transformed equation, with all its signs changed, may be identical with the original equation.

Equations whose coefficients are the same when taken either in direct or reverse order, are, therefore, called *recurring equations*, or, from the form of the roots, *reciprocal equations*.

Corollary 4.—If the sign of the last term of a recurring equation of an odd degree be $+$, one of the roots of such equation will be -1; and if the sign of the last term be —, one root will be $+1$. For the proposed equation and the reciprocal have one root, the same in each, and 1 is the only quantity whose reciprocal is the same quantity; hence, since each of the other roots has the same sign as its reciprocal, the product of each root and its reciprocal must be positive; and, therefore, the last term of the equation, being the product of all the roots with their signs changed, must have a contrary sign to that of the root unity.

Hence a recurring equation of an odd degree may always be depressed to an equation of the next lower degree by dividing it by $x+1$, or $x-1$, according as the sign of the last term is + or —.

Corollary 5.—A recurring equation of an even degree may always be depressed to another of half the dimensions. For let the equation be

$$x^{2n}+A_1x^{2n-1}+A_2x^{2n-2}+\ldots\ldots A_2x^2+A_1x+1=0;$$

dividing by x^n, and placing the first and last, the second and last but one, &c., in juxtaposition, we have

$$x^n+\frac{1}{x^n}+A_1\left(x^{n-1}+\frac{1}{x^{n-1}}\right)+\ldots\ldots A_{n-1}\left(x+\frac{1}{x}\right)+A_n=0 \ldots\ldots [2]$$

Assume $y=x+\frac{1}{x}$, then we have

$$\begin{array}{ll}
x+\frac{1}{x}=y & \therefore x+\frac{1}{x}=y \\
\left(x+\frac{1}{x}\right)^2=x^2+\frac{1}{x^2}+2 & x^2+\frac{1}{x^2}=y^2-2 \\
\left(x+\frac{1}{x}\right)^3=x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right) & x^3+\frac{1}{x^3}=y^3-3y \\
\left(x+\frac{1}{x}\right)^4=x^4+\frac{1}{x^4}+4\left(x^2+\frac{1}{x^2}\right)+6 & x^4+\frac{1}{x^4}=y^4-4(y^2-2)-6 \\
\&c. \qquad \&c. \qquad \&c. & \qquad =y^4-4y^2+2;
\end{array}$$

substituting these values of

$$x+\frac{1}{x},\ x^2+\frac{1}{x^2}\ \ldots\ x^n+\frac{1}{x^n} \text{ in } [2]$$

the resulting equation is of the form

$$y^n+B_1y^{n-1}+B_2y^{n-2}+\ldots\ldots B_{n-1}y+B_n=0;$$

and the original equation is reduced to an equation of half the dimensions.

EXAMPLES.

(1) Transform the equation $x^3-7x+7=0$ into another whose roots shall be less than the reciprocals of those of the given equation by unity.

$$\begin{array}{rrrrl}
7 & -7 & +0 & +1 & (1 \\
 & 7 & 0 & 0 & \\ \hline
 & 0 & 0 & 1 & \\
 & 7 & 7 & & \\ \hline
 & 7 & 7 & & \\
 & 7 & & & \\ \hline
 & 14 & & &
\end{array}$$

∴ $7z^3+14z^2+7z+1=0$ is the equation sought, where $z+1=\frac{1}{x}$, or $x=\frac{1}{z+1}$.

(2) Find the roots of the recurring equation

$$x^5-6x^4+5x^3+5x^2-6x+1=0.$$

By Cor. 4, this equation has one root $x=-1$, and the depressed equation is

$$x^4-7x^3+12x^2-7x+1=0.$$

Divide by x^2, and arrange the terms as in Cor. 5; then

$$x^2+\frac{1}{x^2}-7\left(x+\frac{1}{x}\right)+12=0\ .\ .\ .\ (A)$$

Put $x+\frac{1}{x}=z$; then $x^2+\frac{1}{x^2}=z^2-2$; hence, by substitution, (A) becomes

$$z^2-2-7z+12=0;$$

or

$$z^2-7z+10=0;$$

and, resolving the quadratic, we get

$$z=\frac{7}{2}\pm\sqrt{\frac{49}{4}-10}$$

$$=\frac{7\pm 3}{2}$$

$$=5,\text{ or } z=2.$$

Hence $x+\frac{1}{x}=5$, and $x+\frac{1}{x}=2$, and the resolution of these two quadratics gives

$$x=\tfrac{1}{2}(5\pm\sqrt{21})\text{ and } x=+1,\text{ or }+1,$$

and the five roots are

$$-1,\ +1,\ +1,\ \frac{5+\sqrt{21}}{2},\text{ and }\frac{5-\sqrt{21}}{2};$$

where $\frac{5-\sqrt{21}}{2}=\frac{(5-\sqrt{21})}{2}\cdot\frac{5+\sqrt{21}}{5+\sqrt{21}}=\frac{25-21}{2(5+\sqrt{21})}=\frac{2}{5+\sqrt{21}}$, which is the reciprocal of the root $\frac{5+\sqrt{21}}{2}$.

(3) Give the equation whose roots are the reciprocals of the roots of the equation

$$x^6-3x^5-2x^4+3x^3+12x^2+10x-8=0.$$

(4) Find the roots of the recurring equation

$$5y^5-4y^4+3y^3-3y^2+4y-5=0.$$

(5) Find the roots of the recurring equation

$$x^5+x^4+x^3+x^2+x+1=0.$$

ANSWERS.

(3) $8x^6-10x^5-12x^4-3x^3+2x^2+3x-1=0.$

(4) $1,\ \frac{1+\sqrt{-3}}{2},\ \frac{1-\sqrt{-3}}{2},\ \frac{-3+4\sqrt{-1}}{5}$, and $\frac{-3-4\sqrt{-1}}{5}$.

(5) $-1,\ \sqrt{\frac{-1+\sqrt{-3}}{2}},\ \sqrt{\frac{-1-\sqrt{-3}}{2}},\ -\sqrt{\frac{-1+\sqrt{-3}}{2}}$, and $-\sqrt{\frac{-1-\sqrt{-3}}{2}}$.

PROPOSITION III.

275. *To transform an equation into another whose roots shall be any proposed multiple or submultiple of the roots of the given equation.*

Let $x^n+A_1x^{n-1}+A_2x^{n-2}+\ldots A_{n-1}x+A_n=0$ be any equation; then putting $y=mx$, we have $x=\frac{y}{m}$, and by substituting this value of x in the given equation, and multiplying each term by m^n, we have

$$y^n+mA_1y^{n-1}+m^2A_2y^{n-2}+\ldots m^{n-1}A_{n-1}y+m^nA_n=0;$$

an equation whose roots are m times those of the proposed equation. Hence we have simply to multiply the second term of the given equation by m, the third by m^2, the fourth by m^3, and so on, and the transformation is effected.

Corollary 1.—If the coefficient of the first term be m, then, suppressing m in the first term, making no change in the second, multiplying the third by m, the fourth by m^2, and so on, the resulting equation will have its roots m times those of the given equation.

Corollary 2.—Hence, if an equation have fractional coefficients, it may be changed into another having integral coefficients, by transforming the given equation into another whose roots shall be those of the proposed equation multiplied by the product of the denominators of the fractions.

Corollary 3.—If the coefficients of the second, third, fourth, &c., terms of an equation be divisible by m, m^2, m^3, and so on, respectively, then m is a common measure of the roots of the equation.

EXAMPLES.

(1) Transform the equation $2x^3-4x^2+7x-3=0$ into another whose roots shall be three times those of the proposed equation.

(2) Transform the equation $4x^4-3x^3-12x^2+5x-1=0$ into another whose roots shall be four times those of the given equation.

(3) Transform the equation $x^3+\frac{1}{3}x^2-\frac{1}{4}x+2=0$ into another whose roots shall be 12 times those of the given equation.

ANSWERS.

(1) $2x^3-12x^2+63x-81=0$.
(2) $x^4-3x^3-48x^2+80x-64=0$.
(3) $x^3+4x^2-36x+3456=0$.

PROPOSITION IV.

276. *To transform an equation into another whose roots shall be the squares of the roots of the proposed equation.*

Let $x^n+A_1x^{n-1}+A_2x^{n-2}+\ldots\ldots+A_{n-1}x+A_n=0$ be any equation; then $x^n-A_1x^{n-1}+A_2x^{n-2}-\ldots\ldots\pm A_{n-1}x\mp A_n=0$ is the equation whose roots are the roots of the former, with contrary signs (Prop. VII., Art. 247).

Let a_1, a_2, a_3, &c., be the roots of the former equation, and $-a_1$, $-a_2$, $-a_3$, &c., those of the latter; then we have

$$(x^n+A_2x^{n-2}+\ldots)+(A_1x^{n-1}+A_3x^{n-3}+\ldots)=(x-a_1)(x-a_2)(x-a_3)\ldots$$
$$(x^n+A_2x^{n-2}+\ldots)-(A_1x^{n-1}+A_3x^{n-3}+\ldots)=(x+a_1)(x+a_2)(x+a_3)\ldots$$

Hence, by multiplying these two equations, we have

$$(x^n+A_2x^{n-2}+\ldots)^2-(A_1x^{n-1}+A_3x^{n-3}+\ldots)^2=(x^2-a_1^2)(x^2-a_2^2)(x^2-a_3^2)\ldots$$

Or $x^{2n}-(A_1^2-2A_2)x^{2n-2}+(A_2^2-2A_1A_3+2A_4)x^{2n-4}-\ldots$ &c., $=(x^2-a_1^2)(x^2-a_2^2)(x^2-a_3^2)\ldots$ by actually squaring and arranging according to the powers of x. Now, for x^2 write y, and we have

$$y^n-(A_1^2-2A_2)y^{n-1}+(A_2^2-2A_1A_3+2A_4)y^{n-2}-\ldots \text{\&c.}, =(y-a_1^2)(y-a_2^2)(y-a_3^2)\ldots$$

$\therefore\ y^n-(A_1^2-2A_2)y^{n-1}+(A_2^2-2A_1A_3+2A_4)y^{n-2}-\ldots=0$ is an equation whose roots are the squares of the roots of the given equation.

EXAMPLES.

(1) Transform the equation $x^3+3x^2-6x-8=0$ into another whose roots are the squares of those of the proposed equation.

Here $x^3-6x=-3x^2+8$ by transposition, and by squaring we have

$$x^6-12x^4+36x^2=9x^4-48x^2+64$$
$$\therefore\ x^6-21x^4+84x^2-64=0,$$

or

$$y^3-21y^2+84y-64=0$$

is the required equation.

The roots of the given equation are -1, -4, 2; and those of the transformed equation are 1, 4, 16.

(2) $x^5+x^3+3x^2+16x+15=0$.

The transformed equation is

$$x^5+2x^4+33x^3+23x^2+166x-225=0,$$

which has (Art. 259) only one positive root, and therefore the proposed has only one real root.

(3) Transform the equation $x^3-x^2-7x+15=0$.

(4) Transform the equation $x^4-6x^3+5x^2+2x-10=0$.

(5) Transform the equation $x^4-4x^3-8x+32=0$.

(6) Transform the equation $x^4-3x^3-15x^2+49x-12=0$.

ANSWERS.

(3) $y^3-15y^2+79y-225=0$.

(4) $y^4-26y^3+29y^2-104y+100=0$.

(5) $y^4-16y^3-64y+1024=0$.

(6) $y^4-39y^3+495y^2-2041y+144=0$.

PROPOSITION V.

277. *To transform an equation into another wanting any given term.*

By recurring to the transformed equation in Art. 251, note, in which the roots of the proposed are increased or diminished by a quantity represented by r, it will be seen that in order to know what value r must have to make the coefficient of any power of x disappear, it is only necessary to place the column of quantities by which that power is multiplied equal to zero, and the resulting equation, when resolved, will furnish the proper values of r. This equation will be of the 1° degree when it is required that the second term shall disappear, it will be of the 2° degree when the third is to disappear, and so on. The last term can be made to disappear only by means of an equation of the same degree as the proposed.

By removing the second term from a quadratic equation, we shall be immediately conducted to the well-known formula for its solution. Thus, the equation being

$$x^2+Ax+N=0,$$

the transformed in $x'+r$ will be

$$\left.\begin{array}{l|l} x'^2+2r & x'+r^2 \\ \quad +A & \quad +Ar \\ & \quad +N \end{array}\right\}=0:$$

and, that its second term may vanish, we must have

$$2r+A=0 \therefore r=-\tfrac{1}{2}A,$$

which condition reduces the transformed to

$$x'^2-\tfrac{1}{4}A^2+N=0$$

$$\therefore x'=\pm\sqrt{\tfrac{1}{4}A^2-N}$$

$$\therefore x=x'+r=-\tfrac{1}{2}A\pm\sqrt{\tfrac{1}{4}A^2-N};$$

which is the common formula for the solution of a quadratic equation.

PROPOSITION VI.

278. *To transform an equation into one whose roots are the squares of the differences of the roots of the proposed equation.*

If in the given equation, $f(x)=0$, we make $x=a_1+y$, a_1 being one of the roots, y will be the difference between a_1 and every other root. If we make $x=a_2+y$, y will be the difference between a_2 and every other root, and so on.

But since a_1, a_2, &c., are roots of $f(x)=0$, they must satisfy it; hence

$$f(a_1)=0, f(a_2)=0, \text{ \&c.} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

If we eliminate a_1 or a_2, &c., between either of these equations (1) and the corresponding ones, $f(a_1+y)=0$, $f(a_2+y)=0$, &c., the final equation in y will be in each case the same, and is therefore the equation whose roots are the differences of the roots of the proposed equation. It is evidently the same thing to eliminate between $f(x)$ and $f(x+y)$.

The form of the equation $f(x+y)$ is (Art. 251),

$$f(x)+f_1(x)y+\frac{f_2(x)}{1.2}y^2+ \; . \; . \; . \; . \; . \; y^m.$$

The first term is identical with the proposed equation, and vanishes, and the whole is divisible by y; we thus deduce

$$f_1(x)+\frac{f_2(x)}{1.2}y+ \; . \; . \; . \; y^{m-1} \quad . \; . \; . \; . \; . \quad (2)$$

The equation (2) is of the $m-1$ degree, and by elimination with the proposed equation of the degree m will produce a final equation of the degree $m(m-1)$, as will be hereafter shown. It is evident, indeed, that the roots being of the form a_1-a_2, a_2-a_1, a_1-a_3, a_3-a_1, a_2-a_3, &c., will be equal in number to the permutations of m letters, two and two, which is $m(m-1)$ (Art. 200). The factors m and $m-1$ will the one be even and the other odd, and the product $m(m-1)$ must therefore necessarily be even; moreover, since if one root, a_1-a_2, be represented by β, another, a_2-a_1, will be represented by $-\beta$, and the equation (2) will be composed of factors of the form $(y-\beta)(y+\beta)=y^2-\beta^2$; and hence will contain only even powers of y. It may therefore be written under the form

$$y^{2m}+py^{2m-2}+qy^{2m-4}+, \text{ \&c., } +t^2=0 \quad . \; . \; . \; . \quad (3)$$

and if we make $y^2=z$, we have

$$z^m+pz^{m-1}+qz^{m-2}+, \text{ \&c. } +t=0 \quad . \; . \; . \; . \; . \; . \quad (4)$$

as the equation whose roots are the squares of the differences of the roots of the proposed equation.

279. As an application of the foregoing principles, let us find the equation of the squares of the differences for the equation of the third degree. In the first place, I shall make the general remark, that equations (3) and (4) ought not to change when we augment, or when we diminish, by the same quantity all the roots of equation (1). Consequently, if the second term of a given equation be not wanting, we can cause it to disappear (Art. 273), and then find the equation of the differences for the transformed equation; we shall thus find the same equation as if we had not made the second term vanish, since the differences of the roots will be the same as before, while the calculations will be less complicated. This being premised, I will suppose that the equation of the third degree wants its second term, and has the form

$$x^3+qx+r=0 \quad \ldots \ldots \ldots \ldots \quad [A]$$

Designate the given equation by $f(x)=0$, and the derived polynomials of $f(x)$ by $f_1(x), f_2(x), f_3(x) \ldots$; the rule for finding the equation of the squares of the differences is to eliminate between the two equations

$$f(x)=0,\ f_1(x)+\frac{1}{2}f_2(x)y+\frac{1}{2\,.\,3}f_3(x)y^2+\ \ldots =0 \quad \ldots \ldots \quad [B]$$

But in the case before us we have

$$f(x)=x^3+qx+r,\ f_1(x)=3x^2+q,\ f_2(x)=6x,\ f_3(x)=6.$$

Substituting, therefore, these values in equations [B], we shall readily perceive that the elimination of x ought to be performed between equation [A] and the following equation,

$$3x^2+q+3xy+y^2=0 \quad \ldots \ldots \ldots \quad [C]$$

We shall, therefore, arrange this equation with reference to x, and then eliminate x by proceeding as if we had to find the greatest common divisor of equations [A] and [C].

First Division.

$$\begin{array}{r|l}
x^3+qx+r & 3x^2+3yx+y^2+q \\
3x^3+3qx+3r & \quad x-y \\
+3x^3+3yx^2+(y^2+q)x & \\
\hline
-3yx^2-(y^2-2q)x+3r & \\
-3yx^2-3y^2x-y^3-qy & \\
\hline
2(y^2+q)x+y^3+qy+3r. &
\end{array}$$

Second Division.

$$\begin{array}{r|l}
3x^2+3qx+y^2+q & 2(y^2+q)x+y^3+qy+3r \\
6(y^2+q)x^2+6(y^2+q)yx+2(y^2+q)^2 & 3x+3(y^3+qy-3r) \\
+6(y^2+q)x^2+3(y^3+qy+3r)x & \\
\hline
3(y^3+qy-3r)x+2(y^2+q)^2 & \\
6(y^2+q)(y^3+qy-3r)x+4(y^2+q)^3 & \\
6(y^2+q)(y^3+qy-3r)x+3(y^3+qy+3r)(y^3+qy-3r) & \\
\hline
4(y^2+q)^3-3(y^3+qy+3r)(y^3+qy-3r). &
\end{array}$$

In the last division we have multiplied twice by y^2+q in order to render the divisions possible, but if we take $y^2+q=0$, the divisor reduces to $3r$, a quantity in general differing from 0.

Making the last remainder equal to zero, and performing the operations indicated, the equation of the differences is

$$y^6+6qy^4+9q^2y^2+4q^3+27r^2=0;$$

taking $y^2=z$, the equation of the squares of the differences becomes

$$z^3+6qz^2+9q^2z+4q^3+27r^2=0.$$

For the equation $x^3-7x+7=0$, we have $q=-7$, $r=+7$; and hence the equation in z becomes

$$z^3-42z^2+441z-49=0.$$

BUDAN'S CRITERION

For determining the number of imaginary roots in any equation.

280. If the real positive roots of an equation, taken in the order of their magnitudes, be a_1, a_2, a_3, $a_4 \ldots\ldots a_n$, where a_1 is the smallest, and if we diminish the roots of the equation by a number h greater than a_1, but less than a_2, then the roots will be a_1-h, a_2-h, a_3-h, $\ldots a_n-h$, and the first of these will now be negative. But the number of positive roots is exactly equal to the number of variations of sign in the terms of the equation when the roots are all real; and as we have changed one positive root into a negative one, the transformed equation must have one variation less than the proposed equation.

Again, by reducing all the roots by k, a number greater than a_2, but less than a_3, we shall have two negative roots, a_1-k, a_2-k, in the transformed equation, and, therefore, we shall have two variations of sign less than in the proposed equation, for two positive roots have been reduced so as to become negative ones. Hence it is obvious, that if we reduce the roots by a number greater than a_n, all the positive roots will become negative, and the transformed equation, having all its roots negative, will have the signs of all its terms positive (Art. 259), and all the variations will have entirely disappeared.

We see, then, that if the roots of an equation be reduced until the signs of all the terms of the transformed equation be $+$, we have employed a greater number than the greatest positive root of that equation; and, therefore, its reciprocal must be less than the smallest real root of the reciprocal equation. Now, if we take the reciprocal equation, and reduce its roots by the reciprocal of the former number, we should have as many positive roots *left* in this transformed reciprocal equation as there were positive roots in the proposed equation, unless the equation has imaginary roots; hence the number of variations *lost* in the former case should be exactly equal to the number *left* in the latter, when the roots are all real; and, consequently, if this condition be not fulfilled, the difference of these numbers indicates the number of imaginary roots. To explain this reasoning more clearly, we shall suppose that an equation has three positive roots; as, for instance, 1, 2·5, and 3. Now if the roots of the proposed equation be reduced by 4, a number greater than 3, the greatest positive root, the three positive roots in the original equation will evidently be changed into three negative ones in the transformed one, and hence *three* variations must be lost. Again, the equation whose roots are the reciprocals of the proposed equation must have three positive roots, 1, $\frac{2}{5}$, and $\frac{1}{3}$; and it is evident that if we reduce the roots of the reciprocal equation by $\frac{1}{4}$, the reciprocal of the former reducing number 4, we shall not change the character of the three positive roots, because $\frac{1}{4}$ is less than the least of them, and $1-\frac{1}{4}$,

$\frac{2}{3}-\frac{1}{4}$, $\frac{1}{3}-\frac{1}{4}$ are all positive; hence the *three* variations introduced by the three positive roots must still be found in the transformed reciprocal equation, and, therefore, *three* variations are left in the latter transformation, indicating no imaginary roots. The theorem may, therefore, be stated thus:

If, in transforming an equation by any number r, there be n variations *lost*, and if, in transforming the reciprocal equation by $\frac{1}{r}$ (the reciprocal of r), there be m variations *left*, then there will be at least $n-m$ imaginary roots in the interval 0, r.

For there are as many positive roots in the interval 0, r of the direct equation as there are between $\frac{1}{r}$ and $\frac{1}{0}$ of the reciprocal equation; hence, if n, the number of variations *lost* in the transformation of the direct equation by r, be greater than m, the number of variations *left* in the transformation of the reciprocal equation by $\frac{1}{r}$, there will be a contradiction with respect to the character of a number of the roots, equal to the difference $n-m$. Hence these roots are imaginary.

EXAMPLE.

Find the number of imaginary roots of the equation

$$x^4-x^3+2x^2+x-4=0.$$

Direct.

1	−1	+2	+1	−4 (1
	1	0	2	3
	0	2	3	−1
	1	1	3	
	1	3	6	
	1	2		
	2	5		
	1			
	3			

Reciprocal.

−4	+ 1	+ 2	− 1	+1 (1
	− 4	− 3	− 1	−2
	− 3	− 1	− 2	−1
	− 4	− 7	− 8	
	− 7	− 8	−10	
	− 4	−11		
	−11	−19		
	− 4			
	−15			

Here *two* variations are *lost* in the transformation of the direct equation, and *no* variations are *left* in the transformation of the reciprocal equation; therefore this equation has at least *two* imaginary roots; and it has *only* two, for the sign of the absolute term is negative, implying the existence of two real roots, the one positive and the other negative. (See Art. 248, Pr. VIII., Cor. 5.)

EXAMPLE.

To find the number and situation of the real roots of the equation $x^4+x^3+x^2+3x-100=0$ by Budan's method.

If the roots of this equation be all real, the permanences and variation indicate three negative roots and one positive root.

(1) To find the positive root.

1+1+1+ 3−100 (2	1+1+ 1+ 3−100 (3
3+7+17− 66	4+13+42+ 26

In the transformation by 2, one variation is left, and, in transforming by 3, there is no variation left; therefore the positive root is between 2 and 3.

(2) For the negative roots.

Direct Equation.	*Reciprocal Equation.*
$1-1+1-3-100\ (1$	$-100-\ 3+\ 1-\ 1+\ 1\ (1$
$0+1-2-102$	$-103-102-103-102$
$1+2+0$	
$2+4$	signs all —
3	

Here two variations are lost in the direct transformation, and no variations are left in the reciprocal transformation; therefore the two roots in the interval 0 and —1 are imaginary.

$1-1+1-\ 3-100\ (3$	$1-1+\ 1-\ 3-100\ (4$
$2+7+18-\ 46$	$3+13+49+\ 96$

Hence the negative root is obviously situated between —3 and —4.

DEGUA'S CRITERION.

281. In any equation, if we have a cipher-coefficient, or term wanting, and if the cipher-coefficient be situated between two terms having the same sign, there will be two imaginary roots in that equation.

Let the order of the signs be

$$++-0-+--;$$

and for 0 writing + or — we have either

$$++-+-+--,\ \text{or}\ ++----+--.$$

In the former of these we find *two* permanences and *five* variations, and in the latter we have *four* permanences and only *three* variations; hence, if the roots are all real, we must, in the former case, have *five* positive and *two* negative roots, and in the latter, *three* positive and *four* negative roots (Art. 259); hence we have *two* roots, both *positive* and *negative*, at the same time, and, therefore, these two roots can not be *real* roots. These two roots, which involve the absurdity of being both positive and negative at the same time, must, therefore, be *imaginary* roots.

In nearly the same manner it may be shown that

(1) If between terms having *like* signs, $2n$ or $2n-1$ cipher-coefficients intervene, there will be $2n$ imaginary roots indicated thereby.

(2) If between terms having *different* signs, $2n+1$ or $2n$ cipher-coefficients intervene, there will be $2n$ imaginary roots indicated thereby.

EXAMPLE.

The equation $x^4-x^3+6x^2+24=0$ has two imaginary roots, for the absent term is preceded and succeeded by terms having like signs; and the equation $x^3\pm 1$, having the coefficients $1\pm 0\pm 0\pm 1$, has also two imaginary roots.

EXAMPLES FOR PRACTICE.

(1) How many imaginary roots are in the equation

$$x^5+x^3-2x^2+2x-1=0\ ?$$

(2) Has the equation $x^4-2x^2+6x+10=0$ any imaginary roots?

THE LIMITS OF THE ROOTS OF EQUATIONS.

282. The limits of any group of roots of an equation are two quantities between which the whole group lies; thus, $+\infty$ and 0 are limits of the positive roots of every equation, and 0 and $-\infty$ of the negative roots. But in practice we are required to assign much closer limits than these, usually the two con-

secutive whole numbers between which each root lies, so that the inferior limit is the integral part of the included root. This may be effected without knowing any of the roots of the equation, as will be seen in the following propositions. The roots spoken of in this section are the real roots.

SUPERIOR AND INFERIOR LIMITS OF THE ROOTS.

283. The greatest negative coefficient increased by unity is a superior limit of the positive roots of an equation.

Let $-p$ be the greatest negative coefficient; then any value of x which makes

$$x^n-p(x^{n-1}+x^{n-2}+\ldots+x^2+x+1) \text{ positive},$$

or

$$x^n>p(x^{n-1}+x^{n-2}+\ldots+x^2+x+1)>p\frac{x^n-1}{x-1},^*$$

will, *a fortiori*, make

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n,$$

or $f(x)$ positive, because in the latter, all the terms after x^n will not generally be negative, and of the negative terms not one is greater than the corresponding term in the former expression.

Now the inequality $x^n>p\dfrac{x^n-1}{x-1}$ is satisfied, if

$$x^n= \text{ or } >x^n\frac{p}{x-1}, \text{ or } x-1= \text{ or } >p, \text{ or } x= \text{ or } >p+1.$$

Since, therefore, $p+1$ and every greater number, when substituted for x, will make $f(x)$ positive, the numerical value of the greatest negative coefficient increased by unity is a superior limit of the positive roots.†

284. In any equation, if p_rx^{n-r} be the first term which is negative, and $-p$ the greatest negative coefficient, $1+\sqrt[r]{p}$ is a superior limit of the positive roots.

Any value of x which makes

$$x^n>p(x^{n-r}+x^{n-r-1}+\ldots+x+1)>p\frac{x^{n-r+1}-1}{x-1},$$

will of course make $x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots$ positive.

Now the inequality $x^n>p\dfrac{x^{n-r+1}-1}{x-1}$, is satisfied if

$$x^n>p\frac{x^{n-r+1}}{x-1}, \text{ or } x^{r-1}(x-1)>p, \text{ or if } (x-1)^{r-1}(x-1)= \text{ or } >p, \text{ or } (x-1)=$$
$$\text{or } >p, \text{ or } x= \text{ or } >1+\sqrt[r]{p}.$$

Since, therefore, $1+\sqrt[r]{p}$ and every greater number gives a positive result, $1+\sqrt[r]{p}$ is a superior limit.

This method may be employed when the first term is followed by one or more positive terms.

EXAMPLE.

$$x^4+11x^2-25x-61=0.$$

Here $r=3$, and a limit of the positive roots is

$$1+\sqrt[3]{61}, \text{ or } 5, \text{ taking the next higher integer.}$$

285. If each negative coefficient, taken positively, be divided by the sum of

* See (Art. 23).

† This is commonly known as Maclaurin's limit.

all the positive coefficients which precede it, the greatest of the fractions thus formed, increased by unity, is a superior limit of the positive roots.

Let the equation be

$$x^n+p_1x^{n-1}+p_2x^{n-2}+(-p_3)x^{n-3}+\dots$$
$$\dots+(-p_r)x^{n-r}+\dots+p_n=0;$$

then, since (Art. 23),

$$p_mx^m=p_m(x-1)(x^{m-1}+x^{m-2}+\dots+x+1)+p_m,$$

if we transform every positive term by this formula, and leave the negative terms in their original form, we shall have

$$\begin{aligned}0=(x-1)x^{n-1}+(x-1)x^{n-2}+(x-1)x^{n-3}+\dots+x-1+1&\\ +p_1(x-1)x^{n-2}+p_1(x-1)x^{n-3}+\dots+p_1(x-1)+p_1&\\ +p_2(x-1)x^{n-3}+\dots+p_2(x-1)+p_2&\\ -p_3x^{n-3}\qquad\qquad\qquad&\\ +\ \dots\dots\dots\dots&\end{aligned}$$

Now if such a value be assigned to x that every term is positive, that value will be the superior limit required; in the terms where no negative coefficient enters, it is sufficient to have $x>1$; in the other terms, each of which involves a negative coefficient, we must have

$$(1+p_1+p_2)(x-1)>p_3,$$
$$(1+p_1+p_2+\dots+p_{r-1})(x-1)>p_r, \text{ \&c.},$$

or

$$x>\frac{p_3}{1+p_1+p_2}+1;\ x>\frac{p_r}{1+p_2+p_2+\dots+p_{r-1}}+1, \text{ \&c.}$$

If, then, x be taken equal to the greatest of these fractions increased by unity, this value, and every greater value, will make $f(x)$ positive, and therefore will be a superior limit of the positive roots. This method gives a limit easily calculated, and generally not far from the truth.*

EXAMPLES.

(1) $4x^5-8x^4+23x^3+105x^2-80x+3=0.$

The fractions are $\frac{8}{4}$ and $\frac{80}{4+23+105}$, and $\frac{8}{4}>\frac{80}{132}$; therefore $\frac{8}{4}+1=3$ is a superior limit.

(2) $4x^7-6x^6-7x^5+8x^4+7x^3-23x^2-22x-5=0;$

here 3 is a superior limit.

Observation.—The form of the equation will often suggest artifices, by means of which closer limits may be determined than by any of the preceding methods; thus, writing the equation of Example 1 under the form

$$4x^4(x-2)+23x^3+105x\left(x-\frac{16}{21}\right)+3=0,$$

we see that $x=$ or >2 gives a positive result, therefore 2 is a superior limit. Similarly, by writing the example of Art. 284 under the form

$$x(x^3-25)+11\left(x^2-\frac{61}{11}\right)=0,$$

we see that 3 is a superior limit.

We have seen (Art. 248) that an equation of an even number of dimensions with its last term positive may have no real root; but we shall now show that

* This is the method of Bret.

in any equation whatever, if the absolute term be small compared with the other terms, there will be at least one real root also very small.

286. In the equation

$$p_0x^n+p_1x^{n-1}+, \text{\&c.}, +x-r=0,$$

where r is essentially positive, and which may represent any equation whatever, if $r<\frac{1}{4(1+p)}$, where p is numerically the greatest coefficient, then there is a real positive root, $<2r$.

By dividing by the coefficient of x, and changing the signs of all the terms, and of all the roots, if necessary, every equation may be reduced to the form

$$-r+x+, \text{\&c.}, +p_1x^{n-1}+p_0x^n=0 \quad . \; . \; . \; . \quad (1)$$

where r is essentially positive; let p be numerically the greatest coefficient, then any value of $x<1$ which makes

$$-r+x>p(x^2+x^3+, \text{\&c.}, +x^n)>\frac{px^2(1-x^{n-1})}{1-x},$$

will make the first member of (1) positive; and this condition is fulfilled by

$$-r+x= \text{ or } >\frac{px^2}{1-x},$$

because $1>1-x^{n-1}$, or

$$(1+p)x^2-(1+r)x+r=0,$$

or

$$2(1+p)x=(1+r)-\sqrt{(1+r)^2-4r(1+p)};$$

if, then, $4r(1+p)<1$, the radical will have a real value $>r$, and there will be for x a real value less than $\frac{1}{2(1+p)}$ which makes the first member of (1) positive, while $x=0$ makes it negative; therefore, in any equation reduced to the above form, if $r<\frac{1}{4(1+p)}$, there is a real small positive root, $<2r$.

EXAMPLE.

$$x^4+18x^3-21x^2-12x+1=0$$

has a real root between 0 and $\frac{1}{6}$.

287. To find an inferior limit of the positive roots, we must transform the equation into one whose roots are the reciprocals of the roots of the former; and the reciprocal of the superior limit of the roots of the transformed equation, found by the preceding methods, will be the quantity required.

Hence, if p_r denote the greatest coefficient of a contrary sign to the last term, p_n, an inferior limit of the positive roots is $\frac{p_n}{p_n+p_r}$. For the transformed equation will be (Art. 274)

$$y^n+\frac{p_{n-1}}{p_n}y^{n-1}+\dots+\frac{p_r}{p_n}y^r+\dots+\frac{1}{p_n}=0,$$

of which $\frac{p_r}{p_n}$ is the greatest negative coefficient; therefore $\frac{p_r}{p_n}+1$ is a superior limit of its roots; and, consequently, $\frac{p^n}{p_r+p_n}$ an inferior limit of the positive roots of the proposed equation.

EXAMPLE.

$$x^3-42x^2+441x-49=0.$$

Here $p_n=49$, $p_r=441$, $\therefore \frac{49}{49+441}$, or $\frac{1}{10}$ is an inferior limit of the positive roots. By putting $x=\frac{1}{y}$, we may discover a limit closer to the roots; for the transformed equation is

$$y^3-9y^2+\frac{6}{7}y-\frac{1}{49}=0, \text{ or } y^2(y-9)+\frac{6}{7}\left(y-\frac{1}{42}\right)=0,$$

which evidently has 9 for the superior limit of its positive roots, and, therefore, the proposed has $\frac{1}{9}$ for its inferior limit.

288. To find superior and inferior limits of the negative roots, we must transform the equation into one whose roots are those of the former with contrary signs (Art. 247); and if α, β be limits, found as above, of the positive roots of this equation, then $-\alpha$ and $-\beta$ will be limits of the negative roots of the proposed equation.

EXAMPLE.

$$x^3-7x+7=0;$$

putting $x=-y$, we get $y^3-7y-7=0$, of which $1+\sqrt{7}$ or 4 is a superior limit.

Also, putting $y=\frac{1}{z}$, we get $z^3+z^2-\frac{1}{7}=0$, or $z^3-\frac{1}{28}+z^2-\frac{3}{28}=0$, of which $\frac{1}{3}$ is a superior limit; therefore the negative root of the proposed lies between -4 and -3.

NEWTON'S METHOD OF FINDING LIMITS OF THE ROOTS.

289. The limits, however, deduced by any of the preceding methods seldom approach very near to the roots; the tentative method, depending upon the following proposition, will furnish us with limits which lie much nearer to them.

Every number which, written for x, makes $f(x)$ and all its derived functions positive, is a superior limit of the positive roots.

For, if we diminish the roots a, b, c, &c., of $f(x)=0$ by h, that is (Art. 251), substitute $y+h$ for x, the result is $f(y+h)=0$, or

$$f(h)+f'(h)\frac{y}{1}+f''(h)\frac{y^2}{1.2}+\cdots+f^{n-1}(h)\frac{y^{n-1}}{\underline{|n-1}}+y^n=0.$$

Now, if we give such a value to h that all the coefficients of this equation are positive, then every value of y is negative; that is, all the quantities, $a-h$, $b-h$, $c-h$, &c., are negative, and therefore h is greater than the greatest of the quantities a, b, c, &c., or is a superior limit of the roots of the proposed equation. Similarly, h will be an inferior limit to all the roots, if the coefficients be alternately positive and negative.

EXAMPLE.

To find a superior limit of the roots of

$$x^3-5x^2+7x-1=0.$$

The transformed equation, putting $y+h$ for x, is

$$(h^3-5h^2+7h-1)+(3h^2-10h+7)y+(6h-10)\frac{y^2}{2}+y^3=0;$$

in which, if 3 be put for h, all the coefficients are positive; therefore 3 is a superior limit of the positive roots.

OBSERVATION.—This method of finding a superior limit of the roots by determining by trial what value of x will make $f(x)$ and all its derived functions positive, was proposed by Newton.

WARING'S OR LAGRANGE'S METHOD OF SEPARATING THE ROOTS.

290. If a series of quantities be substituted for x in $f(x)$, then between every two which give results with different signs an odd number of roots of $f(x)=0$ is situated; and between every two which give results with the same sign an even number is situated, or none at all; but we can not assure ourselves that in the former case the number does not exceed unity, or that in the latter it is zero, and that, consequently, the number and situation of all the real roots is ascertained, unless the difference between the quantities successively substituted be less than the least difference between the roots of the proposed equation; since, if it were greater, it is evident that more than one root might be intercepted by two of the quantities giving results with different signs, and that two roots instead of none might be intercepted by two of the quantities giving results with the same sign, and in both cases roots would pass undiscovered. We must, therefore, first find a limit less than the least difference of the roots; this may be done by transforming the equation into one whose roots are the squares of the differences of the roots of the proposed equation. Then, if we find a limit k less than the least positive root of the transformed equation, $\sqrt{k}$ will be less than the least difference of the roots of the proposed equation; and if we substitute successively for x the numbers s, $s-\sqrt{k}$, $s-2\sqrt{k}$, &c. (s being a superior limit of the roots of the proposed), till we come to a superior limit of the negative roots, we are sure that no two real roots lying between the numbers substituted have escaped us, and that every change of signs in the results of the substitutions indicates only one real root. Hence the number of real roots will be known (for it will exactly equal the number of changes), as well as the interval in which each of them is contained.

OBSERVATION.—This method of determining the number and situation of the real roots of an equation was first proposed by Waring; it is, however, of no practical use for equations of a degree exceeding the fourth, on account of the great labor of forming the equation of differences for equations of a higher order.

EXAMPLE.

$$x^3-7x+7=0.$$

The numbers 1 and 2 give each a positive result, but yet two roots lie between them. The equation whose roots are the squares of the differences is (Art. 279) $y^3-42y^2+441y-49=0$, an inferior limit of the positive roots of which is $\frac{1}{9}$ (Art. 287); therefore, $\frac{1}{3}$ is less than the least difference of the roots of $x^3-7x+7=0$, and, substituting 2, $\frac{5}{3}$, $\frac{4}{3}$, the results are $+$, $-$, $+$;

hence, one value of x lies between 2 and $\frac{5}{3}$, and one between $\frac{5}{3}$ and $\frac{4}{3}$; and, similarly, we find the negative root, which necessarily exists, to lie between -3 and $-3\frac{1}{3}$.

METHOD OF DIVISORS.

291. The commensurable roots of $f(x)=0$, which are necessarily whole numbers, may be always found by the following process, called the method of divisors, proposed by Newton.

Suppose a to be an integral root; then, substituting a for x, and reversing the order of the terms, we have

$$p_n+p_{n-1}a+p_{n-2}a^2+\cdots+p_1a^{n-1}+a^n=0;$$

$$\therefore \frac{p_n}{a}+p_{n-1}+p_{n-2}a+\cdots+p_1a^{n-2}+a^{n-1}=0.$$

Hence, $\frac{p_n}{a}$ is an integer which we may denote by q_1; substituting and dividing again by a, we get

$$\frac{q_1+p_{n-1}}{a}+p_{n-2}+\cdots+p_1a^{n-3}+a^{n-2}=0.$$

Similarly, $\frac{q_1+p_{n-1}}{a}$ is an integer $=q_2$ suppose; and proceeding in this manner, we shall at last arrive at

$$\frac{q_{n-1}+p_1}{a}+1=0.$$

Hence, that a may be a root of the equation, the last term, p_n, must be divisible by it, so must the sum of the quotient and next coefficient, q_1+p_{n-1}; and continuing the uniform operation, the sum of each coefficient and the preceding quotient must be divisible by a, the final result being always -1.

If, therefore, we take the quotients of the division of the last term by each of the divisors of the last term which are comprised within the limits of the roots, and add these quotients to the coefficient of the last term but one; divide these sums, some of which may be equal to zero, by the respective divisors, add the new quotients which are integers or zero (neglecting the others) to the next coefficient and divide by the respective divisors, and so on through all the coefficients (dropping every divisor as soon as it gives a fractional quotient), those divisors of the last term which give -1 for a final result are the integral roots of the equation; and we shall thus obtain all the integral roots, unless the equation have equal roots, the test of which will be that some of the roots already found satisfy $f'(x)=0$, and the number of times that any one is repeated will be expressed by the degree of derivation of the first of the derived functions which that root does not reduce to zero, when written in it for x (Art. 253). It is best to ascertain by direct substitution whether $+1$ and -1 are roots, and so to exclude them from the divisors to be tried.

EXAMPLE I.

$$x^3+3x^2-8x+10=0.$$

Here the roots lie between $\frac{8}{4}+1$ and -11 (Arts. 285, 288), and the divisors of the last term are $\pm\{2, 5, 10\}$,

$\therefore a=$	2	-2	-5	-10
$q_1=$	5	-5	-2	-1
$q_1+(-8)=$	-3	-13	-10	-9
$q_2=$			2	
$q_2+3=$			5	
$q_3=$			$-1.$	

Therefore -5, being the only one of the divisors which leads to a last quotient -1, is the only commensurable root, and it is not repeated, since it does not satisfy the equation $f'(x)=3x^2+6x-8=0$.

EXAMPLE II.

$$x^5-5x^4+x^3+16x^2-20x+16=0.$$

Here limits of the roots are 6 and -4; and the commensurable roots are 4, 2, -2.

EXAMPLE III.

$$x^4+5x^3-2x^2-6x+20=0;\ x=-2, \text{ or } -5.$$

292. The number of divisors to be tried may be lessened by observing, that if the roots of $f(x)=0$ were diminished by any whole number, m, the last term of the transformed equation, $f(y+m)=0$, would be $f(m)$; if, therefore, a were an integral value of x, $a-m$ would be an integral value of y, and would be, therefore, a divisor of $f(m)$. Hence, any divisor, a, of the last term of $f(x)$ is to be rejected which does not satisfy the condition $\frac{f(m)}{a-m}=$ an integer, when for m any integer, such as ±1, ±10, &c., is substituted.

EXAMPLE I.

$$x^3-5x^2-18x+72=0.$$

Changing the signs of the alternate terms, we have

$$x^3+5x^2-18x-72=0, \text{ or } x^3-72+5x\left(x-\frac{18}{5}\right)=0;$$

therefore the roots lie between 19 and -5.

But $f(1)=50, f(-1)=84, f(-3)=54$;

and the only admissible divisors of 72, which, when diminished by 1, divide 50, are

$$6,\ 3,\ 2,\ -4;$$

also, all these divisors, when increased by 1, divide 84; but only 6, 3, -4, when increased by 3, divide 54;

$$\therefore 6,\ 3,\ -4,$$

are the only divisors which need to be tried; and they will all be found to be roots.

EXAMPLE II.

$$x^3-6x^2+169x-(42)^2=0.\quad x=9.$$

293. If a proposed equation have fractional coefficients, or if its first term be affected with a coefficient, since (275, Cor. 2) it can be transformed into another equation with first term unity and every coefficient a whole number, this method will enable us to find the commensurable roots of every equation under a rational form. If the coefficients be whole numbers and the first term be p_0x^n, and we only wish to find the roots which are integers, no transforma-

tion will be necessary, only every divisor of the last term which is a root will lead to a result $-p_0$ instead of -1.

EXAMPLE.

$$6x^4-25x^3+26x^2+4x-8=0.$$

It is the same as

$$(x-2)^2(3x-2)(2x+1)=0.$$

NEWTON'S METHOD OF APPROXIMATION.

294. When we know an approximate value of a root, we may easily obtain other values of it, more and more exact, by a method invented by Newton, which rapidly attains its object. We shall give this method, first in the form in which it was proposed by its author, and afterward with the conditions which Fourier has shown to be necessary for its complete success.

Let $f(x)=0$ be an equation having a root c between a and b, the difference of these limits, $b-a$, being a small fraction whose square may be neglected in the process of approximation.

Let c_1, a quantity between a and b, be assumed as the first approximation to c, then $c=c_1+h$, where h is very small;

$$\therefore f(c_1+h)=0,$$

or

$$f(c_1)+f'(c_1)h+f''(c_1)\frac{h^2}{2}+\cdots+h^n=0.$$

Now, since h is very small, h^2, h^3, &c., are very small compared with h; also, none of the quantities $f''(c_1)$, $f'''(c_1)$, &c., can become very great, since they result from substituting a finite value in integral functions of x; therefore, provided $f'(c_1)$ be not very small (that is, provided $f'(x)=0$ have no root nearly equal to c_1 or to c, and, consequently, $f(x)=0$ no other root nearly equal to c besides the one we are approximating to), all the terms in the series after the first two may be neglected in comparison with them; and we have, to determine h, the resulting approximate value of h, the equation

$$f(c_1)+h_1f'(c_1)=0;$$

$$\therefore h_1=-\frac{f(c_1)}{f'(c_1)}=-\left\{\frac{f(x)}{f'(x)}\right\}_{x=c_1};*$$

and the second approximation is

$$c_2=c_1+h_1=c_1-\left\{\frac{f(x)}{f'(x)}\right\}_{x=c_1}.$$

Similarly, starting from c_2 instead of c_1, the third approximate value will be

$$c_3=c_2-\left\{\frac{f(x)}{f'(x)}\right\}_{x=c_2},$$

and so on; and if we can be certain that each new value is nearer to the truth than the preceding, there is no limit to the accuracy which may be obtained.

EXAMPLE I.

$$x^3-2x-5=0.$$

Here one root lies between 2 and 3, and the equation can have only one

* This notation signifies, that after the division indicated is performed, the particular value, c_1, is substituted for x.

positive root; also, upon narrowing the limits, we find that $x=2$ gives a negative, and $x=2\cdot2$ a positive result; therefore, 2·1 differs from the root by a quantity less than 0·1, and we may assume $c_1=2\cdot1$. Hence

$$c_2=2\cdot1-\left(\frac{x^3-2x-5}{3x^2-2}\right)_{x=2\cdot1}=2\cdot1-\frac{0\cdot061}{11\cdot23},$$

or

$$c_2=2\cdot1-0\cdot0054=2\cdot0946.$$

Similarly,

$$c_3=2\cdot09455149.$$

EXAMPLE II.

$$x^3-7x-7=0.$$

There is only one positive root lying between 3 and 3·1, and it equals 3·048917339.

Observation.—To guard against over correction, that is, against applying such a correction to an approximate value as shall make the new value differ more from the root by excess than the original approximate value did by defect, or *vice versâ*, we must be certain that each new value is nearer to the truth than the preceding; this gives rise to the following conditions, first noticed by Fourier.

295. For the complete success of Newton's method of approximation, the following conditions are necessary.

1. The limits between which the required root is known to lie must be so close that no other root of $f(x)=0$, and no root of $f'(x)=0$, or $f''(x)=0$, lies between them.

2. The approximation must be begun and continued from that limit which makes $f(x)$ and $f''(x)$ have the same sign.

Let c be a root of $f(x)=0$ which lies between a and b, $a<b$, c_1 the first approximate value, and h the whole correction, so that $c=c_1+h$; then

$$f(c_1+h)=0, \text{ or } f(c_1)+hf'(\lambda)=0,$$

λ being some quantity between c_1 and c (Art. 239, Note).

Therefore, supposing $\lambda=c_1$, which amounts to neglecting all powers of h above the first, and requires that $f(x)=0$ have no root besides c in that interval, and calling the resulting approximate value of h, h_1, we have

$$f(c_1)+h_1f'(c_1)=0.$$

Now the true value is $c=c_1+h$;

The first approximate value is c_1 with error h;

The second approximate value is $c_2=c_1+h_1$ with error $h-h_1$, which (neglecting signs) must be less than h,

$$i.\ e.,\ h^2-(h-h_1)^2 \text{ must be positive, or } 2hh_1-h_1^2=+,$$

$$\text{or } \frac{h}{h_1}-\tfrac{1}{2}=+, \text{ or } \frac{f'(c_1)}{f'(\lambda)}-\tfrac{1}{2}=+;$$

which condition (since λ is an indeterminate quantity between c_1 and c, or between a and b) can not in all cases be secured unless $f'(x)$ be incapable of changing its sign between a and b, *i. e.*, unless $f'(x)=0$ have no root between a and b.

Moreover, we must have $\frac{f'(c_1)}{f'(\lambda)}>\frac{1}{2}$, or >1, the latter insuring the former.

Now, if $f''(x)$ preserve an invariable sign between a and b, *i. e.*, if $f''(x)=0$

have no root in that interval, then $f'(x)$ will increase or diminish continually from a to b; therefore c_1 must be taken equal to that limit which gives $f'(x)$ its greatest numerical value without regard to sign.

First, let $f'(x), f''(x)$, have the same sign from a to b; then $f'(x)$ increases continually in that interval; therefore we must have $c_1=b$, or we must begin from the greater limit. But $f(b)$ has the same sign as $f(c+h)=f(c)+hf'(c)$ $=hf'(c)$, or as $f'(c)$; therefore we must have c_1 equal to that limit which makes $f(x)$ and $f''(x)$ have the same sign.

Secondly, let $f'(x), f''(x)$, have contrary signs from a to b; then $f'(x)$ diminishes continually in that interval; therefore we must have $c_1=a$, or we must begin from the lesser limit. But $f(a)$ has the same sign as $f(c-h)$ $=f(c)-hf'(c)=-hf'(c)$, or as $-f'(c)$; therefore, in this case, equally as in the former, we must have c_1 equal to that limit which makes $f(x)$ and $f''(x)$ have the same sign.

These conditions being fulfilled, we have

$$\frac{f'(c_1)}{f'(\lambda)}-1=+, \text{ or } \frac{h-h_1}{h_1}=+,$$

or

$$\frac{c-c_2}{c_2-c_1}=+;$$

therefore c_2 lies between c and c_1; hence, the new limit, c_2 fulfills the requisite conditions, and we may with certainty from it continue the approximation.

296. To estimate the rapidity of the approximation, we have

error in first approximate value c_1, $=h$,
error in second approximate value c_2, $=h-h_1$;

But

$$f(c_1)+hf'(c_1)+\tfrac{1}{2}h^2f''(\mu)=0,$$

$$f(c_1)+h_1f'(c_1)=0;$$

$$\therefore\ (h-h_1)f'(c_1)+\tfrac{1}{2}h^2f''(\mu)=0,$$

$$\text{or } h-h_1=-\tfrac{1}{2}h^2\frac{f''(\mu)}{f'(c_1)}.$$

Let the greatest value which $f''(x)$ can assume between a and b (which will be either $f''(a)$ or $f''(b)$, if $f'''(x)=0$ have no root in the interval) be divided by the least value of $2f'(x)$ in that interval which will be either $2f'(a)$ or $2f'(b)$, and let the quotient be denoted by C; then, neglecting signs,

$$h-h_1<h^2\text{C};$$

hence, if the first error h in c_1 be a small decimal, the error $h-h_1$ with which c_2 is affected (since C will not, except in particular cases, be very large) will be very small compared with h; and if the quantity C be less than unity, the number of exact decimals in the result will be doubled by each successive operation. The quantity C, when thus computed for a given interval, preserves the same value throughout the operations which it may be necessary to make in order to approximate to the value of the root lying in that interval; and as we thus know a limit to the difference between the approximate value already found and the true value, we may always avoid calculating decimals which are inexact, and only obtain those which are necessarily correct.

EXAMPLE.

$$6x^3-141x+263=0.$$

This equation has two positive roots, one between 2·7 and 2·8, and the other between 2·8 and 2·9. Now $f'(x)=18x^2-141=0$ has a root $=\sqrt{\frac{47}{6}}$ $=2·798$, between 2·7 and 2·8, therefore these limits are not sufficiently close; but this root is greater than 2·79; also, 2·7 and 2·79, substituted in $f(x)$, give results with different signs; and 2·7, substituted in $f(x)$ and $f''(x)$, gives results with the same sign; therefore, $c_1=2·7$.

With regard to the other interval, 2·8, 2·9, $f'(x)=0$, $f''(x)=0$ have no roots between these limits, and 2·9 makes $f(x)$ and $f''(x)$ have the same sign; therefore, $c_1=2·9$; and starting from these values, we are certain in each case to get a value nearer to the truth.

Again, the greatest value which $\frac{f''(x)}{f'(2·7)}$ can assume in the interval 2·7, 2·79, is nearly equal to 10; hence, if h_1, h_2, be consecutive errors, we have $h_2<\frac{1}{2}(h_1)^2 . 10$.

The same formula will be found to be true for consecutive errors in the interval 2 8, 2·9.

LAGRANGE'S METHOD OF APPROXIMATION BY CONTINUED FRACTIONS.

297. To approximate to the roots of an equation by the method of continued fractions.

Let the equation $f(x)=0$ have only one root between the integers a and $a+1$;* then, writing $a+\frac{1}{y}$ for x, the first transformed equation will be

$$f(a)+f'(a)\frac{1}{y}+f''(a)\frac{1}{1 . 2y^2}+\cdots+\frac{1}{y^n}=0 \quad (1);$$

and, since only one value of $\frac{1}{y}$ lies between 0 and 1, y has only one value greater than 1; if, therefore, we substitute successively 2, 3, 4, &c., for y, stopping at the first which gives a positive result, the integer preceding that is the integral part of the value of y. Let this be b, and in (1) write $b+\frac{1}{z}$ for y; then the second transformed equation will have only one root greater than unity, the integral part of which, as before, will be the whole number next less than the one in the series 2, 3, 4, &c., which first gives a positive result when written for z; let this be c, and in the second transformed equation write $c+\frac{1}{u}$ for z, then the third transformed equation will have only one root greater than unity, the integral part of which may be found as before, and so on. We thus obtain successively the terms of a continued fraction

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}, \text{ \&c.}$$

which expresses the required value of x. The method of reducing such a fraction, called a continued fraction, will be hereafter given.

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* The roots of the equation may be made to differ by at least unity, if we find by means of the equation of the squares of the differences the least limit to the differences of the roots of the proposed equation, and then find a transformed equation whose roots shall be that multiple of those of the proposed, which is expressed by the denominator of the least limit of the differences.

If any of the numbers b, c, d, &c., is an exact root of the corresponding transformed equation, the process terminates, and we find the exact value of x. Also, if one of the transformed equations be identical with a preceding one, the continued fraction expressing the root is periodical; for, after that, the same quotients will recur in the same order; in this case a finite value, in the form of a surd, may be obtained for the root (see Continued Fractions) by solving a quadratic whose coefficients are rational, both of whose roots will be roots of the proposed, since the coefficients of the latter are supposed rational; consequently, the first member of this quadratic will be a factor of the first member of the proposed equation, which may, therefore, be depressed two dimensions.

EXAMPLE.

To find the positive root of $x^3-2x-5=0$ under the form of a continued fraction.

Comparing this with $x^3-qx+r=0$, we find that

$$\frac{r^2}{4}-\frac{q^3}{27}=\frac{25}{4}-\frac{8}{27} \text{ is a positive quantity;}$$

therefore (Art. 258) the equation has two impossible roots; and since its last term is negative, its third root is positive. Substituting 2 and 3, the results are -1 and $+16$; therefore the root lies between 2 and 3. Assume $x=2+\frac{1}{y}$, and the transformed equation is

$$y^3-10y^2-6y-1=0,$$

in which 10 and 11 being substituted, give -61, $+54$. Assume $y=10+\frac{1}{z}$, and we obtain

$$61z^3-94z^2-20z-1=0,$$

whose root lies between 1 and 2. Proceeding in this manner, we find

$$x=2+\frac{1}{10}+\frac{1}{1}+\frac{1}{1}+\frac{1}{2}\cdots$$

the value of the root in a continued fraction; the method of reducing which to a common fraction will be hereafter given.

This method may be combined with Sturm's theorem.

Here finishes our recapitulation of the older methods. What follows belongs to the present more improved state of algebraic science.*

* We shall here point out a method of finding the equal roots of an equation, which avoids the laborious process of seeking the common divisor, and which may be employed when any other than Sturm's process for discovering the roots of an equation is used.

1. If an equation whose coefficients are commensurable have a pair of equal roots and no greater number, these roots must be commensurable; for the common measure of the first member of this equation, and the function derived from it, will be a binomial expression of the first degree with finite coefficients, and which, when equated to zero, will furnish one of the equal roots; these roots, therefore, must be commensurable, that is, either integers or fractions.

2. If the leading coefficient in the supposed equation be unity, and the others integral, the equal roots must be integral, because no fractional root can exist under these conditions (Art. 246).

3. If an equation with commensurable coefficients have three equal roots, and no more, these also must be commensurable; for, in this case, the common measure will be of the second degree, and, when equated to zero, will give *two* of the equal rorts: these roots, as just remarked, must be commensurable; hence all the three roots must be commensurable.

BINOMIAL EQUATIONS.

298. Binomial equations are those which can be reduced to the form

$$x^m = A \text{ or } x^m - A = 0 \text{ . . . . . . } (1)$$

A being any known quantity whatsoever.

And, as before, if the leading coefficient be unity, and the others integral, the equal roots will be integral.

4. By the same reasoning, if an equation with commensurable coefficients have m equal roots, and no other groups of equal roots, these m roots must be commensurable; and they will be integral if the leading coefficient be unity and the other coefficients integers.

5. When the leading coefficient is unity, and the other coefficients whole numbers, and m equal integral roots enter, we may infer, from the formation of the coefficients (245), that the absolute number, and the coefficient of the immediately preceding term, that is, the coefficient of x, will admit of a common measure involving $m-1$ of these roots; that the coefficients of x and x^2 will have a common measure involving $m-2$ of them; and so on till we come to the coefficients of x^{m-2} and x^{m-1}, which will have a common measure involving the multiple root once. For, if the depressed equation containing only the unequal roots be considered, it will involve none but integral coefficients, since its last term is formed from the penult coefficient of the proposed divided by one root; so that if the equal roots be now introduced, they can combine with none but integral factors. Hence, if the root occur twice, it will be found among the integral factors of the common measure of the coefficients A_n (the final coefficient) and A_{n-1}; if it occur three times, it will be found among the factors of the common measure of A_n, A_{n-1}, and A_{n-2}, and so on. And, therefore, by trying several factors of the common measure in question, by actually substituting them for x in the proposed equation, when from any circumstance multiple roots are suspected to exist, we may remove all doubt on the subject. In analyzing an equation, the doubts that may arise as to the entrance of equal roots are confined to certain definite intervals, or within determinate numerical limits; so that, of the factors adverted to above, only those falling within these limits need be regarded.

And further, if the repeated root occur but twice, the square of it must be a factor of x^0 or A_n; if it occur three times, the cube of it must be a factor of A_n, and the square of it a factor of A_{n-1}; if it occur four times, the fourth power of it must be a factor of A_n, the cube of it a factor of A_{n-1}, and the square of it a factor of A_{n-2}, and so on. And thus, of the factors of A_n to be tested, those only need be used whose powers also are factors, entering, as here described, according to the multiplicity of the roots.

6. These inferences may be easily generalized: they apply, whatever be the integral value of the leading coefficient, and whether the repeated root be integral or fractional. Thus, let the repeated root be $x=\frac{a}{b}$, a and b having no common factor; then, if the root enter m times, the original polynomial will be divisible by $(bx-a)^m$, giving a quotient involving the remaining roots, and into which none but integral coefficients enter (253). Let us now return to the original polynomial by multiplying this quotient by $bx-a$ m times: the first multiplication by $bx-a$ will evidently give a product, into the first term of which b must enter as a factor, and into the last of which a must enter; the next multiplication must, therefore, give a product, into the first term of which b^2 must enter, into the second b, into the last a^2, and into the last but one a; the third multiplication, therefore, must give a product whose first three terms involve b^3, b^2, b respectively, and last three a^3, a^2, a, reckoning these last in reverse order, and so on. Hence the coefficients A_1, A_2, A_3, &c., will be divisible by b^m, b^{m-1}, b^{m-2}, &c., respectively, down to b; and the coefficients A_n, A_{n-1}, A_{n-2}, &c., by a^m, a^{m-1}, a^{m-2}, &c., down to a. In other words, the coefficients, taken in order, reckoning from the beginning, will be divisible by the corresponding decreasing powers of the *denominator* of the repeated root; and the coefficients, reckoning from the end, will be divisible by the like powers of the *numerator*.

7. The inferences still have place, whatever be the degree of the multiple factor entering the proposed polynomial, so long as this factor, as well as the original polynomial, have none but integral coefficients. This is plain, from the reasoning in the preceding case, which remains the same, as respects the entrance of the factors b, a, whether the repeated multiplier be $bx-a$ or $bx^m+\ldots+a$.

We perceive immediately that the m roots of this equation are different from one another; for the first member $x^m - A$ has no common factor with its derived function mx^{m-1}, and hence the proposed equation (Art. 253, Schol.) can not have equal roots. The roots, if we raise them to the power m, ought each to produce A, since they are the same as the values embraced in the expression $x = \sqrt[m]{A}$. We know, then, that this radical has m different values; but we shall recur to this subject again, and more at length.

299. When m is any composite number, the solution of equation (1) reduces itself to the solution of several binomial equations, the degrees of which are the factors of m.

Suppose $m = pqr$, instead of the equation $x^{pqr} = 0$, we can take the equations

$$x^p = x', \; x'^q = x'', \; x''^r = A,$$

in which x', x'' are new unknowns.

It is evident that, after we have solved the equation $x''^r = A$, the preceding equation $x'^q = x''$ will make known the values of x', and that then the equation $x^p = x'$ will give all the roots of the proposed equation. This agrees with the formula demonstrated in the theory of radicals (Art. 63), viz.,

$$\sqrt[p]{\sqrt[q]{\sqrt[r]{A}}} = \sqrt[pqr]{A}.$$

300. Designate by a a quantity whose m^{th} power is A, and take $x = ay$. The equation $x^m = A$ becomes $a^m y^m = a^m$; dividing by a^m,

$$y^m = 1;$$

hence $y = \sqrt[m]{1}$, and, consequently, $x = a\sqrt[m]{1}$.

We conclude, therefore, that the roots of the equation $x^m = A$ can be obtained by multiplying one of them by the roots of the equation $y^m = 1$; or, in general, that the different m^{th} roots of a quantity can be obtained by multiplying one of them by the m^{th} roots of unity.

301. Let us consider more particularly the case in which A is a real quantity; and, to distinguish the hypothesis of A being positive or negative, write the binomial equation in this form:

$$x^m = \pm A \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (2)$$

These conclusions will greatly simplify the research after equal roots, and will either enable us wholly to dispense with the laborious process for the common measure, or will, at least, render the more tedious steps of it unnecessary

EXAMPLES.

$$2x^4 - 12x^3 + 19x^2 - 6x + 9 = 0 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (1)$$
$$x^7 + 5x^6 + 6x^5 - 6x^4 - 15x^3 - 3x^2 + 8x + 4 = 0 \; . \; . \; . \; (2)$$

The first of these can have no fractional repeated roots, because the leading coefficient 2 has no factor a perfect power; the equal roots, if any, must, therefore, be integral. Unity, which always has claims to be tried, does not succeed; and from the factors of 9 and 6, it is plain that +3 and —3 are the only other numbers to be tested; and as no higher power of 3 than the square enters 9, we infer that more than two equal roots can not have place in the equation. By testing 3, we find this to be one of a pair of equal roots. Equal quadratic factors could not possibly enter the equation, since, as the first coefficient shows, the polynomial is not a complete square. In the second of the above equations no fractional roots can enter. Applying, therefore, +1 and —1, we discover that +1 is twice a root, and —1 three times. The remaining equal roots —2 and —2 are found from the resulting quadratic obtained by suppressing from the given equation the five factors of the first degree.

We can determine, at least by approximation, a positive quantity a such that we have $a^m = A$. Take, again, $x = ay$, equation (2) will become

$$y^m = \pm 1.$$

This is the equation to which I shall confine myself exclusively.

302. The following remarks may be made with regard to this equation:

1. When m is an odd number, and the equation is $y^m = 1$ or $y^m - 1 = 0$, it evidently has the root $y = 1$; and it has no other real root, for every other positive value of y will give $y^m > 1$ or $y^m < 1$, and a negative value will render y^m negative. To obtain the equation on which the $m-1$ imaginary roots depend, we shall divide $y^m - 1$ by $y - 1$, and thus obtain the equation

$$y^{m-1} + y^{m-2} + y^{m-3} \ldots + y + 1 = 0,$$

which belongs to the class of equations called *reciprocal.*

2. When m is an odd number, and the equation is $y^m = -1$, it has evidently for a root $y = -1$. By a reasoning analogous to the preceding, it may be proved that the other roots are imaginary; and we obtain the equation on which they depend by dividing $y^m + 1 = 0$ by $y + 1$. But to obtain all the roots of the equation $y^m = -1$, it is well to remark that this equation can be derived from $y^m = -1$ by changing y into $-y$. It will suffice, then, to take all the roots of $y^m = 1$ with contrary signs.

3. Suppose m is an even number, and let $m = 2n$, the equation $y^{2n} = 1$, or $y^{2n} - 1 = 0$, has for its roots $y = +1$ and $y = -1$. The other roots are imaginary, and the equation which contains them can be obtained by dividing $y^{2n} - 1 = 0$ by $(y-1)(y+1)$, or $y^2 - 1$; but it will be well to observe that $y^{2n} - 1 = (y^n - 1)(y^n + 1)$, and that, consequently, the equation $y^{2n} - 1 = 0$ can be replaced by two others more simple,

$$y^n - 1 = 0, \quad y^n + 1 = 0.$$

4. Finally, when the equation is $y^{2n} = -1$, or $y^{2n} + 1 = 0$, we know that the even powers of real quantities will always give positive results; we hence conclude that all the roots are imaginary. Taking $y^2 = z$, the equation reduces to the degree n, and becomes simply $z^n = -1$.

303. I now proceed to determine the solutions of the equations $y^m - 1 = 0$, $y^m + 1 = 0$, in some particular cases.

Let $m = 2$; the equations to be resolved are

$$y^2 - 1 = 0, \text{ whence } y = \pm 1;$$

$$y^2 + 1 = 0, \text{ whence } y = \pm \sqrt{-1}.$$

Let $m = 3$; to resolve the equation $y^3 - 1 = 0$, observe that it has for a root $y = 1$; we divide it by $y - 1$, and it becomes

$$y^2 + y + 1 = 0, \text{ whence } y = \frac{-1 \pm \sqrt{-3}}{2}.$$

Hence, the three roots are

$$y = 1, \; y = \frac{-1 + \sqrt{-3}}{2}, \; y = \frac{-1 - \sqrt{-3}}{2}.$$

If we take the equation $y^3 + 1 = 0$, we shall observe that its roots are the same, except as regards sign, with those of $y^3 - 1 = 0$; consequently, they will be

$$y = -1, \; y = \frac{1 - \sqrt{-3}}{2}, \; y = \frac{1 + \sqrt{-3}}{2}.$$

Let $m=4$; the equation $y^4-1=0$ may be decomposed into two others, $y^2-1=0$, $y^2+1=0$; and from these equations we derive the four roots

$$y\pm 1,\ y\pm\sqrt{-1}.$$

The equation y^4+1 will be resolved differently; by adding $2y^2$ to both members of the equation, we can write it thus:

$$(y^2+1)^2=2y^2;$$

we can then decompose it into two others,

$$y^2+1=y\sqrt{2},\ y^2+1=-y\sqrt{2};$$

and, finally, from these we derive the four values of y,

$$y=\tfrac{1}{2}\sqrt{2}\pm\tfrac{1}{2}\sqrt{-2},\ y=-\tfrac{1}{2}\sqrt{2}\pm\tfrac{1}{2}\sqrt{-2}.$$

We could have treated the equation $y^4+1=0$ as a *reciprocal* equation. We might have observed, also, that it gives $y^2=\pm\sqrt{-1}$, and that, taking successively $+\sqrt{-1}$, $-\sqrt{-1}$, we have

$$y=\pm\sqrt{+\sqrt{-1}},\ y=\pm\sqrt{-\sqrt{-1}}.$$

We have then only to reduce these values to the form $\alpha+\beta\sqrt{-1}$ by the process in Art. 104.

By raising the equation $y^m\mp 1=0$ successively to the 10° degree, we shall find that its resolution depends on that of the preceding cases, or on the resolution of reciprocal equations, which reduce it to a degree less than the 5°.

Let us examine, first, the odd degrees. If we have the equation $y^5-1=0$, having observed that it has the root $y=1$, we divide it by $y-1$; it then becomes

$$y^4+y^3+y^2+y+1=0,$$

a reciprocal equation, which we shall reduce to the 2° degree. To do this, we first write it under the form

$$\left(y^2+\frac{1}{y^2}\right)+\left(y+\frac{1}{y}\right)+1=0.$$

Then take $y+\dfrac{1}{y}=z$, which gives $y^2+\dfrac{1}{y^2}=z^2-2$; and, consequently, the equation in y will be changed to the following:

$$z^2+z-1=0,\ \text{whence}\ z=\frac{-1\pm\sqrt{5}}{2}.$$

These values being known, those of y will be by the relation $y+\dfrac{1}{y}=z$, for this relation gives

$$y=\frac{z\pm\sqrt{z^2-4}}{2};$$

and we have only to substitute instead of z successively each of its two values, in order to find the four imaginary values of y. We have then the five values of y,

$$y=1,$$

$$y=\frac{-1+\sqrt{5}}{4}\pm\frac{\sqrt{10+2\sqrt{5}}}{4}\sqrt{-1},$$

$$y=\frac{-1-\sqrt{5}}{4}\pm\frac{\sqrt{10-2\sqrt{5}}}{4}\sqrt{-1}.$$

The equation $y^7-1=0$ will lead to the equation $z^3+z^2-2z-1=0$, and the equation $y^9-1=0$ to the equation $z^4+z^3-3z^2-2z+1=0$.

The equations $y^5+1=0$, $y^7+1=0$, $y^9+1=0$ have, except as regards the signs, the same roots as if their second terms had been -1.

Let us examine the even degrees. The equations $y^6-1=0$, $y^8-1=0$, $y^{10}-1=0$ do not offer any difficulty, because each of them can be decomposed into two others whose roots are known.

Taking $+1$ instead of -1, the analogous equations are

$$y^6+1=0, \text{ whence } y=\sqrt{\sqrt[3]{-1}},$$

$$y^8+1=0, \text{ whence } y=\sqrt{\sqrt[4]{-1}},$$

$$y^{10}+1=0, \text{ whence } y=\sqrt{\sqrt[5]{-1}}.$$

But we know the values of $\sqrt[3]{-1}$, $\sqrt[4]{-1}$, $\sqrt[5]{-1}$; we have, then, only to extract the square roots by the processes in Art. 104. But it will be simpler to treat these equations as reciprocal; for the transformed equations in z, on which they depend, have roots which are real, and are very easy to resolve.

We add some propositions upon binomial equations, preparatory to giving a general method for solving those of all degrees.

PROPOSITION I.

304. If a be one of the imaginary roots of the equation $x^n-1=0$, then any power of a will be also a root.

For, since a is one root of the equation $x^n-1=0$, therefore $a^n=1$, and, consequently,

$$a^{2n}=1,\ a^{3n}=1,\ a^{4n}=1, \text{ \&c., also } a^{-n}=1,\ a^{-2n}=1,\ a^{-3n}=1, \text{ \&c.,}$$

the values

$$a,\ a^2,\ a^3\ldots,\ a^{-1},\ a^{-2},\ a^{-3},\ldots,$$

thus satisfying the conditions of the equation, are roots of it.

Corollary 1.—It hence appears that the roots of the equation $x^n-1=0$ may be represented under an infinite variety of forms, each term in the following series being a root, viz.:

$$\ldots a^{-3},\ a^{-2},\ a^{-1},\ 1,\ a,\ a^2,\ a^3,\ \ldots a^{n-1},\ a^n,\ a^{n+1},\ \ldots a^{2n},\ a^{2n+1},\ldots$$

in which series, however, there can not be more than n quantities essentially different, otherwise the equation would have more than n roots.

PROPOSITION II.

305. If a be one of the imaginary roots of the equation $x^n+1=0$, then any odd power of a will be also a root.

For, since a is one root of the equation $x^n=-1$, therefore $a^n=-1$; and, since every odd power, whether positive or negative, of -1 is also -1, therefore,

$$a^{3n}=-1,\ a^{5n}=-1,\ a^{7n}=-1, \text{ \&c.,}$$

also

$$a^{-3n}=-1,\ a^{-5n}=-1,\ a^{-7n}=-1, \text{ \&c.;}$$

so that the quantities

$$a,\ a^3,\ a^5\ldots,\ a^{-1},\ a^{-3},\ a^{-5},\ldots,$$

are roots of the equation. These roots, therefore, assume an infinite variety of forms, although there can not be more than n essentially different.

PROPOSITION III.

306. To determine the roots of the equation $x^n-1=0$, when n is the square of a prime number p.

Put $x^p=z$, then $x^p-z=0$, and $z^p-1=0$, and let the roots of this last equation be $1, \beta, \beta^2, \beta^3, \ldots \beta^{p-1}$; then, by substitution,

$$x^p-z=\begin{cases} x^p-1=0, \\ x^p-\beta=0, \\ x^p-\beta^2=0, \\ x^p-\beta^3=0, \\ \text{\&c. \quad \&c.} \end{cases}$$

Hence the pp values of x, in these p equations, will evidently be all different, and will be the roots of the equation $x^{pp}-1=0$.

To determine these roots, it will be sufficient to advert to Art. 300, which proves that the roots of $x^p-\beta=0$ are equal to the roots of $x^p-1=0$ multiplied by $\sqrt[p]{\beta}$; and, in a similar manner, the roots of $x^p-\beta^2=0$ are equal to the roots of $x^p-1=0$, multiplied by $\sqrt[p]{\beta^2}$, &c.; therefore, we immediately conclude that the roots of

$$\left.\begin{array}{lll} x^p-1=0 \text{ are} & 1,\ \beta,\ \beta^2,\ \beta^3 \ldots\ldots\ldots\ldots \beta^{p-1} \\ x^p-\beta=0 & \sqrt[p]{\beta},\ \beta\sqrt[p]{\beta},\ \beta^2\sqrt[p]{\beta} \ldots\ldots \beta^{p-1}\sqrt[p]{\beta} \\ x^p-\beta^2=0 & \sqrt[p]{\beta^2},\ \beta\sqrt[p]{\beta^2},\ \beta^2\sqrt[p]{\beta^2} \ldots \beta^{p-1}\sqrt[p]{\beta^2} \\ \text{\&c.} & \text{\&c.} \qquad \text{\&c.} \end{array}\right\} = \text{the } n \text{ roots of } x^n-1=0,$$

For example, let it be required to find the roots of $x^9-1=0$.

The roots of $x^3-1=0$ are

$$1, \frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2};$$

hence the roots of $x^9-1=0$ are

$$1, \frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2},$$

$$\sqrt[3]{\frac{-1+\sqrt{-3}}{2}}, \frac{-1+\sqrt{-3}}{2}\sqrt[3]{\frac{-1+\sqrt{-3}}{2}},$$

$$\frac{-1-\sqrt{-3}}{2}\sqrt[3]{\frac{-1+\sqrt{-3}}{2}}, \sqrt[3]{\frac{-1-\sqrt{-3}}{2}},$$

$$\frac{-1+\sqrt{-3}}{2}\sqrt[3]{\frac{-1-\sqrt{-3}}{2}}, \frac{-1-\sqrt{-3}}{2}\sqrt[3]{\frac{-1-\sqrt{-3}}{2}}.$$

The foregoing propositions have been devoted chiefly to an examination of the properties and relations of these roots, and not to the actual exhibition of their values, although, as in the proposition above, one or two examples of the solution have been given to illustrate the reasoning. To obtain the imaginary roots, however, in their simplest form, that is, in the form $a+b\sqrt{-1}$, and for all values of the exponent, requires the aid of a theorem, borrowed from the science of Trigonometry.

307. The theorem to which we refer is the well-known formula of De Moivre, given in most books on Analytical Trigonometry.

$$(\cos a \pm \sin a \cdot \sqrt{-1})^n = \cos na \pm \sin na \cdot \sqrt{-1};$$

which, if the arc $2k\pi$ (π being a semi-circumference, and k any integer) be substituted for na, becomes

$$\left(\cos\frac{2k\pi}{n}\pm\sin\frac{2k\pi}{n}\,.\sqrt{-1}\right)^n=\cos 2k\pi\pm\sin 2k\pi\,.\sqrt{-1}\,;$$

that is, since

$$\cos 2k\pi=1,\text{ and }\sin 2k\pi=0,$$

$$\left(\cos\frac{2k\pi}{n}\pm\sin\frac{2k\pi}{n}\,.\sqrt{-1}\right)^n=1\,;$$

so that the expression

$$\cos\frac{2k\pi}{n}\pm\sin\frac{2k\pi}{n}\,.\sqrt{-1},$$

comprehends in it all the n roots of unity, or all the particular values of x, which satisfy the equation $x^n-1=0$.

Although, in this general expression, the value of k is quite arbitrary, yet, assume it what we will, the expression can never furnish more than n different values. These different values will arise from the several substitutions of

$$0,\ 1,\ 2,\ 3\ldots.$$

up to the number $\frac{n-1}{2}$, inclusively, if n is odd, and up to $\frac{n}{2}$, if n is even; and for substitutions beyond these limits the preceding results will recur. To prove this, let us actually substitute as proposed; we shall thus have the fol lowing series of results, viz.:

$$\text{for } k=0\ \ldots.\ x=\cos 0\pm\sin 0\,.\sqrt{-1}=1$$

$$k=1\ \ldots.\ x=\cos\frac{2\pi}{n}\pm\sin\frac{2\pi}{n}\,.\sqrt{-1}$$

$$k=2\ \ldots.\ x=\cos\frac{4\pi}{n}\pm\sin\frac{4\pi}{n}\,.\sqrt{-1}$$

$$k=3\ \ldots.\ x=\cos\frac{6\pi}{n}\pm\sin\frac{6\pi}{n}\,.\sqrt{-1}$$

$$\vdots\qquad\qquad\vdots$$

$$k=\frac{n-1}{2}\ \ldots\ x=\cos\frac{(n-1)\pi}{n}\pm\sin\frac{(n-1)\pi}{n}\,.\sqrt{-1}.$$

Each of these expressions, except the first, involves two distinct values; so that, omitting the value given by $k=0$, there are $n-1$ values, and, consequently, altogether, there are n values; and that they are all different is plain, because the arcs

$$0,\ \frac{2\pi}{n},\ \frac{4\pi}{n},\ \frac{6\pi}{n},\ \ldots.\ \frac{(n-1)\pi}{n},$$

being all different, and less than π, have all different cosines. The arcs which would arise from continuing the substitutions are

$$\frac{(n+1)\pi}{n},\ \frac{(n+3)\pi}{n},\ \frac{(n+5)\pi}{n},\ \&c.\,;$$

or, which are the same,

$$2\pi-\frac{(n-1)\pi}{n},\ 2\pi-\frac{(n-3)\pi}{n},\ 2\pi-\frac{(n-5)\pi}{n},\ \&c.,$$

and the sines and cosines of these are respectively the same as the sines and cosines of the arcs

$$\frac{(n-1)\pi}{n}, \frac{(n-3)\pi}{n}, \frac{(n-5)\pi}{n}, \&c.,$$

which have already occurred.*

If n is an even number, the final substitution for k must be $\frac{n}{2}$ instead of $\frac{n-1}{2}$, as above; and, therefore, the final pair of conjugate values for x will be

$$x = \cos\pi \pm \sin\pi \cdot \sqrt{-1} = -1,$$

which values of x differ from all the other values, because in them no arc occurs so great as π.

The arcs which would arise from continuing the substitutions beyond $k = \frac{n}{2}$ are

$$\frac{(n+2)\pi}{n}, \frac{(n+4)\pi}{n}, \frac{(n+6)\pi}{n}, \&c.;$$

or, which are the same,

$$2\pi - \frac{(n-2)\pi}{n}, 2\pi - \frac{(n-4)\pi}{n}, 2\pi - \frac{(n-6)\pi}{n}, \&c.,$$

and the sines and cosines of these are respectively the same as the sines and cosines of the arcs

$$\frac{(n-2)\pi}{n}, \frac{(n-4)\pi}{n}, \frac{(n-6)\pi}{n}, \&c.,$$

which have already occurred.*

It is easy to see that in every pair of conjugate roots, each is the reciprocal of the other. In fact, whatever be k,

$$(\cos\frac{2k\pi}{n} + \sin\frac{2k\pi}{n} \cdot \sqrt{-1})(\cos\frac{2k\pi}{n} - \sin\frac{2k\pi}{n} \cdot \sqrt{-1}) =$$
$$\cos^2\frac{2k\pi}{n} + \sin^2\frac{2k\pi}{n} = 1,$$

which shows that the two factors in the first member are of the form $a, \frac{1}{a}$.

We have proved (Art. 304) that every power of an imaginary root of the binomial equation is also a root; but, unless n be a prime number, we could not infer that all the roots would ever be produced by involving any one of them. Such would not, indeed, be the case. There is always, however, one among the imaginary roots of which the involution will furnish all the others; it is the first imaginary root, or that due to the substitution $k=1$, in the foregoing series of values; for, by De Moivre's formula, the powers of this produce all the others, thus:

$$(\cos\frac{2\pi}{n} + \sin\frac{2\pi}{n} \cdot \sqrt{-1})^2 = \cos\frac{4\pi}{n} + \sin\frac{4\pi}{n} \cdot \sqrt{-1}$$
$$(\cos\frac{2\pi}{n} + \sin\frac{2\pi}{n} \cdot \sqrt{-1})^3 = \cos\frac{6\pi}{n} + \sin\frac{6\pi}{n} \cdot \sqrt{-1}$$
$$\vdots \qquad\qquad \vdots$$
$$(\cos\frac{2\pi}{n} + \sin\frac{2\pi}{n} \cdot \sqrt{-1})^{\frac{n-1}{2}} = \cos\frac{n-1}{n}\pi + \sin\frac{n-1}{n}\pi \cdot \sqrt{-1}.$$

* The *signs* of the sines will, however, be different; but the only effect of this difference is evidently to furnish each pair of conjugate roots in an inverse order.

These powers of the first imaginary root, which we may call a, thus furnish one half of the entire number of imaginary roots, and the reciprocals of these being the other half, all of them are determined from the first; the imaginary roots are, therefore,

$$a,\ a^2,\ a^3,\ \ldots\ a^{\frac{n-1}{2}}$$
$$\frac{1}{a},\ \frac{1}{a^2},\ \frac{1}{a^3},\ \ldots\ \frac{1}{a^{\frac{n-1}{2}}}.$$

When n is even, the last power will be

$$(\cos\frac{2\pi}{n}+\sin\frac{2\pi}{n}\cdot\sqrt{-1})^{\frac{n}{2}}=\cos\pi+\sin\pi\cdot\sqrt{-1};$$

and the imaginary roots are, therefore,

$$a,\ a^2,\ a^3,\ \ldots\ a^{\frac{n}{2}}$$
$$\frac{1}{a},\ \frac{1}{a^2},\ \frac{1}{a^3},\ \ldots\ \frac{1}{a^{\frac{n}{2}}}$$

308. By the general formula (Art. 307), we are enabled to determine all the roots of the equation

$$x^n+1=0;$$

for, since

$$\cos(2k+1)\pi=-1,\text{ and }\sin(2k+1)\pi=0,$$

that formula gives

$$(\cos\frac{2k+1}{n}\pi\pm\sin\frac{2k+1}{n}\pi\cdot\sqrt{-1})^n=$$
$$\cos(2k+1)\pi\pm\sin(2k+1)\pi\cdot\sqrt{-1}=-1;$$

hence the n values of x are all comprised in the general expression

$$x=\cos\frac{2k+1}{n}\pi\pm\sin\frac{2k+1}{n}\pi\cdot\sqrt{-1};$$

which, by putting for k the values 0, 1, 2, 3, &c., in succession, furnishes the following series of separate values, viz.:

$$\text{for } k=0\ \ldots\ x=\cos\frac{\pi}{n}\pm\sin\frac{\pi}{n}\cdot\sqrt{-1}$$
$$k=1\ \ldots\ x=\cos\frac{3\pi}{n}\pm\sin\frac{3\pi}{n}\cdot\sqrt{-1}$$
$$k=2\ \ldots\ x=\cos\frac{5\pi}{n}\pm\sin\frac{5\pi}{n}\cdot\sqrt{-1}$$
$$\vdots$$
$$k=\frac{n-1}{1}\ \ldots\ x=\cos\pi\pm\sin\pi\cdot\sqrt{-1}=-1;$$

or, when n is even,

$$k=\frac{n-2}{2}\ \ldots\ x=\cos\left(\pi-\frac{\pi}{n}\right)\pm\sin(\pi-\frac{\pi}{n}\cdot\sqrt{-1}).$$

Now that the foregoing system of n roots are all different is obvious, since

$$\frac{\pi}{n},\ \frac{3\pi}{n},\ \frac{5\pi}{n}\ \ldots\ \frac{n\pi}{n},\text{ or }\pi-\frac{\pi}{n},$$

are all different arcs, of which the greatest does not exceed a semi-circumference. If the preceding series be extended, it will be easy to prove, after what has been done in Art. 307, that the values formerly obtained will recur.

As in the former case of the general problem, so here, each root may be derived from the first pair of the series; thus, denoting the first root, $\cos\frac{\pi}{n} \pm \sin\frac{\pi}{n} \cdot \sqrt{-1}$, by a or $\frac{1}{a}$, according as the upper or lower sign is taken, we evidently have, for the preceding series, the following equivalent expressions, viz.:

$$\left.\begin{matrix} a, a^3, a^5, \ldots\ldots a^n \\ \frac{1}{a}, \frac{1}{a^3}, \frac{1}{a^5}, \ldots\ldots \frac{1}{a^n} \end{matrix}\right\} \text{when } n \text{ is odd,}$$

and

$$\left.\begin{matrix} a, a^3, a^5, \ldots\ldots\ldots a^{n-1} \\ \frac{1}{a}, \frac{1}{a^3}, \frac{1}{a^5}, \ldots\ldots\ldots \frac{1}{a^{n-1}} \end{matrix}\right\} \text{when } n \text{ is even.}$$

For further researches on the theory of binomial equations, the student may consult Lagrange's Traité de la Résolution des Equations Numériques, Note 14; Legendre's Théorie des Nombres, Part V.; the Disquisitiones Arithmeticæ of Gauss; Barlow's Theory of Numbers; and Ivory's article on Equations, in the Encyclopædia Britannica.

309. We have already frequently had occasion to notice multiple values of radicals, without fixing the precise number which might exist, except for radicals of the second degree. It is time to introduce the following proposition:

Every radical has as many values as there are units in its index, and has no more; in other words, every quantity has as many roots of a given degree as there are units in the index of that degree.

If the given radical be represented by the general form $\sqrt[m]{A}$, this radical designates evidently all the quantities, real or imaginary, which, raised to the power m, reproduce A; consequently they are merely the values of x in the equation $x^m = A$. But we know, from the general theory of equations, that every equation of the m^{th} degree has m values of the unknown quantity, which will each satisfy it; hence the proposition is proved.

This will serve to explain some paradoxes. Let there be the expression $\sqrt[4]{a}\sqrt{-1}$. By reducing the second radical to the index 4, it becomes $\sqrt[4]{(-1)^2}$, and the given expression reduces to $\sqrt[4]{a}$, a result which might be supposed absurd, because, a being positive, the result represents a real quantity, while the proposed expression appears to be imaginary.

There is here a confusion of ideas. If in the expression $\sqrt[4]{a}\sqrt{-1}$ the radical is an arithmetical determination, it is true that this expression is imaginary; but if $\sqrt[4]{a}$ be taken in all its generality, and we represent it by a' multiplied by the four roots of unity, or

$$a', \ -a', \ a'\sqrt{-1}, \ -a'\sqrt{-1},$$

we perceive that some of these values of $\sqrt[4]{a}$, multiplied by $\sqrt{-1}$, cause this imaginary factor to disappear, and the proposed expression becomes real.

I shall terminate this article by the explanation of a paradox which presents itself in the employment of fractional exponents. Let there be the expression $a^{\frac{2}{4}}$. If the fraction $\frac{2}{4}$ be simplified, the expression $a^{\frac{2}{4}}$ becomes $a^{\frac{1}{2}}$. Then, in repassing to the radicals, we have $\sqrt[4]{a^2} = \sqrt{a}$. This equality, however, is not wholly true, because the first member has four values, and the second but two.

The difficulty may be presented in a general manner by placing

$$a^{\frac{mp}{np}}=a^{\frac{m}{n}},$$

and in concluding from thence that

$$\sqrt[mp]{a^{np}}=\sqrt[m]{a^n}.$$

To discover the cause of this error, we must remember that the fractional exponent is but a convention, by means of which we express in another way that the root of a certain power is to be extracted, and, therefore, this exponent must not be regarded in the light of an ordinary fraction.

THE DETERMINATION OF THE IMAGINARY ROOTS OF EQUATIONS.

310. In what relates to the limits of roots at Art. 283 and following, real roots only were in view. We shall show here how the limits may be obtained for the moduli of all roots, whether real or imaginary. Let us consider the equation

$$x^m+Px^{m-1}+Qx^{m-2}\ldots=0 \quad . \; . \; . \; . \; . \; . \; . \; . \; (1)$$

in which P, Q... may be real or imaginary. In order that a value of x may be a root, it is necessary that, after having substituted it in the result, the modulus should be zero.

Call v the modulus of x, and p, q, ... those of the coefficients P, Q, ... According to Art. 239, those of the terms of the equation will be v^m, pv^{m-1}, qv^{m-2}, ..., and that of the part $Px^{m-1}+Qx^{m-2}+\ldots$ can not surpass the sum $pv^{m-1}+qv^{m-2}\ldots$ Then, if we choose for v a value λ such that we have

$$v^m-pv^{m-1}-qv^{m-2}-\ldots=0, \text{ or } >0 \quad . \; . \; . \; . \; (2)$$

we are sure, by virtue of the article just cited, that the modulus of the first member of the equation (1) will not be less than the above difference; and that from this point the modulus will not be zero, or, what is the same thing, the value substituted in place of x will not be a root of the equation. Every value of v above λ will render this difference greater; then λ is a superior limit of the moduli.

The quantity λ will be always easy to determine, because it will be sufficient to substitute in the difference (2) in place of v, increasing positive values until this difference becomes positive. If the coefficients P, Q... are real, the moduli p, q, ... will be these coefficients themselves, but taken positively; and if we designate the greatest of these values by N, we can take at once for the superior limit $\lambda=N+1$.

To have an inferior limit, we make $x=\frac{1}{y}$, determine in the transformed in y the superior limit of the moduli of the roots, and finally divide unity by this limit.

311. It has already been proved that imaginary roots always enter into equations in conjugate pairs of the form $a\pm\beta\sqrt{-1}$. And this previous knowledge of the form which every root must take suggests a method for the actual determination of the proper numerical values for a and β in any proposed case. The method is as follows:

Let $$x^n+A_{n-1}x^{n-1}+\ldots\ldots Ax+N=0$$

be an equation containing imaginary roots; then, by substituting $a+\beta\sqrt{-1}$ for x, we have

$$(a+\beta\sqrt{-1})^n+A_{n-1}(a+\beta\sqrt{-1})^{n-1}+\,..\,A(a+\beta\sqrt{-1})+N=0\,;$$

or, by developing the terms by the binomial theorem, and collecting the real and imaginary quantities separately, we have the form

$$M+N\sqrt{-1}=0,$$

an equation which can not exist except under the conditions

$$M=0,\ N=0\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (1)$$

From these two equations, therefore, in which M, N contain only the quantities a, β, combined with the given coefficients, all the systems of values of a and β may be determined; and these, substituted in the expression $a+\beta\sqrt{-1}$, will make known all the imaginary roots of the proposed equation; those that are real corresponding to $\beta=0$.

It is obvious from the theory of elimination as developed at page 157, and from the method of numerical solution explained in Art. 255, that the labor of deducing from this pair of equations the final equation involving only one of the unknowns a, β, and of afterward solving the equation for that unknown, will in general be very laborious for equations above the third degree. Lagrange, by combining with the principle of this solution the method of the squares of the differences explained at Art. 278, avoids both the elimination and subsequent solution here spoken of. It is easy to see how this may be brought about if we have any independent means of determining one of the unknowns β: for the adoption of these means would enable us to dispense with the elimination; and as the substitution of the value of β in both of the equations (1) would convert those equations into two simultaneous equations involving but one unknown quantity, their first members would necessarily have a common factor of the first degree in a, which, equated to zero, would furnish for a the proper value to accompany β; and thus, instead of solving the final equation referred to, we should only have to find the common measure between the two polynomials M, N containing the unknown quantity a.

Now corresponding to every pair of imaginary roots $a+\beta\sqrt{-1}$, $a-\beta\sqrt{-1}$, there necessarily exists, in the equation of the squares of the differences, a real negative root $-4\beta^2$; so that if all the negative roots of the latter equation be found, the quantity $-4\beta^2$ must appear among them; from which the value of β would be immediately obtained, and thence, by aid of the common measure as just explained, the corresponding value of a.

But the equation of the squares of the differences may have a greater number of negative roots than there are pairs of imaginary roots in the proposed; which, however, can not happen except two non-conjugate imaginary roots have equal real parts, or except a real root be equal to the real part of an imaginary root. Lagrange discusses these peculiarities, and establishes the exactness and generality of the principle in question, as follows:

When the real parts, a, γ, &c., of the imaginaries

$$\begin{array}{ll} a+\beta\sqrt{-1}, & a-\beta\sqrt{-1} \\ \gamma+\delta\sqrt{-1}, & \gamma-\delta\sqrt{-1} \\ \&c. & \&c. \end{array}$$

are unequal, as well when compared with one another as when compared with the real roots a, b, c, &c., it is evident that the equation of the squares of the differences can not have any other negative roots than those furnished by the several pairs of conjugate imaginary roots, and which are

$$-4\beta^2,\ -4\delta^2,\ \&c.$$

All the other roots, not arising from the differences furnished by the real roots, a, b, c, &c., will evidently be imaginary; those between the real and imaginary roots supplying the forms

$$(a-a+\beta\sqrt{-1})^2,\ (a-a-\beta\sqrt{-1})^2$$
$$(a-b+\beta\sqrt{-1})^2,\ (a-b-\beta\sqrt{-1})^2$$
$$\&c. \qquad \&c.$$

and those between the non-conjugate roots the forms

$$\{(a-\gamma)+(\beta-\delta)\sqrt{-1}\}^2,\ \{(a-\gamma)-(\beta-\delta)\sqrt{-1}\}^2$$
$$\{(a-\gamma)+(\beta+\delta)\sqrt{-1}\}^2,\ \{(a-\gamma)-(\beta+\delta)\sqrt{-1}\}^2$$

so that in this case every negative root in the auxiliary equation will indicate a pair of imaginary roots in the proposed, and will, moreover, supply the value of the imaginary part. But if it happen that among the quantities a, γ, &c., there be found any equal among themselves, or equal to any of the quantities a, b, c, &c., then the auxiliary equation will necessarily have negative roots, corresponding to which there can be no imaginary pair in the proposed equation.

For let $a=a$, then the two imaginary roots $(a-a+\beta\sqrt{-1})^2$, $(a-a-\beta\sqrt{-1})^2$ will become $-\beta^2$ and $-\beta^2$, and, consequently, real and negative; so that if the proposed equation contain only two imaginary roots, $a+\beta\sqrt{-1}$ and $a-\beta\sqrt{-1}$, then, in the case of $a=a$, the equation of the squares of the differences will contain, besides the real negative root $-4\beta^2$, the two $-\beta^2$, $-\beta^2$, both negative and equal.

We thus see that when the equation of the squares of the differences has three negative roots, of which two are equal to one another, the proposed may have either three pairs of imaginary roots, or but a single pair.

If the proposed contains four imaginary roots, $a+\beta\sqrt{-1}$, $a-\beta\sqrt{-1}$, $\gamma+\delta\sqrt{-1}$, $\gamma-\delta\sqrt{-1}$, then the equation of the squares of the differences must contain the two negative roots $-4\beta^2$ and $-4\delta^2$; if $a=a$, it must also contain the two equal negative roots $-\beta^2$, $-\beta^2$; and if, moreover, $\gamma=b$, it must contain, in addition to these, the negative pair $-\delta^2$, $-\delta^2$; and lastly, if $a=\gamma$, the four imaginary roots

$$\{(a-\gamma)+(\beta-\delta)\sqrt{-1}\}^2,\ \{(a-\gamma)-(\beta-\delta)\sqrt{-1}\}^2$$
$$\{(a-\gamma)+(\beta+\delta)\sqrt{-1}\}^2,\ \{(a-\gamma)-(\beta+\delta)\sqrt{-1}\}^2$$

will be converted into the two negative pairs

$$-(\beta-\delta)^2,\ -(\beta-\delta)^2;\ -(\beta+\delta)^2,\ -(\beta+\delta)^2.$$

Hence we may deduce the following conclusions, viz.:

(1) When all the real negative roots of the equation of the squares of the differences are unequal, then the proposed will necessarily have so many pairs of imaginary roots.

If in this case we call any one of these negative roots $-w$, we shall have $\beta=\frac{\sqrt{w}}{2}$; and if this value be substituted for β in the two equations (1), and the operation for the common measure of their first members be carried on till we arrive at a remainder of the first degree in a, the proper value of a will be obtained by equating this remainder to zero. Thus, each negative root, $-w$, will furnish two conjugate imaginary roots, $a+\beta\sqrt{-1}$, and $a-\beta\sqrt{-1}$.

(2) If among the negative roots of the equation of the squares of the differences equal roots are found, then each unequal root, if any such occur, will, as in the preceding case, always furnish a pair of imaginary roots. Each pair of equal roots may, however, give either two pairs of imaginary roots or no imaginary roots, so that two equal roots will give either four imaginary roots or none ; three equal roots will give either six imaginary roots or two ; four equal roots will give either eight imaginary roots, or four, or none ; and so on.

Suppose two of the negative roots, $-w$, $-w$, are equal ; then putting, as above, $\beta=\frac{\sqrt{w}}{2}$, we shall substitute this value of β in the two polynomials (1), and shall carry on the process for the common measure between these polynomials till we arrive at a remainder of the second degree in a ; since the polynomials must have a common divisor of the second degree in a, seeing that the equations (1) must have two roots in common, on account of the double value of β.

Equating, then, this quadratic remainder to zero, we shall be furnished with two values for a : these may be either both real or both imaginary. In the former case call the two values a' and a'' ; we shall then have the four imaginary roots

$$a'+\beta\sqrt{-1},\ a'-\beta\sqrt{-1},\ a''+\beta\sqrt{-1},\ a''-\beta\sqrt{-1}.$$

In the second case, the values of a being imaginary, contrary to the conditions by which the fundamental equations (1) are governed, we infer that to the equal negative roots $-w$, $-w$, there can not correspond any imaginary roots in the proposed equation.

If the equation of the squares of the differences have three equal negative roots, $-w$, $-w$, $-w$, then, putting, as before, $\beta=\frac{\sqrt{w}}{2}$, we should operate on the polynomials (1), for the common measure, till we reach a remainder of the third degree in a ; this remainder, equated to zero, will furnish three values of a, which will either be all real, or one real and two imaginary. In the first case six imaginary roots will be implied : in the second only two ; the imaginary values of a being always rejected, as not coming within the conditions implied in (1).

It follows from the above, and from what has been established in Art. 259, that there are at least as many variations of sign in the equation of the squares of differences as there are combinations of two real roots in the proposed equation. Also, it must have at least as many permanences of sign as there are pairs of conjugate imaginary roots in the proposed equation ; or, in other words, it can not have a less number of permanences of sign than half the number of imaginary roots in the proposed equation.

Hence we may infer, that if the equation of the squares of the differences have its terms alternately positive and negative, there can be no imaginary root in the proposed equation.

The foregoing principles are theoretically correct ; but the practical application of them, beyond equations of the third and fourth degrees, is too laborious for them to become available in actual computation. We give the following illustration of them from Lagrange.

312. To determine the imaginary roots of the equation

$$x^3-2x-5=0.$$

Computing the equation of the squares of the differences from the general formula for the third degree at Art. 279, viz.,

$$z^3+6pz^2+9p^2z+4p^3+27q^2=0,$$

in which $p=-2$ and $q=-5$, we have

$$z^3-12z^2+36z+643=0.$$

In order to determine the negative roots of this equation, change the alternate signs, or put $z=-w$, and then change all the signs, converting the equation into

$$w^3+12w^2+36w-643=0,$$

and seek the positive root, which is found by trial to lie between 5 and 6. Adopting Lagrange's development, Art. 297, this root proves to be

$$w=5+\cfrac{1}{6+\cfrac{1}{5+\cfrac{1}{6+,\ \&c.,}}}$$

from which we get the converging fractions (see Continued Fractions)

$$5,\ \frac{31}{6},\ \frac{160}{31},\ \frac{991}{192},\ \&c,$$

Knowing thus an approximate value of w, we know $\beta=\frac{\sqrt{w}}{2}$.

In order now to get the equations (1), p. 385, substitute $a+\beta\sqrt{-1}$ for x in the proposed equation, and form two equations, one with the real terms of the result, the other with the imaginary terms; we shall thus have the equations (1) referred to, viz.,

$$a^3-(3\beta^2+2)a-5=0$$
$$3a^2-\beta^2-2=0,$$

in which β is known.

Seeking now the greatest common measure of the first members of these equations, stopping the operation at the remainder of the first degree in a, and equating that remainder to zero, we have

$$a=-\frac{15}{8\beta^2+4},$$

and thus both a and β are determined in approximate numbers.

313. There is another method of proceeding for the determination of imaginary roots, somewhat different from the preceding, being independent of the equation of the squares of the differences. It is suggested from the following considerations:

Since the quadratic, involving a pair of imaginary conjugate roots, is always of the form

$$x^2-2ax+a^2+\beta^2=0,$$

every equation into which such roots enter must always be accurately divisible by a quadratic divisor of this form; that is, the proper values of a and β are such that the remainder of the first degree in x, resulting from the division, must be zero. This furnishes a condition from which those proper values of a and β may be determined; the condition, namely, that the remainder spoken of, $Ax-B$, must be equal to zero, independent of particular values of x; and this implies the twofold condition

$$A=0,\ B=0,$$

from which α and β, of which A and B are functions, may be determined.

As an example, let the equation proposed be

$$x^4+4x^3+6x^2+4x+5=0.$$

Dividing the first member by

$$x^2-2\alpha x+\alpha^2+\beta^2,$$

we have for quotient

$$x^2+(4+2\alpha)x+6+8\alpha+3\alpha^2-\beta^2,$$

and for the remainder of the first degree in x

$$(4+12\alpha+12\alpha^2+4\alpha^3-4\alpha\beta^2-4\beta^2)x-$$
$$(\alpha^2+\beta^2)(6+8\alpha+3\alpha^2-\beta^2)+5,$$

which, being equal to zero whatever be the value of x, furnishes the two equations

$$4+12\alpha+12\alpha^2+4\alpha^3-4\alpha\beta^2-4\beta^2=0$$
$$(\alpha^2+\beta^2)(6+8\alpha+3\alpha^2-\beta^2)+5=0.$$

From the first of these we get

$$\beta^2=(1+\alpha)^2$$

and this, substituted in the second, gives

$$4\alpha^4+16\alpha^3+24\alpha^2+16\alpha=0,$$

two roots of which are 0 and -2; the other two are imaginary, and must, consequently, be rejected as contrary to the hypothesis as to the form of the indeterminate quadratic divisor.

The two real values of α, substituted in the expression above for β^2, give

$$\text{for } \alpha=\ \ 0,\ \beta^2=1^2 \quad \therefore \beta=+1$$
$$\alpha=-2,\ \beta^2=(-1)^2 \therefore \beta=-1$$

and, consequently, the component factors of the original quadratic divisor, viz., the factors

$$x-\alpha-\beta\sqrt{-1},\ x-\alpha+\beta\sqrt{-1},$$

furnish these two pairs of imaginary roots, viz.,

$$x=\sqrt{-1},\ x=-\sqrt{-1},$$

and

$$x=-2-\sqrt{-1},\ x=-2+\sqrt{-1}.$$

This method, like that before given, is impracticable beyond very narrow limits, because of the high degree to which the final equation in α usually rises. And it is further to be observed of both, and, indeed, of all methods for determining imaginary roots by aid of the real roots of certain numerical equations, that whenever, as is usual, these real roots are obtained only approximately, our results may, under peculiar circumstances, be erroneous. For instance, in the two methods just explained we have two equations, $f(\alpha)=0$, $F(\beta)=0$, where the coefficients of α in the first are functions of β, and the coefficients of β in the second functions of α; hence, whichever of these symbols be computed approximately, in order to furnish determinate values for the coefficients of the other, these coefficients must vary slightly from the true coefficients; and, consequently, under this slight variation of the coefficients, real roots may become converted into imaginary, and imaginary into real.

The terms imaginary and impossible have been thought objectionable when applied to the roots of equations, inasmuch as definite algebraic expressions are always possible for these roots.

A specimen of a strictly impossible equation would be the following:

$$2x-5+\sqrt{x^2-7}=0,$$

when *plus* before the sign $\sqrt{}$ implies the positive root $\sqrt{x^2-7}$. No expression, either real or imaginary, can satisfy the condition or represent a root of this irrational equation.

The terms imaginary and impossible, when used, should be understood rather as applying to the solutions of the problem from which the equation is derived than to the expressions for the roots. The number of solutions which the problem admits will ordinarily be expressed by the degree of the equation, but certain suppositions affecting the values or signs of the coefficients may cause some of these solutions to become absurd or impossible, and these will be indicated by the form $a+b\sqrt{-1}$ for the roots, in which b is not zero.

THEORY OF VANISHING FRACTIONS.

314. From the principles established in (Art. 253), we readily derive the following consequences, viz.:

Since

$$f(x)=(x-a_1)(x-a_2)(x-a_3)(x-a_4)\ .\ .\ .\ .\ .$$

and

$$f_1(x)=(x-a\ )(x-a_2)(x-a_3)\ .\ .\ .\ .+(x-a_1)(x-a_2)(x-a_4)\ .\ .\ .+,\ \&c.,$$

it follows that

$$\frac{f_1(x)}{f(x)}=\ .\ .\ .\ .\ \frac{1}{x-a_4}+\frac{1}{x-a_3}+\frac{1}{x-a_2}+\frac{1}{x-a_1}\ .\ .\ .\ .\ (1)$$

In like manner, for any other equation $F(x)=0$, we have

$$\frac{F_1(x)}{F(x)}=\ .\ .\ .\ .\ \frac{1}{x-b_4}+\frac{1}{x-b_3}+\frac{1}{x-b_2}+\frac{1}{x-b_1}\ .\ .\ .\ .\ (2)$$

Suppose the two equations

$$f(x)=0,\ F(x)=0,$$

have a root in common, viz., $a_1=b_1$, then, dividing (1) by (2), we have

$$\frac{f_1(x)}{F_1(x)}\cdot\frac{F(x)}{f(x)}=\frac{.\ .\ .\ .\ \frac{1}{x-a_4}+\frac{1}{x-a_3}+\frac{1}{x-a_2}+\frac{1}{x-a_1}}{.\ .\ .\ .\ \frac{1}{x-b_4}+\frac{1}{x-b_3}+\frac{1}{x-b_2}+\frac{1}{x-b_1}}$$

Hence, multiplying numerator and denominator of the second member by $x-a_1$, and then substituting for x its value $x=a_1$, we have

$$\frac{f_1(a_1)}{F_1(a_1)}\cdot\frac{F(a_1)}{f(a_1)}=1$$

$$\therefore\ \frac{f_1(a_1)}{F_1(a_1)}=\frac{f(a_1)}{F(a_1)};$$

from which we learn, that if any two equations have a common root a, and their derived equations be taken, the ratio of the original polynomials, when a is put for x, will be equal to the ratio of the derived polynomials when a is put for x.

This property furnishes us with a ready method of determining the value of a fraction, such as $\frac{f(x)}{F(x)}$, when both numerator and denominator vanish for a particular value of x, as, for instance, for $x=a$. For we shall merely have to replace the polynomials in numerator and denominator by their derived polynomials, and then make the substitution of a for x. If, however, the terms of the new fraction should also vanish for this value of x, we must treat it as we did the original, and so on, till we arrive at a fraction of which the terms do not vanish for the proposed value of x. The following examples will sufficiently illustrate this method:

(1) Required the value of

$$\frac{x^2-a^2}{x-a},$$

when $x=a$.

Here $\frac{f_1(a)}{F_1(a)}=\frac{2a}{1}=2a$, the required value

(2) Required the value of

$$\frac{nx^{n+1}-(n+1)x^n+1}{(1-x)^2},*$$

when $x=1$.

$$\frac{f_1(x)}{F_1(x)}=\frac{n(n+1)x^n-n(n+1)x^{n-1}}{-2(1-x)}.$$

This still becomes $\frac{0}{0}$ for $x=1$,

$$\frac{f_2(x)}{F_2(x)}=\frac{n^2(n+1)x^{n-1}-n(n+1)(n-1)x^{n-2}}{2}$$

$$\therefore \frac{f_2(1)}{F_2(1)}=\frac{n(n+1)}{2},$$

the value sought.

(3) Required the value of

$$\frac{1-x^n}{1-x},$$

when $x=1$.

$$\frac{f_1(1)}{F_1(1)}=\frac{-n}{-1}=n.$$

(4) Required the value of

$$\frac{b(a-\sqrt{ax})}{a-x},$$

for $x=a$.

We may here put $\sqrt{x}=y$, and thus change the fraction into

$$\frac{b(a-a^{\frac{1}{2}}y)}{a-y^2}$$

$$\frac{f_1(y)}{F_1(y)}=\frac{-ba^{\frac{1}{2}}}{-2y} \quad \therefore \frac{f_1(a^{\frac{1}{2}})}{F_1(a^{\frac{1}{2}})}=\frac{b}{2}, \text{ the value required.}$$

* This is the expression for the sum of n terms of the series

$$1+2x+3x^2+4x^3+, \&c.$$

(5) Required the value of

$$\frac{f(y)}{\mathrm{F}(y)}=\frac{(a+x)^{\frac{m}{n}}-(a+y)^{\frac{m}{n}}}{x-y},$$

when $x=y$.

Put $a+y=z^n$, then the fraction is changed into

$$\frac{(a+x)^{\frac{m}{n}}-z^m}{x-z^n+a}$$

$$\therefore \frac{f_1(z)}{\mathrm{F}_1(z)}=\frac{-mz^{m-1}}{-nz^{n-1}}=\frac{m}{n}\cdot\frac{z^m}{z^n}=\frac{m}{n}\cdot\frac{(a+y)^{\frac{m}{n}}}{a+y};$$

and, therefore, the value, when $x=y$, is

$$\frac{m}{n}\cdot\frac{(a+x)^{\frac{m}{n}}}{a+x}.$$

ELIMINATION.

RESOLUTION OF EQUATIONS CONTAINING TWO OR MORE UNKNOWN QUANTITIES OF ANY DEGREE WHATEVER.

315. We have already indicated, at p. 157, the possibility of eliminating one of two unknown quantities from two equations by the method of the common divisor. The general theory of equations which has since been unfolded will afford the means of giving a more full development to this subject.

The two given equations may be thus expressed:

$$\mathrm{F}(x, y)=0, f(x, y)=0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

They are said to be *compatible* if they have common values of x and y. This is the case with two equations derived from the same problem, the conditions of which, for the determination of the required quantities, are expressed by the two given equations.

Suppose now that one of the common values of y were known, and substituted for y in the two equations (1), the first members of both would become functions of x, and known quantities; the common value of x, corresponding to this value of y, must have the property of every root of an equation pointed out at Prop. II. of Art. 238; that is to say, if a denote this value of x, each of the equations (1) must be divisible by $(x-a)$; in other words, they must have a common divisor containing x. If, therefore, without knowing and substituting the value of y, we proceed with the two given equations (1), according to the method for finding the greatest common divisor, until we arrive at a divisor of the first degree with respect to x, and to a remainder independent of x, or containing only y, as this remainder would have been zero if the value of y had occupied its place during the process, the value of y ought to be such as to reduce this remainder to zero. The values of y which will do this are found by putting this last remainder equal to zero, and thus forming what is called the final equation in y only. The values of y which satisfy the final equation are the only compatible values of this unknown in the two given equations (1). The corresponding values of x are found by substituting these values of y successively in the last divisor, which will ordinarily be of the first degree with respect to x, and setting this equal to zero; each value of y gives, by means of this divisor, the corresponding value of x, which, substituted with it in the given equations, will satisfy them. Should this divisor reduce to zero by the substitution of the value of y, we must go back to the previous one of the second degree, which, put equal to zero, will furnish two values of x for each of y; if this reduce to 0, we must go to that of the 3° degree, and so on.

316. This conclusion may be arrived at in another manner. Denoting by $A=0$, for simplicity, the first of the two given equations $F(x, y)=0$, and by $B=0$ the second $f(x, y)=0$, by Q the quotient of A by B, and by R the remainder, we have

$$A=BQ+R \quad \ldots\ldots\ldots\ldots \quad (2)$$

It follows from this equality that all the values of the unknown quantities x and y, which give $A=0$ and $B=0$, must also give $R=0$, since the quotient Q can not become infinite for finite values of x and y, the given equations being supposed to be entire functions, or capable of being rendered such with respect to x and y. (See Art. 275, Cor. 2.)

For the same reason, all the values which will give $B=0$ and $R=0$, will also give $A=0$. The system of equations $A=0$, $B=0$ may, therefore, be replaced by the more simple system $B=0$, $R=0$.

If now B be divided by R, and a new remainder, R', be reached, it may be shown in a similar manner that the system $B=0$, $R=0$ can be replaced by the system $R=0$, $R'=0$, R' being of a lower degree with respect to x than R, and so on, till we arrive at a remainder independent of x. Let R'' be this remainder. Then the original equations are replaced by the system $R'=0$, $R''=0$, in which $R''=0$ is the final equation in y only, and R' generally of the 1° degree with respect to x.

317. The same conclusion could not have been arrived at had y been supposed to enter into any of the denominators in the above process. Suppose, for instance, that Q in equation (2) contained denominators functions of y, then Q might possibly become infinite by the values of y reducing these denominators to zero, and BQ thus might be finite (see Art. 156, 3°), though B were zero.

318. If, in order to prevent the occurrence of y in the denominator of the quotient when affecting the division of A by B, it had been necessary to multiply the polynomial A by some function of y, foreign roots might thus be introduced, not belonging to the proposed equation. For, call c this function, and represent by Q still the quotient obtained after this preparation, and by R the remainder, we shall have

$$cA=BQ+R.$$

This equality proves that the solutions of the equations $B=0$, $R=0$ are the same as those of the equations $cA=0$, $B=0$. But this last system divides itself into two others, $A=0$, $B=0$, and $c=0$, $B=0$, consequently the equations $B=0$, $R=0$ will admit all the solutions of the proposed equations; but they will admit, also, all those of the equations $c=0$, $B=0$, which can not belong to the equation $A=0$. The same may be shown for any foreign factor necessary to be introduced to effect any subsequent division.

On the other hand, factors are sometimes suppressed for convenience in the process for finding the common divisor. If these factors were such as would reduce to zero on attributing to y its proper values, the process ought to terminate, since the whole remainder becomes zero with one of its factors, and the preceding divisor would be a common measure of the two polynomials; and yet these values of y which produce this common measure would not have been presented by the final equation arrived at had the factor in question been suppressed without notice.

From the foregoing considerations we see that, to obtain the values of y which belong to the proposed equations, we must equate to zero the remainder which is independent of x, as also each of the factors in y which have been suppressed in the course of the operation, and resolve each equation separately; secondly, that among the values thus obtained there may be some which, on trial in the proposed equations, prove extraneous, and which must, therefore, be rejected.

319. Simplifications may sometimes be employed, the nature of which is explained conveniently by the aid of symbols, as follows: Let the polynomials A and B, the first members of the given equations, be put under the form

$$A=dd'd''uu'u'',\ B=dd'd''vv'v'',$$

in which d represents a common divisor of A and B, containing x only; d' another, containing y only; and d'' a third, containing both x and y. The other factors, u, u', u'', v, v', v'', have a similar meaning, except that they are not common to the two polynomials A and B. The proposed equations may be satisfied by placing $d=0$; this equation contains only x, and, when resolved, furnishes a limited number of values of this unknown quantity, to which may be joined any value whatever of y, and the given equations $A=0$ and $B=0$ will be satisfied. Again, $d'=0$ will satisfy them, which gives similarly limited values for y, unlimited for x. Finally, suppose $d''=0$; as d'' contains both x and y, an arbitrary value may be given to one of the unknown quantities, and this equation will make known a corresponding one for the other.

The other modes of satisfying the given equations consist in equating to zero simultaneously one of the factors u, u', u'' of the first, and one v, v', or v'', of the other. But v and u can not be simultaneously equal to zero, since they each contain only x, and are supposed to have no common divisor, d having been understood to comprise all the common factors depending on x alone. For a similar reason, u' and v' functions of y alone can not at the same time be equal to zero. But u'' and v'', being put equal to zero, are to be proceeded with by the method of the common divisor, as already explained, and will furnish a limited number of values for y, and corresponding values limited also for x.

320. Should the remainder, in seeking for a common divisor, not contain y, but only known quantities, it could not be put equal to zero. In this case the given equations would be incompatible.

EXAMPLES.

(1) Let the equations be

$$(-2x^2+2)y^3+(x^4-2x^3-2x^2+2x+1)y^2+(x^5-2x^3+x)y=0,$$
$$(-x+1)y^5+(-x^2+x)y^4+(x^3-x^2)y^3+(x^4-x^3)y^2=0.$$

There are numerous simplifications of these, for they can be decomposed into factors like the following:

$$y(x-1)(x+y)\times(x+1)(x^2-2y-1)=0,$$
$$y(x-1)(x+y)\times y(x^2-y^2)=0.$$

Equating to zero first the common factors, each in its turn, we obtain

$$\begin{cases} y=0, \\ x \text{ indet.}, \end{cases} \quad \begin{cases} y \text{ indet.}, \\ x=1, \end{cases} \quad \begin{cases} y \text{ indet.}, \\ x=-y; \end{cases}$$

next equating to zero the other factors, we have four systems of equations, viz.,

First system $\begin{cases} y=0 \\ x+1=0 \end{cases}$ whence $\begin{cases} y=0 \\ x=-1. \end{cases}$

Second system $\begin{cases} y=0 \\ x^2-2y-1=0 \end{cases}$ whence $\begin{cases} y=0 \\ x=1, \end{cases} \begin{cases} y=0 \\ x=-1. \end{cases}$

Third system $\begin{cases} x^2-y^2=0 \\ x+1=0 \end{cases}$ whence $\begin{cases} y=\ \ 1 \\ x=-1, \end{cases} \begin{cases} y=-1 \\ x=-1. \end{cases}$

Fourth system $\begin{cases} x^2-y^2=0 \\ x^2-2y-1=0 \end{cases}$ whence $\begin{cases} y=1+\sqrt{2} \\ x=\pm(1+\sqrt{2}) \end{cases} \begin{cases} y=1+\sqrt{2} \\ x=\pm(1-\sqrt{2}) \end{cases}$

In the first three systems, all the solutions, except $x=-1$, $y=-1$, have already been found; in the fourth, those in which we have $x=-y$ are also already known; hence, in reality, we have only determined three new solutions, viz.,

$$\begin{cases} y=-1 \\ x=-1, \end{cases} \quad \begin{cases} y=1+\sqrt{2} \\ x=1+\sqrt{2}, \end{cases} \quad \begin{cases} y=1-\sqrt{2} \\ x=1-\sqrt{2}. \end{cases}$$

(2) To resolve the two equations

$$x^3-3yx^2+(3y^2-3y+1)x-y^3+y^2-2y=0,$$
$$x^2-2yx+y^2-y=0.$$

These equations can not be decomposed into factors; hence we pass immediately to successive divisions. This remark will apply also to equations 3 and 4.

First Division.

$$\begin{array}{l|l} x^3-3yx^2+(3y^2-y+1)x-y^3+y^2-2y & x^2-2yx+y^2-y \\ +x^3-2y--x^2(-y^2+y)x & x-y \\ \hline \quad -yx^2+(2y^2+1)x-y^3+y^2-2y & \\ \quad -yx^2+2y^2x-y^3+\ y^2 & \\ \hline \qquad x-2y & \end{array}$$

Second Division.

$$\begin{array}{l|l} x^2-2yx+y^2-y & x-2y \\ +x^2-2yx & x \\ \hline \qquad y^2-y. & \end{array}$$

Hence, the final equations are $x-2y=0$, $y^2-y=0$. We deduce from these

$$\begin{cases} y=0 \\ x=0 \end{cases} \qquad \begin{cases} y=1 \\ x=2; \end{cases}$$

and as we have neither introduced nor suppressed any factor, these two solutions are those of the proposed equations themselves.

(3) To resolve the two equations,

$$(y-1)x^2+2x-5y+3=0,$$
$$yx^2+9x-10y=0.$$

First Division.

$$\begin{array}{l|l} (y-1)\ x^2+2x\ -5y\ +3 & yx^2+9x-10y \\ (y-1)yx^2+2yx-5y^2+3y & y-1 \\ +(y-1)yx^2-(-9y+9)x-10y^2+10y & \\ \hline \qquad (-7y+9)x+\ 5y^2-\ 7y. & \end{array}$$

As we have multiplied by y, it is necessary to resolve the equations $y=0$, $yx^2+9x-10y=0$, which give $x=0$, $y=0$, and to examine whether these values make the dividend equal to zero. As this is not the case, it follows that they form a foreign solution, which it will be necessary to suppress.

Second Division.

$$\begin{array}{l|l} yx^2+9x-10y & (-7y+9)x+5y^2-7y \\ \hline (-7y+9)yx^2+(-63y+81)x+70y^2-90y & yx+(-5y^3+7y^2-63y+81) \\ (-7y+9)yx^2-(-5y^3+7y)^2x & \\ \hline (-5y^3+7y^2-63y+81)x+70y^2-90y & \\ (-5y^3+7y^2-63y+81)(-7y+9)x-490y^3+1260y^2-810y & \\ (-5y^3+7y^2-63y+81)(-7y+9)x-25y^5+70y^4-364y^3+846y^2-567y & \\ \hline 25y^5-70y^4-126y^3+414y^2-243y. & \end{array}$$

The equations which it is necessary to resolve are

$$(-7y+9)x+5y^2-7y=0,$$
$$25y^5-70y^4-126y^3+414y^2-243y=0.$$

The second gives the results, which may be readily verified,

$$y=0,\ y=1,\ y=3,\ y=\frac{-3\pm3\sqrt{10}}{5}.$$

By substituting these values in the first of the given equations, we obtain for x the corresponding values $x=0$, $x=1$, $x=2$, $x=-5\mp\sqrt{10}$.

In the second division we have been compelled to multiply by $-7y+9$, but no foreign solution has been introduced.

We have, then, only to suppress, in the five solutions above, that which has been introduced by the first division. There remain, then, for the given equations the four following solutions:

$$\begin{cases} y=1 \\ x=1, \end{cases} \quad \begin{cases} y=3 \\ x=2, \end{cases} \quad \begin{cases} y=\frac{-3+3\sqrt{10}}{5} \\ x=-5-\sqrt{10}, \end{cases} \quad \begin{cases} y=\frac{-3-3\sqrt{10}}{5} \\ x=-5+\sqrt{10}. \end{cases}$$

(4) Let the equations be

$$x^2+(8y-13)x+y^2-7y+12=0,$$
$$x^2-(4y+1)x+y^2+5y=0.$$

First Division.

$$\begin{array}{l|l} x^2+(8y-13)x+y^2-7y+12 & x^2-(4y+1)x+y^2+5y \\ \hline +x^2-(4y+1)x+y^2+5y & 1 \\ \hline (12y-12)x-12y+12 & \end{array}$$

This remainder can be decomposed into the factors $12(y-1)(x-1)$; the calculations will be simplified, and we shall have these two systems of equations:

$$\begin{cases} y-1=0 \\ x^2-(4y+1)x+y^2+5y=0, \end{cases} \quad \begin{cases} x-1=0 \\ x^2-(4y+1)x+y^2+5y=0. \end{cases}$$

Each of these can be at once resolved, and we find

$$\begin{cases} y=1 \\ x=3, \end{cases} \begin{cases} y=1 \\ x=2, \end{cases} \begin{cases} y=0 \\ x=1, \end{cases} \begin{cases} y=-1 \\ x=1. \end{cases}$$

(5) $x^3+2yx^2+2y(y-2)x+y^2-4=0.$

Ans. $\begin{cases} y=2 \\ x=-4, \end{cases} \begin{cases} y=3 \\ x=-5. \end{cases}$

(6) $x^3-3yx^2+3x^2+3y^2x-6yx-x-y^3+3y^2+y-3=0$,
$x^3+3yx^2-3x^2+3y^2x-6yx-x+y^3-3y^2-y+3=0$.

$$\text{Ans.}\begin{cases}\text{First system} & \begin{cases}y=1\\x=0,\end{cases}\begin{cases}y=1\\x=2,\end{cases}\begin{cases}y=1\\x=-2.\end{cases}\\ \text{Second system} & \begin{cases}y=0\\x=1,\end{cases}\begin{cases}y=0\\x=-1,\end{cases}\begin{cases}y=2\\x=1,\end{cases}\begin{cases}y=2\\x=-1.\end{cases}\\ \text{Third system} & \begin{cases}y=3\\x=0,\end{cases}\begin{cases}y=-1\\x=0.\end{cases}\end{cases}$$

(7) $x^3+yx^2-(y^2+1)x+y-y^3=0$,
$x^3-yx^2-(y^2+6y+9)x+y^3+6y^2+9y=0$.

The first division gives the remainder

$$yx^2+(3y+4)x-(y^3+3y^2+4y).$$

To be able to perform the second division, we multiply the dividend by y, in the same way we prepare the first remainder to be divided. We thus arrive at a remainder of the first degree in x, which can be put under the form

$$32(y^2+3y+2)(x-y).$$

Dividing, then, the remainder of the second degree by $x-y$, we obtain the quotient

$$yx+y^2+3y+4=0,$$

and there is no remainder.

From these calculations we conclude that the first members of the proposed equations are divisible by $x-y$, so that they can be verified by all the solutions of the indeterminate equation $x-y=0$. The other solutions are furnished by the system of two equations,

$$y^2+3y+2=0,\ yx+y^2+3y+4=0;$$

hence we obtain the solutions

$$y=-1,\ x=+2;\ y=-2,\ x=+1.$$

METHOD OF LABATIE.

321. Having thus stated the principles on which the ordinary method of elimination depends, we shall now proceed to show how this method has lately been perfected by Labatie and Sarrus. By the aid of the theory which they have introduced, we shall be able to perform the required eliminations without introducing any foreign solutions.

Suppose that A and B represent the quotients which we obtain by dividing the first members of the given equations by all of their factors which depend only on y.

Let c be the factor by which it is necessary to multiply A, in order that we may be able to divide it by B; represent by q the quotient that we obtain in this division, and by $\mathrm{R}r$ the remainder, r designating those factors of this remainder that are not dependent on x. Let c_1 be the factor by which we must multiply B to render it divisible by R; represent by q_1 the quotient, and by R_1r_1 the quotient that we obtain in this second division, r_1 designating the product of those factors of this remainder which do not depend on x, and so on. Finally, suppose, for the sake of simplicity, that at the fourth division we obtain a remainder independent of x, and designate this remainder by r_3. We have the equalities

$$\left\{\begin{array}{l} c\,A = B\,q + R\,r \\ c_1B = R\,q_1 + R_1r_1 \\ c_2R = R_1q_2 + R_2r_2 \\ c_3R_1 = R_2q_3 + r_3 \end{array}\right. \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Let d be the greatest common divisor of c and r, d_1 the greatest common divisor of $\frac{cc_1}{d}$ and r_1, d_2 that of $\frac{cc_1c_2}{dd_1}$ and r_2, d_3 that of $\frac{cc_1c_2c_3}{dd_1d_2}$ and r_3. We shall proceed to prove that we can obtain all the solutions of the system $A=0$, $B=0$, without any foreign solution, by resolving the following systems :

$$\left\{\begin{array}{l} \frac{r}{d}=0 \\ B=0, \end{array}\right. \quad \left\{\begin{array}{l} \frac{r_1}{d_1}=0 \\ R=0, \end{array}\right. \quad \left\{\begin{array}{l} \frac{r_2}{d_2}=0 \\ R_1=0, \end{array}\right. \quad \left\{\begin{array}{l} \frac{r_3}{d_3}=0 \\ R_2=0 \end{array}\right. \quad \ldots\ldots (2)$$

To establish this proposition, we shall first prove that the solutions of the systems (2) all agree with those of the system $A=0$, $B=0$; we shall afterward show that the solutions of the system $A=0$, $B=0$, are all comprised in those of the systems (2).

[a] Dividing by d the two members of the first equation of system (1), it becomes

$$\frac{c}{d}A=\frac{q}{d}B+\frac{r}{d}R \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$\frac{q}{d}$ is entire, for c and r, by hypothesis, are divisible by d; hence, qB is divisible by d; but B, by hypothesis, is prime with respect to d; therefore, d divides q.

Equation (3) shows that the values of x and y, which satisfy the equations $B=0$, $\frac{r}{d}=0$, destroy also $\frac{c}{d}A$; but $\frac{c}{d}$ and $\frac{r}{d}$ are prime with respect to each other. Consequently, 1°, *all the solutions of the system* $B=0$, $\frac{r}{d}=0$, *agree with those of the system* $A=0$, $B=0$.

[b] To obtain a relation between A, R, and $\frac{r_1}{d_1}$, we multiply equation (3) by c_1, and in the resulting equations place, instead of c_1B, its value as found in the second member of the second equation of system (1); we thus obtain

$$\frac{cc_1}{d}A=\left(\frac{c_1r+qq_1}{d}\right)R+\frac{q}{d}r_1R_1.$$

The quantity $\frac{c_1r+qq_1}{d}$ is entire, because r and q are divisible by d; moreover, this quantity is divisible by d_1; for d_1 divides $\frac{cc_1}{d}$ and r_1, and it is prime with respect to R. Dividing the two members of the above equation by d_1, and taking, to abridge, $\frac{q}{d}=M$, $\frac{c_1r+qq_1}{dd_1}=M_1$, it becomes

$$\frac{cc_1}{dd_1}A=M_1R+MR_1\frac{r_1}{d_1} \quad \ldots\ldots\ldots\ldots\ldots (4)$$

To obtain a relation between B, R, and $\frac{r_1}{d_1}$, we first multiply the second equation of system (1) by $\frac{c}{d}$, which gives $\frac{cc_1}{d}B=\frac{cq_1}{d}R+\frac{c}{d}R_1r_1$. Since $\frac{cc_1}{d}$ and

r_1 are, by hypothesis, divisible by d_1, it follows that d_1 divides also $\frac{cq_1}{d}R$; but d_1 is prime with respect to R; hence, d_1 divides $\frac{cq_1}{d}$. Dividing all the terms of the equation by d_1, and taking, to abridge, $\frac{c}{d}=N$, $\frac{cq_1}{dd_1}=N_1$, it becomes

$$\frac{cc_1}{dd_1}B=N_1R+NR_1\frac{r_1}{d_1} \quad \ldots\ldots\ldots\ldots (5)$$

Equations (4) and (5) prove that all the values of x and y, which reduce the polynomials R and $\frac{r_1}{d_1}$ to zero, destroy also $\frac{cc_1}{dd_1}A$ and $\frac{cc_1}{dd_1}B$; but $\frac{cc_1}{dd_1}$ and $\frac{r_1}{d_1}$ are prime with respect to each other; consequently, 2°, *all the solutions of the system* $R=0$, $\frac{r_1}{d_1}=0$, *agree with those of the given system*, $A=0$, $B=0$.

[c] We obtain a relation between A, R_1, and $\frac{r_2}{d_2}$, by multiplying equation (4) by c_2, and placing, instead of c_2R, its value found in the second member of the third equation of system (1); we thus find

$$\frac{cc_1c_2}{dd_1}A=R_1\left(M_1q_2+Mc_2\frac{r_1}{d_1}\right)+M_1R_2r_2.$$

By hypothesis, d_2 divides the first member of this equation, it also divides r_2; it ought, then, to divide $R_1\left(M_1q_2+Mc_2\frac{r_1}{d_1}\right)$; but R_1 and d_2 are prime with respect to each other; d_2 then divides the term by which R_1 in the above equation is multiplied. Designating the quotient by M_2 the equation becomes

$$\frac{cc_1c_2}{dd_1d_2}A=M_2R_1+M_1R_2\frac{r_2}{d_2} \quad \ldots\ldots\ldots\ldots (6)$$

Multiplying equation (5) by c_2, and then placing, instead of c_2R, its value found in the second member of the third equation of system (1), it becomes

$$\frac{cc_1c_2}{dd}B=R_1\left(N_1q_2+Nc_2\frac{r_1}{d_1}\right)+N_1R_2r_2.$$

We can demonstrate as before that the multiplier of R_1 is divisible by d_2, and, representing the quotient by N_2, we find

$$\frac{cc_1c_2}{dd_1d_2}B=N_2R_1+N_1R_2\frac{r_2}{d_2} \quad \ldots\ldots\ldots\ldots (7)$$

Equations (6) and (7) show that all the values of x and y, which reduce the polynomials R_1 and $\frac{r_2}{d_2}$ to zero, destroy also the first members of these two equations; but $\frac{cc_1c_2}{dd_1d_2}$ and $\frac{r_2}{d_2}$ are prime with respect to each other; consequently. 3°, *all the solutions of the system* $R_1=0$, $\frac{r_2}{d_2}=0$, *suit those of the proposed system*, $A=0$, $B=0$.

[d] The equation which gives a relation between A, R_2, and $\frac{r^3}{d_3}$, can be obtained by multiplying equation (6) by c_3, and placing, instead of c_3R_1, its value as given in the second member of the fourth equation of system (1); we thus find

$$\frac{cc_1c_2c_3}{dd_1d_2}A=R_2\left(M_2q_3+c_3M_1\frac{r_2}{d_2}\right)+M_2r_3.$$

Dividing the two members of this equation by d_3, and designating by M_3 the quotient obtained by dividing the entire polynomial $M_2q_3+c_3M_1\frac{r_2}{d_2}$ by d_3, there results

$$\frac{cc_1c_2c_3}{dd_1d_2d_3}A=M_3R_2+M_2\frac{r_3}{d_3} \qquad (8)$$

To obtain a relation between B, R_2, and $\frac{r_3}{d_3}$, we multiply equation (7) by c_3, and put in the place of c_3R_1 the second member of the fourth equation of the system (1), which gives

$$\frac{cc_1c_2c_3}{dd_1d_2}B=R_2\left(N_2q_3+c_3N_1\frac{r_2}{d_2}\right)+N_2r_3.$$

Dividing both members by d_3, and designating by N_3 the quotient obtained by dividing the entire polynomial $N_2q_3+c_3N_1\frac{r_2}{d_2}$ by d_3, it becomes

$$\frac{cc_1c_2c_3}{dd_1d_2d_3}B=N_3R_2+N_2\frac{r_3}{d_3} \quad \ldots\ldots \quad (9)$$

Equations (8) and (9) show that all the values of x and y, which reduce the polynomials R_2 and $\frac{r_3}{d_3}$ to zero, destroy also the first members of those equations; but $\frac{cc_1c_2c_3}{dd_1d_2d_3}$ and $\frac{r_3}{d_3}$ are prime with respect to each other; consequently, 4°, *all the solutions of the system* $R_2=0$, $\frac{r_3}{d_3}=0$ *concur with those of the proposed system*, $A=0$, $B=0$.

(II.) It remains still to be proved *that any system whatsoever of values which satisfy the equations* $A=0$, $B=0$, *is a part of the systems of values which furnish equations* (2).

To form the equations which demonstrate this second part of the theorem, let us first place in equation (3) N instead of $\frac{c}{d}$, and M instead of $\frac{q}{d}$; it will become, transposing the term MB,

$$NA-MB=R\frac{r}{d} \quad \ldots\ldots\ldots \quad (10)$$

Eliminate now R between equations (4) and (5). We can effect this elimination by subtracting one of these equations from the other, after we have multiplied the first by N_1, the second by M_1, remembering the values previously given to N_1 and M_1; but the calculations will be simpler if we multiply equation (4) by B and equation (5) by A. Subtracting the two resulting equations the one from the other, we find

$$(M_1B-N_1A)R+(MB-NA)R_1\frac{r_1}{d_1}=0.$$

Placing instead of $MB-NA$ its value previously determined, $-R\frac{r}{d}$, and suppressing the factor R_1, this equation becomes

$$N_1A-M_1B=-R_1\frac{rr_1}{dd_1} \quad \ldots . \quad (11)$$

Finally, we eliminate R_1 between equations (6) and (7). To do this, multiply equation (6) by B and equation (7) by A; then subtract the one of the resulting equations from the other, we thus obtain

$$(M_2B-N_2A)R_1+(M_1B-N_1A)R_2\frac{r_2}{d_2}=0.$$

Placing in this equation, instead of M_1B-N_1A, its value, determined in (11), $R_1\frac{rr_1}{dd_1}$, and suppressing the factor R_1, it becomes

$$N_2A-M_2B=R_2\frac{rr_1r_2}{dd_1d_2}. \quad . \quad . \quad . \quad . \quad (12)$$

In the same manner we obtain the equation

$$N_3A-M_3B=-\frac{rr_1r_2r_3}{dd_1d_2d_3}.$$

Equation 13 shows that every system of values of x and y which gives $A=0$, $B=0$, ought also to satisfy the equation

$$\frac{r}{d}\cdot\frac{r_1}{d_1}\cdot\frac{r_2}{d_2}\cdot\frac{r_3}{d_3}=0,$$

an equation which requires that one of its factors equal zero, whence it follows that the equations

$$\frac{r}{d}=0,\ \frac{r_1}{d_1}=0,\ \frac{r_2}{d_2}=0,\ \frac{r_3}{d_3}=0,$$

give all the correct values of y.

This being established, let $x=\alpha$, $y=\beta$ be a system of correct values of the equations $A=0$, $B=0$.

If the value $y=\beta$ is a root of the equation $\frac{r}{d}=0$, it is clear that the system $x=\alpha$, $y=\beta$ will be a solution of the system $B=0$, $\frac{r}{d}=0$.

If the value $y=\beta$ does not verify the equation $\frac{r}{d}=0$, and if it is a root of the equation $\frac{r_1}{d_1}=0$, we perceive, by equation (10), that the system $x=\alpha$, $y=\beta$ will give $R=0$; consequently, it will be a solution of the system $R=0$, $\frac{r_1}{d_1}=0$.

If the value $y=\beta$ verifies neither the equation $\frac{r}{d}=0$ nor the equation $\frac{r_1}{d_1}=0$, and is a root of the equation $\frac{r_2}{d_2}=0$, we see, by equation (11), that the system $x=\alpha$, $y=\beta$ will give $R_1=0$; consequently, it will be a solution of the system $R_1=0$, $\frac{r_2}{d_2}=0$.

If the value $y=\beta$ does not verify any one of the equations $\frac{r}{d}=0$, $\frac{r_1}{d_1}=0$, $\frac{r_2}{d_2}=0$, and is a root of the equation $\frac{r_3}{d_3}=0$, we see by equation (12) that the system $x=\alpha$, $y=\beta$, will give $R_2=0$; consequently, it will be a solution of the system $R_2=0$, $\frac{r_3}{d_3}=0$.

Hence, *all the systems of values which satisfy the equations* $A=0$, $B=0$, *form part of the values which furnish equations* (2).

The equation $\frac{r}{d}\cdot\frac{r_1}{d_1}\cdot\frac{r_2}{d_2}\cdot\frac{r_3}{d_3}=0$, which gives all the correct values of y, is called the *final equation* in y.

REMARKS ON THE PRECEDING METHOD.

It may chance that in one of the equations of system (2), for example, $\frac{r_1}{d_1}=0$, $R=0$, a value of y, derived from the first equation, destroys some of the coefficients of the powers of x in the second equation, after the highest power of x; in this case we only obtain a number of values of x inferior to the degree of the equation $R=0$; and if the substitution of the value of y should destroy all the multipliers of the powers of x in R, the equation $R=0$ would not give any value of x. In fact, it can be proved, by a method similar to that which we have employed with reference to the general equation of the second degree (Art. 191), that if in an equation of the form $Sx^n+Hx^{n-1}+Kx^{n-2}+\ldots=0$, we suppose that the quantities which enter into the coefficients S, H, K, &c., are of such a nature that we have $S=0$, $H=0$, &c., the equation has infinite roots equal in number to the consecutive coefficients which are reduced to zero. But it should be remarked that the theory by which we have proved that the solutions of systems (2) are the same with those of the system $A=0$, $B=0$, only applies to solutions expressed by finite values of x and y.

To prove that the solutions of systems (2), in which the value of x is infinity, also suit the proposed equations $A=0$, $B=0$, suppose that $y=\beta$, verifying the equation $\frac{r_1}{d_1}=0$, causes one or more of the multipliers of the higher powers of x in R to vanish. If, in the two members of the equality (4) we make $y=\beta$, the term $MR_1\frac{r_1}{d_1}$ will be reduced to zero, and the degree of the term M_1R will be lowered with respect to x one or more units.

Again, we can not suppose that the terms of M_1R, which are reduced to zero, have been destroyed, until we have made $y=\beta$ in the terms of $MR_1\frac{r_1}{d_1}$, because the degrees of A, B, R, R_1, &c., are decreasing, and we see without difficulty, from the relations which exist between M, M_1, M_2, &c., that the degrees of these quantities with respect to x go on increasing. It will be necessary, then, in order that y may have the value β, that the degree of $\frac{cc_1}{dd_1}A$ with respect to x be lowered as many units as the degree of R is lowered. We can prove, in the same manner, that the value $y=\beta$ ought also to cause one or more of the coefficients of the higher powers of x in B to vanish. The equations $A=0$, $B=0$ will give then for $y=\beta$ one or more infinite values of x.

As to the reciprocal proposition, that the solutions of the equations $A=0$, $B=0$, in which x is infinite, ought to be found among the solutions of systems (2), it is not the fact, as will be seen in the second example following.

EXAMPLE I.

$$(y-1)x^3+y(y+1)x^2+(3y^2+y-2)x+2y=0,$$
$$(y-1)x^2+y(y+1)x+3y^2-1=0.$$

The first division gives at once the remainder $(y-1)x+2y$; taking this remainder for a divisor, we obtain, without any preparation, the remainder y^2-1. We shall obtain then all the solutions of the proposed system by resolving the equations

$$y^2-1=0,\ (y-1)x+2y=0.$$

The first equation gives $y=\pm1$. For the value $y=-1$ we find $x=-1$, and this system will satisfy the proposed equations. For the value $y=+1$ we find $x=\infty$. This system, also, will satisfy the proposed equations; for dividing each of these equations by the highest power of x, and taking $x=\infty$, the two equations will be reduced to $y-1=0$.

EXAMPLE II.

$$(y-1)x^2+yx+y^2-2y=0,$$
$$(y-1)x+y=0.$$

The division gives the remainder $y^2-2y=0$; the solutions, therefore, of the proposed equations depend on the system

$$y^2-2y=0,\ (y-1)x+y=0.$$

These equations give the two systems

$$y=0,\ x=0;\ y=2,\ x=-2.$$

But the proposed equations possess, besides, another solution, $y=1$, $x=\infty$, since the value $y=1$ causes the multiplier of the highest power of x in each of these equations to vanish.

322. The following method of elimination avoids the introduction of foreign roots, and enables us to determine the degree of the final equation:

Let equation A or $x^m+Px^{m-1}+Qx^{m-2}\ldots\ldots+Tx+V$ be supposed equal to

$$(x-a)(x^{m-1}+Ax^{m-2}+Bx^{m-3}+,\ \&c.)\ \ldots\ldots\ C;$$

and equation B or $x^n+P'x^{n-1}+Q'x^{n-2}\ldots+T'x+V'$ to

$$(x-a)(x^{n-1}+A'x^{n-2}+B'x^{n-3}+,\ \&c.)\ \ldots\ldots\ D;$$

also, let equation A be multiplied by $x^{n-1}+A'x^{n-2}+B'x^{n-3}$, &c., and equation B be multiplied by $x^{m-1}+Ax^{m-2}+Bx^{m-3}$, &c., it is evident that the products must be equal; therefore,

$$(x^m+Px^{m-1}+Qx^{m-2}+,\ \&c.)(x^{n-1}+A'x^{n-2}+B'x^{n-3}+,\ \&c.)=(x^n+P'x^{n-1}+Q'x^{n-2}+,\ \&c.)(x^{m-1}+Ax^{m-2}+Bx^{m-3}+,\ \&c.)\ \ldots\ldots\ldots\ E.$$

Performing the multiplications and making equal to each other, the coefficients of the same powers of x (Art. 209), $m+n-1$ equations are obtained between the indeterminate quantities A, B, C, ... A′, B′, C′, ... Now, the number of indeterminate quantities in equation C is $m-1$, and in equation D, $n-1$; therefore, the number in equation E is $m+n-2$. Of the $m+n-1$ equations $m+n-2$ suffice to determine A, B, C, . . .A′, B′, C′, ...; and one equation remains between P, Q, R . . . . P′, Q′, R′ . . . ., which it is necessary to satisfy in such a manner that the equations C, D may have a common divisor, $x-a$; this equation of condition is the final equation required.

Since none of the indeterminate quantities A, B, C . . . A′, B′, C′ . . . . is multiplied by itself, the equations by means of which these quantities are determined are of the first degree.

The final equation being resolved, and the values of y successively substituted in A, B, C, ... A′, B′, C′, . . ., the results are obtained from the division of the polynomials C, D by the common divisor $x-a$.

If the equations A, B are incomplete, the two products E can not be complete polynomials of the degree $m+n-1$; but the terms which are deficient in the one are found in the other. For, taking the least favorable case, viz.,

$$x^m+P=0;\ x^n+P'=0;$$

the identity which results from the equality of the two products is

$$(x^m+P)(x^{n-1}+A'x^{n-2}+,\ \&c.)=(x^n+P')(x^{m-1}+Ax^{m-2}+,\ \&c.)$$

EXAMPLE.

Let
$$x^2+P\,x+Q\ =0;$$
$$x^2+P'x+Q'=0.$$

Denoting by $x-a$ the factor which is to be rendered common to these equations by the suitable determination of y, the first equation may be considered the product of $x-a$ by a factor, $x+A$, of the first degree; and the second the product of $x-a$ by a factor, $x+A'$, also of the first degree.

$$\therefore\ x^2+P\,x+Q\ =(x-a)(x+A),$$
$$x^2+P'x+Q'=(x-a)(x+A'),$$

and
$$(x^2+Px+Q)(x+A')=(x^2+P'x+Q')(x+A),$$

or
$$x^3+\left.\begin{matrix}P\\+A'\end{matrix}\right|x^2+\left.\begin{matrix}Q\\+PA'\end{matrix}\right|x+QA'=x^3+\left.\begin{matrix}P'\\+A\end{matrix}\right|x^2+\left.\begin{matrix}Q'\\+AP'\end{matrix}\right|x+AQ'.$$

Making the coefficients of the same powers of x equal to each other,

$$P+A'=P'+A \quad \text{or} \quad A-A'=P-P' \ \ldots\ldots\ (1)$$
$$Q+PA'=Q'+AP' \quad \text{or} \quad AP'-PA'=Q-Q' \ \ldots\ldots\ (2)$$
$$QA'=AQ' \quad \text{or} \quad AQ'-QA'=0 \ \ldots\ldots\ldots\ (3)$$

By mean of these three equations of the first degree the two indeterminate quantities A, A′ can be eliminated, and a single equation obtained in terms of the quantities P, Q, P′, Q′.

For, if from equation (1), multiplied by P, or $AP-PA'=(P-P')P$, equation (2) be subtracted, or $AP'-PA'=Q-Q'$, the remainder is

$$AP-AP'=(P-P')P-(Q-Q').$$

Whence
$$A=\frac{(P-P')P-(Q-Q')}{P-P'}.$$

Similarly,
$$A'=\frac{(P-P')P'-(Q-Q')}{P-P'}.$$

If these values of A, A′ are substituted in equation (3),

$$\frac{(P-P')P-(Q-Q')}{P-P'}\times Q'-\frac{(P-P')P'-(Q-Q')}{P-P'}\times Q=0,$$

or
$$(P-P')PQ'-(Q-Q')Q'-(P-P')QP'+(Q-Q')Q=0,$$

or
$$(P-P')(PQ'-QP')+(Q-Q')(Q-Q')=0,$$

or
$$(P-P')(PQ'-QP')+(Q-Q')^2=0.$$

The quantities P, P′, Q, Q′, containing only y and known quantities, this is the final equation in y.

It has been already noticed that, if this equation is identical, the proposed equations have at least one common factor of the form $x-a$, whatever be the value of y; and that, if it contains only known quantities, these equations are contradictory.

When the final equation has the proper form, the factor $x-a$ is obtained by dividing the first of the proposed equations by $x+A$; thus,

$$
\begin{array}{l}
x+A)\ x^2+Px+Q\ (x+P-A \\
\quad\ \underline{x^2+Ax} \\
\quad\quad (P-A)x+Q \\
\quad\quad \underline{(P-A)x+(P-A)A} \\
\quad\quad\quad\quad Q-(P-A)A.
\end{array}
$$

The quotient is $x+P-A$, and the remainder is considered equal to zero, because it is reduced to zero by the substitution, for y, of a value deduced from the final equation.

Making the quotient $x+P-A$ equal to zero, the value of x is $x=A-P$, and by substituting the value of A,

$$x=\frac{(P-P')P-(Q-Q')}{P-P'}-P,$$

or

$$x=-\frac{Q-Q'}{P-P'}.$$

This example is given as an illustration of the general method. From its particular form it admits of resolution by another and a much shorter process.

For if from $\quad x^2+P\,x+Q=0$

$\quad x^2+P'x+Q'=0$ is subtracted,

the remainder is

$$(P-P')x+Q-Q'=0\,;$$

$$\therefore\ x=-\frac{Q-Q'}{P-P'}.$$

OF THE DEGREE OF THE FINAL EQUATION.

323. The degree of the final equation can not be depressed by the reduction of each of the coefficients P, Q, R . . . P′, Q′, R′ . . . in the equations

$$x^m+P\,x^{m-1}+Q\,x^{m-2}\ \ldots\ +T\,x+V=0,$$
$$x^n+P'x^{n-1}+Q'x^{n-2}\ \ldots\ +T'x+V'=0,$$

to the term of the highest exponent in y which it contains; for the degree of each of the equations is not changed by the reduction. Therefore, the reasoning may be applied to the equations

$$x^m+ayx^{m-1}+by^2x^{m-2}\ \ldots\ +ty^{m-1}x+vy^m=0\ \ldots\ (1)$$
$$x^n+a'yx^{n-1}+b'y^2x^{n-2}\ \ldots\ +t'y^{n-1}x+v'y^n=0\ \ldots\ (2)$$

which are of the same degree respectively as the preceding equations. The latter are reducible to

$$\left(\frac{x}{y}\right)^m+a\left(\frac{x}{y}\right)^{m-1}+b\left(\frac{x}{y}\right)^{m-2}\ \ldots\ +t\frac{x}{y}+v=0\ \ldots\ (3)$$
$$\left(\frac{x}{y}\right)^n+a'\left(\frac{x}{y}\right)^{n-1}+b'\left(\frac{x}{y}\right)^{n-2}\ \ldots\ +t'\frac{x}{y}+v'=0\ \ldots\ (4)$$

in which the unknown quantity is $\frac{x}{y}$, and a, b, . . . t, v; a', b', . . . t', v', are numbers.

Denoting by α, β, γ . . . the numerical roots of equation (3)

and by α', β', γ' . . . the numerical roots of equation (4)

these equations may be decomposed into

$$\left(\frac{x}{y}-\alpha\right)\left(\frac{x}{y}-\beta\right)\left(\frac{x}{y}-\gamma\right),\ \&c.=0,$$
$$\left(\frac{x}{y}-\alpha'\right)\left(\frac{x}{y}-\beta'\right)\left(\frac{x}{y}-\gamma'\right),\ \&c.=0.$$

Whence $(x-\alpha y)(x-\beta y)(x-\gamma y)$, &c. $=0$ . . . . (5)

$(x-\alpha' y)(x-\beta' y)(x-\gamma' y)$, &c. $=0$ . . . . (6)

Substituting in equation (5) the roots of x from equation (6), viz., $\alpha' y$, $\beta' y$, &c.,

$$(\alpha' y-\alpha y)(\alpha' y-\beta y)(\alpha' y-\gamma y), \text{ \&c. } =0,$$
$$(\beta' y-\alpha y)(\beta' y-\beta y)(\beta' y-\gamma y), \text{ \&c. } =0,$$
$$(\gamma' y-\alpha y)(\gamma' y-\beta y)(\gamma' y-\gamma y), \text{ \&c. } =0.$$

Or, since the number of factors in equation (5) is m, and the number of roots in equation (6) is n,

$$y^m(\alpha'-\alpha)(\alpha'-\beta)(\alpha'-\gamma), \text{ \&c. } =0,$$
$$y^m(\beta'-\alpha)(\beta'-\beta)(\beta'-\gamma), \text{ \&c. } =0,$$
$$y^m(\gamma'-\alpha)(\gamma'-\beta)(\gamma'-\gamma), \text{ \&c. } =0.$$

Consequently, there are n equations, each of the degree m; these give all the solutions in y. The product of these roots (or solutions) of y is the final equation, since it becomes zero for all the values of y which render its factors zero, and only for these values. Now, this product is evidently of the degree mn. Consequently, the degree of the final equation (unless roots not belonging to the proposed equations are introduced by the process of elimination) can not exceed the product of the degrees of the proposed equations.

It ought to be observed that the numerical values of the roots of y are changed by this process, but that their number remains undisturbed by it.

IRRATIONAL EQUATIONS.

324. All the *direct* methods employed for the solution of equations suppose that the unknown quantities in them are not affected with any radical sign; when, therefore, the unknown is found under a radical sign, it will be necessary, before applying the process of solution, to employ some preparatory method of rendering the equation rational. Such a method is at once suggested by the theory of elimination. For, if we equate each of the irrational terms with an unknown quantity, and remove the radical from each of these new equations by involution, we shall have a series of equations (including the original one, with its irrational terms replaced by the new symbols) without radicals, from which the quantities, temporarily introduced, may be eliminated, and thence a rational equation obtained, involving only the original unknown quantities.

The following examples will fully illustrate the mode of proceeding:

(1) Let the equation be

$$x-\sqrt{x-1}+\sqrt[3]{x+1}=0.$$

Put

$$y=\sqrt{x-1},\ z=\sqrt[3]{x+1};$$

and we then have the three following rational equations, from which we may eliminate y and z, viz.,

$$y^2=x-1,\ z^3=x+1,\ x-y+z=0.$$

From the last equation we get $y=x+z$, and, by substituting this value in the first, y becomes eliminated, and we have these two equations in x and z, viz.,

$$z^3-x+1=0$$
$$z^2+2xz+x^2-x+1=0;$$

and, to eliminate z from these, we apply the process explained in the preceding articles, and thus get the final equation

$$x^6-3x^5+8x^4+x^3+7x^2-7x+2=0.$$

(2) Let the equation be

$$\sqrt[3]{4x+7}+2\sqrt{x-4}=1.$$

Putting

$$y=\sqrt[3]{4x+7},\ z=\sqrt{x-4},$$

we have the system of equations

$$y^3=4x+7,\ z^2=x-4,$$
$$y+2z=1.$$

EXPONENTIAL EQUATIONS.

325. An *exponential equation* is an equation in which the unknown appears in the form of an exponent or index; thus, the following are exponential equations:

$$a^x=b,\ x^x=a,\ a^{b^x}=c,\ x^{x^x}=a,\ \&c.*$$

To resolve the equation

$$10^x=2$$

put $x=\frac{1}{x'}$, then

$$10^{\frac{1}{x'}}=2\ \therefore\ 10=2^{x'}.$$

The value of x' lies evidently between 3 and 4; place it, therefore, equal to 3 plus an unknown fraction, and we shall have

$$10=2^{3+\frac{1}{x''}},\ \text{or}\ 10=2^3\times 2^{\frac{1}{x''}}$$
$$\therefore\ \frac{10}{8}=2^{\frac{1}{x''}}\ \therefore\ \left(\frac{5}{4}\right)^{x''}=2.$$

The value of x'' lies evidently between 3 and 4, $\therefore$ place

$$\left(\frac{5}{4}\right)^{3+\frac{1}{x'''}}=2,$$

and proceed as before. The value of x is thus obtained in a continued fraction.

$$x=\frac{1}{x'}=\frac{1}{3+\frac{1}{x''}}=\frac{1}{3+\frac{1}{3+x'''}},\ \&c.,$$

which may be carried to any extent at pleasure, and the value found by the method explained hereafter. (See Continued Fractions.)

When the equation is of the form $a^x=b$, or $a^{b^x}=c$, the value of x is readily obtained by logarithms, as we have already seen in Art. 220. But if the equation be of the form $x^x=a$, the value of x may be obtained by the rule of *double position*, as in the following

EXAMPLE.

Given $x^x=100$, to find an approximate value of x.

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* Exponential equations, and those in which logarithms of unknown quantities enter, belong to a class called transcendental.

The value of x is evidently between 3 and 4, since $3^3=27$ and $4^4=256$; hence, taking the logarithms of both sides of the equation, we have

$$x \log. x = \log. 100 = 2.*$$

First, let $x_1 = 3{\cdot}5$; then

$$3{\cdot}5 \log. 3{\cdot}5 = 1{\cdot}9042380$$
$$\text{true no.} = 2{\cdot}0000000$$
$$\text{error} = -{\cdot}0957620$$

Second, let $x_2 = 3{\cdot}6$; then

$$3{\cdot}6 \log. 3{\cdot}6 = 2{\cdot}0026890$$
$$\text{true no.} = 2{\cdot}0000000$$
$$\text{error} = +{\cdot}0026890$$

Then, as the difference of the results is to the difference of the assumed numbers, so is the least error to a correction of the assumed number corresponding to the least error; that is,

$${\cdot}098451 : {\cdot}1 :: {\cdot}002689 : {\cdot}00273;$$

hence $x=3{\cdot}6-{\cdot}00273=3{\cdot}59727$, nearly.

Again, by forming the value of x^x for $x=3{\cdot}5972$, we find the error to be $-{\cdot}0000841$, and for $x=3{\cdot}5973$, the error is $+{\cdot}0000149$;

hence, as ${\cdot}000099 : {\cdot}0001 :: {\cdot}0000149 : {\cdot}0000151$;

therefore, $x=3{\cdot}5973-{\cdot}0000151=3{\cdot}5972849$, the value nearly.

EXAMPLES FOR PRACTICE.

(1) Find x from the equation $x^x=5$. Ans. 2·129372.

(2) Solve the equation $x^x=123456789$. Ans. 8·6400268.

(3) Find x from the equation $x^x=2000$. Ans. 4·8278226.

DEMONSTRATION OF THE BINOMIAL THEOREM.

CASE I.

326. If, at Prop. V., Art. 245, we suppose the second terms a_1, a_2, a_3, &c., of the binomials to be all positive instead of negative, and all equal to a, then the products two and two will all become a^2; those three and three, a^3, and so on; and, by recurring to Art. 203, we perceive that the number of combinations or products two and two, if we suppose that there are n binomials, will be expressed by $\frac{n(n-1)}{1\,.\,2}$, the number three and three by $\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}$, and so on. Hence, where n is a whole number,

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1\,.\,2}a^2x^{n-2} +, \&c., + a^n,$$

or

$$(a+x)^n = a^n + na^{n-1}x + \mathrm{A}a^{n-2}x^2 + \mathrm{B}a^{n-3}x^3 +, \&c. \;.\;.\;.\;.\;.\;.\;.\; (1)$$

by reversing the order of the terms, and disregarding the particular form of the coefficients after the second term.

CASE II.

If the exponent be fractional, we have

$$(a+x)^{\frac{m}{n}} = \sqrt[n]{(a+x)^m} = \sqrt[n]{a^m + ma^{m-1}x + \mathrm{A}a^{m-2}x^2 +, \&c.}$$

* In equations of this kind the following method may be adopted: Let $x^x=a$; then $x \log. x = \log. a$; put $\log. x = y$, and $\log. a = b$; then $xy=b$, and $\log. x + \log. y = \log b$; hence $y + \log. y = \log. b$. Now y may be found by double position, as above, and then x becomes known. When a is less than unity, put $x=\frac{1}{y}$ and $a=\frac{1}{b}$; then we have $b^y=y$ $\therefore y \log. b = \log. y$, and if $\log. b = c$, and $\log. y = z$; then $cy=z$, and $\log. c + \log. y = \log. z$, or $\log. c + z = \log. z$. Hence z may be found by the preceding method, and then y and x become known.

Applying the rule at Art. 113 for extracting the root of a polynomial, the first term of the root will be $a^{\frac{m}{n}}$; the divisor of the second term of the given polynomial, $n\left(a^{\frac{m}{n}}\right)^{n-1}=na^{m-\frac{m}{n}}$; and the quotient or second term of the root will be $\frac{m}{n}a^{m-1-\left(m-\frac{m}{n}\right)}x=\frac{m}{n}a^{\frac{m}{n}-1}x$. When the two terms of the root thus found are raised to the n^{th} power, and subtracted from the given polynomial according to the rule, the first two terms of the latter will be canceled, and the next highest power of a to be divided by the constant divisor $na^{m-\frac{m}{n}}$ will be a^{m-2} multiplied by x^2, and the quotient, which is the third term of the root, will contain a to the power $n-2-\left(m-\frac{m}{n}\right)=\frac{m}{n}-2$ with x^2, and so on, so that the root may be written under the form

$$a^{\frac{m}{n}}+\frac{m}{n}a^{\frac{m}{n}-1}x+A'a^{\frac{m}{n}-2}x^2+B'a^{\frac{m}{n}-3}x^3+,\ \&c.,$$

the same form, so far as regards the exponents, as when the exponent is a whole number. The coefficients will be examined for this and the next case together.

CASE III.

When the exponent is negative, either entire or fractional, as a consequence of what has just been demonstrated, we have

$$(a+x)^{-m}=\frac{1}{(a+x)^m}=\frac{1}{a^m+ma^{m-1}x+,\ \&c.}$$

But if the division be effected according to the ordinary rules, the quotient will be indefinite, and of the form

$$a^{-m}-ma^{-m-1}x+A''a^{-m-2}x^2+,\ \&c.;$$

then, whatever be the exponent of a binomial, its development, as to the coefficients of the first two terms and the exponents of all, is of the same form, viz., that indicated by equation (1).

Now, to examine the coefficients of the other terms, for the sake of generality, I shall consider two consecutive terms of any rank whatever, and I shall write

$$(a+x)^m=a^m+ma^{m-1}x\ldots+Ma^{m-n}x^n+Na^{m-n-1}x^{n+1}+,\ \&c.$$

Let us change throughout x into $x+y$; as the unknown coefficients contain neither a nor x, the above expression becomes

$$(a+x+y)^m=a^m+ma^{m-1}(x+y)\ldots.$$
$$\ldots+Ma^{m-n}(x+y)^n+Na^{m-n-1}(x+y)^{n+1}+,\ \&c.$$

By changing a into $a+y$, we should have found

$$(a+y+x)^m=(a+y)^m+m(a+y)^{m-1}x\ldots$$
$$\ldots.+M(a+y)^{m-n}x^n+N(a+y)^{m-n-1}x^{n+1}+,\ \&c.$$

In the two preceding equalities the first members are equal, therefore the second members must be equal also; and this is the case whatever values x and y may have. Then, if they be arranged according to the powers of y, they must be identical. It is true, they contain binomials, but we know the first two terms of each of these binomials, so that we can form the part which, in each second member, contains y to the first degree, and that will suffice for our purpose. Designating it by Yy in the one and by $Y'y$ in the other, it is easy to find

$$Y = ma^{m-1}\ldots.+Mna^{m-n}x^{n-1}+N(n+1)a^{m-n-1}x^{n}\ldots.$$
$$Y' = ma^{m-1}\ldots.+M(m-n)a^{m-n-1}x^{n}+N(m-n-1)a^{m-n-2}x^{n+1}\ldots..$$

These two quantities must be equal, whatever be the value of x; the coefficients, therefore, of the same powers of x must be equal. Considering only those which pertain to $a^{m-n-1}x^{n}$, we have

$$N(n+1)=M(m-n) \therefore N=\frac{M(m-n)}{n+1}.$$

We see by this according to what law, in the development (1), any coefficient whatever is formed from the preceding. It is the same that we have found for the case of a positive exponent (Art. 107, IV.); and as we have seen that the first two terms are composed in the same manner, whatever be the exponent m, it will be so also with all the other terms.

An abbreviate notation, sometimes employed to express the coefficients of the binomial formula, is the initial letter B of the word binomial, with the exponent of the power of the binomial before it, and the order of the coefficient above. Thus, the coefficient of the 1° term, if the exponent be n, is expressed by ${}^{n}\overset{0}{B}$; of the 2°, ${}^{n}\overset{1}{B}$; of the 3°, ${}^{n}\overset{2}{B}$, &c.; of the k^{th} term $\frac{n(n-1)\ldots(n-k+1)}{1.2.3\ldots k}$ by ${}^{n}\overset{k}{B}$, or otherwise simply n_k.

SERIES.

RECURRING SERIES.

327. To develop the expression $\frac{a'}{a+bx}$ in a series, place

$$\frac{a'}{a+bx}=A+Bx+Cx^2+, \text{\&c.},$$

and proceeding by the method of undetermined coefficients, explained at Art. 209, we find

$$A=\frac{a'}{a}, B=-\frac{b}{a}A, C=-\frac{b}{a}B, D=-\frac{b}{a}C, \text{\&c.}$$

From which we perceive that each coefficient is obtained by multiplying the preceding by $-\frac{b}{a}$. Here the series is a simple geometrical progression.

Proceeding in a similar manner with the fraction

$$\frac{a'+b'x}{a+bx+cx^2}=A+Bx+Cx^2+, \text{\&c.},$$

we obtain

$$A=\frac{a'}{a}, B=\frac{b'-Ab}{a}, C=-\frac{c}{a}A-\frac{b}{a}B, D=-\frac{c}{a}B-\frac{b}{a}C, \text{\&c.}$$

Here each coefficient from the 3° is the sum of the two preceding, multiplied respectively by $-\frac{c}{a}$ and $-\frac{b}{a}$, or each term is the sum of the two preceding multiplied by $-\frac{cx^2}{a}$ and $-\frac{bx}{a}$.

Again, in the development of

$$\frac{a'+b'x+c'x^2}{a+bx+cx^2+dx^3}=\text{A}+\text{B}x+, \text{ \&c.},$$

each term will be composed of the three preceding, multiplied respectively by $-\frac{dx^3}{a}$, $-\frac{cx^2}{a}$, $-\frac{bx}{a}$.

Finally, it becomes now evident that in general a fraction of the form

$$\frac{a'+b'x+c'x^2 \ldots +h'x^{m-1}}{a+b\,x+c\,x^2 \ldots +k\,x^m}$$

produces a series, each term of which from the $(m+1)^{\text{th}}$ is composed of the m preceding, multiplied respectively by $-\frac{k}{a}x^m$, $-\frac{h}{a}x^{m-1}$, ... $-\frac{c}{a}x^2$, $-\frac{b}{a}x$.

Series of this form are called recurrent, and the assemblage of quantities by which it is necessary to multiply several consecutive terms to obtain the following term, is called the *scale of relation* of the terms.

328. Problem.—*A recurring series being given, to return to the generating fraction.*

In this enunciation it is supposed that the recurring series is arranged with respect to an indeterminate x. Let

$$\text{S}=\text{A}+\text{B}x+\text{C}x^2+\ldots$$

be such a series, having for a scale of relation $[px^3, qx^2, rx]$. Since this scale contains three terms, the generating fraction is of the form

$$\frac{a'+b'x+c'x^2}{a+bx+cx^2+dx^3}.$$

If this fraction had been given, we have seen that the scale of relation would be $\left[-\frac{d}{a}x^3, -\frac{c}{a}x^2, -\frac{b}{a}x\right]$. But the generating fraction can be written thus,

$$\frac{\frac{a'}{a}+\frac{b'}{a}x+\frac{c'}{a}x^2}{1+\frac{b}{a}x+\frac{c}{a}x^2+\frac{d}{a}x^3};$$

and then we perceive that the three terms in x of the denominator can be at once obtained by taking those of the scale of relation with contrary signs. Thus, we can put the generating fraction under the form

$$\frac{\alpha+\beta x+\gamma x^2}{1-rx-qx^2-px^3},$$

and we shall only have to determine α, β, γ. To do this, place

$$\frac{\alpha+\beta x+\gamma x^2}{1-rx-qx^2-px^3}=\text{A}+\text{B}x+\text{C}x^2+\ldots;$$

and since, after clearing it of fractions, the equation ought to be identical in form, we derive from it, having regard only to the first three terms,

$$\alpha+\beta x+\gamma x^2=\text{A}+\left.\begin{array}{r}\text{B}\\-\text{A}r\end{array}\right|x+\left.\begin{array}{r}\text{C}\\-\text{B}r\\-\text{A}q\end{array}\right|x^2$$

Consequently, we shall have for the generating fraction

$$\text{S}=\frac{\text{A}+(\text{B}-\text{A}r)x+(c-\text{B}r-\text{A}q)x^2}{1-rx-qx^2-px^3}.$$

For example, let $S=1-2x-x^2-5x^3+4x^4-\ldots$ be a recurring series, whose scale of relation is $[+x^3, +4x^2, -2x]$. Taking the above formula, we shall have

$$A=1,\ B=-2,\ c=-1,\ p=1,\ q=4,\ r=-2,$$

and we shall find

$$S=\frac{1-9x^2}{1+2x-4x^2-x^3}.$$

329. Problem.—*A series being given, to determine whether it be recurring, and, in this case, to return to the generating fraction.*

Let the given series be

$$S=A+Bx+Cx^2+Dx^3+\ldots.$$

Let us determine first whether it be equal to a fraction of the form $\frac{a'}{a+bx}$, and place $S=\frac{a'}{a+bx}$. From this equation we derive

$$\frac{1}{S}=\frac{a+bx}{a'}=\frac{a}{a'}+\frac{b}{a'}x;$$

the quotient, therefore, of (1), divided by the series, ought to be exact, and of the form $p+qx$. Then the generating fraction will be expressed thus:

$$S=\frac{1}{p+qx}.$$

If the division does not stop at the second term this series will not be recurring, or else it will arise from a more complicated fraction.

Place $S=\frac{a'+b'x}{a+bx+cx^2}$, we shall have

$$\frac{1}{S}=\frac{a+bx+cx^2}{a'+b'x}=p+qx+\frac{a''x^2}{a'+b'x};$$

that is to say, dividing (1) by the series S, if we stop the division after we have obtained as a quotient terms of the form $p+qx$, the series S_1x^2, which is the remainder that we then have, and which is always divisible by x^2, will be such that, after we have removed this factor, we must have $\frac{S_1}{S}=\frac{a''}{a'+b'x}$.

Hence we derive

$$\frac{S}{S_1}=\frac{a'+b'x}{a''}=p_1+q_1x;$$

that is to say, the new division ought to terminate at the second term in the quotient; and then, to find the generating fraction, we shall have the two equations,

$$\frac{1}{S}=p+qx+\frac{S_1}{S}x^2,\quad \frac{S}{S_1}=p_1+q_1x,$$

whence

$$S=\frac{1}{p+qx+\frac{S_1}{S}x^2},\quad \frac{S_1}{S}=\frac{1}{p_1+q_1x}.$$

Consequently, the generating fraction will be

$$S=\frac{p_1+q_1x}{(p+qx)(p_1+q_1x)+x^2}.$$

Suppose that the quotient of S by S_1 is not exactly p_1+q_1x; if the series is recurring, it will be of an order superior to the second. Let us examine if we can have $S=\frac{a'+b'x+c'x^2}{a+bx+cx^2+dx^3}$.

We derive from this equation

$$\frac{1}{S}=p+qx+\frac{a''+b''x}{a'+b'x+c'x^2}x^2;$$

that is to say, after having obtained the first two terms of the quotient of 1, divided by the series S_1, we shall find for a remainder a series, all of whose terms will contain x^2; and if we designate this remainder by S_1x^2, we shall have

$$\frac{S_1}{S}=\frac{a''+b''x}{a'+b'x+c'x^2}.$$

This equality gives

$$\frac{S}{S_1}=p_1+q_1x+\frac{a'''}{a''+b''x}x^2;$$

hence, designating by S_2x^2 the series which we find for a remainder after having carried the division of the series S by the series S_1 to the terms of the quotient p_1+q_1x, we should have

$$\frac{S_2}{S_1}=\frac{a'''}{a''+b''x}.$$

From this last equality we derive

$$\frac{S_1}{S_2}=p_2+q_2x.$$

Here the operations stop; for, returning to the generating fraction, we shall have the equations

$$\frac{1}{S}=p+qx+\frac{S_1}{S}x^2,\ \frac{S}{S_1}=p_1+q_1x+\frac{S_2}{S_1}x^2,\ \frac{S_1}{S_2}=p_2+q_2x;$$

and from these equations we derive

$$S=\frac{1}{p+qx+\frac{S_1}{S}x^2},\ \frac{S_1}{S}=\frac{1}{p_1+q_1x+\frac{S_2}{S_1}x^2},\ \frac{S_2}{S_1}=\frac{1}{p_2+q_2x}.$$

We have, then, only a few substitutions to make in order to obtain a fraction equal to S.

Without proceeding further, the reader will doubtless perceive that the successive operations for finding the quotients $p+qx$, p_1+q_1x, &c., and for returning to the generating fraction, bear a striking analogy to those which are necessary in reducing an ordinary fraction to a continued fraction, and in returning to the ordinary fraction. This observation will take the place of a general rule. If we arrive at a division which gives an exact quotient of the form $p+qx$, we know that the series is recurring. (See Contin. Fractions.)

EXAMPLE.

Suppose we wish to determine whether the series of numbers 1, 2, 3, &c., be recurring. It is not this numerical series which we must consider, but the equation

$$S=1+2x+3x^2+4x^3+\cdots$$

We perceive that the operations will be performed as follows:

Division of 1 *by* S

$$\begin{array}{l|l} 1 & 1+2x+3x^2+4x^3+\dots \\ \cline{2-2} 1+2x+3x^2+4x^3+\dots & 1-2x \\ \hline \quad -2x-3x^2-4x^3-5x^4-\dots & \\ \quad -2x-4x^2-6x^3-8x^4-\dots & \\ \hline \qquad x^2+2x^3+3x^4+\dots=S_1x^2. & \end{array}$$

Division of S *by* S_1.

$$\begin{array}{l|l} 1+2x+3x^2+4x^3+\dots & 1+2x+3x^2+4x^3+\dots \\ \cline{2-2} 1+2x+3x^2+4x^3+\dots & 1 \\ \hline \qquad 0 & \end{array}$$

Hence, the series S is recurring, and we have $\frac{1}{S}=1-2x+\frac{S_1}{S}x^2$, $\frac{S}{S^1}=1$.

We derive from this $S=\frac{1}{1-2x+\frac{S_1}{S}x^2}$, $\frac{S_1}{S}=1$; consequently, $S=\frac{1}{1-2x+x^2}$

$=\frac{1}{(1-x)^2}$.

Remark.—In finding a rule to determine whether a series is recurring, we have considered the series as derived from a fraction whose numerator is of a degree inferior to the denominator. But even if this last condition does not have place, it is easy to perceive that the same explications, and, consequently, the same rule, will always subsist.

329. Problem.—*To find the general term of a recurring series.*

Suppose that the series has for a generating fraction

$$F=\frac{a'+b'x+\dots+h'x^{m-1}}{a+bx+\dots+kx^m}.$$

We can write this fraction thus:

$$F=(a'+b'x\dots+h'x^{m-1})(a+bx+\dots+kx^m)^{-1}.$$

It is evident, then, that by developing the power -1, obtaining the product of the two factors in this equation, and taking in this product the part which contains x to any power whatsoever, we shall have the general term of the recurring series. But the problem is resolved ordinarily by another process, which I proceed to exhibit.

We divide first all the terms of the fraction F by k, and write it under the form

$$\frac{U}{V}=\frac{\alpha'x^{m-1}+\beta'x^{m-2}+\dots}{x^m+\alpha x^{m-1}+\beta x^{m-2}+\dots}.$$

The fraction is supposed in all cases to be reduced to its most simple form, so that U has no common factor with V.

We decompose, then, the denominator into binomial factors, such as $x+a$, whether it be by equating this denominator to zero, or by some other method, and then the fraction is regarded as resulting from the addition of many others, which have for denominators these different factors. We determine all these partial fractions, and then form the general term of the development of each; then, taking the sum of these general terms, we shall have the general term of the recurring series.

In this decomposition into partial fractions it is necessary carefully to distinguish in V the simple factors from those which are raised to powers For each simple factor $x+a$ we shall take a fraction of the form

$$\frac{M}{x+a}.$$

For each factor, such as $(x+b)^n$, we might take one of the form

$$\frac{Ax^{n-1}+Bx^{n-2}+\ldots}{(x+b)^n};$$

but it is more convenient to have only fractions with monomial numerators; instead, therefore, of a fraction like the preceding, we take n, like the following:

$$\frac{N}{(x+b)^n}+\frac{N_1}{(x+b)^{n-1}}+\frac{N_2}{(x+b)^{n-2}}\cdots+\frac{N_{n-1}}{x+b},$$

M, N, $N_1\ldots$ representing quantities independent of x.

Consequently, if we suppose that $V=(x+a)(x+b)^n\ldots$, we can place

$$\frac{U}{V}=\frac{M}{x+a}+\frac{N}{(x+b)^n}+\frac{N_1}{(x+b)^{n-1}}\cdots+\frac{N_{n-1}}{x+b}+\cdots,$$

and the question will be reduced, for the present, to the determination of the numerators M, N, N_1, &c. But these have been determined in Art. 209, (3).

The preceding decomposition being effected, the determination of the general term of the recurring series does not offer any difficulty.

Each partial fraction can be put under the form $P(p+x)^{-\lambda}$, designating by λ an entire positive number, which can be equal to 1. If we develop this power, we readily find that the term affected with x^n is

$$\frac{-\lambda(-\lambda-1)(-\lambda-2)\ldots(-\lambda-n+1)}{1.2.3\ldots n}Pp^{-\lambda-n}x^n.$$

It is the sum of like expressions, all containing x^n, and resulting from the different partial fractions which compose the general term required.

When the denominator of the generating fraction contains imaginary factors, these factors introduce imaginary quantities into the general term. If we suppose, however, that the coefficients of the numerator and denominator of the proposed fraction are all real (and they are always taken so), it is evident, *à priori*, that, as we find the development of this fraction by division, the general term can not embrace any imaginary factors; consequently, we are sure that all the imaginary quantities which arise from the factors of the denominator will disappear.

SUMMATION OF SERIES.

The summation of series is the finding of a finite expression equal to the proposed series, even when the series is infinite, and in many cases this finite expression is found by subtraction.

EXAMPLES.

(1) Required the sum of the series $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots$ to infinity.

$$\text{Let } S=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots\ldots\ldots\ldots \text{ ad infinitum.}$$

$$\therefore S-1=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\ldots\ldots\ldots\ldots \text{ ad infinitum.}$$

Hence, by subtracting the latter from the former, we have the required sum:

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}+\ldots\ldots=1.$$

(2) Required the sum of the series $\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\ldots\ldots$ to n terms.

$$\text{Let } S=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\ldots\ \frac{1}{n}\ \ldots\ldots (a)$$

$$\therefore S-1-\frac{1}{2}+\frac{1}{n+1}+\frac{1}{n+2}=\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots\ldots\frac{1}{n+2}\ \ldots\ldots (b)$$

Subtracting (b) from (a), we have

$$1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}=\frac{2}{1.3}+\frac{2}{2.4}+\frac{2}{3.5}+\frac{2}{4.6}+\ldots\ldots\frac{2}{n(n+2)}$$

$$\therefore \frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\frac{1}{4.6}+\ldots\frac{1}{n(n+2)}=\frac{1}{2}\left\{1+\frac{1}{2}-\left(\frac{1}{n+1}+\frac{1}{n+2}\right)\right\}$$

$$=\frac{1}{2}\left\{1-\frac{1}{n+1}+\frac{1}{2}-\frac{1}{n+2}\right\}$$

$$=\frac{n}{2n+2}+\frac{n}{4n+8}.$$

When n is infinitely great, then we have

$$\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\frac{1}{4.6}+\ldots \text{ ad infinitum }=\frac{1}{2}\left(1+\frac{1}{2}\right)-\frac{1}{\infty}-\frac{1}{\infty}=\frac{3}{4}.$$

(3) Sum the series $\frac{1}{1.3}-\frac{1}{2.4}+\frac{1}{3.5}-\frac{1}{4.6}+\ldots\ldots$ ad infinitum.

Ans. $\frac{1}{4}$.

(4) Sum the series $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}+\ldots\ldots$ ad infinitum.

Ans. $\frac{1}{18}$.

(5) Sum the series $\frac{5}{1.2.3}+\frac{6}{2.3.4}+\frac{7}{3.4.5}+\ldots\ldots$ to n terms.

Ans. $\frac{3}{2}-\frac{2}{n+1}+\frac{1}{n+2}$.

(6) Sum the series $a+2ar+3ar^2+4ar^3+\ldots\ldots$ to n terms.

Ans. $a\left\{\frac{1-r^n}{(1-r)^2}-\frac{nr^n}{1-r}\right\}$.

(7) Sum the series $1+3x+5x^2+7x^3+9x^4\ldots\ldots$ ad infinitum.

Ans. $\frac{1+x}{(1-x)^2}$.

DIFFERENCE SERIES.

330. Let there be the arithmetical progression

$$a,\ a+\delta,\ a+2\delta,\ a+3\delta\ldots\ldots$$

If we begin with a new term, b, and add to it successively each term of the above, we obtain

$$b,\ b+a,\ b+2a+\delta,\ b+3a+3\delta,\ b+4a+6\delta\ldots,$$

which is called a difference series of the 2° order, and so on, as in the following scheme:

Order of Series.	1° term.	2° term.	3° term		nth term.
I.	a,	$a+\delta$,	$a+2\delta$,	. . .	$a+(n-1)\delta$.
II.	b,	$b+a$,	$b+2a+\delta$	. . .	$b+(n-1)a+\frac{(n-2)(n-1)}{1.2}\delta$.
III.	c,	$c+b$,	$c+2b+a$	. . .	$c+(n-1)b+\frac{(n-2)(n-1)}{1.2}a+\frac{(n-3)(n-2)(n-1)}{1.2.3}\delta$.
&c.	&c.				

EXAMPLE.

I. order, 2, 5, 8, 11, 14 . .
II. order, 4, 6, 11, 19, 30 . .
III. order, 5, 9, 15, 26, 45 . .

331. From the manner in which these difference series are formed, it is evident that if we subtract from one another the successive terms of any order, we obtain the terms of the preceding, and continuing in this way till we subtract the successive terms of the first from one another, we obtain between them the constant difference δ.

332. If the order of a series be unknown, its order may be found from what has been said above. Thus the series

$$5,\ 9,\ 15,\ 26,\ 45;$$

taking the difference of the consecutive terms,

$$4,\ 6,\ 11,\ 19$$
$$2,\ 5,\ 8$$
$$3,\ 3,\ 3,$$

after three subtractions of consecutive terms presents a constant difference, and is, therefore, a series of the 3° order.

333. *To separate the roots of an equation by means of difference series.*

The x^{th} term of a series of the order m would be expressed by

$$k+(x-1)f+\frac{(x-2)(x-1)}{1.2}g+\ldots.+\frac{(x-m)\ldots(x-2)(x-1)}{1.2\ldots m}\delta,$$

which, arranged according to the powers of x, would be of the form

$$Mx^m+Ax^{m-1}+Bx^{m-2}\ldots.+Gx+K;$$

that is, of the form of the first member of an equation of the m^{th} degree, $X=0$.

If, now, we give to x the values . . . -4, -3, -2, -1, -0, 1, 2, 3, 4, ... representing the values which the polynomial X assumes by

$$X_{-4},\ X_{-3},\ X_{-2},\ X_{-1},\ X_0,\ X_1,\ X_2,\ \&c.\ \ldots\ldots\ (1)$$

these quantities will form a difference series, since x denotes the order of the term in a series of which X is the general term. There is no objection to x being negative, as a series may be continued below as well as above the first term, observing the same law in a contrary sense.

Taking a sufficient number of terms of the series (1) to obtain, by subtraction of its successive terms, the series of next lower order, and from this, in the same manner, that of the next lower order still, till we arrive at constant differences, the terms of the series (1) may be extended indefinitely to the right and left by forming them according to (Art. 330), without the trouble of substituting numerical values for x, and calculating the corresponding values of X. Those values of X which have contrary signs will (Art. 252, Cor. 1) have one or an odd number of roots between them.

Take, for example, the equation

$$9x^4-3x^3-130x^2-17x+260=0.$$

Giving x the values $-2, -1, 0, 1, 2$, we have the following results inclosed in the parentheses :

$$\begin{array}{ccccccccc} X_{-4} & X_{-3} & X_{-2} & X_{-1} & X_0 & X_1 & X_2 & X_3 & X_4 \\ +744 & -49(& -58 & +159 & +260 & +119 & -174) & -313 & +224, \end{array}$$

forming a series of the fourth order. The series of the third order is

$$-793 \quad -9(+217 \quad +101 \quad -141 \quad -293)-139+537\,;$$

of the second, $\quad +784 \quad +226(-116 \quad -242 \quad -152)+154+676\,;$

of the first, $\quad -558 \quad -342(-126 + 90)+306+522\,;$

equal differences, $\quad +216 \quad +216(+216)+216+216.$

By substituting other values, as $-3, -4, -5, -6$, and $+3, +4, +5, +6$, &c., we may extend the top series to any length.

To save the time and trouble of substituting consecutive numbers and calculating the result, the method of difference series is employed, thus :

Substitute a number of consecutive values one more than the degree of the equation; the smallest numbers, being more easily substituted, are preferred. In the present example, substituting $-2, -1, 0, 1, 2$, we obtain that portion of the first series which is of the $3°$ order, included in brackets ; from this, by subtracting its consecutive terms, the corresponding portions of the series of the $2°$ order, and so on ; and, finally, the difference, 216. Using this difference, we may extend the top series at pleasure, according to the method in Art. 330.

The roots of the equation lie between those numbers the substitutions of which produce unlike signs in the result ; thus, in the above there is one root between -3 and -4, one between -1 and -2, one between 1 and 2, and one between 3 and 4.

334. There exists between the coefficients of two consecutive powers of $x+a$ relations from which many useful consequences may be deduced.

Suppose the m^{th} power of $x+a$ to be

$$x^m + \mathrm{A}ax^{m-1} + \mathrm{B}a^2x^{m-2} + \mathrm{C}a^3x^{m-3} +, \text{ \&c.}$$

Multiplying the polynomial by $x+a$, there results

$$\begin{array}{l} x^{m+1} + \mathrm{A}ax^m + \mathrm{B}a^2x^{m-1} + \mathrm{C}a^3x^{m-2} + \ldots \\ \qquad\quad + \; ax^m + \mathrm{A}a\,x^{m-1} + \mathrm{B}a\,x^{m-2} + \ldots \end{array}$$

From which we conclude that, *to obtain the coefficient of any term of the* $(m+1)^{th}$ *power of* $x+a$, *it is only necessary to add to the coefficient of the term of the same rank in the* m^{th} *power that of the preceding term.*

335. According to this rule, we can form the coefficients of the successive powers of $x+a$, as may be seen in the following table :

$$\begin{array}{rrrrrrrrrl} 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1 & \ldots \\ & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8 & \ldots \\ & & 1, & 3, & 6, & 10, & 15, & 21, & 28 & \ldots \\ & & & 1, & 4, & 10, & 20, & 35, & 56 & \ldots \\ & & & & 1, & 5, & 15, & 35, & 70 & \ldots \\ & & & & & 1, & 6, & 21, & 56 & \ldots \\ & & & & & & 1, & 7, & 28 & \ldots \\ & & & & & & & 1, & 8 & \ldots \\ & & & & & & & & 1 & \ldots \end{array}$$

The first vertical column of this table is formed of the single number 1. The second column is formed of the number 1 written twice. We form the third column by placing at the side of each term in the second column the number obtained by adding it to the term above it; we find thus, for the first term of the third column $1+0$ or 1; the second term is $1+1$ or 2, and the third $0+1$ or 1. The fourth column is deduced from the third in the same manner that that is from the second, and so on. The two terms of the second column may be considered as the coefficients of the first power of $x+a$. It results from the above rule that the terms of the third column are the coefficients of the development of $(x+a)^2$, those of the fourth column of $(x+a)^3$, &c.

This table, which may be indefinitely extended, is called the *Arithmetical Triangle of Pascal.*

336. It is easy to see from the composition of the arithmetical triangle that the p^{th} term of any horizontal line is the sum of the p first terms of the preceding horizontal line. Because if we consider, for example, the term 56, which is the sixth of the fourth line, this term is formed by adding the two numbers 21 and 35, which are placed at its left in the third and fourth lines; but the second of these two numbers, 35, is the sum of 15 and 20; the last number, 20, is the sum of 10 and 10, and the last number, 10, the sum of 6 and 4; finally, 4 is the sum of the two numbers 3 and 1; we have, therefore, $56=21+15+10+6+3+1$.

THE DIFFERENTIAL METHOD OF SUMMING SERIES.

337. Let $a, b, c, d, e, \ldots$ be a series of terms, in which each term is less than the succeeding one; and, taking the successive differences, we have

	a	b	c	d	e, &c.
(d_1)		$b-a$	$c-b$	$d-c$	$e-d$, &c.
(d_2)			$c-2b+a$	$d-2c+b$	$e-2d+c$, &c.
(d_3)				$d-3c+3b-a$	$e-3d+3c-b$, &c.
(d_4)					$e-4d+6c-4b+a$, &c.

Putting $d_1, d_2, d_3, d_4, \ldots$ for the first terms of the first, second, third, fourth, ... differences, we have

$$\begin{aligned} b-a &= d_1 \;\therefore\; b=a+d_1 \\ c-2b+a &= d_2 \;\therefore\; c=a+2d_1+d_2 \\ d-3c+3b-a &= d_3 \;\therefore\; d=a+3d_1+3d_2+d_3 \\ e-4d+6c-4b+a &= d_4 \;\therefore\; e=a+4d_1+6d_2+4d_3+d_4, \\ \&c. &\qquad\qquad \&c. \end{aligned}$$

Hence the $(n+1)^{\text{th}}$ term of the proposed series is evidently

$$a+nd_1+n\frac{(n-1)}{1.2}d_2+\frac{n(n-1)(n-2)}{1.2.3}d_3+\ldots.$$

and, therefore, the n^{th} term is (by writing $n-1$ for n)

$$a+(n-1)d_1+\frac{(n-1)(n-2)}{1.2}d_2+\frac{(n-1)(n-2)(n-3)}{1.2.3}d_3+\ldots \quad (1)$$

338. *To find* (S) *the sum of* n *terms of a series.*

Let	a,	b,	c,	d,	e,	&c.
and	0,	a,	$a+b$,	$a+b+c$,	$a+b+c+d$,	&c.,

be two series, of which the $(n+1)^{\text{th}}$ term of the latter is obviously the sum of n terms of the former; but the first terms of the first, second, third, fourth, ... differences in the latter, are

$$a,\ b-a=d_1,\text{*}\ c-2b+a=d_2,\ d-3c+3b-a,\ =d_3,\ \&c.;$$

hence the $(n+1)^{\text{th}}$ term of the latter series, or the sum of n terms of the former, is, by (1) in the last article,

$$0+na+\frac{n(n-1)}{1.2}d_1+\frac{n(n-1)(n-2)}{1.2.3}d_2+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}d_3+\ldots,$$

or

$$S=na+\frac{n(n-1)}{1.2}d_1+\frac{n(n-1)(n-2)}{1.2.3}d_2+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}d_3+\ldots \quad (2)$$

EXAMPLES.

(1) To what is $1.2+2.3+3.4+4.5+\ldots n(n+1)$ equal?

2, 6, 12, 20, 30, is the given series;
4, 6, 8, 10, differences of the consecutive terms;
2, 2, 2, differences of these again, d_2;
0, 0.

Hence, $a=2$, $d_1=4$, $d_2=2$, and d_3, d_4, &c. $=0$; therefore

$$S=na+\frac{n(n-1)}{2}d_1+\frac{n(n-1)(n-2)}{2.3}d_2;$$
$$=2n+2n(n-1)+\tfrac{1}{3}n(n-1)(n-2)$$
$$=\tfrac{1}{3}n(n+1)(n+2).$$

Proceed always in this way till the differences become the same.†

(2) Find the sum of n terms of the series 1, 2^3, 3^3, 4^3, 5^3, &c.

(3) Find the sum of n terms of the series 1, 4, 10, 20, 35, &c.

(4) To what is $1.2.3+2.3.4+3.4.5+\ldots\ldots n(n+1)(n+2)$ equal?

(5) Sum n terms of the series 1, 3, 5, 7, 9, 11, &c. ...

(6) Find the sum of 15 terms of the series 1, 4, 8, 13, 19, &c.

(7) Sum 8 terms of the series 1, 2^4, 3^4, 4^4, 5^4, 6^4, &c.

ANSWERS.

(2) $\dfrac{n^2(n+1)^2}{4}$.

(3) $\dfrac{n(n+1)(n+2)(n+3)}{1.2.3.4}$.

(4) $\tfrac{1}{4}n(n+1)(n+2)(n+3)$.

(5) n^2.

(6) $\tfrac{1}{6}n(n^2+6n-1)=785$.

(7) $\dfrac{n^5}{5}+\dfrac{n^4}{2}+\dfrac{n^3}{3}-\dfrac{n}{30}=8772$.

POWERS OF THE TERMS OF PROGRESSIONS.

339. If all the terms of a geometrical progression

$$\div\!\div a:aq:aq^2:aq^3\ldots aq^{n-1}$$

are raised to the same power m, the result is the series

$$a^m,\ a^mq^m,\ a^mq^{2m},\ a^mq^{3m}\ldots\ldots a^mq^{m(n-1)},$$

which is a geometrical progression, of which the first term is a^m, the ratio q^m, and the number of terms n.

340. If the terms of a progression by differences, whose first term is a and common difference δ, be each raised to the m^{th} power, we have

* This is the d_1 of the former series, but the d_2 of the latter.

† The terms of the formula (2), containing those orders of differences which become zero, like d_3, d_4, &c., in example 1, will all vanish, and the expression for S will be composed only of the preceding terms.

$$a^m = a^m.$$

$$(a+\delta)^m = a^m + ma^{m-1}\delta + \frac{m(m-1)}{1\,.\,2}a^{m-2}\delta^2 +, \&c.$$

$$(a-2\delta)^m = a^m + ma^{m-1}2\delta + \frac{m(m-1)}{1\,.\,2}a^{m-2}4\delta^2 +, \&c.$$

$$(a-3\delta)^m = a^m + ma^{m-1}3\delta + \frac{m(m-1)}{1\,.\,2}a^{m-2}9\delta^2 +, \&c.$$

&c. &c.

Taking the differences of the consecutive terms,

$$(a+\delta)^m - a^m = ma^{m-1}\delta + \frac{m(m-1)}{1\,.\,2}a^{m-2}\delta^2 +, \&c.$$

$$(a+2\delta)^m - (a+\delta)^m = ma^{m-1}\delta + \frac{m(m-1)}{1\,.\,2}a^{m-2}3\delta^2 +, \&c.$$

$$(a+3\delta)^m - (a+2\delta)^m = ma^{m-1}\delta + \frac{m(m-1)}{1\,.\,2}a^{m-2}5\delta^2 +, \&c.$$

These differences being not the same, the same powers of the terms of an arithmetical progression do not form an arithmetical progression.

341. To find the sum of the m^{th} powers of an arithmetical progression. Let

$$\div a\,.\,b\,.\,c\,.\,d\ldots k\,.\,l$$

be any arithmetical progression, of which the common difference is δ. Then

$$b = a+\delta,\ c = b+\delta,\ \ldots\ldots l = k+\delta.$$

Raising these equalities to the power $m+1$,

$$b^{m+1} = a^{m+1} + (m+1)a^m\delta + \frac{(m+1)m}{1\,.\,2}a^{m-1}\delta^2 +, \&c.$$

$$c^{m+1} = b^{m+1} + (m+1)b^m\delta + \frac{(m+1)m}{1\,.\,2}b^{m-1}\delta^2 +, \&c.$$

. . . .

$$l^{m+1} = k^{m+1} + (m+1)k^m\delta + \frac{(m+1)m}{1\,.\,2}k^{m-1}\delta^2 +, \&c.$$

Adding all these equalities, suppressing the common terms in the two equa sums, viz., b^{m+1}, c^{m+1}, &c., and transposing a^{m+1}, we have

$$l^{m+1} - a^{m+1} = (m+1)\delta(a^m + b^m \ldots\ldots + k^m),$$
$$+\frac{(m+1)m}{1\,.\,2}\delta^2(a^{m-1} + b^{m-1}\ldots\ldots + k^{m-1}),$$
$$+, \&c.$$

To abridge, let

$$a + b + c + d \ldots\ldots + k + l = S_1,$$
$$a^2 + b^2 \ldots\ldots\ldots\ldots + k^2 + l^2 = S_2,$$
$$\ldots\ldots\ldots$$
$$a^m + b^m + \ldots\ldots\ldots + k^m + l^m = S_m.$$

Then the last expression becomes

$$l^{m+1} - a^{m+1} = \frac{m+1}{1}\delta(S_m - l^m) + \frac{(m+1)m}{1\,.\,2}\delta^2(S_{m-1} - l^{m-1}) +, \&c.$$

The value of S_m deduced from this is

$$S_m = l^m + \frac{l^{m+1} - a^{m+1}}{(m+1)\delta} - \frac{m}{2}\delta(S_{m-1} - l^{m-1}) - \frac{m(m-1)}{2\,.\,3}\delta^2(S_{m-2} - l^{m-2}) -, \&c. \quad (1)$$

The law of the unwritten terms is sufficiently apparent, and the series must evidently end with the term preceding that which contains the factor $m-m$ or 0.

By formula (1) the sum S_m can be found, when the sums of the inferior powers are known; for this purpose, make $m=0$, the formula gives S_0; making $m=1$, it gives S_1, and so on to the sum of the powers required.

If the progression $\div a.a+\delta.a+2\delta\ldots.$ is replaced by $\div 1.2.3\ldots.N$ (or the series of natural numbers from 1 to N), *i. e.*, $a=1$, $\delta=1$, $l=N$, then formula (1) becomes

$$S_m=N^m+\frac{N^{m+1}-1}{m+1}-\frac{m}{2}(S_{m-1}-N^{m-1})-\frac{m(m-1)}{2.3}(S_{m-2}-N^{m-2})-, \text{\&c.} \quad (2)$$

If $m=0$, (2) becomes

$$S_0=N^0+\frac{N^{0+1}-1}{0+1}=1+\frac{N-1}{1}=N \ldots\ldots\ldots\ldots (3)$$

If $m=1$,

$$S_1=\frac{N(N+1)}{2}\ldots\ldots\ldots\ldots (4)$$

If $m=2$,

$$S_2=N^2+\frac{N^3-1}{3}-(S_1-N)-\frac{1}{3}(S_0-N^0),$$

$$=N^2+\frac{N^3-1}{3}-\left(\frac{N^2+N}{2}-N\right)-\frac{1}{3}(N-1),$$

$$=N^2+\frac{N^3}{3}-\frac{1}{3}-\frac{N^2}{2}-\frac{N}{2}+N-\frac{N}{3}+\frac{1}{3},$$

$$=\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}=\frac{2N^3+3N^2+N}{6}.$$

$$S_2=\frac{N(N+1)(2N+1)}{6}\ldots\ldots\ldots\ldots (5)$$

Formula (3) expresses the sum of $1^0+2^0+3^0\ldots.$ to N terms, or of $1+1+1\ldots$ to N.

EXAMPLES.

(1) If $m=0$ and $N=10$, $S_0=N=10$.

Formula (4) expresses the sum of $1+2+3\ldots.+N$.

(2) If $m=1$ and $N=10$, $S_1=\frac{10(10+1)}{2}=\frac{110}{2}=55$.

Formula (5) expresses the sum of $1^2+2^2+3^2\ldots..+N^2$.

(3) If $m=2$ and $N=10$, $S_2=\frac{10\times 11\times 21}{6}=385$.

PILING OF BALLS AND SHELLS.

342. Balls and shells are usually piled in three different forms, called triangular, square, or rectangular, according as the figure on which the pile rests is triangular, square, or rectangular.

(1) A triangular pile is formed by continued horizontal courses of balls or shells laid one above another, and these courses or rows are usually equilateral triangles whose sides decrease by unity from the bottom to the top row, which is composed simply of *one* shot.

Denoting by N the number of balls contained in one side of the equilateral triangle which forms the base of the triangular pile, it is evident that the number of balls in the base will be expressed by $1+2+3\ldots+N$ or S_1, which by (4) is equal to

$$\frac{N^2+N}{2}.$$

If in this expression N is successively replaced by the numbers 1, 2, 3 . . . ., the number of balls in the successive layers, beginning at the top, will be obtained. These are,

in the first, $\frac{1^2+1}{2}=1$;

in the second, $\frac{2^2+2}{2}=3$;

in the third, $\frac{3^2+3}{2}=6$;

in the fourth, $\frac{4^2+4}{2}=10$.

Whence the sum of the whole number of balls contained in the pile is

$$\frac{1^2+1}{2}+\frac{2^2+2}{2}+\frac{3^2+3}{2}\dots+\frac{N^2+N}{2},$$

which is sometimes used. A better form may be obtained from this by writing it first

$$\frac{1^2+2^2+3^2\dots+N^2}{2}+\frac{1+2+3\dots+N}{2},$$

or

$$\frac{S_2+S_1}{2}=\frac{1}{2}\left(\frac{2N^3+3N^2+N}{6}+\frac{N^2+N}{2}\right)=\frac{N^3+3N^2+2N}{6},$$

or

$$\frac{N(N+1)(N+2)}{6},$$

the most convenient expression for the number of balls in a triangular pile.

EXAMPLE.

How many balls in a triangular pile, the side of whose base contains 35 ?

Ans. $\frac{35(35+1)(35+2)}{6}=7770$.

(2) A square pile is formed by continued horizontal courses of shot laid one above another, and these courses are squares whose sides decrease by unity from the bottom to the top row, which is also composed simply of *one* shot; and hence the series of balls composing a square pile is

$$1+4+9+16+25+\dots\dots+N^2=S_2=\frac{N(N+1)(2N+1)}{6},$$

where N denotes the number of courses in a pile.

EXAMPLE.

If a side of the base of a quadrangular pile contains 35 balls, how many in the pile ?

Ans. $\frac{35\times36\times71}{6}=14910$.

(3) A rectangular pile is one in which the layers, except the uppermost, are arranged in rectangles. Representing by $m+1$ the number of balls in the top row, the layer below it must contain 2 rows of $m+2$ balls, the next layer 3 rows of $m+3$ balls, and so on, to the N^{th}, which contains N rows of $m+N$ balls each ; and the number in this pile is

$$(m+1)+2(m+2)+3(m+3)+4(m+4)+\ldots\ldots N(m+N)$$
$$=m+2m+3m+4m+\ldots\ldots Nm+1^2+2^2+3^2+4^2+\ldots\ldots N^2$$
$$=m(1+2+3+4+\ldots\ldots N)+\text{ square pile}$$
$$=\frac{N(N+1)}{2}\cdot m+\text{ square pile.}$$

(4) The number of balls in a complete triangular or square pile must evidently depend on the number of courses or rows; and the number of balls in a complete rectangular pile depends on the number of courses, and also on the number of shot in the top row, or the amount of shot in the latter pile depends on the length and breadth of the bottom row; for the number of courses is equal to the number of shot in the breadth of the bottom row of the pile. Therefore, the number of shot in a triangular or square pile is a function of N, and the number of shot in a rectangular pile is a function of N and m.

The expression for a rectangular pile,

$$\frac{N(N+1)}{2}m+\frac{N(N+1)(2N+1)}{6},$$

may be written

$$\frac{N(N+1)(3m+2N+1)}{6}=\frac{1}{6}N(N+1)[2(m+N)+m+1].$$

But $m+1$ is the number of balls in the top row, N is the number in the smaller side of the base, and $m+N$ the number in the greater side, $2(m+N)$ the number in the two parallel greater sides; moreover, $\frac{N(N+1)}{2}$ is the number of balls in the triangular face of each pile; hence we have also this general rule for rectangular or square piles.

RULE.

Add to the number of balls or shells in the top row the numbers in its two parallels at bottom, and the sum multiplied by one third of the slant end or face gives the number of balls in the pile.

EXAMPLES.

(1) How many balls are in a triangular pile of 15 courses? Ans. 680.

(2) A complete square pile has 14 courses: how many balls are in the pile, and how many remain after the removal of 5 courses? Ans. 609 and 554.

(3) In an incomplete rectangular pile, the length and breadth at bottom are respectively 46 and 20, and the length and breadth at top are 35 and 9: how many balls does it contain? Ans. 7190.

(4) The number of balls in an incomplete square pile is equal to 6 times the number removed, and the number of courses left is equal to the number of courses taken away: how many balls were in the complete pile?

Ans. 385.

(5) Let h and k denote the length and breadth at top of a rectangular truncated pile, and N the number of balls in each of the slanting edges; then if B be the number of balls in the truncated pile, prove that

$$B=\frac{N}{6}\left\{2N^2+3N(h+k)+6hk-3(h+k+N)+1\right\}.$$

VARIATION.

343. Let a denote a constant quantity, or one which does not change its value, and x a variable which is supposed to increase or diminish.

The product of the quantities a and x being denoted by X, if x is increased or diminished, X will be increased or diminished in the same proportion. Thus, if x become x', and, consequently, X become X′, we shall have

$$x : x' :: X : X',$$

for

$$ax = X \text{ and } ax' = X' \therefore \frac{ax}{ax'} = \frac{x}{x'} = \frac{X}{X'}, \text{ or } x : x' :: X : X'.$$

Under these circumstances X is said to vary directly as x.

The symbol of variation is $\propto$; and the expression X varies directly as x, is indicated by the combination of symbols $X \propto x$.

344. If the product of x and y be constant, and x, y both variable, since $xy = x'y' = C$,

$$x : x' :: y' : y :: \frac{1}{y} : \frac{1}{y'}.$$

In this case as x varies as the reciprocal of y, x is said to vary inversely as y, and the symbolical expression is

$$x \propto \frac{1}{y}.$$

If $xy = X$ and $x'y' = X'$, then $X : X' :: xy : x'y'$.

The variation of X in this case depends on the variation of two quantities x and y, which is expressed thus,

$$X \propto xy.$$

345. If $xy = X$ and $x'y' = X'$, then, $x = \frac{X}{y}$ and $x' = \frac{X'}{y'} \therefore x : x' :: \frac{X}{y} : \frac{X'}{y'}$.
In this case x is said to vary as X directly, and as y inversely. The symbol is

$$x \propto \frac{X}{y}.$$

346. Let $x \propto y$, i. e., $x : x' :: y : y'$ or $\frac{x}{x'} = \frac{y}{y'}$, and let $y \propto z$, i. e., $y : y' : z :: z'$ or $\frac{y}{y'} = \frac{z}{z'}$

$$\therefore \frac{x}{x'} = \frac{z}{z'} \text{ or } x : x' :: z : z', \text{ i. e., } x \propto z;$$

that is, if one quantity vary as a second and the second as a third, the first varies as the third.

347. In like manner, if $x \propto y$ and $y \propto \frac{1}{z}$, $x \propto \frac{1}{z}$.

Again, let $x \propto y$ and $z \propto y \therefore x \propto z$, or $x : x' :: z : z'$, or $x : z :: x' : z'$;
$\therefore x \pm z : z :: x' \pm z' : z'$; or $x \pm z : x' \pm z' :: z : z'$.
But $z : z' :: y : y'$, $\therefore x \pm z : x' \pm z' :: y : y'$, i. e., $y \propto x \pm z$.

Again, since $x \propto y$, $x : x' :: y : y'$, and since $z \propto y$, $z : z' :: y : y'$, $\therefore xz : x'z' :: y^2 : y'^2$, and $\sqrt{xz} : \sqrt{x'z'} :: y : y'$, or $y \propto \sqrt{xz}$; that is, if two quantities vary respectively as a third, their sum, difference, or square root of their product, varies as this third quantity.

348. If $x \propto y$ and m be a constant quantity, integer or fractional, since $x : y ::$

$x' : y'$, $\therefore x : y :: mx' : my'$ (Art. 127), i. e., $x \propto my$; that is, if one quantity vary as another, it varies as any multiple or part of this other.

When $x \propto y$, and, consequently, $x \propto my$, so that $x : x' :: my : my'$ or $x : my :: x' : my'$, then, if $x = my$, x' will be equal to my' in all cases; whence, if x vary as y, x is equal to y multiplied by some constant quantity.

349. If X and Y are two corresponding values of x, y,

$$X = mY, \therefore m = \frac{X}{Y};$$

from which it follows that, when two corresponding values of x, y are known, the constant m may be found.

350. Let $x \propto y \therefore x : x' :: y : y' \therefore x^m : x'^m :: y^m : y'^m \therefore x^m \propto y^m$;

m being any exponent integer or fractional. Whence, if one quantity vary as another, any power or root of the first quantity will vary as the same power or root of the second quantity.

351. Let $x \propto y$, and let t be another quantity, either variable or constant, and of which t, t' are either equal or different values. Then, since

$$x \propto y,\ x : x' :: y : y', \text{ and } t : t' :: t : t';$$
$$\therefore xt : x't' :: yt : y't', \text{ or } xt \propto yt;$$
$$\frac{x}{t} : \frac{x'}{t'} :: \frac{y}{t} : \frac{y'}{t'}, \text{ or } \frac{x}{t} \propto \frac{y}{t},$$

that is, if one quantity vary as another, and if each of them be multiplied or divided by any quantity, variable or constant, the products or quotients will vary as each other.

Consequently, if $x \propto y$, $\frac{x}{y} \propto \frac{y}{y}$, or $\frac{x}{y} \propto 1$.

Whence, if $x \propto y$, $\frac{x}{y}$ is constant.

352. Let $xy \propto X$, i. e., $xy : x'y' :: X : X'$;

by alternation, $xy : X :: x'y' : X'$;

$$\therefore y : \frac{X}{x} :: y' : \frac{X'}{x'}, \therefore y \propto \frac{X}{x};$$

and similarly, $x \propto \frac{X}{y}$;

that is, if the product of two quantities vary as a third quantity, each of the two quantities varies as the third directly, and as the other inversely.

353. If $X = X' =$ constant, $xy : 1 :: x'y' : 1$;

$$\therefore x : \frac{1}{y} :: x' : \frac{1}{y'}, \text{ or } x \propto \frac{1}{y};$$

that is, if the product of two variable quantities be constant, these quantities vary inversely as each other.

354. Let a be a constant, and x, y, z variables, and let

$$a : x :: y : z,\ a : x' :: y' : z', \text{ \&c.};$$
$$\therefore az = xy,\ az' = x'y', \text{ \&c.};$$
$$\therefore az : az' :: xy : x'y', \text{ or } z : z' :: xy : x'y'$$
$$\therefore z \propto xy;$$

that is, if four quantities are always proportional, and one or two of them are constant, the others being variable, it can be found how the latter vary.

355. Let x, y, z be three quantities, of which, $x \propto y$ when z is constant, and

$x \propto z$ when y is constant; it is required to determine the variation of x when y, z are both variable.

Suppose, first, that x is made to vary as y, and that when y becomes y', x becomes x'.

Next, that x' (varied from x by the variation of y) is made further to vary as z, and that when z becomes z', x' becomes x''. Then, since

$$x : x' :: y : y', \text{ and } x' : x'' : z : z'$$

$$\therefore\ xx' : x'x'' :: yz : y'z',$$

or

$$x : x'' :: yz : y'z';$$

$$\text{i. e., } x \propto yz.$$

Therefore, if x vary as y when z is constant, and as z when y is constant, when y, z are both variable, x varies as the product yz.

Similarly, it can be proved, that if t vary as v, x, y, z separately, the others being constant when v, x, y, z are all variable, t varies as the product $vxyz$.

SYMMETRICAL FUNCTIONS OF THE ROOTS OF AN EQUATION.

356. There are certain functions of the roots of an equation which may be expressed, in a general manner, by means of the coefficients of that equation, without the equation itself being resolved.

These functions, which form a very extensive class, are termed *rational and symmetric functions*, or simply *symmetric functions*.

They are called *rational*, because the roots do not enter into them under the radical sign, nor with fractional exponents; the roots are combined only by addition, subtraction, multiplication, and division. These functions are called *symmetric*, because the roots are combined in such a way that any two of them may be interchanged without altering the value of the function.

For example, the expressions

$$ac+bc+ab,\ a^2+b^2+c^2,\ \frac{ab}{2c^2}+\frac{ac}{2b^2}+\frac{bc}{2a^2}-3abc$$

are rational and symmetric functions of a, b, c.

All the coefficients of an equation are symmetric functions of its roots, as may be seen in the expressions for the coefficients in Art. 245; for, in these expressions, if a_1 were written in every place where a_2 occurs, instead of a_2, and a_2 in every place where a_1 occurs, instead of a_1, or if any other two of the roots were interchanged, the values of the expressions would not be altered.

Several quantities, a, b, c, &c., being given, if we arrange them two and two, in every possible way, and if in each arrangement, e. g., ab, we give the exponent α to the first factor and the exponent β to the second, we have a series of products such as $a^\alpha b \beta$, whose sum is evidently a symmetric function of the quantities a, b, c, &c. This function is called a *double* function, because each term contains two of the given quantities; it is represented, abridged, by $S(a^\alpha b^\beta)$, the letter S being here employed to denote the word *sum*. In like manner, *triple*, *quadruple*, &c., symmetric functions are represented by $S(a^\alpha b^\beta c^\gamma)$, $S(a^\alpha b^\beta c^\gamma d^\delta)$, &c.

In accordance with this notation, *simple* symmetric functions, as $a^\alpha+b^\alpha$

$+c^{\alpha}+\ldots$, will be represented by $S(a^{\alpha})$, which, for the sake of abridgment, is ordinarily written S_{α}. In like manner, we have

$$\begin{aligned} S_1 &= a+b+c+\ldots \\ S_2 &= a^2+b^2+c^2+\ldots \\ S_3 &= a^3+b^3+c^3+\ldots \\ &\&c. \qquad \&c. \end{aligned}$$

The notation of which we have been speaking applies to *entire* symmetric functions; but when the terms of a symmetric function are fractional, we can, by reducing them to a common denominator, express the function by a single fraction, whose numerator and denominator are integral symmetric functions. Thus:

$$\frac{ab}{3c^3}+\frac{ac}{3b^3}+\frac{bc}{3a^3}-2abc,$$

which is a fractional symmetric function of a, b, c, becomes, by reduction,

$$\frac{a^4b^4+a^4c^4+b^4c^4-6a^4b^4c^4}{3a^3b^3c^3}.$$

357. *An equation being given, to find the sums* S_1, S_2, &c., *of the like and entire powers of its roots.*

Let the equation be $X=0$,

or $$x^m+Px^{m-1}+Qx^{m-2}+Rx^{m-3}\ldots+Tx+U=0 \quad \ldots \quad (1)$$

and call the m roots a, b, c, d.

We can find by Art. 238 the quotients obtained by dividing X by each of its factors, $x-a$, $x-b$, $x-c$, &c.; and we know (Art. 253) that by adding these m quotients together, the sum must be equal to the derived polynomial X′, or

$$mx^{m-1}+(m-1)Px^{m-2}+(m-2)Qx^{m-3}+(m-3)Rx^{m-4}\ldots+T.$$

The coefficients, therefore, of the powers of x, in this sum, must be equal to the coefficients of the same powers of x in the derived polynomial X′, each to each. In this manner the required sums can be determined.

Let us take, then, the quotient of X divided by $x-a$,

$$\frac{X}{x-a}=x^{m-1}+\left.\begin{matrix}a\\+P\end{matrix}\right|x^{m-2}+\left.\begin{matrix}a^2\\+Pa\\+Q\end{matrix}\right|x^{m-3}+\left.\begin{matrix}a^3\\+Pa^2\\+Qa\\+R\end{matrix}\right|x^{m-4}\ldots\begin{matrix}+a^{m-1}\\+Pa^{m-2}\\+Qa^{m-3}\\+Ra^{m-4}\\ \ldots\ldots\\+T.\end{matrix}$$

In order to have the other quotients, it will be sufficient simply to substitute for a, in this expression, successively b, c, d, &c. If we add these quotients, and put S_1, S_2, S_3, &c., instead of the sums $a+b+c+\ldots$, $a^2+b^2+c^2+\ldots$ $a^3+b^3+c^3+\ldots$, we shall have

$$mx^{m-1}+\left.\begin{matrix}S_1\\+mP\end{matrix}\right|x^{m-2}+\left.\begin{matrix}S_2\\+PS_1\\+mQ\end{matrix}\right|x^{m-3}+\left.\begin{matrix}S_3\\+PS_2\\+QS_1\\+mR\end{matrix}\right|x^{m-4}\ldots\begin{matrix}+S_{m-1}\\+PS_{m-2}\\+QS_{m-3}\\+RS_{m-4}\\ \ldots\ldots\\+mT.\end{matrix}$$

Hence, equating the coefficients of corresponding terms in these identical expressions, we get

$$S_1+mP=(m-1)P,$$
$$S_2+PS_1+mQ=(m-2)Q,$$
$$S_3+PS_2+QS_1+mR=(m-3)R,$$
$$\dots\dots\dots\dots\dots$$
$$S_{m-1}+PS_{m-2}+QS_{m-3}\dots\dots+mT=T,$$

or, simplifying,

$$S_1+P=0,$$
$$S_2+PS_1+2Q=0,$$
$$S_3+PS_2+QS_1+3R=0, \qquad (2)$$
$$\dots\dots\dots\dots\dots$$
$$S_{m-1}+PS_{m-2}+QS_{m-3}\dots+(m-1)T=0.$$

By means of these equations it will be easy to calculate successively S_1, S_2, S_3, &c., and, finally, S_{m-1}, *i. e.*, the sums of all the similar powers of the roots whose index is less than the degree of the equation. In order to determine the sums of the higher powers, expressed by S_m, S_{m+1}, S_{m+2}, &c., we substitute successively $a, b, c, \dots$ in equation (1), and thus obtain

$$a^m+Pa^{m-1}+Qa^{m-2}\dots\dots+Ta+U=0$$
$$b^m+Pb^{m-1}+Qb^{m-2}\dots\dots+Tb+U=0$$
&c.

We multiply these m equalities respectively by a^n, b^n, &c., and then add them; we thus obtain

$$S_{m+n}+PS_{m+n-1}+QS_{m+n-2}\dots\dots+TS_{n+1}+US_n=0.$$

We can make successively $n=0, 1, 2,$ &c., and thus determine S_m, S_{m+1}, $S_{m+2}, \dots\dots$; we find

$$S_m\ \ +PS_{m-1}+QS_{m-2}\dots+TS_1+US_0=0$$
$$S_{m+1}+PS_m\ \ +QS_{m-1}\dots+TS_2+US_1=0 \qquad (3)$$
$$S_{m+2}+PS_{m+1}+QS_m\ \ \dots+TS_3+US_2=0$$

In the first of these equations we can put in place of US_0, mU, for $S_0 =a^0+b^0+c^0+\dots=m$; we shall thus find that these formulas follow the same law with those in (2). By means of the first of these we can determine S_m, and, passing successively to each of the succeeding formulas, we shall be able to determine each new sum by means of the sums already calculated.

It may be well to observe that all the sums, S_1, S_2, S_3, &c., may be expressed without any denominator in functions of P, Q, R, &c. This results from the fact that the first term in each of the relations (2) and (3) has unity for its coefficient.

EXAMPLES.

(1) For a numerical application take the equation $x^3-7x+7=0$. Here $P=0$, $Q=-7$, $R=7$. Since $P=0$, the relation $S_1+P=0$ gives $S_1=0$. The relations, then, which determine the sums $S_1, S_2, \dots S_6$, reduce themselves to

$$S_1=0,\ S_2+2Q=0,\ S_3+3R=0,$$
$$S_4+QS_2=0,\ S_5+QS_3+RS_2=0,\ S_6+QS_4+RS_3=0;$$

and, by substituting the values of Q and R, we readily find

$$S_1=0,\ S_2=14,\ S_3=-21,\ S_4=98,\ S_5=-245,\ S_6=833.$$

(2) Calculate the sums of the similar and entire powers of the roots of the equation $x^4-x^3-19x^2+49x-30=0$.

Ans. $S_1=1$, $S_2=39$, $S_3=-89$, $S_4=723$, $S_5=-2849$, $S_6=16419$, &c.

(3) $x^4+rx+s=0$.

Ans. $S_1=0$, $S_2=0$, $S_3=-3r$, $S_4=-4s$, $S_5=0$, $S_6=3r^2$.

358. In the equation $S_{m+n}+PS_{m+n-1}+QS_{m+n-2}\ldots\ldots+TS_{n+1}+US_n=0$, n can be a negative number, and thus the sums of the negative powers of the roots can be determined. But it will be more simple to change x into $\frac{1}{x}$ in the proposed equation, and to find successively, by means of formulas (2) and (3), the sums of the positive powers of the roots of the transformed equation.

It is evident that these powers are the negative powers of a, b, c, ...

359. *To determine double, triple, &c., functions*, represented by $S(a^\alpha b^\beta)$, $S(a^\alpha b^\beta c^\gamma)$, &c.

In order to find $S(a^\alpha b^\beta)$ we multiply together the two sums

$$a^\alpha+b^\alpha+c^\alpha+\ldots=S_\alpha,$$
$$a^\beta+b^\beta+c^\beta+\ldots=S_\beta,$$

we have

$$S_\alpha S_\beta=a^{\alpha+\beta}+b^{\alpha+\beta}+c^{\alpha+\beta}+\ldots\ldots$$
$$+a^\alpha b^\beta+a^\alpha c^\beta+b^\alpha c^\beta+\ldots\ldots$$

This product contains two series of terms. The first series is the sum of all the powers $\alpha+\beta$ of the roots, and may be expressed by $S_{\alpha+\beta}$; the second series is the sum of all the products which are formed by multiplying the power α of any root whatsoever by the power β of any other root, and may be expressed by $S(a^\alpha b^\beta)$. We have, then,

$$S_{\alpha+\beta}+S(a^\alpha b^\beta)=S_\alpha S_\beta;$$

and from this equation we derive, for double functions, the formula

$$S(a^\alpha b^\beta)=S_\alpha S_\beta-S_{\alpha+\beta}.$$

To find the triple function $S(a^\alpha b^\beta c^\gamma)$, multiply together the three sums

$$a^\alpha+b^\alpha+c^\alpha+\ldots=S_\alpha,$$
$$a^\beta+b^\beta+c^\beta+\ldots=S_\beta,$$
$$a^\gamma+b^\gamma+c^\gamma+\ldots=S_\gamma.$$

The product is a symmetric function, which evidently comprises all the terms contained in each of the five forms

$$a^{\alpha+\beta+\gamma},\ a^{\alpha+\beta}b^\gamma,\ a^{\alpha+\gamma}b^\beta,\ a^{\beta+\gamma}b^\alpha,\ a^\alpha b^\beta c^\gamma;$$

hence we have

$$\left.\begin{matrix} S_{\alpha+\beta+\gamma}+S(a^{\alpha+\beta}b^\gamma)+S(a^{\alpha+\gamma}b^\beta) \\ +S(a^{\beta+\gamma}b^\alpha)+S(a^\alpha b^\beta c^\gamma) \end{matrix}\right\}=S_\alpha S_\beta S_\gamma.$$

But the formula for double functions gives

$$S(a^{\alpha+\beta}b^\gamma)=S_{\alpha+\beta}S_\gamma-S_{\alpha+\beta+\gamma},$$
$$S(a^{\alpha+\gamma}b^\beta)=S_{\alpha+\gamma}S_\beta-S_{\alpha+\beta+\gamma},$$
$$S(a^{\beta+\gamma}b^\alpha)=S_{\beta+\gamma}S_\alpha-S_{\alpha+\beta+\gamma}.$$

By substituting these values in the preceding equality, and then deriving from this equality the value of $S(a^\alpha b^\beta c^\gamma)$, we obtain for triple functions the formula

$$S(a^\alpha b^\beta c^\gamma)=S_\alpha S_\beta S_\gamma-S_{\alpha+\beta}S_\gamma-S_{\alpha+\gamma}S_\beta-S_{\beta+\gamma}S_\alpha+2S_{\alpha+\beta+\gamma}.$$

In the same manner might the quadruple function $S(a^\alpha b^\beta c^\gamma d^\delta)$, or the sum of any succeeding combinations, be expressed by the sums of the powers.

360. *Every rational and symmetric algebraic function of the roots of an equation can be expressed rationally by the coefficients of that equation.*

Since S_1, S_2, S_3, &c., can be expressed without denominators (Art. 357) in functions of the coefficients of the proposed equation, and the double, triple, quadruple, &c., functions can be expressed by the sums of the powers, it follows that all these symmetrical functions can be expressed by integral functions of the coefficients. And as every symmetrical polynomial in $a, b, c \ldots$ must be composed of the assemblage, by addition or subtraction, of several symmetric functions of the form $S(a^\alpha b^\beta c^\gamma d^\delta \ldots)$, it follows that the value of every rational symmetric function whatever of the roots of an equation (without the roots being known) can be expressed by the coefficients of the equation.

USE OF SYMMETRIC FUNCTIONS IN THE TRANSFORMATION OF EQUATIONS.

361. Symmetric functions present themselves in the transformation of equations, whenever the roots of the transformed equation must be rational functions of the roots of the given equation.

Let $a, b, c \ldots$ be the roots of the given equation; for the sake of definiteness, I suppose that two of its roots enter into the composition of each root of the transformed equation, and I represent by $F(a, b)$ the rational function which expresses the law of this composition.

Suppose that, after we have made all these combinations, two and two, of $a, b, c \ldots$ we put successively in $F(a, b)$ instead of a and b, the two roots of each arrangement, it is clear that we shall thus have all the roots of the transformed equation, to wit:

$$F(a, b),\ F(a, c), \ldots, \qquad F(b, a),\ F(b, c) \ldots \quad \&c.$$

Consequently, this equation, decomposed into factors, will be

$$[z-F(a, b)]\ [z-F(a, c)] \ldots = 0.$$

This product does not vary in making between $a, b, c \ldots$ the proposed exchange; for, if we make the change, the factors can only place themselves in some other order. We are sure, then, that, after the multiplication, the coefficients of the different powers of z will be symmetric and rational functions of $a, b, c \ldots$

Thus, by following the method of procedure hitherto explained, we can express these coefficients by means of those of the proposed equation.

362. But there exists another method, often preferable, of employing symmetric functions.

It is founded on the observation that the relations [2] and [3] in Art. 357, existing between the coefficients of an equation and the sums of the similar powers of its roots, can be used to discover the coefficients of the equation when they are unknown, provided we know these sums as far as that sum of the powers whose order is equal to the number of unknown coefficients, *i. e.*, to the degree of the equation.

Hence, to arrive at the transformed equation, we determine, first, of what degree this equation is to be. We next find the sums of the first, second, &c., powers of its roots, as far as the sum of the powers whose order is equal to the degree of this transformed equation; then, by means of these sums, we calculate the unknown coefficients. It is clear that these different sums are symmetric functions of the roots of the proposed equation, and that they can be expressed by the coefficients of this equation. Hence they can readily be determined.

363. As an illustration of the preceding method, I will resume here the question of the equation of the squares of the differences, already treated of in Art. 278. Symmetric functions give the most simple and elegant solution of which it is susceptible. The question is this:

To find the equation whose roots are the squares of the differences of the roots of a given equation,

$$x^m+Px^{m-1}+Qx^{m-2}+\ldots=0 \ldots\ldots\ldots [A]$$

Represent the transformed equation by

$$z^n+pz^{n-1}+qz^{n-2}+rz^{n-3}+\ldots+tz+u=0 \ldots [B]$$

The m roots of [A] being $a, b, c \ldots$ those of [B] will be

$$(a-b)^2,\ (a-c)^2,\ (a-d)^2, \ldots (b-c)^2, \ldots (b-d)^2,\ (c-d)^2, \ldots \text{\&c.}$$

The number of these squares is evidently that of the combinations, two and two, that can be made with the m quantities, $a, b, c \ldots$; hence the degree of the required transformed equation will be $n=\frac{1}{2}m(m-1)$.

The coefficients $p, q, r \ldots$ may easily be found when we know the sums of the similar and entire powers of the roots of equation [B]; since the sum of the first powers is equal to that of the n^{th} powers. Let us designate these new sums, then, by f_1, f_2, f_3, &c., and find the general value of f_α, α being any entire and positive number whatsoever.

The roots of the equation [B] are, as has already been stated, $(a-b)^2$, &c. Raising these roots, then, to the power α, we have

$$f_\alpha=(a-b)^{2\alpha}+(a-c)^{2\alpha}+(a-d)^{2\alpha} \ldots +(b-c)^{2\alpha}+, \text{\&c.}$$

In order to find this sum, consider the expression

$$\phi(x)=(x-a)^{2\alpha}+(x-b)^{2\alpha}+(x-c)^{2\alpha}+\ldots$$

which contains the m binomials $x-a$, $x-b$, $x-c \ldots\ldots$ If we make in this expression successively $x=a, b, c, \ldots$, and add the m results, we evidently obtain

$$2f_\alpha=\phi(a)+\phi(b)+\phi(c)+\ldots$$

If we develop the powers which compose $\phi(x)$, we find

$$\phi(x)=\begin{cases} x^{2\alpha}-2\alpha ax^{2\alpha-1}+\dfrac{2\alpha(2\alpha-1)}{1.2}a^2x^{2\alpha-2}+a^{2\alpha} \\ +x^{2\alpha}-2\alpha bx^{2\alpha-1}+\dfrac{2\alpha(2\alpha-1)}{1.2}b^2x^{2\alpha-2}+b^{2\alpha} \\ +, \text{\&c.,} \end{cases}$$

or, more simply, by using the notation S_1, S_2, &c.,

$$\phi(x)=mx^{2\alpha}-2\alpha S_1x^{2\alpha-1}+\frac{2\alpha(2\alpha-1)}{1.2}S_2x^{2\alpha-2}\ldots+S_{2\alpha}.$$

Substituting $a, b, c \ldots$ in this expression instead of x, and adding the results, we obtain

$$2f_\alpha=mS_{2\alpha}-2\alpha S_1S_{2\alpha-1}+\frac{2\alpha(2\alpha-1)}{1.2}S_2S_{2\alpha-2}\ldots+mS_{2\alpha}.$$

In this second member it will be perceived that the terms at an equal distance from the extremes are equal; consequently, stopping at the middle term of the expression, and taking only the half of that term, we have the general value of f_α, to wit,

$$f_a = mS_{2a} - 2aS_1S_{2a-1} + \frac{2a(2a-1)}{1.2}S_2S_{2a-2}\ldots$$
$$\pm\tfrac{1}{2}\frac{2a(2a-1)(2a-2)\ldots(a+1)}{1.2.3\ldots a}S_aS_a.$$

As the signs are alternately $+$ and $-$, there will never be any uncertainty as regards this last term. Let us view, then, the operations which must be performed.

1°. We calculate the sums S_1, S_2, S_3.. up to S_{2a} by means of the known relations $S_1+P=0$, $S_2+PS_1+2Q=0$, &c.

2°. In the formula which expresses f_a we make successively $a=1$, 2, 3, ..n, and we thus have, to determine the n sums $f_1, f_2, f_3, \ldots f_n$,

$$f_1 = mS_2 - S_1S_1,\ f_2 = mS_4 - 4S_1S_3 + 3S_2S_2, \text{ \&c.}$$

3°. Finally, the relations existing between these n sums and the n coefficients p, q, r, ... will give the values of these coefficients, viz.,

$$p = -f_1,\ q = -\tfrac{1}{2}(f_2 + pf_1),\ r = -\tfrac{1}{3}(f_3 + pf_2 + qf_1), \text{ \&c.}$$

364. A method entirely analogous to that which has been employed in finding the equation of the squares of the differences can be employed in a great number of cases, and particularly in those where the roots of the transformed equation are similar, and entire powers of the difference, of the sum, of the product, or of the quotient of any two roots whatsoever of the given equation.

For example, suppose that each new root is to be the power k of the sum $a+b$ of two roots of equation [A]. Taking $n=\frac{1}{2}m(m-1)$, the transformed equation ought to have the form

$$z^n + pz^{n-1} + qz^{n-2} + \ldots + tz + u = 0 \quad \ldots\ldots \text{[C]}$$

and if we make

$$f_a = (a+b)^{ka} + (a+c)^{ka} + \ldots + (b+c)^{ka} +, \text{ \&c.},$$

the calculation will reduce itself to expressing f_a by a general formula. To do this, we take the function

$$\phi(x) = (x+a)^{ka} + (x+b)^{ka} + (x+c)^{ka} +, \text{ \&c.},$$

the development of which is

$$\phi(x) = mx^{ka} + kaS_1x^{ka-1} + \frac{ka(ka-1)}{1.2}S_2x^{ka-2} + \ldots + S_{ka}.$$

But if, before the development, we substitute in $\phi(x)$ successively a, b, c, ..., instead of x, the sum of the resultants will be equal to $2f_a + 2^{ka}S_{ka}$; hence it is easy to perceive that by making the same substitutions in the development, we shall have

$$2f_a + 2^{ka}S_{ka} = mS_{ka} + kaS_1S_{ka-1} \ldots + mS_{ka}.$$

Finally, we derive from this equation the required formula,

$$f_a = (m - 2^{ka-1})S_{ka} + kaS_1S_{ka-1} + \frac{ka(ka-1)}{1.2}S_2S_{ka-2} +, \text{ \&c.}$$

When ka is even, we stop at the term which contains S with two equal indices, and we take only the half of it; but when ka is uneven, we stop at the term in which the two indices are $\frac{1}{2}(ka-1)$ and $\frac{1}{2}(ka+1)$, and we take the entire term.

QUADRATIC FACTORS OF EQUATIONS.

365. **Every equation of an even degree has at least one real quadratic factor.**

Let the proposed equation be

$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0$, having roots a, b, c, &c., and let $n=2\mu$, μ being an odd number. Let it be transformed (Art. 362) into an equation whose roots are the combinations of every two of its roots, of the form $y=a+b+mab$, m being any number; also, let the transformed equation be $\phi_m(y)=0$; then its coefficients will be symmetrical functions of a, b, c, &c., and, therefore, rational and known functions of p_1, p_2, &c.; and its degree will be $\frac{2\mu(2\mu-1)}{2}$, which is odd; therefore, $\phi_m(y)=0$ will have at least one real root, whatever be the value of m. Hence, making $m=1, 2, 3, \ldots \{\mu(2\mu-1)+1\}$, successively, each of the equations $\phi_1(y)=0$, $\phi_2(y)=0$, &c., will have at least one real root; that is, we shall have $\mu(2\mu-1)+1$ real values for combinations of two roots of the proposed equation, of the form $a+b+mab$; but there are only $\mu(2\mu-1)$ such combinations which are differently composed of the roots a, b, c, &c.; therefore, two of these combinations, for which we have obtained real values, must involve the same pair of the quantities a, b, c, &c.; let this pair of roots be a, b, and α, α', the real roots of the corresponding equations $\phi_m(y)=0$, $\phi_{m'}(y)=0$, so that

$$a+b+mab=\alpha,\quad a+b+m'ab=\alpha';$$

therefore, $a+b$ and ab are real, and the proposed equation has at least one real quadratic factor, and two roots, either real, or of the form $\alpha\pm\beta\sqrt{-1}$. Hence every equation whose degree is only once divisible by 2 has at least one real quadratic factor.

We shall now prove that if it be true that every equation has at least one real quadratic factor when its degree is r times divisible by 2, or when $n=2^r\mu$, where μ is odd, the same is true when the degree of the equation is $r+1$ times divisible by 2. For, let $n=2^{r+1}\mu$; then the degree of the transformed equation will be $2^r\mu(2^{r+1}\mu-1)$, which is only r times divisible by 2; therefore, by supposition, the transformed equation, $\phi_m(y)=0$, will have two roots, either real or imaginary. If they are real, then, exactly in the same way as for the preceding case of the index being only once divisible by 2, it may be shown that the proposed equation has at least one real quadratic factor. If they are imaginary, we shall have $y=\alpha\pm\beta\sqrt{-1}$, each of which expresses the value of some one of the combinations $a+b+mab$, $a+c+mac$, &c. Suppose, therefore, that we have $a+b+mab=\alpha+\beta\sqrt{-1}$; then, as shown above, we can give m such a value m', that $\phi_{m'}(y)=0$ shall have a root corresponding to the combination of the same letters, so that $a+b+m'ab=\alpha'+\beta'\sqrt{-1}$; from which equations we can obtain values of ab and $a+b$ under the forms

$$a+b=\gamma+\delta\sqrt{-1},$$
$$ab=\gamma'+\delta'\sqrt{-1},$$
$$\therefore\ x^2-(\gamma+\delta\sqrt{-1})x+\gamma'+\delta'\sqrt{-1} \text{ is a factor of } f(x);$$

but if any real expression have a factor of the form $\text{M}+\text{N}\sqrt{-1}$, it must also have one of the form $\text{M}-\text{N}\sqrt{-1}$;

$$\therefore\ x^2-(\gamma-\delta\sqrt{-1})x+\gamma'-\delta'\sqrt{-1} \text{ is a factor of } f(x);$$

if, therefore, these two expressions have no simple factor in common, their product will be a biquadratic factor of $f(x)$,

$$(x^2-\gamma x+\gamma')^2+(\delta x-\delta')^2,$$

which can always be resolved into two real quadratic factors. (See solution of Biquadratics.) If they have a factor in common, since they may be written

$$x^2-\gamma x+\gamma'-\sqrt{-1}(\delta x-\delta'),\ x^2-\gamma x+\gamma'+\sqrt{-1}(\delta x-\delta'),$$

it can only be of the form $x-\varepsilon$; and the factors themselves become

$$(x-\kappa+\lambda\sqrt{-1})(x-\varepsilon),\ (x-\kappa-\lambda\sqrt{-1})(x-\varepsilon);$$

and, therefore, the proposed equation admits the real quadratic factor

$$(x-\kappa)^2+\lambda^2.$$

Hence an equation whose degree $=2^{r+1}\mu$ will have a real quadratic factor, provided an equation whose degree $=2^r\mu$ has one; but we have proved this to be the case when $r=1$; therefore it is universally true that every equation of an even degree has at least one real quadratic factor. If now this factor be expelled, the depressed equation will have its coefficients real and its degree even, and will, therefore, as before, have one real quadratic factor. Hence the first member of every equation of an even degree may be resolved into real quadratic factors.

366. Hence if we divide the first member of any equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ \dots\ +p_n=0$$

by x^2+ax+b, admitting no terms into the quotient that have x in the denominator, we shall at last obtain a remainder of the form $Ax+B$, A and B being rational functions of a and b; and in order that x^2+ax+b may be a quadratic factor of the proposed equation, it is necessary and sufficient that this remainder should equal zero for all values of x, which requires that we separately have $A=0$, $B=0$. The different pairs of values, real or imaginary, of a and b which satisfy these equations will give all the quadratic factors of the proposed; and as the number of these factors is $\frac{1}{2}n(n-1)$ (Art. 244, Cor. 2), the final equation for determining one of the quantities a, b, obtained by eliminating the other between the two preceding equations, will be of the degree $\frac{1}{2}n(n-1)$, which exceeds n, if $n>3$; therefore, the determination of the quadratic factors of an equation will generally present greater difficulties than the solution of the equation.

As the proposed equation has necessarily $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ real quadratic factors, according as n is even or odd, there will always exist the same number of pairs of real values of a and b, satisfying the equations $A=0$, $B=0$; and if any of these pairs of real values be commensurable, they may be easily found; and the commensurable quadratic factors being known, the equation may be depressed.

EXAMPLES.

(1) To resolve $x^4-6x^2+nx-3=0$ into its factors. Dividing by x^2+ax+b, we find a remainder,

$$(n+2ab+6a-a^3)x-(a^2b-b^2-6b+3);$$

therefore, to determine a and b, we have

$$n+2ab+6a-a^3=0,$$
$$a^2b-b^2-6b+3=0.$$

Solving the former with respect to b, and substituting in the latter, we find $(a^2-4)^3=n^2-64$, or $a=\sqrt{4+\sqrt[3]{n^2-64}}$; from whence b, and the other quadratic factor,

$$x^2-ax+a^2-b-6,$$

may be determined.

(2) The resolution of $x^4+px^3+qx^2+rx+s$ into its two quadratic factors, x^2+mx+n, $x^2+m'x+n$, may be effected by the following formulæ:

$$m=\tfrac{1}{2}(p+\sqrt{z}),\ m'=\tfrac{1}{2}(p-\sqrt{z}),$$

$$n=\frac{r-qm+pm^2-m^3}{p-2m},\ n'=\frac{r-qm'+pm'^2-m'^3}{p-2m'},$$

where z is a root of the equation,

$$z^3-(3p^2-8q)z^2+(3p^4-16p^2q+16q^2+16pr-64s)z-(8r-4pq+p^3)^2=0,$$

which has necessarily a real root.

ELIMINATION BY SYMMETRIC FUNCTIONS.

367. Symmetric functions furnish a method of elimination which has the advantage of making known the degree of the final equation.

Let the two equations be

$$x^m+Px^{m-1}+Qx^{m-2}+Rx^{m-3}\ldots=0 \quad \ldots\ldots (1)$$
$$x^n+P'x^{n-1}+Q'x^{n-2}+R'x^{n-3}\ldots=0 \quad \ldots\ldots (2)$$

in which P, Q..., P′, Q′... are functions of y. If we could resolve (1) with respect to x, we would derive from it m values, a, b, c..., of x, which would be functions of y; and, by substituting these values of x in equation (2), we would have, for determining the values of y, m equations free from x, viz.,

$$\left.\begin{array}{l} a^n+P'a^{n-1}+Q'a^{n-2}+R'a^{n-3}\ldots=0 \\ b^n+P'b^{n-1}+Q'b^{n-2}+R'b^{n-3}\ldots=0 \\ c^n+P'c^{n-1}+Q'c^{n-2}+R'c^{n-3}\ldots=0 \\ \&c. \qquad\qquad \&c. \end{array}\right\} \ldots\ldots (3)$$

But, in general, the resolution of equation (1) is impossible, and the problem is to obtain a final equation which embraces all the values of y without distinction.

We shall have an equation which will fulfill this condition by multiplying together the m equations (3), for the resulting equation will be satisfied by each value of y derived from any one of them, and it can not be satisfied in any other way. But the factors of this resultant can only change places, whatever permutations we may make between the quantities a, b, c ...; the product, then, will only contain entire and rational symmetric functions of these quantities; hence we shall be able to express these factors by means of the coefficients of equation (1), and in this way we shall have the final equation in y.

This method of elimination leads, in general, to very tedious calculations, but it has the advantage of giving a final equation containing all the roots that it ought to embrace, without any complication of foreign roots.

368. This method has also the advantage of leading to a general theorem with respect to *the degree of the final equation.* In the preceding article the first equation is of the degree m, the second of the degree n, and P, Q..., P′, Q′... are any functions whatsoever of y; but, for the theorem in question, these functions must evidently be polynomes, such that the sum of the exponents of x and y shall be, at most, equal to m in each term of equation (1), and, at most, equal to n in each term of equation (2). We have, then, to determine to what degree y can be raised in the symmetric functions which compose the product of equations (3).

Each term of this product is the product of m terms taken respectively from the m equations (3); hence, designating these terms by Ya^{α}, $Y'b^{\beta}$, $Y''c^{\gamma}$, the term of the product will be $YY'Y''\ldots a^{\alpha}b^{\beta}c^{\gamma}\ldots$. But the product of these m equations being symmetric with respect to the quantities a, b, $c\ldots$, all the terms should have the same form with the one that we have given above; consequently, we know that the product embraces all the terms represented by

$$YY'Y''\ldots\times S(a^{\alpha}b^{\beta}c^{\gamma}\ldots)\ \ldots\ldots\ldots\ldots\ (4)$$

We have now to determine the degree of y in this expression. Observing that the degree of y in Y is, at most, equal to $n-\alpha$, in Y' to $n-\beta$, in Y'' to $n-\gamma$, &c., we shall readily see that in $YY'Y''\ldots$ its degree will be, at most, equal to $mn-\alpha-\beta-\gamma\ldots$. On the other hand, if we refer back to the relations (Art. 356) from which the sums S_1, S_2, S_3, &c., are derived, we shall see that, P being, at most, of the first degree in y, Q of the second, R of the third, and so on, the degree of y in these sums can not surpass the subscript number of S; and, in like manner, if we refer (Art. 359) to the formulas which express double, triple, &c., functions, we shall perceive that in $S(a^{\alpha}b^{\beta}c^{\gamma}\ldots)$ the degree of y can not surpass $\alpha+\beta+\gamma\ldots$ Hence in expression (4) the degree of y will be, at most, equal to mn.

The same remark will apply to all the symmetric functions whose sum composes the product of the m equations (3); therefore, lastly, *the final equation can not be of a degree superior to* mn.

The demonstration seems to require that equation (1) contain m. But we can suppose that at first x^m had a coefficient, A, independent of y, and that we have divided the whole equation by A. The final equation ought to subsist, whatever may be the value of A; we can make $A=0$, and it is evident that this supposition will not raise the degree of the final equation. Finally, the theorem is to be thus understood: that the elimination between two general equations, the one of the degree m, the other of the degree n, ought to give a final equation of the degree mn; but that, in particular cases, the degree of the final equation can be less than mn.

EXAMPLES.

The two equations, $x-y^m=0$, $x^n+ay^n+by+c=0$, although very simple, will give a final equation fully of the degree mn; for, by substituting in the second the value of x derived from the first, it becomes $y^{mn}+ay^n+by+c=0$.

On the other hand, in eliminating x between the equations $x^n-y^m=0$, $x^n+ay^n+by+c=0$, we obtain a final equation of a degree less than mn, viz., $y^m+ay^n+by+c=0$.

369. For extending the theorem to any number whatsoever of equations, we have the general theorem given by Bezout, viz., that *If, between equations equal in number to that of the unknowns, we eliminate all the unknowns, except one, the degree of the final equation will be, at most, equal to the product of the degrees of these equations.*

Before Bezout, the theorem had been known for the case of two equations; and Cramer, in the appendix to his Introduction to the Analysis of Right Lines, has given a very simple demonstration, which, in reality, does not differ from that which we have stated. It has been a desideratum that the same demonstration should be capable of being applied to all other cases; this has been accomplished by Poisson, in a memoir which appeared in the eleventh volume of the *Journal de l'École Polytechnique.*

METHOD OF TSCHIRNHAUSEN FOR SOLVING EQUATIONS.

370. As another application of the theory of elimination, we shall briefly illustrate the principle upon which Tschirnhausen proposed to accomplish the general solution of equations, but which, as observed at Art. 277, was soon found to be of but very limited application, not extending beyond equations of the fourth degree ; and, even within this extent, too laborious for general use. The principle consists in connecting with the proposed an auxiliary equation of inferior degree with undetermined coefficients, and of as simple a form as possible consistently with the office it is to perform, but involving, besides the unknown quantity x, a second unknown y. The unknown, common to both equations, is then eliminated according to the method at Art. 315, and a final equation in y thus obtained, of which the coefficients are functions of the undetermined coefficients in the auxiliary equation. The arbitrary quantities, thus entering the coefficients of the final equation in y, are then determined so as to cause certain of these coefficients to vanish ; by which means the equation is ultimately reduced to a prescribed form, supposed to be solvable by known methods.

371. As an example, let it be required to reduce the cubic equation

$$x^3+ax^2+bx+c=0 \quad \dots\dots \quad (1)$$

to the binomial form

$$y^3+k=0.$$

Assume an auxiliary equation

$$x^2+a'x+b'+y=0 \quad \dots\dots\dots \quad (2)$$

and eliminate x from (1) and (2) in the usual way. The remainder arising from dividing the first member of (1) by the first member of (2) is

$$(a'^2-aa'+b-b'-y)x+(a'-a)(b'+y)+c,$$

which, equated to zero, gives

$$x=\frac{(a-a')(b'+y)-c}{a'^2-aa'+b-b'-y};$$

and this value of x, substituted in the proposed equation, transforms it, after reduction, into the form

$$y^3+hy^2+iy+k=0 \quad \dots\dots \quad (3)$$

where

$$\begin{aligned} h&=3b'-aa'+a^2-2b \\ i&=3b'^2-2b'(aa'-a^2+2b)+a'^2b \\ &\quad +(3c-ab)a'+b^2-2ac \\ k&=b'^3-ab'^2a'+bb'a'^2-ca'^3+(a^2-2b)b'^2+ \\ &\quad (3c-ab)a'b'+aca'^2+(b^2-2ac)b'-bca'+c^2. \end{aligned}$$

Hence, in order to reduce (3) to the prescribed form, we must determine the arbitrary quantities a', b' conformably to the conditions $h=0$, $i=0$; that is, these quantities must satisfy the equations

$$\begin{gathered} 3b'-aa'+a^2-2b=0 \\ 3b'^2-2b'(aa'-a^2+2b)+a'^2b+ \\ (3c-ab)a'+b^2-2ac=0, \end{gathered}$$

of which the first is of the first degree with respect to a' and b', and the other of the second degree, so that their values may be determined by a quadratic equation. And these values, or, rather, the expression for them in terms of the given coefficients, being substituted in the preceding expression for k, render that symbol known; and thus the required form

$$y^3+k=0$$

is obtained.

372. In a similar manner may the general equation of the fourth degree

$$x^4+ax^3+bx^2+cx+d=0$$

be transformed into one of the form

$$y^4+hy^2+k=0,$$

which is virtually a quadratic, by eliminating x from the pair of equations

$$x^4+ax^3+bx^2+cx+d=0,$$
$$x^2+a'x+b' \ +y=0,$$

which elimination will conduct to a final equation in y of the form

$$y^4+gy^3+hy^2+iy+k=0,$$

from which the second and fourth terms will vanish by the equations of condition

$$g=0, \ i=0,$$

the first of which will be of the first degree as regards the arbitrary quantities a', b', and the second of the third; both quantities are, therefore, determinable by means of an equation of the third degree, and thence the quantities h, k, which are known functions of them.

All this is very laborious, but it really does effect the object proposed thus far; that is, it reduces the solution of equations of the third and fourth degrees to those of inferior degrees; but beyond this point the method fails, as the conditional equations resolve themselves ultimately into a final equation that exceeds in degree that which they are intended to simplify.

On this subject we may add that Mr. Jerrard has greatly extended the principle of Tschirnhausen, and has succeeded in reducing the general equation of the fifth degree

$$x^5+A_4x^4+A_3x^3+A_2x^2+Ax+N=0$$

to the remarkably simple forms

$$x^5+ax^4+b=0$$
$$x^5+ax^3+b=0$$
$$x^5+ax^2+b=0$$
$$x^5+ax \ +b=0;$$

so that the solution of the general equation of the fifth degree might be considered as accomplished if either of the above forms could be solved in general terms.

For a very masterly analysis of Mr. Jerrard's researches, the reader is referred to the paper of Sir W. R. Hamilton in the Report of the sixth meeting of the British Association.

METHOD OF LAGRANGE FOR SOLVING EQUATIONS.

373. A remarkable application of the theory of symmetrical functions is that made by Lagrange to the general solution of equations; by that means he solves the general equations of the first four degrees by a uniform process, and one which includes all others that have been proposed for that purpose. the common relation of which to one another is thus made apparent.

It consists in employing an auxiliary equation, called a reducing equation. whose root is of the form

$$x_1+ax_2+a^2x_3+\dots+a^{n-1}x_n,$$

denoting by $x_1, x_2, \dots x_n$ the n roots of the proposed equation, and by a one of the n^{th} roots of unity; and the principle on which it is based is as follows: Let y be the unknown quantity in the reducing equation, and let

$$y=a_1x_1+a_2x_2+\dots+a_nx_n,$$

$a_1, a_2, \dots a_n$ denoting certain constant quantities; then, if $n-1$ values of y, and suitable values of the constants $a_1, a_2, \dots a_n$, can be found, so that we may have $n-1$ simple equations, these, together with the equation

$$-p_1=x_1+x_2+\dots+x_n,$$

will enable us to determine the n roots.

Now, supposing the constants in the value of y to preserve an invariable order, a_1, a_2, &c., since the number of ways in which the n roots may be combined with them to form the expression $a_1x_1+a_2x_2+$, &c., is the same as the number of permutations of n things taken all together; therefore, the expression for y will have $n(n-1)\dots3.2.1$ values, and the equation for determining y will rise to the same number of dimensions, or will be of a degree higher than that of the proposed equation; hence the method will be of no use, unless such values can be assumed for the constants $a_1, a_2, \dots a_n$ as shall make the solution of the equation in y depend upon that of an equation, at most, of $n-1$ dimensions. Now this may be done (at least when n does not exceed 4) by taking the n^{th} roots of unity $a^0, a, a^2, a^3, \dots a^{n-1}$ for $a_1, a_2, \dots a_n$, so that

$$y=a^0x_1+ax_2+\dots+a^{r-1}x_r+a^rx_{r+1}+\dots+a^{n-1}x_n.$$

For, in the first place, with this assumption, the reducing equation will contain only powers of y which are multiples of n; for, since $a^n=1$,

$$a^{n-r}y=a^{n-r}x_1+a^{n-r+1}x_2+\dots+x_{r+1}+ax_{r+2}+\dots+a^{n-r-1}x_n,$$

or

$$a^{n-r}y=a^0x_{r+1}+ax_{r+2}+\dots+a^{n-1}x_r,$$

which is the same result as if we had interchanged x_1 and x_{r+1}, x_2 and x_{r+2}, &c., so that if y be a root of the reducing equation, $a^{n-r}y$ is also a root; therefore, the reducing equation, since it remains unaltered when $a^{n-r}y$ is written for y, contains only powers of y which are multiples of n; if, therefore, we make $y^n=z$, we shall have a reducing equation in z of only $1.2.3\dots(n-1$ dimensions, whose roots will be the different values of z which result from the permutations of the $n-1$ roots $x_2, x_3, \dots x_n$ among themselves. We shall now have, expanding and reducing,

$$z=y^n=u_0+u_1a+u_2a^2+\dots+u_{n-1}a^{n-1},$$

in which $u_0, u_1, u_2, \dots u_{n-1}$ are determinate functions of the roots, which will be invariable for the simultaneous changes of x_1 into x_{r+1}, x_2 into x_{r+2}, &c., since $z=(a^ry)^n$; and when their values are known in terms of the coefficients of the proposed equation, we shall immediately know the values of the roots. For let $z_0, z_1, z_2, \dots z_{n-1}$ be the different values of z, when $1, a, \beta, \gamma, \dots \lambda$, the roots of $y^n-1=0$, are substituted for a; then, since $y=\sqrt[n]{z}$, we have

$$x_1+x_2+\dots+x_n=\sqrt[n]{z_0}$$
$$x_1+ax_2+\dots+a^{n-1}x_n=\sqrt[n]{z_1}$$
$$\dots\dots\dots\dots\dots=\dots$$
$$x_1+\lambda x_2+\dots+\lambda^{n-1}x_n=\sqrt[n]{z_{n-1}};$$

therefore, adding, and taking account of the properties of the sums of the powers of $1, a, \beta, \gamma$, &c., (Art. 357, [2]), we get

$$nx_1=\sqrt[n]{z_0}+\sqrt[n]{z_1}+\ldots+\sqrt[n]{z_{n-1}}.$$

Again, multiplying the above system of equations respectively by 1, a^{n-1}, β^{n-1}, ... λ^{n-1}, we get

$$nx_2=\sqrt[n]{z_0}+a^{n-1}\sqrt[n]{z_1}+\beta^{n-1}\sqrt[n]{z_2}+\ldots+\lambda^{n-1}\sqrt[n]{z_{n-1}},$$

and so on for the rest. Hence, since $-p_1=\sqrt[n]{z_0}$, and $\therefore (-p_1)^n=z_0=u_0+u_1+\ldots+u_{n-1}$, the problem is reduced to finding the values of $u_1, u_2, \ldots u_{n-1}$.

374. When n is a composite number, the above general method admits of simplifications. For let n have a divisor m, so that $n=mp$, and let a be a root of $y^m-1=0$; then, since $a^m=1$, $a^{m+1}=a$, $a^{m+2}=a^2$, &c., $a^{2m}=1$, $a^{2m+1}=a$, &c., we have

$$\begin{aligned} y&=x_1+ax_2+a^2x_3+\ldots+a^{n-1}x_n\\ &=X_1+aX_2+a^2X_3+\ldots+a^{m-1}X_m,\end{aligned}$$

where $X_r=x_r+x_{m+r}+x_{2m+r}+\ldots+x_{n-m+r}$, and consists of p roots;

$$\therefore z=y^n=u_0+u_1a+u_2a^2+\ldots+u_{m-1}a^{m-1},$$

where u_0, u_1, &c., are known functions of X_1, X_2, &c.; and when they are found in terms of the coefficients of the proposed equation, we shall be able to determine immediately the values of X_1, X_2, &c., as before. To deduce the values of the primitive roots $x_1, x_2, x_3, \ldots x_n$, we must regard separately those which compose each of the quantities X_1, X_2, &c., as the roots of an equation of p dimensions. Thus, let the roots whose sum is X_1 be those of the equation

$$x^p-X_1x^{p-1}+Lx^{p-2}-Mx^{p-3}+\ldots=0,$$

where L, M, &c., are unknown; then the first member of this equation is a divisor of the first member of the proposed, since all its roots belong to the latter. Hence, effecting the division and equating to zero the coefficients of x^{p-1}, x^{p-2}, &c., in the remainder, we shall have p equations in X_1, L, M, &c., of which the first $p-1$ will give the values of L, M, &c., in terms of X_1 by linear equations. It will then remain to solve the equation so formed of p dimensions. Similarly, substituting the value of X_2 in place of that of X_1, we shall have an equation giving the next group of roots x_2, x_{m+2}, &c.; and so on

EXAMPLE I.

$$x^3-px^2+qx-r=0.$$

Let the roots be a, b, c, and let

$$y=a+ab+a^2c;$$

$$\begin{aligned}\therefore z=y^3&=a^3+b^3+c^3+6abc+3(a^2b+b^2c+c^2a)a+3(a^2c+b^2a+c^2b)a^2,\\ &=u_0+u_1a+u_2a^2.\end{aligned}$$

But u_1, u_2 are roots of the quadratic

$$u^2-(u_1+u_2)u+u_1u_2=0,$$

and $u_1+u_2=3\Sigma(a^2b)=3pq-9r$ (Arts. 357, 359),

$$\begin{aligned}u_1u_2&=9\{abcS_3+\Sigma(a^3b^3)+3a^2b^2c^2\}\\ &=9q^3+9(p^3-6pq)r+81r^2.\end{aligned}$$

Hence u_1, u_2 are known,

and $\therefore u_0=p^3-(u_1+u_2)$, is known.

Hence, denoting by z_1, z_2, the values of z when a and a^2 are respectively written for a, we have

$$a+b+c=p$$
$$a+a\,b+a^2c=\sqrt[3]{z_1}$$
$$a+a^2b+a\,c=\sqrt[3]{z_2};$$

from which we obtain the values of a, b, and c, viz.,

$$a=\tfrac{1}{3}(p+\sqrt[3]{z_1}+\sqrt[3]{z_2})$$
$$b=\tfrac{1}{3}(p+a^2\sqrt[3]{z_1}+a\sqrt[3]{z_2})$$
$$c=\tfrac{1}{3}(p+a\sqrt[3]{z_1}+a^2\sqrt[3]{z_2}).$$

EXAMPLE II.

$$x^4-px^3+qx^2-rx+s=0.$$

Since $4=2.2$, let a be a root of $y^2-1=0$, so that $a^2=1$;

then $$y=x_1+ax_2+x_3+ax_4=X_1+aX_2,$$

if $$X_1=x_1+x_3, \qquad X_2=x_2+x_4;$$

$$\therefore\ z=y^2=u_0+au_1$$

where $$u_0=X_1^2+X_2^2,\ u_1=2X_1X_2, \text{ and } u_0+u_1=z_0=p^2.$$

Hence $u_1=2(x_1+x_3)(x_2+x_4)$, by interchanging the roots among themselves, will admit the two other values $2(x_1+x_2)(x_3+x_4)$, and $2(x_1+x_4)(x_2+x_3)$, and will, therefore, be a root of an equation of the form

$$u_1^3-Mu_1^2+Nu_1-P=0;$$

the coefficients being symmetrical functions of x_1, x_2, x_3, x_4, and, consequently, assignable in terms of p, q, r, s. It is easily seen that if we make $u_1=2q-2u$, we shall have an equation in u whose roots are

$$x_1x_3+x_2x_4,\ x_1x_2+x_3x_4,\ x_1x_4+x_2x_3;$$

and the transformed equation is (Art. 362)

$$u^3-qu^2+(pr-4s)u-(p^2-4q)s-r^2=0.$$

Let u' be a root of this equation, then $u_1=2q-2u'$; hence, making

$$a=-1,\ z_1=u_0-u_1=p^2-2u_1=p^2-4q+4u';$$
$$\therefore\ X_1+X_2=p,\ X_1-X_2=\sqrt{z_1};$$
$$\therefore\ X_1=\tfrac{1}{2}(p+\sqrt{z_1}),\ X_2=\tfrac{1}{2}(p-\sqrt{z_1}).$$

Hence x_1, x_3 may be regarded as roots of a quadratic $x^2-X_1x+L=0$; dividing the proposed by this, and putting the first term of the remainder equal to zero, we find

$$L=\frac{X_1^3-pX_1^2+qX_1-r}{2X_1-p};$$

therefore, x_1, x_3 are known; and x_2, x_4 will result from the same formulæ by interchanging X_1 and X_2, or by changing the sign of the radical $\sqrt{z_1}$.

EXAMPLE III.

$$\frac{x^n-1}{x-1}=0,\ n \text{ being a prime number.}$$

If r be one of the roots, and a be a primitive root of the prime number n (that is, a number whose several powers from 1 to $n-1$, when divided by n, leave different remainders), it will be proved hereafter that all the roots of this equation may be represented by

$$r,\ r^a,\ r^{a^2},\ r^{a^3},\ \ldots\ r^{a^{n-2}}.$$

Let $$y=r+ar^a+a^2r^{a^2}+\ldots+a^{n-2}r^{a^{n-2}},$$

a being a root of the equation $y^{n-1}-1=0$. Therefore, observing that $a^{n-1}=1$ and $r^n=1$,

$$z=y^{n-1}=u_0+au_1+a^2u_2+\ldots+a^{n-2}u_{n-2}, \quad \ldots \quad (1)$$

u_0, u_1, &c., being rational and integral functions of r which do not change by the substitution of r^a, r^{a^2}, r^{a^3}, &c., in the place of r; for these quantities, regarded as functions of x_1, x_2, x_3, &c., do not alter by the simultaneous changes of x_1 into x_2, x_2 into x_3, &c., nor by the simultaneous changes of x_1 into x_3, x_2 into x_4, &c., to which correspond the changes of r into r^a, into r^{a^2}, &c.

Now every rational and integral function of r, in which $r^n=1$ may be reduced to the form

$$A+Br+Cr^2+Dr^3+\ldots+Nr^{n-1},$$

the coefficients A, B, C, ... N being given quantities independent of r; or, since in this case the powers r, r^2, r^3, ... r^{n-1} may be represented, although in a different order, by r, r^a, r^{a^2}, ... $r^{a^{n-2}}$, we may reduce every rational function of r to the form

$$A+Br+Cr^a+Dr^{a^2}+\ldots+Nr^{a^{n-2}}.$$

Therefore, if this function is such that it remains unaltered when r is changed into r^a, it follows that the new form

$$A+Br^a+Cr^{a^2}+Dr^{a^3}+\ldots+Nr,$$

coincides with the preceding;

$$\therefore B=C,\ C=D,\ D=E, \text{ \&c., } N=B,$$

and therefore the function is reduced to the form

$$A+B(r+r^a+r^{a^2}+\ldots+r^{a^{n-2}}), \text{ or } A-B,$$

since the sum of the roots $=-1$; hence each of the quantities u_0, u_1, u_2, &c., will be of the form A—B, and its value will be found by the actual development of $z=y^{n-1}$; so that we have the case where the values of u_0, u_1, u_2, &c., are known immediately, without depending upon the solution of any equation. Hence, if we denote by 1, a, β, γ, &c., the $n-1$ roots of the equation $x^{n-1}-1=0$, and by z_0, z_1, z_2, &c., the value of z answering to the substitution of these roots in the place of a in equation (1), we shall have, as in the former cases,

$$r=\frac{\sqrt[n-1]{z_0}+\sqrt[n-1]{z_1}+\sqrt[n-1]{z_2}+\ldots+\sqrt[n-1]{z_{n-i}}}{n-1},$$

an expression for one of the roots of the equation $x^n-1=0$; and the other roots are r^2, r^3, &c.

Thus, the solution of $x^n-1=0$ is reduced to that of the inferior equation $y^{n-1}-1=0$, of which 1, a, β, γ, &c., are the roots; also, since $n-1$ is a composite number, the determination of a, β, γ, &c., will not require the solution of an equation of a higher degree than the greatest prime number in $n-1$; that is, the solution of $x^n-1=0$ (n prime) may be made to depend upon the solution of equations whose degrees do not exceed the greatest prime number, which is a divisor of $n-1$.

EXAMPLE IV.

$$x^5-1=0.$$

The least primitive root of 5 is 2; for the powers of 2 from 1 to 4, when divided by 5, leave remainders 2, 4, 3, 1;

$$\therefore\ y=r+ar^2+a^2r^4+a^3r^3;$$

also $$a^4=1,\ r^5=1,\ \text{and}\ r+r^2+r^4+r^3=-1;$$

$$\therefore\ z=y^4=-1+4a+14a^2-16a^3.$$

But the four roots of $y^4-1=0$ are

$$1,\ -1,\ \sqrt{-1},\ -\sqrt{-1};$$

$$\therefore\ z_0=1,\ z_1=25,\ z_2=-15+20\sqrt{-1},$$

$$z_3=-15-20\sqrt{-1};$$

$$\therefore\ x=\tfrac{1}{4}\{-1+\sqrt{5}+\sqrt[4]{-15+20\sqrt{-1}}+\sqrt[4]{-15-20\sqrt{-1}}\}.$$

375. For the proof that, in the general equation of the n^{th} degree, the formation of the reducing equation will require the solution of an equation of $1.2.3\ldots(n-2)$ dimensions, when n is prime; and of $\dfrac{1.2.3\ldots n}{(m-1)m(1.2.3\ldots p)^m}$ dimensions, when n is a composite number, and $=mp$, where m is prime; and that, consequently, the method fails when n exceeds 4, the reader is referred to Lagrange's *Traité de la resolution des équations numériques*, note xiii., from which the matter of this section is taken.

RESOLUTION OF THE GENERAL EQUATIONS OF THE THIRD AND FOURTH DEGREES.

RESOLUTION OF THE EQUATION OF THE THIRD DEGREE.

376. I shall suppose that we have made the second term of the equation of the third degree disappear, and, to avoid fractions, I will write this equation under the form

$$x^3+3px+2q=0\ \ldots\ldots\ldots\ (1)$$

Among the different modes of resolving it, the most simple consists in forming *a priori* an equation of the third degree, without a second term, which admits of one known root, but expressed with indeterminates, and to make use afterward of these indeterminates to render the equation identical with the proposed equation (1). To establish this identity, it will be necessary to write two equalities, and for this reason we employ two indeterminates.

Let there be made $x=a+b$: the cube will be $x^3=a^3+b^3+3ab(a+b)$; then, replacing $a+b$ by x, and transposing, we shall have

$$x^3-3abx-a^3-b^3=0\ \ldots\ldots\ (2)$$

an equation which admits the root $x=a+b$, and which it is necessary to render identical with equation (1). Therefore we place

$$ab=-p,\ a^3+b^3=-2q\ \ldots\ (3)$$

The first of these equalities gives $a^3b^3=-p^3$. Thus we know the sum a^3+b^3, and the product a^3b^3. Then the values of a^3 and b^3 are roots of an equation of the second degree, in which the coefficient of the second term is equal to $+2q$, and the last term equal to $-p^3$ (see Art. 191); so that this equation will be, calling z the unknown,

$$z^2+2qz-p^3=0.$$

This is called the *reduced* equation.

Its two roots represent the values of a^3 and b^3; moreover, we can take either of them indifferently for the value of a^3, because this amounts to changing a into b, and b into a, in the value $x=a+b$. I will take

$$a^3=-q+\sqrt{q^2+p^3},\quad b^3=-q-\sqrt{q^2+p^3}.$$

$$\therefore\ a=\sqrt[3]{-q+\sqrt{q^2+p^3}},\quad b=\sqrt[3]{-q-\sqrt{q^2+p^3}}.$$

Each radical of the second degree here has but one value, but each one of the third degree has three. If we could satisfy equation (3) without making any choice between these values, we could also, by the same values, render equation (1) identical with equation (2); and since $a+b$ is a root of the second, the first ought to be satisfied by taking

$$x=\sqrt[3]{-q+\sqrt{q^2+p^3}}+\sqrt[3]{-q-\sqrt{q^2+p^3}}\quad . \quad . \quad (4)$$

which is the formula of Cardan.

But an important remark presents itself: it is, that since each radical of the third degree has three values, the above expression must have nine, while the equation (1) ought to have but three roots. It is necessary to explain, then, whence comes this multiplicity of values, and to discern among them which ought to be true roots of the equation (1).

For this purpose, let us observe that, properly speaking, it is not the resolution of equations (3) which has given a and b, but rather the equations

$$a^3b^3=-p^3,\ a^3+b^3=-2q\quad . \quad . \quad . \quad (5)$$

Now if we designate by α and α^2 the two imaginary cubic roots of unity, which, as we know, are the one the square of the other, it will be readily seen that the equation $a^3b^3=-p^3$ may result indifferently, from raising to the cube these following:

$$ab=-p,\ ab=-\alpha p,\ ab=-\alpha^2p.$$

Hence it follows that the nine values contained in formula (4) ought to give the roots of the three equations,

$$x^3+3px+2q=0,\ x^3+3\alpha px+2q=0,\ x^3+3\alpha^2px+2q=0\quad . \quad . \quad . \quad . \quad (6)$$

We can, moreover, consider these nine values as the roots of the equation of the 9° degree, which would be obtained by multiplying together the three equations (6). But it will be more simple, and will amount to the same thing, to raise to the cube either one of these equations, after transposing to the second member the term which contains p. In this manner we find at once

$$(x^3+2q)^3=-27p^3x^3.$$

As to the roots which belong especially to each of the three equations, what precedes furnishes the means of distinguishing them; because, according as the coefficient of x shall be $3p$, $3\alpha p$, or $3\alpha^2p$, it is clear that we ought to add only the values of a and b, for which we have $ab=-p$, or $ab=-\alpha p$, or $ab=-\alpha^2p$.

By this rule it will be easy to form the roots of the proposed equation $x^3+3px+2q=0$, the only one with which we have to do. Designate by A one of the values of the first cubic radical, and by B one of the values of the second; the values of a and b will be

$$a=\mathrm{A},\ \alpha\mathrm{A},\ \alpha^2\mathrm{A};\ b=\mathrm{B},\ \alpha\mathrm{B},\ \alpha^2\mathrm{B}.$$

Moreover, suppose, for this is admissible, that A and B represent the values, the product of which is $-p$. From what has just been said we ought to add only the values, the product of which is AB; then, recollecting that $\alpha^3=1$, we must take

$$x=A+B,\ x=\alpha A+\alpha^2 B,\ x=\alpha^2 A+\alpha B;$$

and, besides, we know (303) that we have

$$\alpha=\frac{-1+\sqrt{-3}}{2},\ \alpha^2=\frac{-1-\sqrt{-3}}{2}.$$

If we replace A and B by the two cubic radicals, and α and α^2 by their values, we shall have

$$x=\sqrt[3]{-q+\sqrt{q^2+p^3}}+\sqrt[3]{-q-\sqrt{q^2+p^3}},$$

$$x=\frac{-1+\sqrt{-3}}{2}\sqrt[3]{-q+\sqrt{q^2+p^3}}+\frac{-1-\sqrt{-3}}{2}\sqrt[3]{-q-\sqrt{q^2+p^3}},$$

$$x=\frac{-1-\sqrt{-3}}{2}\sqrt[3]{-q+\sqrt{q^2+p^3}}+\frac{-1+\sqrt{-3}}{2}\sqrt[3]{-q-\sqrt{q^2+p^3}}.$$

These are the roots of the proposed equation, but we must take care to attach to the two cubic radicals the same restricted sense as to A and B, without which we should find false roots.

377. To discuss these values, it will be more convenient to leave A and B substituted for the cubic radicals, and to isolate the one which is multiplied by $\sqrt{-3}$. By this means we have

$$x=A+B,$$

$$x=-\frac{A+B}{2}+\frac{A-B}{2}\sqrt{\overline{3}},$$

$$x=-\frac{A+B}{2}-\frac{A-B}{2}\sqrt{\overline{3}}.$$

I shall suppose, also, as is done ordinarily, that the coefficients $3p$ and $2q$ represent real quantities. Then equation (1), being of an uneven degree, has always one real root, and it is admissible to suppose that A and B are the values of a and b, which give this root; so that $A+B$ will be a real quantity. This being premised, let us return to the two radicals

$$a=\sqrt[3]{-q+\sqrt{q^2+p^3}},\ b=\sqrt[3]{-q-\sqrt{q^2+p^3}}.$$

If $q^2+p^3>0$, each of them has one real value; then we can suppose A and B real. Consequently, $A+B$ and $A-B$ will be so also; then the first root $x=A+B$ is real, and the other two are imaginary.

If $q^2+p^3=0$, we have $A=B$, and then the three roots will be $x=2A$, $x=-A$, $x=-A$. They are all three real, and the last two are equal with one another.

Finally, let $q^2+p^3<0$, which requires p to be negative. Then a and b have no longer any real determination, and, consequently, the three values of x are found complicated with imaginary quantities. However, we know that one of them must be real, and, indeed, it is evident that the cases in which the three roots of equation (1) are real and unequal can only be found on the hypothesis in question, that $q^2+p^3<0$, as may be seen by referring to the supposition just above of $q^2+p^3>0$. It would be wrong, then, to affirm that the values of x are imaginary. I will prove, in fact, that neither of them are so; and as we can always suppose that A and B are determinations such that the sum $A+B$ represents the real root, the existence of which is demonstrated, the whole is reduced to showing that the part $\frac{1}{2}(A-B)\sqrt{-3}$, which is found in the other two values of x, must be real. By the rules of algebra alone we have $(A-B)(A^2+AB+B^2)=A^3-B^3$; then

$$A-B=\frac{A^3-B^3}{A^2+AB+B^2}=\frac{A^3-B^3}{(A+B)^2-AB}.$$

But, because of the values of a^3 and of b^3, we have $A^3-B^3=2\sqrt{q^2+p^3}$; and, by the manner in which A and B have been chosen, we have $AB=-p$; then, making $A+B=x'$, there results $A-B=\frac{2\sqrt{q^2+p^3}}{x'^2+p}$; consequently

$$\frac{A-B}{2}\sqrt{-3}=\frac{\sqrt{-3(q^2+p^3)}}{x'^2+p}.$$

But by hypothesis we have $q^2+p^3<0$; then the quantity above is real; then the three values of x are also.

It is thus demonstrated that, upon the hypothesis of $q^2+p^3<0$, the imaginary quantities which affect the three values of x must destroy one another. It would seem, therefore, that analysis ought to furnish the means of making them disappear, but as yet it has not been found capable of effecting this reduction. For this reason, the case under examination has been called the *irreducible case*. Whenever the equation falls under this case, the general expressions of the roots will be of no use in calculating their numerical values, and then we can recur to the methods of Arts. 290–297.

EXAMPLES.

(1) $$x^3-6x-9=0.$$

We have $p=-2$, $q=-\frac{9}{2}$ $\therefore$ $\sqrt{q^2+p^3}=\frac{7}{2}$, which gives

$$A=\sqrt[3]{-q+\sqrt{q^2+p^3}}=\sqrt[3]{\frac{16}{2}}=2,$$

$$B=\sqrt[3]{-q-\sqrt{q^2+p^3}}=\sqrt[3]{\frac{2}{2}}=1.$$

Thus the three roots are

$$x=3,$$

$$x=-1+\sqrt{3}\sqrt{-1}+\tfrac{1}{2}(-1-\sqrt{3}\sqrt{-1})=\tfrac{1}{2}(-3+\sqrt{3}\sqrt{-1}),$$

$$x=-1-\sqrt{3}\sqrt{-1}+\tfrac{1}{2}(-1+\sqrt{3}\sqrt{-1})=\tfrac{1}{2}(-3-\sqrt{3}\sqrt{-1}).$$

(2) $$x^3-21x+20=0.$$

Here $$p=-7,\ q=10;$$

$$\therefore x=\sqrt[3]{-10+9\sqrt{-3}}+\sqrt[3]{-10-9\sqrt{-3}}.$$

This example is one of the irreducible case. The general value of x appears in an imaginary form, and yet the roots are real, being the numbers 1, 4, and -5, which, by substitution, will be found to verify the given equation.

378. The solution of the irreducible case may be obtained, also, by the help of a table of sines and cosines. We subjoin the method, for the benefit of the student acquainted with trigonometry.

Solution of the irreducible case by trigonometry.

$$\cos 2\theta=2\cos^2\theta-1$$

$$\cos 3\theta=2\cos 2\theta\cos\theta-\cos\theta$$

Substituting the first expression in the second,

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta.$$

Whence

$$\cos^3 \theta - \frac{3}{4} \cos \theta - \frac{1}{4} \cos 3\theta = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

In the proposed cubic equation, which we may write under the form

$$x^3 + 3px + 2q = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

put the unknown $r \cos \theta$ for x; or, which is the same thing, put $\frac{x}{r}$ for $\cos \theta$, and (1) becomes

$$x^3 - \frac{3}{4} r^2 x - \frac{1}{4} r^3 \cos 3\theta = 0.$$

Comparing this with (2), we have

$$\frac{1}{4} r^3 \cos 3\theta = -2q,$$

and

$$\frac{3}{4} r^2 = -3p \therefore r = 2\sqrt{-p}, \text{ which is real, } p \text{ being negative},$$

$$\therefore \cos 3\theta = \frac{2q}{pr} = \frac{q}{\sqrt{-p^3}}$$

Consequently, the trigonometrical solution of the proposed cubic equation, that is, the determination of θ, and thence of $r \cos \theta$, depends upon *the trisection of an arc*, or the determination of $\cos \theta$ from $\cos 3\theta$.

The mode of proceeding by aid of trigonometrical tables is obvious; we are to seek in the table of cosines for the angle whose cosine is $q\sqrt{\frac{1}{-p^3}}$; this will be the angle 3θ, and, consequently, one third of it will be θ; and the cosine of this, multiplied by r, or $2\sqrt{-p}$, will give $r \cos \theta = x$ for one of the real roots of equation (2). As the given cosine, $q\sqrt{\frac{1}{-p^3}}$, belongs equally to *three* arcs, viz., 3θ, $2\pi + 3\theta$, and $2\pi - 3\theta$, by taking the cosine of one third of each of the latter two, we shall have the values of the remaining roots. Thus all the three roots will be expressed as follows:

$$2\sqrt{-p} \cos \theta, \; 2\sqrt{-p} \cos \frac{1}{3}(2\pi + 3\theta), \; 2\sqrt{-p} \cos \frac{1}{3}(2\pi - 3\theta).$$

Or, using the supplements of the two latter arcs instead of the arcs themselves, and remembering that the cosine of an arc is equal to minus the cosine of its supplement, we have somewhat more simply the three values of x in the following form:

$$2\sqrt{-p} \cos \theta, \; -2\sqrt{-p} \cos (60° - \theta), \; -2\sqrt{-p} \cos (60° + \theta).$$

This method, with a single exception, applies to the irreducible case; for, as the trigonometrical cosine of an arc is always less than unity, except when that arc is a multiple of 180°, we must have

$$q\sqrt{\frac{1}{-p^3}} < 1 \therefore q^2 < -p^3,$$

or

$$q^2 + p^3 < 0.$$

When 3θ is a multiple of 180°, two roots must be equal.

The reducible case may also employ the aid of trigonometry.

379. If in the expression

$$\left(-\frac{r}{2}\pm\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}\right)^{\frac{1}{3}}=\sqrt{\frac{q}{3}}\left\{-\frac{r}{2}\left(\frac{3}{q}\right)^{\frac{3}{2}}\pm\sqrt{\frac{r^2}{4}\left(\frac{3}{q}\right)^3+1}\right\}^{\frac{1}{3}},$$

we put $\cot\phi=\frac{r}{2}\left(\frac{3}{q}\right)^{\frac{3}{2}}$, it becomes $\sqrt{\frac{q}{3}}(-\cot\phi\pm\operatorname{cosec}\phi)^{\frac{1}{3}}$.

Hence, reducing, the real root of $x^3+qx+r=0$ is

$$\sqrt{\frac{q}{3}}\left(\tan^{\frac{1}{3}}\frac{\phi}{2}-\cot^{\frac{1}{3}}\frac{\phi}{2}\right);$$

which, by putting $\tan\frac{\phi}{2}=\tan^3\theta$, may be further transformed into

$$-2\sqrt{\frac{q}{3}}\cot 2\theta.$$

Similarly, the real root of $x^3-qx+r=0$, $\frac{q^3}{27}<\frac{r^2}{4}$, becomes (by putting $\operatorname{cosec}\phi=\frac{r}{2}\left(\frac{3}{q}\right)^{\frac{3}{2}}$, $\tan\frac{\phi}{2}=\tan^3\theta$),

$$-2\sqrt{\frac{q}{3}}\operatorname{cosec} 2\theta.$$

380. The following method of arriving at a new and valuable formula for the solution of cubic equations will be found an excellent exercise for the student :*

Let the given equation be

$$x^3+px+q=0 \quad \ldots\ldots \quad (1)$$

Placing

$$x=m+y \quad \ldots\ldots \quad (2)$$

we obtain

$$y^3+3my^2+(3m^2+p)y+m^3+pm+q=0 \quad \ldots\ldots \quad (3)$$

Taking

$$y=\frac{1}{z} \quad \ldots\ldots \quad (4)$$

we obtain

$$\left(\frac{1}{z}\right)^3+3m\left(\frac{1}{z}\right)^2+(3m^2+p)\frac{1}{z}+m^3+pm+q=0;$$

which gives

$$z^3+\frac{3m^2+p}{m^3+pm+q}z^2+\frac{3m}{m^3+pm+q}z+\frac{1}{m^3+pm+q}=0 \quad \ldots\ldots \quad (5)$$

Placing

$$z=w-\frac{3m^2+p}{3(m^3+pm+q)} \quad \ldots\ldots \quad (6)$$

we find

$$w^3+\frac{3pm^2+9qm-p^2}{3(m^3+pm+q)^2}w+\frac{-27qm^3+18p^2m^2+27pqm+27q^2+2p^3}{27(m^3+pm+q)^3}=0 \quad \ldots \quad (7)$$

* It is the production of an old pupil of the author's, Mr. James S. Woolley, whom ill health, and other discouraging circumstances, have not prevented from making some important discoveries in algebra, which it would be premature at present to publish to the world.

The value of m, which renders the coefficient of w zero, may be found thus

$$3pm^2+9qm-p^2=0.$$

Then

$$m=-\frac{3q}{2p}\pm\frac{3}{p}\times\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

The value of w in (7), substituting the value of m, found in (8), is expressed in the following four equations, (9), (9, a), (9, b), (9, c), the last three being obtained by decomposing (9) into factors.

$$w=\frac{\sqrt[3]{-\frac{729}{2}\frac{q^4}{p^3}-108q^2-8p^3\pm\left(729\frac{q^3}{p^3}+108q\right)\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}}}{-\frac{81q^3}{2p^3}-6q\pm\left(81\frac{q^2}{p^3}+12\right)\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}} \quad \ldots\ldots (9)$$

$$w=\frac{\sqrt[3]{+\left(729\frac{q^3}{p^3}+108q\right)\left(-\frac{q}{2}-\frac{2}{27}\frac{p^3}{q}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)}}{\left(81\frac{q^2}{p^3}+12\right)\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)} \quad \ldots\ldots\ldots (9, a)$$

$$w=\frac{\pm\sqrt[3]{\left(729\frac{q^3}{p^3}+108q\right)\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\left(-\frac{2}{q}\right)}}{\left(81\frac{q^2}{p^3}+12\right)\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)} \quad \ldots (9, b)$$

$$w=\frac{\mp\sqrt[3]{-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}}\times\sqrt{12p+81\frac{q^2}{p^2}}}{\left(81\frac{q^2}{p^3}+12\right)\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)} \quad \ldots\ldots\ldots\ldots\ldots (9, c)$$

Substituting in (6) the values of m and w, found in (8) and (9, c), we shall have

$$z=\frac{-\frac{27}{2}\frac{q^2}{p^2}-2p\pm\frac{27q}{p^2}\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\mp\sqrt[3]{-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}}\times\sqrt{12p+81\frac{q^2}{p^2}}}{\left(81\frac{q^2}{p^3}+12\right)\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)} . (10)$$

Substituting in (4) the values of z, given in (10), and decomposing one more of its terms into factors, we shall have

$$y=\frac{\left(81\frac{q^2}{p^3}+12\right)\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)}{\mp\frac{54}{p^2}\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)\mp\left(\sqrt[3]{-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}}\right)\sqrt{12p+81\frac{q^2}{p^2}}} . (11)$$

Hence

$$x=\frac{\mp\frac{162}{p^3}\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)^2\mp\frac{3}{p}\sqrt{12p+\frac{81q^2}{p^2}}\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)^{\frac{4}{3}}+}{\mp\frac{54}{p^2}\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)\mp\sqrt{12p+81\frac{q^2}{p^2}}\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)^{\frac{1}{3}}}$$

(continuing the numerator) $\left(81\frac{q^2}{p^3}+12\right)\left(-\frac{q}{2}\pm\sqrt{\frac{p^3}{27}+\frac{q^2}{4}}\right)$ . . . . (12)

But the first term in the numerator of (12) may be transformed thus:

$$\mp\frac{162}{p^3}\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)^2=\left[-\left(\frac{81}{2}\frac{q^2}{p^3}+6\right)\pm\frac{81q}{p^3}\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right]\times$$
$$\left(-\frac{q}{2}\pm\sqrt{\frac{p^3}{27}+\frac{q^2}{4}}\right).$$

And the last term in the numerator of equation (12) is

$$\left(81\frac{q^2}{p^3}+12\right)\cdot\left(-\frac{q}{2}\pm\sqrt{\frac{p^3}{27}+\frac{q^2}{4}}\right).$$

Therefore the sum of the first and last terms of the numerator of (12) is

$$\pm\sqrt{\frac{4}{3}p^3+9q^2}.$$

Therefore,

$$x=\frac{\mp\frac{3}{p}\sqrt{12p+81\frac{q^2}{p^2}}\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)^{\frac{4}{3}}\pm\sqrt{\frac{4}{3}p^3+9q^2}}{\mp\frac{54}{p^2}\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)\mp\sqrt{12p+81\frac{q^2}{p^2}}\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)^{\frac{1}{3}}}$$

Dividing both numerator and denominator by $\mp\frac{54}{p^2}\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}$, we have

$$x=\frac{\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)^{\frac{4}{3}}-\frac{p^2}{9}}{\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)+\frac{p}{3}\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)^{\frac{1}{3}}}$$

The numerator of this value of x is equal to

$$\left[\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)^{\frac{2}{3}}+\frac{p}{3}\right]\cdot\left[\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)^{\frac{2}{3}}-\frac{p}{3}\right]$$

The denominator is equal to

$$\left[\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)^{\frac{2}{3}}+\frac{p}{3}\right]\cdot\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)^{\frac{1}{3}}$$

Dividing numerator and denominator by the common factor, we have

$$x=\frac{\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)^{\frac{2}{3}}-\frac{p}{3}}{\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)^{\frac{1}{3}}}$$

This formula may be reduced to that of Cardan by dividing the numerator by the denominator, and observing that

$$-\frac{p}{3}=\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)^{\frac{1}{3}}\cdot\left(-\frac{q}{2}\mp\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)^{\frac{1}{3}};$$

we thus obtain

$$x=\left(-\frac{q}{2}\pm\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)^{\frac{1}{3}}+\left(-\frac{q}{2}\mp\sqrt{\frac{1}{27}p^3+\frac{q^2}{4}}\right)^{\frac{1}{3}}.$$

But the first form is preferable, as it gives only the three values which satisfy equation (1), whereas Cardan's formula gives nine values, six of which have to be rejected.

A partial division gives

$$x=\left(-\frac{q}{2}\pm\sqrt{\frac{p^3}{27}+\frac{q^2}{4}}\right)^{\frac{1}{3}}-\frac{p}{3\left(-\frac{q}{2}\pm\sqrt{\frac{p^3}{27}+\frac{q^2}{4}}\right)^{\frac{1}{3}}}$$

which is an advantageous form, inasmuch as but one third root has to be extracted, both radicals having the same form.

A shorter solution of the above might be given, but we have already extended our article on cubics sufficiently far.

IRRATIONAL EXPRESSIONS ANALOGOUS TO THOSE OBTAINED IN THE RESOLUTION OF EQUATIONS OF THE THIRD DEGREE.

381. One of these expressions is $\sqrt[n]{A\pm\sqrt{B}}$; but it frequently happens that A and B are rational numbers, and then it may be possible to reduce these radicals to simpler expressions, in which there are no longer radicals over radicals. This problem has already been resolved for radicals of the second degree, and it is now proposed to resolve it with reference to radicals of the higher degrees.

I shall commence with the cubic radical $\sqrt[3]{A+\sqrt{B}}$. We can not suppose for this root a quantity of the form $\sqrt{a}+\sqrt{b}$, for we have

$$(\sqrt{a}+\sqrt{b})^3=a\sqrt{a}+3a\sqrt{b}+3b\sqrt{a}+b\sqrt{b}$$
$$=(a+3b)\sqrt{a}+(3a+b)\sqrt{b},$$

a result which contains the radicals $\sqrt{a}$ and $\sqrt{b}$. But the preceding calculation shows that we should have a result of the form $A+\sqrt{B}$, by raising to the third power the expression $a+\sqrt{b}$ and $(a+\sqrt{b})\sqrt[3]{c}$. I will choose this last expression as the more general; we shall then have

$$\sqrt[3]{A+\sqrt{B}}=(a+\sqrt{b})\sqrt[3]{c}\quad\ldots\ldots\ldots\ldots\quad(1)$$

Raising both members to the third power, it becomes $A+\sqrt{B}=c(a^3+3ab)+c(3a^2+b)\sqrt{b}$; equating the rational parts together, and the irrational parts by themselves,

$$A=c(a^3+3ab)\quad\ldots\ldots\ldots\ldots\quad(2)$$
$$\sqrt{B}=c(3a^2+b)\sqrt{b}\quad\ldots\ldots\ldots\ldots\quad(3)$$

The problem, then, is, to find for a, b, c rational values which satisfy these two equations. But squaring these equations, and then subtracting the one from the other, we have

$$A^2-B=c^2(a^6-3a^4b+3a^2b^2-b^3)=c^2(a^2-b)^3;$$

hence
$$a^2-b=\frac{\sqrt[3]{(A^2-B)c}}{c}.$$

Since a and b ought to be rational, it will be necessary to take c such that $(A^2-B)c$ be an entire or fractional cube, which is always possible. Calling the second member of the above equation M, we shall have $a^2-b=M$, whence $b=a^2-M$. By substituting this value of b in equation (2), it will become

$$4ca^3-3Mca-A=0 \ldots\ldots\ldots\ldots \quad (4)$$

This equation must give for a at least a commensurable value, without which the transformation (1) will be impossible.

If, instead of $\sqrt[3]{A+\sqrt{B}}$, we should have to reduce $\sqrt[3]{A-\sqrt{B}}$, it would suffice to change throughout in the preceding method the sign of $\sqrt{b}$.

For example, let the expression be $\sqrt[3]{14\pm\sqrt{200}}$. We shall have $A=14$, $B=200$, $A^2-B=-4$; hence $(A^2-B)c=-4c$; we shall then have the perfect cube -8, by taking $c=2$. Consequently, $M=-1$, $b=a^2+1$, and equation (4) becomes $8a^3+6a-14=0$. It can be satisfied by the commensurable value $a=1$, which gives $b=2$. Again, we have already obtained $c=2$; hence, finally,

$$\sqrt[3]{14\pm\sqrt{200}}=(1\pm\sqrt{2})\sqrt[3]{2}.$$

Again, let the expression be $\sqrt[3]{-11\pm2\sqrt{-1}}$. We will pass 2 under the radical of the second degree; we shall then have $A=-11$, $B=-4$, $A^2-B=125$. As 125 is already the cube of 5, it will suffice to make $c=1$. Consequently, we have $M=5$, $b=a^2-5$, and equation (4) becomes $4a^3-15a+11=0$. But this equation is satisfied by the value $a=1$; hence $b=-4$, and, consequently,

$$\sqrt[3]{-11\pm2\sqrt{-1}}=(1\pm\sqrt{-4})\sqrt[3]{1}.$$

382. Let us consider the more general expression $\sqrt[n]{A\pm\sqrt{B}}$, and take

$$\sqrt[n]{A\pm\sqrt{B}}=(a\pm\sqrt{b})\sqrt[n]{c} \ldots\ldots\ldots\ldots \quad (5)$$

The problem, again, is to determine rational numbers for a, b, c, if it be possible.

Raising (5) to the power n, and equating separately the rational parts, we obtain

$$A=c[a^n+\frac{n(n-1)}{1.2}a^{n-2}b+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}a^{n-4}b^2+, \&c.] \quad (6)$$

$$\sqrt{B}=c[na^{n-1}+\frac{n(n-1)(n-2)}{1.2.3}a^{n-3}b+, \&c.]\sqrt{b} \quad . \, . \, . \quad (7)$$

We can, as in the case of the cubic radical, square these two equalities, and subtract the one from the other; but the reductions will be immediately perceived by observing that we ought to have, at the same time,

$$A+\sqrt{B}=c(a+\sqrt{b})^n,\ A-\sqrt{B}=c(a-\sqrt{b})^n;$$

and that, consequently,

$$A^2-B=c^2(a+\sqrt{b})^n(a-\sqrt{b})^n=c^2(a^2-b)^n;$$

whence
$$a^2-b=\frac{\sqrt[n]{(A^2-B)c^{n-2}}}{c}.$$

We see from this that it will be necessary to take c of such a value that the second member of this last equation shall be rational. Calling this second member M, we shall have $a^2-b=M$, whence $b=a^2-M$; substituting this value of b in (6), the resulting equation in a will have a commensurable root every time that the transformation (5) is possible.

383. In the resolution of equations of the third degree, what renders the irreducible case so remarkable is, that although we are assured that the three roots are real, it is, nevertheless, impossible to make the imaginary quantities disappear otherwise than by means of series. This difficulty is not confined to the equation of the third degree; it will be encountered equally in the general formula

$$\sqrt[n]{A+B\sqrt{-1}}+\sqrt[n]{A-B\sqrt{-1}} \quad \dots\dots\dots\dots\dots \quad (8)$$

which formula I shall stop to consider for a moment.

To consider this expression in its most general sense, we ought to combine the n determinations of the first part with the n determinations of the second, so that we shall have, in all, n^2 values. But the expression is rarely taken in so general a sense, and I proceed to define that which we ordinarily attach to it.

As the two radicals which have the index n represent the roots of the binomial equation, their determinations are equal in number to the quantities which have the form $f+g\sqrt{-1}$. Moreover, it is manifest that to each determination of the first radical there corresponds one of the second, which only differs by the sign of $\sqrt{-1}$. But we suppose that these corresponding values are those which ought to be added in formula (8); and, with these restrictions, the values of x are all real, and only n in number.

The product of these two radical values, thus taken in a same pair, is real and positive; but for the product of the two radicals we have, in general,

$$\sqrt[n]{A+B\sqrt{-1}}\times\sqrt[n]{A-B\sqrt{-1}}=\sqrt[n]{A^2+B^2},$$

and the radical which expresses this product can only have a single real and positive value; hence, if we represent it by K^2, we ought to be able to characterize the conjugate values, which must be added in formula (8), by the condition that their product be equal to K^2.

Formula (8) can be regarded as a general expression of the roots of an equation whose degree is marked by the number of values of which the equation is susceptible; hence, provided that it be taken in its greatest extension, or with the restriction which we have just mentioned, the degree of the equation must be either n^2 or n.

This last remark leads us to explain how we form an equation, when we know the expression for its root; that is to say, that an equation being given, susceptible of taking different values, by reason of the multiple values of the radicals which it contains, it is required to find an equation free from radicals which has these values for roots. I will take, for example, the same expression (8).

To abridge, let us make

$$A+B\sqrt{-1}=a,\ A-B\sqrt{-1}=b;$$

the problem reduces itself to eliminating y and z between the three equations

$$y+z=x,\ y^n=a,\ z^n=b.$$

But here the elimination can be conducted according to a very simp[illegible] process, analogous to that which has been employed for reciprocal equations. By the rules of multiplication we have

$$(y^m+z^m)(y+z)=y^{m+1}+z^{m+1}+yz(y^{m-1}+z^{m-1}).$$

But $y+z=x$ and $yz=\sqrt[n]{ab}$; hence, making $\sqrt[n]{ab}=c$, the equation will become

$$y^{m+1}+z^{m+1}=x(y^m+z^m)-c(y^{m-1}+z^{m-1}).$$

By means of this formula we express, in function of x and c, successively all the quantities y^2+z^2, y^3+z^3, &c. When we have arrived at y^n+z^n, we replace y^n+z^n by $a+b$, and then we shall have the required equation, which will be of the degree n in x.

This equation contains c; but we have $c=\sqrt[n]{ab}=\sqrt[n]{A^2+B^2}$; hence, c is, in general, susceptible of n different values. By putting in the equation each of these n values in its turn, we shall have n equations, and, consequently, $n\times n$, or n^2 values of x. This, in fact, ought to be the case, from what has been said at the close of the preceding article. If we should wish to have a single equation which has all these values for roots, it would be still necessary to eliminate c between the equation of the degree n in x and the equation $c^n=ab$.

But if in formula (8) we only wish to associate the radical values whose product is real, it is this real value solely which we must choose for c, and we shall only have a single equation of the degree n for determining all the values of x.

RESOLUTION OF THE EQUATION OF THE FOURTH DEGREE.

384. After having made the second term disappear, the general equation of the 4° degree is

$$x^4+px^2+qx+r=0 \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

If we make $x=a+b+c$, squaring, there results

$$x^2=a^2+b^2+c^2+2(ab+ac+bc),$$

or, transposing,

$$x^2-(a^2+b^2+c^2)=2(ab+ac+bc);$$

raising anew to the square, we have

$$x^4-2(a^2+b^2+c^2)x^2+(a^2+b^2+c^2)^2=2(a^2b^2+a^2c^2+b^2c^2)+8abc(a+b+c);$$

then, replacing $a+b+c$ by x, and transposing, we obtain

$$x^4-2(a^2+b^2+c^2)x^2-8abcx+(a^2+b^2+c^2)^2$$
$$-4(a^2b^2+a^2c^2+b^2c^2)=0.$$

This equation is without a second term, and by the manner in which it has been formed, we know that it admits of the root $x=a+b+c$. Thus, we resolve equation (1) in determining a, b, c, by the condition that it shall be identical with the preceding, which gives

$$-2(a^2+b^2+c^2)=p$$
$$-8abc=q$$
$$(a^2+b^2+c^2)^2-4(a^2b^2+a^2c^2+b^2c^2)=r.$$

These equalities show that, by taking a^2, b^2, c^2 for unknowns, these three quantities are the roots of an equation of the 3° degree, the coefficients of which are (see Art. 245)

$$-(a^2+b^2+c^2)=\frac{p}{2}$$

$$a^2b^2+a^2c^2+b^2c^2=\frac{p^2-4r}{16}$$

$$-a^2b^2c^2=-\frac{q^2}{64}.$$

Consequently, this equation of the 3° degree is

$$z^3+\frac{p}{2}z^2+\frac{p^2-4r}{16}z-\frac{q^2}{64}=0 \quad \ldots\ldots\ldots\ldots \quad (2)$$

Such is the *reduced equation* upon which the solution of equation (1) depends.

Suppose that the three values of z have been determined, which designate by z', z'', z''', we shall have

$$a=\pm\sqrt{z'},\ b=\pm\sqrt{z''},\ c=\pm\sqrt{z'''}.$$

If the signs be combined in all possible ways, there will result eight values for $a+b+c$ or x. But as the last term of the *reduced equation* (2) was formed by squaring the equation $abc=-\frac{1}{8}q$, it follows that the values contain not only the roots of the proposed equation, but also those of an equation which would differ from it in the sign of q.

At the same time it may be perceived that, to have only the roots of the proposed, it is necessary to add only the values of a, b, c, for which $abc=-\frac{1}{8}q$, and the product of which has, consequently, the contrary sign to q. In each particular case it will be easy to determine for the radicals three values, A, B, C, which shall fulfill this condition; and afterward, with these values, we form the four roots of the proposed, to wit,

$$x=+A+B+C,\ x=+A-B-C,$$
$$x=-A+B-C,\ x=-A-B+C.$$

Generally, instead of A, B, C, the three radicals are placed, and the values of x are written thus:

$$x=+\sqrt{z'}+\sqrt{z''}-\sqrt{z'''},\ x=+\sqrt{z'}-\sqrt{z''}+\sqrt{z'''},$$
$$x=-\sqrt{z'}+\sqrt{z''}+\sqrt{z'''},\ x=-\sqrt{z'}-\sqrt{z''}-\sqrt{z'''}.$$

But it is necessary to understand that in applying these formulas to particular cases there must be taken for $\sqrt{z'}$, $\sqrt{z''}$, $\sqrt{z'''}$ three determinations, the product of which shall be of the same sign as q. This observation is important; failing to have regard to it, we might find false roots.

385. The nature of the roots of the reduced equation will make known the nature of the roots of the proposed. But the reduced having its last term negative, has always one positive root (see Art. 248, Prop. VIII., Cor. 4), and the product of the other two roots should be positive; then, if these last are not imaginary, they will be both positive or both negative. I pass over the case in which $q=0$, because then the proposed would be solved by the rules for the second degree. Consequently, there are three cases only to be examined.*

1°. *Case where the three roots of the reduced equation are positive.* There the four values of x are evidently real, and if the radicals $\sqrt{z'}$, $\sqrt{z''}$, $\sqrt{z'''}$ be regarded as representing positive determinations, their product will be positive;

* This explains an operation in Art. 365.

then the preceding formulas will be specially applicable to the case of $q>0$. For $q<0$ it would be necessary to change the sign of one of the radicals.

2°. *Case where the reduced has one root* z' *positive, and two* z'', z''' *negative.* The radical $\sqrt{z'}$ will be real, but the radicals $\sqrt{z''}$ and $\sqrt{z'''}$ will be imaginary; consequently, the four values of x will be imaginary also, unless $z''=z'''$. When $z''=z'''$, one of the two quantities $\sqrt{z''}+\sqrt{z'''}$ and $\sqrt{z''}-\sqrt{z'''}$ will become zero, and supposing it to be the latter, the values of x will be simply

$$x=\sqrt{z'},\ x=\sqrt{z''},\ x=-\sqrt{z'}+2\sqrt{z''},\ x=-\sqrt{z'}-2\sqrt{z''}.$$

The first two are real, since z' is positive, and the other two are imaginary, since z'' is negative. Besides, as in the reduction, we have supposed $\sqrt{z''}=\sqrt{z'''}$, we ought to have here $\sqrt{z'}\sqrt{z''}\sqrt{z'''}=z''\sqrt{z'}$; so that this product can only have the sign of q by choosing for $\sqrt{z'}$ a sign contrary to that of q, since, by hypothesis, z'' is negative.

3°. *Case in which the reduced has one root* z' *positive, and two roots* z'', z''' *imaginary.* The positive root z' being known, we can divide the reduced by $x-z'$, and we shall have an equation of the second degree, which will give for z'' and z''' imaginary values of the form

$$z''=f+g\sqrt{-1},\ z'''=f-g\sqrt{-1}.$$

Consequently, two of the values of x will contain the sum

$$\sqrt{f+g\sqrt{-1}}+\sqrt{f-g\sqrt{-1}};$$

and the other two will contain the difference

$$\sqrt{f+g\sqrt{-1}}-\sqrt{f-g\sqrt{-1}}.$$

THE DIOPHANTINE ANALYSIS.

386. THIS branch of analysis derives its name from its inventor, Diophantus, of Alexandria, in Egypt, who flourished about the year 360, A.D. It relates chiefly to the finding of square and cube numbers.

The solutions of the questions must frequently be left, notwithstanding the various rules that have been given for this purpose, to the talents and ingenuity of the learner, who, in pursuing these inquiries, will soon perceive that nothing less than the most refined algebra, applied with great skill and judgment, can surmount the various difficulties which attend them; and, in this respect, no one, perhaps, has ever excelled *Diophantus*, or discovered a greater knowledge of the extent and resources of the analytic art.

When we consider his work with attention, we are at a loss which to admire most, his singular sagacity, and the peculiar artifices he employs in forming such positions as the nature of the problems requires, or the more than ordinary subtilty of his reasoning upon them.

Every particular question puts us upon a new way of thinking, and furnishes a fresh vein of analytical treasure, which can not but prove highly useful to the mind in conducting it through other difficulties of this kind whenever they occur, and also in enabling it to encounter more readily those that may arise in subjects of a different nature.

The following directions for resolving questions in the Diophantine analysis will be found useful; but no general rule can be given, and, therefore, the student must often be left to depend solely upon his own ingenuity and skill.

RULE.

Substitute for the root of the square or cube required, one or more letters, such, that, when they are involved, either the given number or the highest power of the unknown quantity may vanish from the equation; and then, if the unknown quantity be of the first degree, the problem will be solved by reducing the equation. But if the unknown quantity be still a square or a higher power, some other new letters must be assumed to denote the root, with which proceed as before, and so on till the unknown quantity is but of the first degree, and from this all the rest will be determined.

EXAMPLES.*

(1) To find two square numbers whose sum is a square.

Let x^2 and y^2 be the two squares; let $3z$ and $4z$ be the roots.

Then $$25z^2 = \square\dagger = \overline{n-5z}|^2 = n^2 - 10nz + 25z^2;$$

$$\therefore z = \frac{n^2}{10n}; \text{ if } n=10,\ z=1, \text{ then } z\times 3=3 \text{ and } z\times 4=4;$$

and the two squares are 9 and 16, whose sum is 25, a square, if $n=20$, $z=2$; and from this we get another value of x and y, and so on.

(2) To find two square numbers whose difference is a square.

Let x^2 and y^2 be the two squares.

Assume $x^2 - y^2 = (x - ny)^2 = x^2 - 2nxy + n^2y^2$.

Then $$-y^2 = -2nxy + n^2y^2,$$

or $$2nx = (n^2+1)\cdot y;$$

$$\therefore x = \frac{n^2+1}{2n}\cdot y.$$

Suppose $y=2n$, then $x=n^2+1$. If $n=2$, $y=4$, and $x=5$; also $x^2-y^2 = 25-16=9$, a square number. If $n=3$, $y=6$, and $x=10$; also $x^2-y^2 = 100-36=64$, a square number.

(3) To change the sum of two squares into the sum of two others any number of ways at pleasure; for example, in three different ways.

Let a^2 and b^2 be the given squares, and let $a-x$ and $cx-b$ be the roots of the required squares; then, by the question, we get

$$\overline{a-x}|^2 + \overline{cx-b}|^2 = a^2 + b^2;$$

by involution, $$a^2 - 2ax + x^2 + c^2x^2 - 2bcx + b^2 = a^2 + b^2;$$

by transposing and dividing,

$$-2a + x + c^2x - 2cb = 0,$$

or $$c^2x + x = 2bc + 2a \text{ and } x = \frac{2bc+2a}{1+c^2},$$

where c may be taken at pleasure; for example,

$$c = 2, 3, \text{ and } 4;$$

then, $$x = \frac{4b+2a}{5},\ \frac{6b+2a}{10}, \text{ and } \frac{8b+2a}{17}.$$

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* Many of these problems are selected from the Arithmetical Questions of Diophantus, of which six out of thirteen books now remain. The best edition is that published at Paris, by *Bachet*, in the year 1670, with notes by *Fermat*.

† This sign $\square$ denotes that the number placed equal to it is a perfect square.

(4) To divide a number which is the product of the sum of two squares by the sum of two others, into two squares two different ways.

Let a^2+b^2 be the sum of two squares, and c^2+d^2 the sum of two others, whose product $(a^2+b^2)\cdot(c^2+d^2)=(ac+bd)^2+(bc-ad)^2=(ac-bd)^2+(bc+ad)^2$, as required.

(5) To find a number, x, such, that $x+1$ and $x-1$ shall be squares.

Let $x+1=a^2$,

and $x-1=b^2$

$\therefore\ 2=a^2-b^2$ by subtraction;

$$\therefore 2\times 1=(a+b)(a-b),$$

or $3=2a$, and $a=\frac{3}{2}$,

and $a^2=\frac{9}{4}$;

$$\therefore x+1=\frac{9}{4}, \text{ and } x=\frac{5}{4}.$$

Or thus:

$$x+1=y^2$$
$$x\ \ =y^2-1$$
$$x-1=y^2-2=\square=\overline{s-y}|^2=s^2-2sy+y^2$$
$$\therefore s^2-2sy=-2$$
$$2sy=s^2+2, \text{ and } y=\frac{s^2+2}{2s};$$

take $s=1\ \therefore y=\frac{3}{2}$, and $x=y^2-1=\frac{9}{4}-1=\frac{5}{4}$, as before.

(6) Required to find four square numbers whose sum shall be a $\square$.

Let 1, 4, 9, and x^2 be the required squares; then, by the question, we get

$$14+x^2=\square=\overline{n-x}|^2=n^2-2nx+x^2,$$

and $x=\frac{n^2-14}{2n}$,

where n may be any number at pleasure, if $n=3$, $x=-\frac{5}{6}$, $x^2=\frac{25}{36}$, or if $n=4$, $x=\frac{1}{4}$, and the numbers are 1, 4, 9, and $\frac{1}{16}$; then $1+4+9+\frac{1}{16}=\frac{225}{16}=\overline{\frac{15}{4}}|^2$ as required.

(7) Divide 2 into three rational squares.

Let x, $2x-1$, and $3x-1$ be the roots of the three squares respectively;

then $x^2+4x^2-4x+1+9x^2-6x+1=2$;

by transposing and dividing,

$$x=\frac{5}{7},\ 2x-1=\frac{3}{7},\ 3x-1=\frac{8}{7}, \text{ the roots;}$$

and the $\square$'s will be

$$x^2=\frac{25}{49},\ \overline{2x-1}|^2=\frac{9}{49}, \text{ and } \overline{3x-1}|^2=\frac{64}{49},$$

the sum of which is $\frac{25}{49}+\frac{9}{49}+\frac{64}{49}=\frac{98}{49}=2$, the proof.

Or thus:

Let 1, x^2, and y^2 be the squares; then

$$1+x^2+y^2=2 \text{ and } x^2+y^2=1,$$

or
$$x^2=1-y^2=\square=\overline{1-ny}|^2=1-2ny+n^2y^2$$

$$\therefore y=\frac{2n}{n^2+1},$$

where n may be taken any number greater than 1; if $n=2$, then $y=\frac{4}{5}$ and $y^2=\frac{16}{25}$; then will $x^2=\frac{9}{25}$, and the sum of these plus 1 is evidently 2.

(8) Divide $\frac{1}{2}$ into three rational squares.

Let x, $2x-\frac{1}{2}$, and $3x-\frac{1}{2}$, be the roots of the rational squares, and their squares are

$$x^2,\ 4x^2-2x+\frac{1}{4},\ 9x^2-3x+\frac{1}{4},$$

and
$$x^2+4x^2-2x+\frac{1}{4}+9x^2-3x+\frac{1}{4}=\frac{1}{2},$$

and x will be found to be $\frac{5}{14}$, from which we get the three squares, viz., $\frac{25}{196}$, $\frac{9}{196}$, $\frac{64}{196}$, and their sum is evidently $\frac{1}{2}$, as required.

(9) To divide a given square number, 100, into two such parts, that each of them may be a square number.

Let x^2 be one of the parts, then $100-x^2$, the other part, will be a square number.

Assume $100-x^2=(2x-10)^2=4x^2-40x+100$.

$\therefore x=8$, and $2x-10=6$; hence 64 and 36 are the parts required.

The same problem may be resolved generally in the following manner:

Let a^2 be the given square, $x^2=$ one of its parts, and a^2-x^2 the other.

Assume $$a^2-x^2=(nx-a)^2=n^2x^2-2anx+a^2;$$

Then $$-x^2=n^2x^2-2anx;$$

$$\therefore x=\frac{2na}{n^2+1}, \text{ and } nx-a=\frac{an^2-a}{n^2+1};$$

$$\therefore \left(\frac{2na}{n^2+1}\right)^2 \text{ and } \left(\frac{an^2-a}{n^2+1}\right)^2$$

are the two squares required; in which expressions a and n may be any whole numbers whatever, provided n be greater than unity.

(10) To find a number, x, such, that $x+128$ and $x+192$ shall be both square numbers.

Assume $x+128=z^2$ $\therefore x=z^2-128$, which is one condition answered; then $z^2-128+192=z^2+64=\square=a^2$ $\therefore z^2=a^2-64$; then we have only to assume such a value for a as will make a^2-64 a square; but it is plain that if a be taken $=10$, then $a^2-64=36=\square$, and $z^2=36$; but this would make the value of x negative; then, in order to find values for z that will make x positive, take $a=17$, and then $a^2=289$, and $\therefore a^2-64=225=\square$ $\therefore z^2=225$ and $\therefore x=225-128=97$, the value required.

(11) To divide a given number, 13, consisting of two known squares, 9 and 4, into two other square numbers.*

* In the solution given of the above problem, n and m may be taken equal to any num-

Let $nx-3$ be the root of the first square sought, and $mx-2$ the root of the other square.

Then $$\overline{nx-3}|^2+\overline{mx-2}|^2=13,$$
or $$(n^2+m^2)\cdot x^2=(4m+6n)\cdot x;$$
$$\therefore x=\frac{6n+4m}{n^2+m^2};$$
whence $$nx-3=\frac{3n^2+4mn-3m^2}{n^2+m^2}=\text{ the root of the first square,}$$
and $$mx-2=\frac{6mn-2n^2+2m^2}{n^2+m^2}=\text{ the root of the second.}$$

If $n=2$ and $m=1$, we have $\frac{3n^2+4mn-3m^2}{n^2+m^2}=\frac{17}{5}=$ the root of one square, and $\frac{6mn-2n^2+2m^2}{n^2+m^2}=\frac{6}{5}=$ the root of the other square.

(12) Let 14 be divided into three rational squares. It is well known that the least three squares in whole numbers are 1, 4, and 9, which will answer the question; but to give a general solution,

Let 1, $3x-2$, $2x-3$, be the roots of the required squares;

then $$1+(3x-2)^2+(2x-3)^2=14,\text{ or } x=\frac{24}{13};$$
then $$\frac{24}{13}\times 3=\frac{72}{13},\text{ from which subtract 2;}$$
then $$\left(\frac{46}{13}\right)^2=\frac{2116}{169};\ \frac{24}{13}\times 2=\frac{48}{13},\text{ from which subtract 3;}$$
then $$\left(\frac{9}{13}\right)^2=\frac{81}{169}\ \therefore\ 1+\frac{2116}{169}+\frac{81}{169}=14.$$

(13) To find two square numbers whose difference shall be equal to any given number.

Let x be the root of the lesser square sought; and let d, the given difference of the squares, be resolved into any two unequal factors a and b, of which a is the greater.

Let $x+b$ be the root of the greater square;

then $$(x+b)^2-x^2=d=ab,$$
$$i.\ e.,\ 2x+b=a.$$
Whence $$x=\frac{a-b}{2}=\text{ the root of the lesser }\square,$$
and $$x+b=\frac{a+b}{2}=\text{ the root of the greater.}$$

If $d=60$, and $a\times b=30\times 2$, we have
$$\frac{30-2}{2}=14,\text{ and }\frac{30+2}{2}=16;$$
whence 16^2 and 14^2 are the squares required whose difference $=60$.

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bers whatever, provided their ratio be not that of 3 : 2. For if n were to m as 3 to 2, the roots of the squares sought would be found the same as the roots of the known squares.

If it were required to divide a given square, x^2, into two other squares,
$$\text{Since } (m^2+n^2)^2=(m^2-n^2)^2+(2mn)^2,$$
$$\therefore (m^2+n^2)^2\cdot x^2=(m^2-n^2)^2\cdot x^2+(2mn)^2\cdot x^2,$$
$$x^2=\left\{\frac{m^2-n^2}{m^2+n^2}\right\}^2\cdot x^2+\left\{\frac{2mn}{m^2+n^2}\right\}^2\cdot x^2,$$
where m and n may be assumed at pleasure, m being greater than n.

(14) To find two numbers, such, that if either of them be added to the square of the other, the sum shall be a square number.

Let x^2+2xy and y be the required numbers;

then $$x^2+2xy+y^2=\square=\overline{x+y}|^2;$$

hence it only remains to make

$$y+\overline{2xy+x^2}|^2=\square=\overline{x^2+ny}|^2=x^4+2nx^2y+n^2y^2,$$

$$\therefore\ y=\frac{1+4x^3-2nx^2}{n^2-4x^2}.$$

If $n=2\frac{1}{3}$, and $x=1$, then $y=\frac{3}{13}$ and $x^2+2xy=\frac{19}{13}$, which are two numbers that will answer the conditions; for

$$\overline{\frac{3}{13}}\Big|^2+\frac{19}{13}=\frac{256}{169}=\overline{\frac{16}{13}}\Big|^2 \text{ and } \overline{\frac{19}{13}}\Big|^2+\frac{3}{13}=\frac{400}{169}=\overline{\frac{20}{13}}\Big|^2.$$

Or thus:

Put $\frac{1}{4}-x$ and x for the numbers; then $\frac{1}{4}-x+x^2=\left(\frac{1}{2}-x\right)^2$, a square, and $\overline{\frac{1}{4}-x}\Big|^2+x=\frac{1}{16}-\frac{x}{2}+x+x^2=\frac{1}{16}+\frac{x}{2}+x^2=\overline{\frac{1}{4}+x}\Big|^2=\square$, where x may be taken at pleasure, provided it be less than $\frac{1}{4}$.

(15) To find two numbers whose sum and difference shall be both square numbers.

Let x and y be the two numbers; then, by the question,

$$x+y=\square=a^2 \text{ and } x-y=\square=b^2;$$

add both squares, and we get

$$2x=a^2+b^2;$$

hence $$x=\frac{a^2+b^2}{2}.$$

Again, by subtraction,

$$2y=a^2-b^2 \text{ and } y=\frac{a^2-b^2}{2},$$

where a and b may be taken at pleasure, provided a be greater than b; if $a=3$ and $b=1$, then $\frac{9+1}{2}=5$ and $\frac{9-1}{2}=4$, whose sum and difference are both squares. Or thus:

Let x and x^2-x be the numbers.

It is evident that their sum is a square; and, in order to satisfy the other condition in the question,

Assume $\overline{x-n}|^2=x^2-2x$, the difference of the numbers;

whence $$x=\frac{n^2}{2n-2};$$

$$\therefore\ x^2-x=\left\{\frac{n^2}{2n-2}\right\}^2-\frac{n^2}{2n-2}.$$

Hence the two numbers are $\frac{n^2}{2n-2}$ and $\left\{\frac{n^2}{2n-2}\right\}^2-\frac{n^2}{2n-2}$, in which n may be taken at pleasure, provided it be greater than 1. If $n=3$, $x=\frac{9}{4}$, and $x^2-x=\frac{45}{16}$.

(16) Find two numbers whose sum is a square, the sum of their squares a square, and either added to the square of the other a square.

Let $\frac{1}{4}-x$ and x be the numbers; then their sum $\frac{1}{4}$ is a square, and $\frac{1}{4}-x$ $+x^2=\square=\overline{\frac{1}{2}-x}|^2$ a square, and $\frac{1}{16}-\frac{x}{2}+x+x^2=\square=\overline{\frac{1}{4}+x}|^2$ a square; and, in order to satisfy the other condition, we assume

$$\frac{1}{16}-\frac{x}{2}+2x^2=\overline{nx-\frac{1}{4}}|^2=n^2x^2-\frac{nx}{2}+\frac{1}{16},$$

which, solved, gives $x=\frac{n-1}{2n^2-4}$, if $n=4$, $x=\frac{3}{28}$, and $\frac{1}{4}-x=\frac{4}{28}$, so that $\frac{4}{28}$ and $\frac{3}{28}$ are numbers that answer the conditions as follows:

$$\overline{\frac{3}{28}}|^2+\overline{\frac{4}{28}}|^2=\frac{25}{28|^2}=\overline{\frac{5}{28}}|^2 \text{ and } \overline{\frac{3}{28}}|^2+\frac{4}{28}=\frac{9}{28|^2}+\frac{112}{28\,|^2}=\frac{121}{28\,|^2}=\overline{\frac{11}{28}}|^2;$$

also,

$$\overline{\frac{4}{28}}|^2+\frac{3}{28}=\frac{16}{28|^2}+\frac{84}{28|^2}=\frac{100}{28\,|^2}=\overline{\frac{10}{28}}|^2.$$

(17) Find two such numbers, that if their product be added to the sum of their squares, the sum shall be a square.

Let $2x$ be their sum and $2y$ be their difference; then the greater will be $x+y$ and the less $x-y$; hence $x^2-y^2=$ their product, and $2x^2+2y^2=$ the sum of their squares; then, by the question, $3x^2+y^2=\square=\overline{nx-y}|^2$ and $x=\frac{2ny}{n^2-3}$; if $n=2$ and $y=2$, $\therefore x=8$, which will answer the conditions.

(18) To find two square numbers, such, that the difference of their cube roots shall be a square number.

Let x^6 and y^6 be the required numbers. Then $x^2-y^2=\square$; consequently, x and y may be any two numbers which are the hypotenuse and one leg of a right-angled triangle, and the least numbers of this description are 5 and 3, and the numbers themselves $15625=125^2$ and $729=27^2$.

(19) Find three numbers, such, that not only the sum of all three of them, but also the sum of every two, shall be a $\square$.

Put $4x$, x^2-4x, and $2x+1$ for the three numbers; then it only remains to render $6x+1=\square$.

Assume its root $n-1$;

then $$6x+1=\overline{n-1}|^2=n^2-2n+1;$$

whence $$x=\frac{n^2-2n}{6},$$

if $n=12$, $x=20$, which will answer the conditions of the problem.

(20) Find two numbers, such, that the sum of their squares and the sum of their cubes shall be both squares.

Let b be the base, p the perpendicular, and h the hypotenuse of a rational right-angled triangle, x any multiplier of b, p, and h; then $(bx)^2+(px)^2=(hx)^2$, but $(bx)^3+(px)^3=$ a rational square $=r^2x^2$; hence $(b^3+p^3).x=r^2$, or $x=\frac{r^2}{b^3+p^3}$; now if $r=b^3+p^3$, $\therefore x=b^3+p^3$, and $\therefore bx=b(b^3+p^3)$, $px=p\times$

(b^3+p^3); now let $b=3$, $p=4$; then is $x=91$, $bx=273$, and $px=364$. If $b=6$ and $p=8$, then $x=728$, $bx=4368$, and $px=5824$, and so on in general.

(21) Find a number to which if 8 be added, the sum shall be a cube, and from which if 1 be subtracted, the remainder shall be a cube.

Let x be the number; $b=2$, $c=1$; then $x+b^3=$ a cube and $x-c^3=$ a cube;

hence $$x+b^3=(b+\frac{c^2}{b^2}a)^3=b^3+3c^2a+\frac{3c^4}{b^3}a^2+\frac{c^6}{b^6}a^3;$$

$$\therefore x=3c^2a+\frac{3c^4}{b^3}a^2+\frac{c^6}{b^6}a^3.$$

Assume $x-c^3=(a-c)^3=$ a cube $=a^3-3a^2c+3ac^2-c^3$, and $\therefore x=a^3-3a^2c+3ac^2$; and, equating both values of x, we get

$$a^3-3a^2c+3ac^2=3ac^2+\frac{3c^4}{b^3}a^2+\frac{c^6}{b^6}a^3,$$

whence $$a=\frac{b^3+c^3}{b^6-c^6}\times 3cb^3=\frac{3cb^3}{b^3-c^3};$$

and, putting the right-hand member of this equation into numbers, we get

$$a=\frac{3\times 8}{8-1}=\frac{24}{7};$$

hence $$x=\frac{5256}{343}.$$

(22) To find three square numbers, such, that the sum of every two of them shall be a square number.

Let x^2, y^2, and z^2 be the numbers sought.

Then x^2+z^2, y^2+z^2, and x^2+y^2 are the three numbers; *i. e.*,

$$\frac{x^2}{z^2}+1,\ \frac{y^2}{z^2}+1, \text{ and } \frac{x^2}{z^2}+\frac{y^2}{z^2}$$

are three square numbers.

Assume $$\frac{x}{z}=\frac{m^2-1}{2m}, \text{ and } \frac{y}{z}=\frac{n^2-1}{2n},$$

we have

$$\frac{x^2}{z^2}+1=\frac{m^4+2m^2+1}{4m^2}, \text{ and } \frac{y^2}{z^2}+1=\frac{n^4+2n^2+1}{4n^2},$$

which are evidently two squares; and therefore it remains to make $\frac{x^2+y^2}{z^2}$ a square number.

Now

$$\frac{x^2+y^2}{z^2}=\left\{\frac{m^2-1}{2m}\right\}^2+\left\{\frac{n^2-1}{2n}\right\}^2=\frac{(m^2-1)^2}{4m^2}+\frac{(n^2-1)^2}{4n^2}=\frac{(m^2-1)^2.n^2+(n^2-1)^2.m^2}{4m^2n^2},$$

a square number.

Hence

$(m^2-1)^2.n^2+(n^2-1)^2.m^2$, or $(m+1)^2.(m-1)^2.n^2+(n+1)^2.(n-1)^2.m^2=$ a square number.

Let $$m+1=n-1 \therefore n=m+2.$$

Hence $$(m+1)^2.(m-1)^2.(m+2)^2+m^2.(m+3)^2\times(m+1)^2,$$

or $$(m-1)^2 \cdot (m+2)^2 + m^2 \cdot (m+3)^2,$$

or $$2m^4+8m^3+6m^2-4m+4,$$

is a square number.

Let the root of this quantity be assumed $=\dfrac{5m^2}{4}-m+2$.

Then $$\left(\frac{5m^2}{4}-m+2\right)^2=2m^4+8m^3+6m^2-4m+4;$$

whence $$m=-24, \text{ and } n=-22.$$

Also, $$\frac{x}{z}=\frac{m^2-1}{2m}=\frac{575}{-48}, \text{ and } \frac{y}{z}=\frac{n^2-1}{2n}=\frac{483}{-44};$$

hence $$x=-\frac{575z}{48}, \text{ and } y=-\frac{483z}{44}.$$

To obtain the answer in whole numbers, let $z=528$;* then $x=-6325$, and $y=-5796$. Hence 528, -5796, -6325 are the roots of the squares, and 528^2, 5796^2, 6325^2 are the squares required.

(23) To find three cube numbers, such, that if from every one of them a given number 1, be subtracted, the sum of the remainders shall be a square.

Let $1+x$, $2-x$, and 2 represent the required roots.

Then, per question, $(1+x)^3-1+(2-x)^3-1+8-1=\square$;

or $$(1+x)^3+(2-x)^3+8-3=\square;$$

$$x^3+3x^2+3x+1+8-12x+6x^2-x^3+8-3=\square;$$

$$9x^2-9x+14=\square, =(a-3x)^2=a^2-6ax+9x^2;$$

$$14-9x=a^2-6ax;$$

and $$6ax-9x=a^2-14 \therefore x=\frac{a^2-14}{6a-9}.$$

Suppose $a=4$; then $x=\dfrac{16-14}{15}=\dfrac{2}{15}$, and $1+x=\dfrac{17}{15}$, and $2-x=\dfrac{28}{15}$;

$$\therefore (1+x)^3=\frac{4913}{3375}, (2-x)^3=\left(\frac{28}{15}\right)^3=\frac{21952}{3375}, \text{ and } 8$$

are the numbers.

(24) It is required to find three integral square numbers, such, that the difference of every two of them shall be a square number.

Let the roots of the required numbers be denoted by

$$s^2+y^2, s^2-y^2, \text{ and } r^2+x^2.$$

Assume $$r^2-x^2=s^2+y^2;$$

then $$r^2-x^2-s^2=y^2=\square$$

and $$y^4=r^4-2r^2x^2-2r^2s^2+x^4+2x^2s^2+s^4;$$

but $$(r^2+x^2)^2-(s^2-y^2)^2=\square$$

$$=(r^2+x^2)^2-(s^2-r^2+x^2+s^2)^2=r^4+2r^2x^2+x^4-s^4+2r^2s^2-2s^2x^2-2s^4-r^4$$
$$+2r^2x^2+2r^2s^2-x^4-2s^2x^2-s^4=\square$$
$$=4r^2x^2+4r^2s^2-4s^2x^2-4s^4=\square$$
$$=4(r^2x^2+r^2s^2-s^2x^2-s^4)=\square,$$
$$\therefore r^2x^2+r^2s^2-s^2x^2-s^4=\square=a^2,$$

and $$(r^2-s^2)\cdot x^2=a^2-r^2s^2+s^4,$$

and $$x^2=\frac{a^2-r^2s^2+s^4}{r^2-s^2}=\frac{a^2}{r^2-s^2}-s^2;$$

take $$r=21 \text{ and } s=13,$$

* The least common multiple of the denominators, 48 and 44.

then $$x^2=\frac{a^2}{441-169}-169$$
$$=\frac{a^2}{272}-169.$$

Take $$a=340,$$
then $x^2=256$ and $y^2=r^2-s^2-x^2=441-256-169=16$,
$$\therefore\ (r^2+x^2)^2=(441+256)^2=(697)^2=\text{ one number},$$
and $(r^2-x^2)^2=(s^2+y^2)^2=(441-256)^2=(185)^2=$ the second number;
and $(s^2-y^2)^2=(169-16)^2=(153)^2$, which is the other number.

(25) To find three square numbers such, that their sum, being severally added to their three roots, shall make square numbers.

Let $2x$, $6x$, and $9x$ denote the three roots; $\therefore$ by the question,
$$121x^2+2x=\square,$$
$$121x^2+6x=\square,$$
$$121x^2+9x=\square.$$

Assume $x=\frac{y}{121}$; then $121x=y$; and $\therefore$ $121x^2=\frac{y^2}{121}$, and $121x^2+2x=\frac{y^2}{121}+\frac{2y}{121}$.

Hence, we get
$$y^2+2y=\square,$$
$$y^2+6y=\square,$$
$$y^2+9y=\square.$$

Assume $y^2+2y=\left(\frac{z^2-1}{2z}\right)^2$; and $\therefore$ $y^2+2y+1=\left(\frac{z^2-1}{2z}\right)^2+1=\frac{z^4-2z^2+1}{4z^2}+1=\frac{z^4-2z^2+4z^2+1}{4z^2}=\frac{z^4+2z^2+1}{4z^2}=\left(\frac{z^2+1}{2z}\right)^2$; and, consequently, $y+1=\frac{z^2+1}{2z}$ $\therefore =y\frac{z^2+1}{2z}-1=\frac{z^2-2z+1}{2z}=\frac{(z-1)^2}{2z}$; hence, by substitution in the second equation above, we have
$$\frac{(z-1)^4}{4z^2}+6\times\frac{(z-1)^2}{2z}=\square=\frac{(z-1)^4}{4z^2}+12z\times\frac{(z-1)^2}{4z^2}=\square.$$

But $4z^2$ is a square number;
$$\therefore\ (z-1)^4+12z\times(z-1)^2=\square$$
$$=(z-1)^2\times(z-1)^2+12z\,.\,(z-1)^2=(z-1)^2\times\{(z-1)^2+12z\}.$$

But $$(z-1)^2 \text{ is } \square,$$
$$\therefore\ (z-1)^2+12z=\square=z^2+10z+1=\square.$$

Again, by substitution in the third, we have
$$\frac{(z-1)^4}{4z^2}+9\times\frac{(z-1)^2}{2z}=\square=\frac{(z-1)^4}{4z^2}+\frac{18z\times(z-1)^2}{4z^2}=\square,$$

$\therefore$ $(z-1)^4+18z\times(z-1)^2=\square$, and $\therefore$ $(z-1)^2\,.\,(z-1)^2+18z\,.\,(z-1)^2=\square$

Hence $$(z-1)^2\times\{(z-1)^2+18z\}=\square,$$
and $$\therefore\ (z-1)^2+18z=\square=z^2+16z+1;$$
hence $$(z^2+16z+1)-(z^2+10z+1)=6z=3z\times 2,$$
the $\frac{1}{2}$ sum of which factors is $\frac{3z+2}{2}=\frac{3z}{2}+1$, the root of the greater $\square$.

$$\therefore\ z^2+16z+1=\left(\frac{3z}{2}+1\right)^2=\frac{9z^2}{4}+3z+1,$$

$$\therefore z^2+16z=\frac{9z^2}{4}+3z, \text{ and } 4z^2+64z=9z^2+12z,$$

and $$\therefore 4z+64=9z+12,\ 5z=52, \text{ and } z=\frac{52}{5};$$

$$\therefore y=\frac{\left(\frac{52}{5}-1\right)^2}{2\times\frac{52}{5}}=\frac{\left(\frac{47}{5}\right)^2}{2\times\frac{52}{5}}=\frac{\frac{2209}{25}}{2.\frac{52}{5}}=\frac{\frac{2209}{25}}{\frac{104}{5}}$$

$$=\frac{2209}{520} \text{ and } x=\frac{y}{121}, =\frac{2209}{62920};$$

$\therefore$ we see that $\frac{4418}{62920}$, $\frac{13254}{62920}$, and $\frac{19881}{62920}$ are the roots.

QUESTIONS FOR EXERCISE.

(1) Required six numbers whose sum and product shall be equal.

Ans. 1, 2, 3, 4, 5, and $\frac{15}{119}$.

(2) Required five square numbers whose sum shall be a square.

Ans. 1, 4, 9, 16, and $\frac{1}{4}$.

(3) Divide the number 3 into four rational squares.

Ans. $\frac{16}{25}$, $\frac{1}{25}$, $\frac{9}{25}$, and $\frac{49}{25}$.

(4) Divide unity into three rational squares.

Ans. $\frac{9}{49}$, $\frac{4}{49}$, and $\frac{36}{49}$.

(5) Find two numbers whose sum is a cube, and difference a square.

Ans. 1512 and 216.

(6) Find two numbers whose product plus their sum or difference is each a square.

Ans. $\frac{5}{12}$ and $4\frac{5}{12}$.

(7) To find two numbers, such, that when each is multiplied into the cube of the other, the products will be squares.

Ans. 2 and 8.

(8) To find two square numbers whose difference is 40.

Ans. 49 and 9.

(9) To find two square numbers, such, that their sum added to their product may be a square number.

Ans. $\frac{1}{9}$ and $\frac{4}{9}$.

(10) It is required to find two whole numbers, such, that their difference, the difference of their squares, and the difference of their cubes shall be squares.

Ans. 10 and 6.

(11) Find two numbers, such, that the sum of their squares shall be both a square and a cube.

Ans. 75 and 100.

(12) Find two numbers whose sum shall be a cube, but their product and quotient squares.

Ans. 25 and 100.

(13) It is required to find three integral square numbers that shall be in arithmetical progression.

Ans. 1, 25, and 49.

(14) To find three square integral numbers in harmonical progression.

Ans. 1225, 49, and 25.

(15) To find three numbers, such, that if to the square of each of them the sum of the other two be added, the three sums shall be all squares.

Ans. 1, $\frac{8}{3}$, and $\frac{16}{3}$.

(16) It is required to find three whole numbers, such, that if to the square of each of them the product of the other two be added, the sums shall be squares.

Ans. 9, 73, and 328.

(17) It is required to find three whole numbers in geometrical progression, such, that the difference of every two of them shall be a square number.

Ans. 567, 1008, and 1792.

(18) It is required to find three integral square numbers, such, that the difference between every two of them and the third shall be a square number.

Ans. 149^2, 241^2, and 269^2.

(19) To find three square numbers, such, that the sum of their squares shall also be a square number.

Ans. 9, 16, and $\frac{144}{25}$.

(20) To find three biquadrate numbers the sum of which shall be a square.

Ans. 12^4, 15^4, and 20^4.

For generalization of Diophantine problems in certain cases, see Bonnycastle's Algebra. See, also, Theory of Numbers.

THEORY OF NUMBERS.

387. We have already had occasion to demonstrate some propositions which fall under this head, and which would have been reserved for this place had they not been required for the elucidation of previous parts of the work.

We recur to one or two of these for the purpose of exhibiting some of the other methods by which they may be established.

I. To prove that $a\times b=b\times a$. Suppose $a>b$ and c their difference;

$$\therefore a\times b=(b+c)b=b^2+cb;$$

i. e., b taken b times and c taken b times, and

$$b\times a=b(b+c)=b^2+bc;$$

i. e., b taken b times and also c times.

We perceive that the product $a\times b$ will be the same as $b\times a$, if the partial product $c\times b$ is equal to $b\times c$. But, by similar reasoning, the equality of cb and bc will be proved by the equality of two smaller products, cd and dc; and continuing thus, we arrive necessarily at the case where the two factors are equal, or at the case where one of them is equal to unity. In the first case, the equality is manifest; in the second, it will follow, from the fact that $h\times 1$ is h as well as $1\times h$. Then the product $a\times b$ is always equal to the product $b\times a$.

II. To demonstrate that $N \times a \times b = N \times ab$, I observe, first, that the product ab is nothing else than $a+a+a+$, &c., the number of these terms being b. Then $N \times ab = Na + Na + Na +$, &c., to b terms, $= Na \times b$. Q. E. D.

III. $Nab = Nba$; for $Na = N+N+N+ \ldots$ to a terms; then, to multiply Na by b, it is necessary to take each of the terms b times,

thus $$Nab = Nb + Nb + Nb \ldots = Nba. \qquad \text{Q. E. D.}$$

Corollary 1.—If all the factors of N be 1, then $1 \times ab = 1 \times ba$, or $ab = ba$, according to I.

Corollary 2.—The above reasoning applies only to entire factors. The principle is equally true, however, when some of the factors are fractions; because, if the entire factors, which are combined with the fractional ones, be written in a fractional form by placing unity under them, all the factors to be multiplied together will be fractions; the product of these, we know, is obtained by taking the product of the numerators and denominators separately, which are entire numbers, and therefore the order is immaterial, from what has been proved above.

Corollary 3.—If the factors be incommensurable, it is to be observed that the product of two incommensurable quantities has no precise meaning.

But by regarding the incommensurables as limits to which approximating commensurables tend, since the above reasoning applies to the latter, and their order is immaterial, we may infer that the order is immaterial also in a product of incommensurable factors.

Corollary 4.—We have seen that, from the above proposition, it follows that the order of factors in a product is immaterial; hence it follows that if a number, P, contains the factors a, b, c, &c., it is divisible by their product.

Corollary 5.—If a number, P, is divisible by another, $Q = abc$, then is P divisible by each of the factors a, b, c.

THE FORMS AND RELATIONS OF INTEGRAL NUMBERS, AND OF THEIR SUMS, DIFFERENCES, AND PRODUCTS.

388. I. The sum or difference of any two even numbers is an even number. For, let $A = 2n$ and $B = 2n'$ be any two even numbers; then

$$A \pm B = 2n \pm 2n' = 2(n \pm n') = 2n'',$$

which, being of the form $2n$, is an even number.

II. The sum or difference of two odd numbers is even, but the sum of three odd numbers is odd.

Let $A = 2n+1$, $B = 2n'+1$, and $C = 2n''+1$, be three odd numbers; then

$$A + B = 2n + 2n' + 2 = 2n'',$$

and $$A + B + C = 2n + 2n' + 2n'' + 3 = 2n''' + 1;$$

the former having the form of an even, and the latter of an odd number.

In a similar way it may be shown,

(1) That the sum of any number of even numbers is even.

(2) That any even number of odd numbers is even, but that any odd number of odd numbers is an odd number.

(3) That the sum of an even and odd number is an odd number.

(4) That the product of any number of factors, one of which is even, will be an even number, but the product of any number of odd numbers is odd; and hence, again,

(5) Every power of an even number is even, and every power of an odd number is an odd number.

(6) Hence the sum and difference of any power and its root is an even number.

For the power and root will be either both even or both odd, and the sum or difference in either case is an even number.

III. If an odd number divide an even number, it will also divide the half of it.

Let $A=2n$, $B=2n'+1$ be any even and odd number, such that B is a divisor of A; let the division be made, and call the quotient p; then we have

$$2n=p(2n'+1);$$

consequently (4), p is even, or of the form $2n''$;

hence $$2n=2n''(2n'+1),$$

and $$\frac{n}{2n'+1}=n'';$$

that is, $n=\frac{1}{2}A$ is divisible by B, if A itself be so.

DEFINITIONS.

389. (1) A perfect number is that which is equal to the sum of all its aliquot parts, or of all its divisors.

Thus, $6=\frac{6}{2}+\frac{6}{3}+\frac{6}{6}$, and is, therefore, a perfect number.

(2) Amicable numbers are those pairs of numbers each of which is equal to all the aliquot parts of the other. Thus, 284 and 220 are a pair of amicable numbers, for it will be found that all the aliquot parts of 284 are equal to 220, and all the aliquot parts of 220 are equal to 284.

(3) Figurate numbers are all those that fall under the general expression

$$\frac{n(n+1)(n+2)(n+3)\ldots.(n+m)}{1.2.3.4\ldots.(m+1)};$$

and they are said to be of the 1°, 2°, 3°, &c., order, according as $m=1$, 2, 3, &c.

(4) Polygonal numbers are the sums of different and independent arithmetical series, and are termed lineal or natural, triangular, quadrangular or square, pentagonal, &c., according to the series from which they are generated.

(5) Natural numbers are formed from a series of units; thus:

Units,	1, 1, 1, 1, 1, &c.
Natural numbers,	1, 2, 3, 4, 5, &c.

(6) Triangular numbers are the successive sums of an arithmetical series, beginning with unity, the common difference of which is 1; thus:

Arithmetical series,	1, 2, 3, 4, 5, &c.
Triangular numbers,	1, 3, 6, 10, 15, &c.

(7) Quadrangular or square numbers are the sums of an arithmetical series, beginning with unity, and the common difference of which is 2; thus:

Arithmetical series,	1, 3, 5, 7, 9, 11, &c.
Quadrangular or square numbers,	1, 4, 9, 16, 25, 36, &c.

(8) Pentagonal numbers are the sums of an arithmetical series, beginning with unity, the common difference of which is 3; thus:

Arithmetical series,	1, 4, 7, 10, 13, 16, &c.
Pentagonal numbers,	1, 5, 12, 22, 35, 51, &c.

And, universally, the m—gonal series of numbers is formed from the successive sums of an arithmetical progression, beginning with unity, the common difference of which is $m-2$.

DIVISIBILITY OF NUMBERS.

390. I. *The product of two numbers,* a *and* b, *is divisible by every number which exactly divides one of the two factors* a *and* b.

For let θ be a number which divides b, so that $b=c\theta$, we have by the foregoing $ab=ac\times\theta$. Then ab, divided by θ, gives the exact quotient ac.

Corollary.—To divide a product of several factors, divide one of the factors and multiply the quotient by the others.

On this subject we must observe that a number may sometimes divide a product when it will not divide any factor. Thus, 20 divides neither 12 nor 15, but does their product, 180. This is because 20 is composed of factors some of which are found in 12 and others in 15. But if the number 20 had no common factor with one of the factors, it must divide the other. (See Art. 84, note.)

II *If there be* n *numbers, each of them divisible by* k, *then is their product divisible by* k^n.

For $a=kq,\ b=kq',\ c=kq''\ldots\quad\therefore\quad abc\ldots=k^n.w,$

w being equal to $q\times q'\times q''\times\ \ldots$

III. *The sum of several numbers,* a+b+c+d, *is divisible by a number,* k, *when the sum of the remainders obtained by dividing each by* k *is divisible by this number.*

For $a=kq+r,\ b=kq'+r',\ c=kq''+r''$, &c.

$$\therefore\ a+b+c+d=k(q+q'+q''+,\ \&c.)+r+r'+r''+,\ \&c.$$

Whence it is evident that $a+b+c$, &c., is divisible by k when $r+r'+r''$, &c., is.

IV. *The difference of two numbers,* a *and* b, *is divisible by a number,* k, *when, if each be divided by* k, *the remainders are equal.*

For $a=kq+r$, and $b=kq'+r$

$$\therefore\ a-b=k(q-q').$$

V. *Every number consisting of units, tens, hundreds, &c., is divisible by a number,* k, *when the sum of the products of the number of units, tens, &c., by the remainder, after dividing the units, tens, &c., each by* k, *is divisible by this number.*

For, representing by A, B, C, &c., the quotients, and by α, β, γ, &c., the remainders of the units, tens, &c., by k, we have

$$\begin{array}{lll}
10^n=Ak+\alpha & \therefore & a.10^n=aAk+a\alpha \\
10^{n-1}=Bk+\beta & & b.10^{n-1}=bBk+b\beta \\
10^{n-2}=Ck+\gamma & & c.10^{n-2}=cCk+c\gamma \\
\vdots & & \vdots \\
10^2=Dk+\delta & & d.10^2=dDk+d\delta \\
10^1=Ek+\varepsilon & & e.10^1=eEk+e\varepsilon \\
10^0=\ldots 1 & & f.10^0=\ldots.f
\end{array}$$

VI. *The product*, P, *of several numbers*, a, b, c, d, . . . *is divisible by a number*, k, *only when the product of the remainders, after dividing each of the factors by* k, *is so divisible.*

For, let $a=kq+\alpha,\ b=kq'+\beta,\ c=kq''+\gamma$, &c.,

$$\therefore\ ab=kz+\alpha.\beta.$$
$$abc=kz+\alpha\beta.\gamma, \text{ \&c.}$$

VII. *The product*, P, *of several factors*, a, b, c, d, . . . *is divisible by a prime number*, k′, *only when one of the factors is divisible by this prime number.*

For, let $a=k'q+\alpha,\ b=k'q'+\beta,\ c=k'q''+\gamma$, &c.,

$$\therefore\ \mathrm{P}=k'z+\alpha.\beta.\gamma\ldots$$

Therefore, if k' divide P, it must divide $\alpha, \beta, \gamma \ldots$

But k' is not found among the factors $\alpha, \beta, \gamma, \ldots$ since, being remainders to the divisor k', they are all less than it. Neither is k' any combination of them, since it is supposed to be a prime number. Hence $\alpha, \beta, \gamma, \ldots$ and therefore P is divisible by k' only when one of the remainders $=0$.

VIII. *If the factors*, a, b, c, . . . *of a product*, P, *are prime to* k, *then is the product not divisible by* k.

For, if k be an absolute prime number, this follows from VII. Again, if k be a multiple of a prime number, as $p'v$; then, if P be divisible by k, we have

$$\frac{\mathrm{P}}{k}=\frac{a.b.c\ldots.}{p'.v}=m\ \therefore\ a.b.c\ldots.=mp'v;$$

therefore $a.b.c\ldots.$ must be divisible by p', which by VII. is impossible.

391. I. Problem.—*To find all the divisors of any number whatever.* The first thought which presents itself is to try successively as divisors each of the numbers 1, 2, 3, &c., to N. But this groping process may be abridged. Let D be a divisor of N, and D′ the quotient, we have $\mathrm{DD'}=\mathrm{N}$, or, under another form, $\mathrm{DD'}=\sqrt{\mathrm{N}}\times\sqrt{\mathrm{N}}$; then, if D is $<\sqrt{\mathrm{N}}$, D′ will be $>\sqrt{\mathrm{N}}$. Then, after having found all the divisors $<\sqrt{\mathrm{N}}$, the quotients which shall have been obtained in dividing N by these divisors will be the divisors $>\sqrt{\mathrm{N}}$.

For example, let $\mathrm{N}=360$. The square root of 360 is comprised between 18 and 19; thus, we divide 360 only by the numbers 1, 2, 3 . . . 18. In this manner we find all the divisors of 360, to wit:

1,	2,	3,	4,	5,	6,	8,	9,	10,	12,	15,	18.
360,	180,	120,	90,	72,	60,	45,	40,	36,	30,	24,	20.

392. II. Problem.—*To form a table of prime numbers.* When the above proceeding produces no divisor, the number is a prime number. To avoid the long calculations necessary in these cases, tables have been constructed which contain the prime numbers up to certain limits.*

The most simple manner of constructing it is to write in succession the series of uneven numbers 3, 5, 7, &c., to such a limit as we seek, and to efface all the multiples of 3, of 5, of 7, &c. It is evident that the prime numbers are all that remain. At the head of these numbers it must not be forgotten to place 1 and 2.

Nothing is easier than to know what multiples to efface. Those of 3 are

* The student is referred to the tables of Burckhardt, in which the prime numbers extend to 3036000.

found by counting the numbers 3, 5, 7, &c., in threes, setting out from 5; those of 5 in counting them in fives, beginning with 7, and so on.*

393. Remark I.—The series of prime numbers is unlimited. For, suppose it to be otherwise, and that n is the greatest: if we form the product $P=2.3.5\ldots n$, which contains all the prime numbers, then $P+1$, which $>n$, must be divisible by some one of these numbers; but this is impossible, because there will always be the remainder 1. Then it is impossible that the series of prime numbers should be limited.

II. In comparing all numbers with multiples of the same number, we are led to present them under different forms, of which use is often made. For example, if we compare them with multiples of 6, they may be represented first, by one of the six formulas,

$$6x,\ 6x+1,\ 6x+2,\ 6x+3,\ 6x+4,\ 6x+5,$$

in which x is any whole number whatever.

But if we wish to consider only prime numbers, it is necessary to preserve only the two formulas,

$$6x+1 \text{ and } 6x+5;$$

because the others give numbers divisible by 2 or by 3.

We can also, in place of $6x+5$, write $6(x+1)-1$ or $6x-1$, since x is any entire number whatever. Thus all the prime numbers except 2 and 3, which are divisors of 6, are comprised in the formula

$$N=6x\pm 1.$$

The reasoning would be analogous for any other number than 6.

394. III. Problem.—*To decompose a number into prime factors, and to find afterward all its divisors.*

A number N, if it be not a prime number, can be represented by the product of several prime numbers a, b, c, &c., raised each to a certain power, so that we can always suppose $N=a^m b^n c^p\ldots$ This is the decomposition which it is required to effect.

Take, for example, the number 504. Divide it first by 2 as many times as possible; we find thus,

$$504=252\times 2=126\times 2\times 2=63\times 2\times 2\times 2.$$

Then divide 63 as many times as possible by 3, which is the smallest prime number greater than 2:

$$63=21\times 3=7\times 3\times 3.$$

Then we have

$$504=7\times 3\times 3\times 2\times 2\times 2,$$

or, rather, under another form,

$$504=2^3\times 3^2\times 7.$$

The divisions by 3 have led to the quotient 7. If the quotient had not been a prime number, we should have continued the operations by trying successively the other prime numbers, 5, 7, &c.

We can now readily form all the divisors of 504. They are, in fact, the numbers which we obtain in taking all the prime factors one by one, two by

* Conceive a board pierced with holes in which the numbers 3, 5, 7, &c., are placed in order. Then, as we arrive, in counting them by threes, fives, &c., at the multiples to be effaced, suppose these multiples to fall through the holes, there will remain only prime numbers. Such was the famous sieve of Eratosthenes, of Alexandria, who lived 280 B.C.

two, &c. That we may be sure not to omit any divisor, we adopt the following arrangement:

		1,					
504	2	2,					
252	2	4,					
126	2	8,					
63	3	3,	6,	12,	24,		
21	3	9,	18,	36,	72,		
7	7	7,	14,	28,	56,	21,	42,
		84,	168,	63,	126,	252,	504.

The first column on the left contains the given number and the quotient of the successive divisions. By the side of these numbers, in a second column, are written the prime numbers, which we employ as divisors, and which are the prime factors of the number 504. Finally, we place at the right of this column all the divisors of 504; and I now proceed to state how we obtain them.

At the top of the third column, but on the line above that which contains 504, we write unity, which may be regarded as the first divisor of 504. We multiply this unity by the first number of the second column, and thus obtain the divisor 2, which we write by the side of this first prime number. We next multiply 1 and 2, the divisors already found, by the second number of the second column, and, neglecting the product 1×2, or 2, which has already been found, we obtain the new divisor 4, which is written on a line with the last multiplier. We proceed in the same manner, multiplying the number of the second column on the horizontal line which we are forming by each of the numbers above it in the third column successively, until we multiply, finally, by the last number of the second column, which gives a last series of divisors, which series will always be terminated by the given number.

When we know the prime factors of a number, we can find its divisors by another process. Suppose that a number N, when decomposed into prime factors, gives

$$N=a^m b^n c^p \ldots;$$

the divisors of N will be represented by the formula $a^{m'}b^{n'}c^{p'}\ldots$, in which the exponents m', n', $p'\ldots$ can not surpass m, n, $p\ldots$

Hence we know that these divisors will be the different terms which we obtain in effecting the product

$$P=(1+a+a^2+\ldots a^m)(1+b+b^2+\ldots b^n)(1+c+c^2+\ldots c^p)\ldots\ldots$$

395. Remarks.—The multiplication of the first two polynomes gives a number of terms equal to $(m+1)(n+1)$; consequently, that of the first three polynomes gives a number equal to $(m+1)(n+1)(p+1)$, and so on; hence, the number of all the divisors of N is expressed by the formula

$$(m+1)(n+1)(p+1)\ldots\ldots$$

We also see that P is the sum of all these divisors. But we know that the polynomes which compose P are respectively equal (Art. 23) to $\frac{a^{m+1}-1}{a-1}$, $\frac{b^{n+1}-1}{b-1}$, &c.; hence, the sum of all the divisors of N can be expressed by the formula

$$P=\frac{a^{m+1}-1}{a-1}\times\frac{b^{n+1}-1}{b-1}\times\frac{c^{p+1}-1}{c-1}\times\ldots\ldots$$

For example, taking $N=504=2^3\times3^2\times7$, we shall have $m=3$, $n=2$, $p=1$. Hence the number of divisors of 504 will be $4\times3\times2=24$, and the sum of all the divisors will be

$$\frac{2^4-1}{2-1}\times\frac{3^3-1}{3-1}\times\frac{7^2-1}{7-1}=15\times13\times8=1560.$$

396. IV. Problem.—*How many times is a prime number, θ, factor in a series of natural numbers, from* 1 *to* n? or, in other words, *what is the highest power of θ which divides the product* $1.2.3\ldots n$?

Let n' be the entire part of the quotient of n by θ. In the proposed series of natural numbers we find the n' factors, θ, 2θ, $3\theta\ldots$, of the product $\theta.2\theta.3\theta\ldots n'\theta$; and it is clear that they are the only numbers of the series which are divisible by θ. This product can be written thus:

$$1.2.3\ldots n'\times\theta^{n'}.$$

Hence we shall obtain the required power of θ by multiplying $\theta^{n'}$ by the highest power of θ, contained in the product $1.2.3\ldots n'$.

The same reasoning may be repeated with reference to this product; hence, calling n'' the entire part of the quotient of n' by θ, we readily perceive that the highest power of θ contained in the last of the above products is composed of the power $\theta^{n''}$ multiplied by the highest power of θ which is contained in the series $1.2.3\ldots n''$.

In like manner, calling n''' the entire part of the quotient of n'' by θ, we are led to seek the highest power of θ contained in the product $1.2.3\ldots n'''$.

We continue this process till we arrive at a quotient $<\theta$. For the sake of definiteness, suppose that n''' is this quotient; then we conclude that the highest power of θ contained in the given product $1.2.3\ldots n$ is $\theta^{n'+n''+n'''}$.

For example, suppose we wish to know what is the highest power of 7 which divides the product $1.2.3\ldots1000$.

We make $n=1000$, and taking only the entire parts of the quotients, we shall have

$$\frac{1000}{7}=142,\ \frac{142}{7}=20,\ \frac{20}{7}=2.$$

The sum of these quotients being 164, it follows that the required power is 7^{164}.

397. *Corollary.*—Let m, n, p, q be entire numbers, such that we have $m=n+p+q+\ldots$; the expression

$$\frac{1.2.3.4.m}{1.2\ldots n\times1.2\ldots p\times1.2\ldots q\times,\ \&c.}\quad\ldots\ldots\ (1)$$

will always represent an entire number. To prove this, let θ be a prime factor of the denominator; we shall have

$$\frac{m}{\theta}=\frac{n}{\theta}+\frac{p}{\theta}+\frac{q}{\theta}+,\ \&c.$$

Calling these entire quotients m', n', p', $q'\ldots$, we shall have also

$$m'=\text{ or }>n'+p'+q'+,\ \&c.$$

If we divide again by θ, and call the new entire quotients m'', $n''\ldots$, we shall, in like manner, have

$$m''=\text{ or }>n''+p''+q''+,\ \&c.$$

We continue this process as long as the quotients are not all less than θ.

Then adding, we shall have

$(m'+m''+\ldots)=$ or $>(n'+n''+\ldots)+(p'+p''+\ldots)+(q'+q''+\ldots)+$, &c.
But these different sums make known the highest powers of θ, by which we can divide the products which compose expression (1); hence there is no prime factor in the denominator which does not exist of a power at least equal in the numerator of the fraction. This expression, therefore, represents an entire number.

398. Perfect numbers are expressed or determined as follows:

Find 2^n-1, a prime number, then will $N=2^{n-1}(2^n-1)$ be a perfect number. For, from what has been demonstrated in the preceding section, the sum of all the divisors of this formula will be represented by $\frac{2^n-1}{2-1}\times\frac{(2^n-1)^2-1}{(2^n-1)-1}$; because 2^n-1 is a prime by hypothesis. But in this expression 1 is included as a divisor, which must be excluded in the case of perfect numbers; exclusive of this, therefore, the formula will be

$$\frac{2^n-1}{2-1}\times\frac{(2^n-1)^2-1}{(2^n-1)-1}-2^{n-2}-1(2^n-1)=$$
$$(2^n-1)\times(2^n-1+1)-2^{n-1}(2^n-1)=$$
$$2(2^n-1)2^{n+1}-2^{n-1}(2^n-1)=2^{n-1}(2^n-1)=N,$$

that is, the sum of all the aliquot parts of N, exclusive of itself, or of 1 as a divisor, is equal to N, and is, therefore, by the definition a perfect number.

The only perfect numbers known are the following eight:

6, 33550336,
28, 8589869056,
496, 137438691328,
8128, 2305843008139952128.

399. To find a pair of amicable numbers N and M, or such a pair that each shall be respectively equal to all the divisors of the other.

Make $N=a^m b^n c^p$, &c., and $M=a^\mu\beta^\nu\gamma^\pi$; then, according to the definition and from what has been demonstrated in the last section, we must have

$$\frac{a^{m+1}-1}{a-1}\times\frac{b^{n+1}-1}{b-1}\times\frac{c^{p+1}-1}{c-1}=N+M,$$
$$\frac{a^{\mu+1}-1}{a-1}\times\frac{\beta^{\nu+1}-1}{\beta-1}\times\frac{\gamma^{\pi+1}-1}{\gamma-1}=M+N.$$

Find, therefore, such a power of 2, as 2^r, that

$$3\,.\,2^r-1,\ 6\,.\,2^r-1,\ \text{and}\ 18\,.\,2^r-1$$

may be all prime numbers; then will

$$N=2^{r+1}d \text{ and } M=2^{r+1}bc$$

be the pair of amicable numbers sought.

The least three pair of amicable numbers are

284, 220,
17296, 18416,
9363583, 9437056.

400. We shall here introduce the student to the nomenclature and notation of Gauss, given in his Disquisitiones Arithmeticæ, which is now generally adopted by writers upon the theory of numbers.

CONGRUOUS NUMBERS IN GENERAL.

401. If a number a divide the difference of the numbers b and c, b and c are said to be *congruous* with reference to a; if not, *incongruous*. The quantity a is called the *modulus*; each of the numbers b and c a *residue* of the other in the first case, a *non-residue* in the second.

The numbers may be either positive or negative, but entire. As to the modulus, it ought evidently to be taken without regard to the sign.

Thus, -9 and $+16$ are congruous with reference to the modulus 5; -7 is a residue of 15 with reference to the modulus 11, and not a residue with reference to the modulus 3.

Zero being divisible by all numbers, every number may be regarded as congruous with itself with reference to any modulus whatever.

All the residues of a given number, a, with reference to a given number, m, are comprised in the formula $a+km$, k being an entire indeterminate number. This is self-evident.

The congruence of two numbers is expressed by the sign $\equiv$, joining to it the modulus, when necessary, in a parenthesis, thus:*

$$-16\equiv 9(\text{mod. } 5),\quad -7\equiv 15(\text{mod. } 11).$$

402. Theorem.—*Let there be* m *entire successive numbers*, a, a+1, a+2, ... a+m−1, *and another*, A; *one of the former will be congruous with* A, *with reference to the modulus* m, *and but one.*

For if $\frac{a-A}{m}$ is entire, $a\equiv A$; if it is fractional, let k be the nearest entire number; above, if $\frac{a-A}{m}$ be positive; below, if it be negative; $A+km$ will fall between a and $a+m$,† and will be the number sought; but it is evident that the quotients $\frac{a-A}{m}$, $\frac{a+1-A}{m}$, &c., are comprised between $k-1$ and $k+1$,‡ therefore one of them only can be entire.

403. It follows from this that every number will have a residue as well in the series $0, 1, 2 \ldots m-1$, as in the series $0, -1, -2 \ldots -(m-1)$. They are called *minima* residues; and it is evident that, unless zero is the residue, there will be two, the one positive and the other negative. If they are unequal, the one will be $<\frac{m}{2}$; if they are equal, each of them $=\frac{m}{2}$, without regard to the sign; from which it follows, that any number whatever has a residue which does not surpass the half of the modulus; this is called the *absolute minimum* residue.

For example: -13 relative to the modulus 5, has for a positive *minimum* residue 2, which is at the same time its *absolute minimum*, and -3 for its negative *minimum* residue; $+5$, with reference to the modulus 7, is itself its

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* The analogy between equality and congruence led Legendre to employ the sign of equality itself. This modification of it has been introduced by Gauss to avoid ambiguity.

† This may be seen from the equality $\frac{a-A}{m}=k-n$, where $n<m$.

‡ This may be seen by observing that $\frac{a+1-A}{m}=\frac{a-A}{m}+\frac{1}{m}$, and it is not till the numerator of $\frac{1}{m}$ increases to m that the quotient k increases to $k+1$.

positive *minimum* residue; -2 is the negative *minimum* residue, and, at the same time, the *absolute minimum*.

404. The following consequences follow from the above:

Numbers which are congruous with reference to a composite modulus are so with reference to any of its divisors.

If several numbers are congruous with the same number with reference to the same modulus, they will be congruous with each other with reference to this modulus.

The same modulus must be supposed in what follows:

Congruous numbers have the same *minima* residues; *incongruous have different.*

405. *If the numbers* A, B, C, &c.; a, b, c, &c., *are congruous each to each, i. e.*, $A \equiv a$, $B \equiv b$, &c, we shall have

$$A+B+C \ldots \equiv a+b+c \ldots$$

If $A \equiv a$, $B \equiv b$, *we have also* $A-B \equiv a-b$.

406. *If* $A \equiv a$, *we have also* $kA \equiv ka$.

If k is positive, this is but a particular case of the preceding article, in which $A \equiv B \equiv C \ldots$ and $a \equiv b \equiv c \ldots$

If k is negative, $-k$ will be positive; then $-kA \equiv ka \therefore kA \equiv ka$.

If $A \equiv a$, $B \equiv b$, then $AB \equiv ab$; because $AB \equiv AB \equiv Ab \equiv ba$.

407. If the numbers A, B, C $\ldots \equiv a, b, c \ldots$, each to each, then

$$ABC \ldots \equiv abc \ldots$$

for, by the preceding article, $AB \equiv ab$; for the same reason, $ABC \equiv abc$, and so on.

By taking all the terms, A, B, C . . . equal, and a, b, c . . . also equal, if $A \equiv a$, $A^k \equiv a^k$.

408. Let X *be a function of the indeterminate* x *of the form*

$$Ax^a + Bx^b + Cx^c +, \&c.,$$

A, B, C . . . *being any entire numbers whatever. If we give to* x *congruous values with reference to a certain modulus, the resulting values for* X *will be congruous also.*

Let f and g be congruous values of x; by the preceding articles, $f^a \equiv g^a$, and $Af^a \equiv Ag^a$; in the same way we have $Bf^b \equiv Bg^b$, &c.

This theorem may be easily extended to functions of several indeterminates.

409. If, then, we substitute in place of x all entire consecutive numbers, and seek the *minima* residues of the values of X, they will form a series in which, after an interval of m terms (m being the modulus), the same terms will be again presented; that is to say, this series will be formed of a period of m terms repeated indefinitely.

Let there be, for example, $X = x^3 - 8x + 6$, and $m = 5$; for $x = 0, 1, 2, 3$, &c.; the values of X give for positive minima residues 1, 4, 3, 4, 3, 1, 4, &c., or the five, 1, 4, 3, 4, 3, are repeated indefinitely; and if we continue the series in the contrary direction, that is, if we give to x negative values, the same period will reappear in an inverse order; whence it follows that the series contains no other terms than those which compose the period.

410. Then, in this example, X can not become $\equiv 0$, nor $\equiv 2$(mod. 5); and still less $=0$ or $=2$; from which it follows that the equations $x^3 - 8x + 6 = 0$ and $x^3 - 8x + 4 = 0$ have not entire roots, and, consequently, not rational roots. We see, in general, that when X is of the form

$$X^n + Ax^{n-1} + Bx^{n-2} +, \&c., +N,$$

A, B, C . . . being entire quantities, and n entire and positive, the equation $X=0$ (a form to which every algebraic equation may be reduced) will have no rational root, if it happen that, for a certain modulus, the congruence $X \equiv 0$ be not satisfied.

411. Many arithmetic theorems may be demonstrated by the aid of the foregoing principles, as, for instance, the rule for determining whether a number is divisible by 9, 11, or any other number.

With reference to the modulus 9, all the powers of 10 are congruous with unity; then, if the number is of the form $a+10b+100c+1000d+$, &c., it will have, with reference to the modulus 9, the same *minimum* residue as $a+b+c+$, &c. It is clear from this, that if we add the figures of the number without regarding their place value, the sum obtained and the proposed number will have the same *minimum* residue. If, then, this last is divisible by 9, the sum of the figures will be also, and only in this case. It is the same with the divisor 3.

Many of the properties of prime numbers, the divisibility of products already given, &c., may be demonstrated by the aid of this system, but we shall not repeat them.

412. The term *congruence* is analogous to *equation*, and the determination of such values, for an indeterminate x, as to produce congruence in expression, is called *resolving* them. There are congruences *resolvable* and irresolvable.

Congruences are also divided, like equations, into *algebraic* and *transcendental.* Those which are *algebraic* are divided, again, into congruences of the first, second, and higher degrees. There are congruences, also, containing different unknown quantities, of the elimination of which Gauss treats.

413. The congruence $ax+b \equiv c$ may be solved when its modulus m is prime with a; thus, let e be the positive minimum residue of $c-b$. We find necessarily a value of $x<m$, such that the minimum residue of the product ax, with reference to the modulus m, shall be e. Call v this value, and we shall have

$$av \equiv e \equiv c-b;$$

then $$av+b \equiv c (\text{mod. } m).$$

Here v is called the root of the congruence. It is evident that all the numbers congruous with v, with reference to the modulus of the congruence, will also be roots (Art. 408). It is also evident that all the roots should be congruous with v; in fact, if t be another root, we have $av+b \equiv at+b$; then $at \equiv av$; and therefore $v \equiv t$. We may therefore conclude that the congruence $x \equiv v (\text{mod. } m)$ gives the complete resolution of the congruence $ax+b \equiv c$.

The foregoing exposition will serve to show how the algorithm of Gauss connects itself with the indeterminate analysis, and we shall here quit the subject.

414. No algebraical formula can contain prime numbers only.

Let $$p+qx+rx^2+sx^3, \&c.,$$

represent any general algebraical formula. It is to be demonstrated that such values may be given to x, that the formula in question shall not with that value produce a prime number, whatever values are given to p, q, r, &c.

For suppose, in the first place, that by making $x=m$, the formula

$$P = p + qm + rm^2 + sm^3 +, \&c.,$$

is a prime number.

And if now we assume $x = m + \phi P$, we have

$$\begin{array}{ll} p = \ldots & p \\ qx = \ldots & qm + q\phi P \\ rx^2 = \ldots & rm^2 + 2rm\phi P + r\phi^2 P^2 \\ sx^3 = \ldots & sm^3 + 3sm^2\phi P + 3sm\phi^2 P^2 + s\phi^3 P^3 \\ \&c. & \&c. \end{array}$$

Or

$$\begin{aligned} p + qx + rx^2 + sx^3 &= (p + qm + rm^2 + sm^3 +, \&c.,) + \\ &P(q\phi + 2rm\phi + 3sm^2\phi) + P^2(r\phi^2 + 3sm\phi^2) + s\phi^3 P^3 \\ &= P + P(q\phi + 2rm\phi + 3sm^2\phi) + \\ &P^2(r\phi^2 + 3sm\phi^2) + s\phi^3 P^3. \end{aligned}$$

But this last quantity is divisible by P; and, consequently, the equal quantity

$$p + qx + rx^2 + sx^3, \&c.,$$

is also divisible by P, and can not, therefore, be a prime number.

Hence, then, it appears, that in any algebraical formula such a value may be given to the indeterminate quantity as will render it divisible by some other number; and, therefore, no algebraical formula can be found that contains prime numbers only.

But, although no algebraical formula can be found that contains prime numbers only, there are several remarkable ones that contain a great many; thus, $x^2 + x + 41$, by making successively $x = 0, 1, 2, 3, 4$, &c., will give a series 41, 43, 47, 53, 61, 71, &c., the first forty terms of which are prime numbers. The above formula is mentioned by Euler in the Memoirs of Berlin (1772, p. 36).

To the above we may add the following: $x^2 + x + 17$ and $2x^2 + 29$; the former has 17 of its first terms prime, and the latter 29.

Fermat asserted that the formula $2^m + 1$ was always a prime, while m was taken any term in the series 1, 2, 4, 8, 16, &c.; but Euler found that $2^{32} + 1 = 641 \times 6700417$ was not a prime.

415. If a and b be any two numbers prime to each other, and each of the terms of the series

$$b,\ 2b,\ 3b,\ 4b,\ \&c.,\ (a-1)b$$

be divided by a, they will each leave a different remainder. For if any two of these terms, when divided by a, leave the same remainder, let them be represented by xb, yb; then it is obvious that $xb - yb$ would be divisible by a, or $(x-y)b$ would be divisible by a. But this is impossible, because a is prime to b, and $x-y$ is less than a; therefore $b(x-y)$ is not divisible by a, but it would be so divisible if the terms xb, yb left the same remainder; these do not, therefore, leave the same remainder; consequently, every term of the series

$$b,\ 2b,\ 3b,\ \&c.,\ (a-1)b,$$

divided by a, will leave a different remainder.

DEDUCTIONS.

416. Since the remainders arising from the division of each term in the series

$$b,\ 2b,\ 3b,\ \&c.,\ (a-1)b$$

by a are different from each other, and $a-1$ in number, and each of them necessarily less than a, it follows that these remainders include all numbers from 1 to $a-1$.

417. Hence, again, it appears that some one of the above terms will leave a remainder 1; and that, therefore, if b and a be any two numbers prime to each other, a number $x<a$ may be found that will render $bx-1$ divisible by a, or the equation $bx-ay=1$ is always possible if a and b are numbers prime to each other.

And it is always impossible if a and b have any common measure, as is evident, because one side of the equation $bx-ay=1$ would be divisible by this common measure, but the other side, 1, would not be so; therefore, in this case the equation is impossible.

418. If a be any prime number, then will the formula

$$1.2.3.4.5, \&c., (a-1)+1$$

be divisible by a; for it is demonstrated in our preceding second deduction, that if a and b be any two numbers prime to each other, another number x may be found $<a$, that renders the product $bx-1 \div\!\!\div^* a$, or, which is the same thing, $bx=ya+1$; and that there is only one such value of $x<a$, may be shown as follows:

The foregoing equation gives, by transposition,

$$bx-ay=1;$$

and, if it be possible, let also

$$bx'-ay'=1;$$

and make $x'=x\pm m$ and $y'=y\pm n$, where m is necessarily less than a, because both x and x' are so by the supposition.

Now, by this substitution, we have

$$(bx\pm bm)-(ay\pm an)=1;$$

but

$$bx-ay=1;$$

therefore $\pm bm=\mp an$, or $bm \div\!\!\div a$; but this is impossible, since b is prime to a and $m<a$, as in Art. 415. There can not, therefore, be two values of x less than a, that render the equation $bx-ay=1$ possible.

But, in the series of integers

$$1.2.3.4.5 \ldots\ldots a-1,$$

every term is prime to a except the first, a being itself a prime; if, therefore, we write successively $b=2$, $b'=3$, $b''=4$, &c., a corresponding term x, in the same series, may be found for each distinct value of b, that renders the product $xb \mathrel{\text{工}} ay+1$, $x'b' \mathrel{\text{工}} ay'+1$, $x''b'' \mathrel{\text{工}} ay''+1$, &c.; and it is evident that no one of these values of x can be equal either to 1 or $a-1$; for, in the first case, we should have $1\times b=ay+1$, which is impossible, because $b<a$; and the second would give $(a-1)b=ay+1$, or $a(b-y)=b+1$; that is, $b+1 \div\!\!\div a$, which can only be when $b=a-1$, or when $b=x$, which case is excepted, because we suppose two different terms of the series. In fact, since $(a-1)^2 \mathrel{\text{工}} ay+1$, there can be no other term in the same series that is of this form; for if $x^2 \mathrel{\text{工}} ay'+1$, then $(a-1)^2-x^2$ would be divisible by a, or $(a-1+x)\times(a-1-x) \div\!\!\div a$, which is impossible, since each of these factors is prime to

* To save the repetition of the words "divisible by," which frequently occur, the sign $\div\!\!\div$ is used to express them; and, for the same reason, the symbol 工 is introduced, to express the words "of the form of," which are also of frequent occurrence.

a, as is evident, because $x<a$, and a is a prime number. Hence our product

$$1.2.3.4.5\ldots.(a-1)$$

becomes

$$1.bx.b'x'.b''x''\ldots.a-1;$$

but each of these products, bx, $b'x'$, $b''x''$, &c., is, as we have seen, of the form $ay+1$; therefore their continued product will have the same form, and the whole product, including 1 and $a-1$, will be

$$=(ay+1)\times(a-1)=a^2y+ay+a-1,$$

to which if unity be added, the result will be evidently divisible by a; that is, the formula

$$1.2.3.4.5\ldots\ldots(a-1)+ .$$

is always divisible by a when a is a prime number.

DEDUCTIONS.

(1) The product

$$1.2.3.4.5\ldots\ldots(a-1)$$

is the same as

$$1(a-1)2(a-2)3(a-3),\ \&c.,\ \left(\frac{a-1}{2}\right)^2;$$

and this product, as regards remainder, when divided by a, is the same as

$$\pm 1^2.2^2.3^2.4^2\ldots\ldots\left(\frac{a-1}{2}\right)^2;$$

the ambiguous sign being $+$ when $a-1$ is even, and $-$ when $a-1$ is odd; *i. e.*, $+$ when a is a prime of the form $4n+1$, and $-$ when a is a prime of the form $4n-1$; also, this last product is the same as

$$\pm\left(1.2.3.4\ldots\ldots\frac{a-1}{2}\right)^2;$$

therefore, from what is said above relating to the ambiguous sign, we shall have

$$\left\{\left(1.2.3.4\ldots\ldots\frac{a-1}{2}\right)^2+1\right\}\div a$$

when $a=4n+1$; and

$$\left\{\left(1.2.3.4\ldots\ldots\frac{a-1}{2}\right)^2-1\right\}\div a$$

when $a=4n-1$.

Hence every prime of the form $4n+1$ is a divisor of the sum of two squares.

Again, the latter form may be resolved into the two factors

$$\left\{\left(1.2.3.4\ldots\ldots\frac{a-1}{2}\right)+1\right\}\times$$

$$\left\{\left(1.2.3.4\ldots\ldots\frac{a-1}{2}\right)-1\right\},$$

which product being divisible by a, it follows that a is a divisor of one or other of these factors when it is a prime number of the form $4n-1$.

(2) From the first product, which we have shown to be divisible by a, viz.,

$$\frac{1.2.3.4,\ \&c.,\ (a-1)+1}{a}=e, \text{ an integer,}$$

we may derive a great many others, as

$$\frac{1^2.2^2.3.4,\ \&c.,\ (a-3)(a-1)+1}{a}=e, \text{ an integer,}$$

$$\frac{1^2.2^2.3^2.4.5,\ \&c.,\ (a-4)(a-1)+1}{a}=e, \text{ an integer,}$$

and so on till we arrive at the same form as that in the first deduction.

PRIMITIVE ROOTS.

419. Theorem.—*If* p, *a number prime to* a, *divide the successive powers* 1, a, a^2, a^3 . . . *there will be one at least, before arriving at* a^p, *which will leave the remainder* 1.

The remainders being each less than p, there can be but $p-1$ different ones, and, therefore, in the p first terms of the series 1, a, a^2, a^3 . . . a^{p-1}, there are at least two which will give the same remainder. Representing them by a^m, $a^{m'}$, and their common remainder by r, suppose

$$a^m = Ep + r,\ a^{m'} = E'p + r \ . \ . \ . \ . \ . \ (1)$$
$$\therefore\ a^{m'} - a^m = (E' - E)p, \text{ or } a^m(a^{m'-m} - 1) = (E' - E)p;$$

and, as p is prime to a, it must divide $a^{m'-m}-1$. Therefore we have unity for remainder in dividing by p the power $a^{m'-m}$, which is $< a^p$. Q. E. D.

420. Let a^n designate the lowest power other than a^0, which gives the remainder 1. All the preceding remainders are unequal. For, if for two powers, a^m, $a^{m'}$ less than a^n, we could have the equalities (1), we might conclude, as just now, that $a^{m'-m}$ would give the remainder 1. Consequently, a^n would not be the lowest power to which this property belonged.

THEOREM OF FERMAT.

421. *If* p *be a prime number which will not divide* a, *the division of* a^{p-1} *by* p *will give* 1 *for a remainder. In other words,* $a^{p-1}-1$ *is exactly divisible by* p.

It must be carefully observed that p is an absolute prime number, and not simply prime to a.

Call q, q', q'', . . ., and r, r', r'', . . . the quotients and remainders of the $p-1$ quantities a, $2a$, $3a$ . . . $(p-1)a$, divided by p. If we multiply these quantities, and suppose E to be an entire number, we have

$$a \cdot 2a \cdot 3a \ . \ . \ . \ . \ (p-1)a = (qp+r)(q'p+r')(q''p+r'') \ . \ . \ .$$
$$= E + rr'r'' \ . \ . \ .$$

The first member is equal to

$$1 \cdot 2 \cdot 3 \ . \ . \ . \ . \ (p-1)a^{p-1}$$

and, as the remainders r, r', r'' . . . are all different (Art. 415), the product $rr'r''$ . . . must evidently be that of the whole series of natural numbers, 1, 2, 3 . . . $(p-1)$, from 1 to $(p-1)$. Hence the above equality becomes

$$1 \cdot 2 \cdot 3 \ . \ . \ . \ . \ (p-1) \times a^{p-1} = Ep + 1 \cdot 2 \cdot 3 \ . \ . \ . \ (p-1)$$
$$\therefore\ 1 \cdot 2 \cdot 3 \ . \ . \ . \ (p-1)(a^{p-1}-1) = Ep.$$

The 1° member of this equality is, therefore, divisible by p; but since p is a prime number, it can not divide any of the factors 1 . 2 . 3 . . . $(p-1)$; it must, therefore, divide $a^{p-1}-1$. Q. E. D.

Suppose that we take for p only prime numbers; if we wish that the powers a^0, a^1 . . . a^{p-1} should give for remainders all the numbers inferior to p, it is necessary to choose a, such that a^{p-1} should be the lowest power above a^0, which gives the remainder 1; and if, among those which fulfill this condition, we take for a only numbers below p, we have those which Euler calls *primitive roots.*

For the best method of calculating them, the student is referred to the article by Mr. Ivory, in the fourth volume of Supplement to Encyclopedia Britannica. We shall limit ourselves to setting down here the primitive roots of numbers as far as 37.

Numbers p.	*Primitive roots of* p.
3	2
5	2 . 3
7	3 . 5
11	2 . 6 . 7 . 8
13	2 . 6 . 7 . 11
17	3 . 5 . 6 . 7 . 10 . 11 . 12 . 14
19	2 . 3 . 10 . 13 . 14 . 15
23	5 . 7 . 10 . 11 . 13 . 14 . 15 . 17 . 20 . 21
29	2 . 3 . 8 . 10 . 11 . 14 . 15 . 18 . 19 . 21 . 26 . 27
31	3 . 11 . 12 . 13 . 17 . 21 . 22 . 24
37	2 . 5 . 13 . 15 . 17 . 18 . 19 . 20 . 22 . 24 . 32 . 35

THE FORMS OF SQUARE NUMBERS.

422. Every square number is of one of the forms $4n$ or $4n+1$.

Every number is either even or odd; that is, every number is of one of the forms $2n$ or $2n+1$; and, consequently, every square is of one of the forms

$$4n^2 \equiv 4n,$$
$$4n^2+4n+1 \equiv 4n+1.$$

DEDUCTIONS.

(1) Every even square number is divisible by 4.

(2) Since every odd square by the above is of the form $4(n^2+n)+1$, and since n^2+n is necessarily even, it follows that every odd square is of the form $8n+1$; and, consequently, no number of the forms $8n+3$, $8n+5$, $8n+7$ can be a square number.

(3) The sum of two odd squares can not be a square; for

$$(8n+1)+(8n+1) \equiv 4n+2,$$

which is an impossible form.

423. Every square number is of one of the forms $5n$ or $5n\pm1$. For all numbers, compared by the modulus 5, are of one of the forms

$$5n,\ 5n\pm1,\ 5n\pm2;$$

and all squares, therefore, are of one of the forms

$$25n^2 \equiv 5n,$$
$$25n^2\pm10n+1 \equiv 5n+1,$$
$$25n^2+20n+4 \equiv 5n+4 \text{ or } 5n-1.$$

Therefore all squares are of one of the forms $5n$ or $5n\pm1$.

DEDUCTIONS.

(1) If a square number be divisible by 5, it is also divisible by 25; and if a number be divisible by 5 and not by 25, it is not a square.

(2) No number of the form $5n+2$ or $5n+3$ is a square number.

(3) If the sum of two squares be a square, one of the three is divisible by 5, and, consequently, also by 25; for all the possible combinations of the three forms $5n$, $5n+1$, and $5n-1$ are as follows:

$$(5n+1)+(5n'+1) \equiv 5n+2,$$
$$(5n-1)+(5n'-1) \equiv 5n-2 \equiv 5n+3,$$
$$5n + 5n' \equiv 5n,$$
$$5n + (5n'+1) \equiv 5n+1,$$
$$5n + (5n'-1) \equiv 5n-1,$$
$$(5n+1)+(5n'-1) \equiv 5n.$$

Now, of these six forms, the latter four have one of the squares divisible by 5, and, therefore, also by 25. And the first two are each impossible forms for square numbers; that is, neither of these two combinations can produce squares; therefore, if the sum of two squares be a square, one of the three squares is divisible by 25.

(4) In a similar way, it may be shown that all square numbers compared by modulus 10 are of one of the forms

$$10n,\ 10n+5,\ 10n+1,\ 10n+6,\ 10n+4, \text{ or } 10n+9.$$

Therefore, all square numbers terminate with one of the digits 0, 1, 4, 5, 6, or 9; and hence, again, no number terminating with 2, 3, 7, or 8 can be a square number.

(5) By examining, in like manner, the forms of squares to modulus 100, we may deduce the following properties:

(6) A square number can not terminate with an odd number of ciphers.

(7) If a square number terminate with a 4, the last figure but one must be even.

(8) If a square number terminate with a 5, it must terminate with 25.

(9) If the last digit of a square be odd, the last digit but one must be even; and if it terminate with any even digit except 4, the last but one must be odd.

(10) A square number can not terminate with more than three equal digits, unless they are 0's; nor can it terminate with three, unless they are 4's.

424. All square numbers are of the same form with regard to any modulus, a, as the squares

$$0^2,\ 1^2,\ 2^2,\ 3^2, \text{ \&c. } (\tfrac{1}{2}a)^2, \ a \text{ being even;}$$

and as

$$0^2,\ 1^2,\ 2^2,\ 3^2, \text{ \&c. } \left(\frac{a-1}{2}\right)^2, \ a \text{ being odd.}$$

For every number may be represented by the formula $an \pm r$, in which r shall never exceed $\frac{1}{2}a$.

Now $$(an \pm r)^2 = a^2n^2 \pm 2arn + r^2,$$

where it is obvious that r^2 and $(an \pm r)^2$ will leave the same remainder when divided by a; therefore, $(an \pm r)^2$ and r^2 will be of the same form compared by modulus a; but r never exceeds $\frac{1}{2}a$, therefore all numbers compared by modulus a are of the same forms as

$$0^2,\ 1^2,\ 2^2,\ 3^2, \text{ \&c., } r^2,$$

or, as the squares,

$$0^2,\ 1^2,\ 2^2,\ 3^2, \text{ \&c., } (\tfrac{1}{2}a)^2, \text{ when } a \text{ is even,}$$

and as

$$0^2,\ 1^2,\ 2^2,\ 3^2, \text{ \&c., } \left(\frac{a-1}{2}\right)^2, \text{ when } a \text{ is odd.}$$

DEDUCTIONS.

(1) When a is even, the general formula

$$a^2n^2 \pm 2anr + r^2$$

becomes

$$4a'^2n^2 \pm 4a'nr + r^2$$
$$= 4a'(a'n^2 \pm nr) + r^2.$$

Therefore, all square numbers are of the same form to modulus $4a$ as the squares

$$0^2,\ 1^2,\ 2^2,\ 3^2, \text{ \&c., } a^2;$$

and hence we see immediately that all square numbers to modulus 8 must be of the same forms as the squares

$$0^2, 1^2, 2^2$$

that is, they are all of the form

$$8n, 8n+1, 8n+4,$$

as we have already demonstrated.

(2) The following tables exhibit the possible and impossible forms of square numbers for all moduli from 2 to 10.

Possible Formulæ.

$2n$,	$2n+1$,		
$3n$,	$3n+1$,		
$4n$,	$4n+1$,		
$5n$,	$5n\pm1$,		
$6n$,	$6n+1$,	$6n+3$,	$6n+4$,
$7n$,	$7n+1$,	$7n+2$,	$7n+4$,
$8n$,	$8n+1$,	$8n+4$,	
$9n$,	$9n+1$,	$9n+4$,	$9n+7$,
$10n$,	$10n\pm1$,	$10n\pm4$,	$10n\pm5$.

Impossible Formulæ.

$3n$,			
$4n$,	$4n+3$,		
$5n$,	$5n+3$,		
$6n$,	$6n+5$,		
$7n$,	$7n+5$,	$7n+6$,	
$8n$,	$8n\pm3$,	$8n+7$,	
$9n$,	$9n\pm3$,	$9n+5$,	$9n+8$,
$10n$,	$10n\pm3$.		

CONTINUED FRACTIONS.

425. The name continued fraction is given to an expression of the form

$$\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{6+\cfrac{1}{8}}}} \quad \text{or} \quad \cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+}}}}, \text{ \&c.},$$

i. e., a fraction whose denominator is a whole number and a fraction, and which latter fraction has also for its denominator a whole number plus a fraction, and so on.

An expression whose numerators and denominators are any quantities whatever, may have the form of a continued fraction; but continued fractions, of which the numerators are 1 and the denominators whole positive numbers, are the kind which most usually occur.

These expressions arise in various ways, and are of great use in finding the approximate values of fractions and ratios that are expressed in large numbers, as well as in the resolution of certain unlimited problems of the first and second degrees; in the latter of which the answer can not be easily obtained in whole numbers by any other method.

Thus, in order to represent the irreducible fraction or ratio $\frac{a}{b}$ by a continued fraction, let b be contained in a, p times with a remainder c; also, let c be contained in b, q times with a remainder d, and so on, according to the following scheme:

$$\begin{array}{l} b)\ a\ (p \\ \quad c)\ b\ (q \\ \quad\quad d)\ c\ (r \\ \quad\quad\quad e)\ d\ (s \\ \quad\quad\quad\quad f, \text{\&c.,} \end{array}$$

and we shall have, by the principles of division,

$$\frac{a}{b}=p+\frac{c}{b},\ \frac{b}{c}=q+\frac{d}{c},\ \frac{c}{d}=r+\frac{e}{d},\ \text{\&c.;}$$

p, q, r, &c., are called *partial quotients*, and $p+\frac{c}{b}$, $q+\frac{d}{c}$, &c., complete quotients.

By taking the reciprocals of the second, third, &c., of the above equations, we have

$$\frac{c}{b}=\frac{1}{q+\frac{d}{c}},\ \frac{d}{c}=\frac{1}{r+\frac{e}{d}},\ \text{\&c.}$$

$$\therefore \frac{a}{b}=p+\frac{c}{b}=p+\cfrac{1}{q+\frac{d}{c}}=p+\cfrac{1}{q+\cfrac{1}{r+\frac{e}{d}}},\ \text{\&c.}$$

Whence, by extending the number of terms and generalizing the formula, we shall have

$$\frac{a}{b}=p+\cfrac{1}{q+\cfrac{1}{r+\cfrac{1}{s}}},\ \text{\&c.,} \qquad \text{or } \frac{a}{b}=\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r+\cfrac{1}{s}}}},\ \text{\&c.,}$$

according as the numerator is greater or less than the denominator; for in the latter case we should invert the first as well as the second, third, &c., equations.

To convert a given irreducible fraction into a continued one, we have the following

RULE.

Divide the greater of the two terms of the fraction by the less, and the last divisor continually by the last remainder, till nothing remains, as in finding their greatest common measure; then the successive quotients thus found will be the denominators of the several terms of the continued fraction, the numerators of which are always 1.

EXAMPLES.

(1) Reduce $\frac{2431}{1051}$ to a continued fraction.

```
1051) 2431 (2
      2102
       329) 1051 (3
             987
              64) 329 (5
                  320
                    9) 64 (7
                       63
                        1) 9 (9.
```

Hence
$$\frac{2431}{1051}=2+\frac{1}{3+}\frac{1}{5+}\frac{1}{7+}\frac{1}{9}.$$

(2)
$$\frac{1096}{9119}=\frac{1}{8+}\frac{1}{3+}\frac{1}{8+}\frac{1}{6+}\frac{1}{7}.$$

(3)
$$\frac{421}{972}=\frac{1}{2+}\frac{1}{3+}\frac{1}{4+}\frac{1}{5+}\frac{1}{6}.$$

As the fraction $\frac{a}{b}$, in every case of this kind, is supposed to be irreducible, or in its lowest terms, it is evident, by following the above process (which is similar to the method used for finding the common measure of true numbers), that we shall necessarily arrive at a remainder equal to 1; or otherwise a and b would have a common divisor, which is contrary to the hypothesis.

The continued fraction obtained will consist of a greater or less number of terms, according as the fraction $\frac{a}{b}$ is more or less complicated; but they will always terminate when $\frac{a}{b}$ is rational.

426. A continued fraction may be converted into a series of vulgar fractions by finding the successive sums of its several terms, after the manner of reducing complex fractions to simple ones, in common arithmetic; and the result will be more or less accurate, according to the number of terms of the continued fraction employed.

Each of these results is called a *convergent*, and they are numbered in order.

Thus, if it were required to reduce the following continued fraction,

$$a+\frac{1}{b+}\frac{1}{c+}\frac{1}{d}, \text{ \&c.,}$$

to a series of common vulgar fractions, the operation will stand thus:

$$a=\frac{a}{1}\ (1),\ a+\frac{1}{b}=\frac{ab+1}{b}\ (2),\ a+\frac{1}{b+}\frac{1}{c}=a+\frac{c}{bc+1}=\frac{abc+a+c}{bc+1},$$

or

$$\frac{(ab+1)c+a}{bc+1}\ (3),\ a+\frac{1}{b+}\frac{1}{c+}\frac{1}{d}=a+\frac{1}{b+}\frac{d}{cd+1}=a+\frac{cd+1}{bcd+b+d}$$

$$=\frac{abcd+ab+ad+cd+1}{bcd+b+d}=\frac{[(ab+1)c+a]d+ab+1}{(bc+1)d+b} \qquad (4)$$

(1), (2), (3), and (4) are called the first, second, third, and fourth convergents.

427. By inspecting the above convergents, we perceive that each may be formed from the preceding by the following

RULE.

Add the product of the numerator of the convergent already found by the denominator of the next term of the continued fraction, to the preceding numerator, for the next numerator and follow the same process for the denominator.*

EXAMPLE I.

$$3+\cfrac{1}{5+\cfrac{1}{2+\cfrac{1}{7}}},$$

denominators or quotients 3, 5, 2, 7, arranged in horizontal line;

convergents $\frac{3}{1}, \frac{16}{5}, \frac{35}{11}, \frac{261}{82}$.

After having formed the convergents $\frac{3}{1}$ and $\frac{16}{5}$, the rule applies. Then multiply 16, the second numerator, by 2, the third quotient, and add 3, the preceding numerator, it gives 35; and multiplying 5, the second denominator, by the same quotient 2, and adding 1, the preceding denominator, it gives 11; and so on. This method may proceed from the commencement, if we write $\frac{1}{0}$ before the first convergence.

Thus, 3, 5, 2, 7,

$$\frac{1}{0}\quad \frac{3}{1}\quad \frac{16}{5}\quad \frac{35}{11}\quad \frac{261}{82}.$$

When the continued fraction is not terminated, the numerators and denominators form two series increasing to infinity.

428. The convergents are alternately less and greater than the value of the continued fraction; for the first in the general form is equal to a, and as the fractional part which is added is neglected, this is too small. The second convergent is $a+\frac{1}{b}$, and, since b is too small by $\frac{1}{c}$, the fraction $\frac{1}{b}$ is too great, and, consequently, the whole convergent; and so on.

EXAMPLE II.

It is shown in geometry that the ratio of the circumference of a circle to its diameter is $\frac{31415926535}{10000000000}$, which, by being converted into a continued fraction, and the successive convergents found, will be as follows:

$$\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \text{\&c.};†$$

---

* The generality of this rule may be proved as follows:

Let $\frac{N}{D}, \frac{N'}{D'}, \frac{N''}{D''}$ be three consecutive convergents, m the quotient, of the same rank as the convergent $\frac{N''}{D''}$, and $\frac{1}{n}$ the partial fraction which follows $\frac{1}{m}$; and let $N''=N'm+N$ and $D''=D'm+D$, according to the rule. The convergent which follows $\frac{N''}{D''}$ is formed by substituting $m+\frac{1}{n}$ for m in $\frac{N''}{D''}$. Making this substitution in its equivalent $\frac{N'm+N}{D'm+D}$, we have

$$\frac{N'''}{D'''}=\frac{N'\left(m+\frac{1}{n}\right)+N}{D'\left(m+\frac{1}{n}\right)+D}=\frac{(N'm+N)n+N'}{(D'm+D)n+D'}=\frac{N''n+N'}{D''n+D'}$$

† The second, $\frac{22}{7}$, was the ratio assigned by Archimedes; the third, which is much more accurate, that by Metius.

and either of these will be the approximate value of the ratio, more and more accurate as we advance.

429. The difference between two convergents is equal to 1 divided by the product of the denominators of the two convergents. Thus, in the above example, the difference between the first and second convergents is $\frac{1}{7}$, between the second and third it is $\frac{1}{7\times 106}$, or $\frac{1}{742}$, between the third and fourth $\frac{1}{11978}$; and as the true value of the continued fraction is somewhere between any two consecutive convergents, we have its value to within less than the fraction $\frac{1}{7}$, $\frac{1}{742}$, or $\frac{1}{11978}$, &c., according to the convergent which we take.

To prove this in a general way, let

$$\frac{N}{D}, \frac{N'}{D'}, \frac{N''}{D''}$$

be three consecutive convergents, and m the quotient, of the same rank as the convergent $\frac{N''}{D''}$; then $N''=N'm+N$; $D''=D'm+D$.

But $$\frac{N'}{D'}-\frac{N}{D}=\frac{DN'-D'N}{DD'}\ldots\ldots (1)$$

$$\therefore \frac{N''}{D''}-\frac{N'}{D'}=\frac{N'm+N}{D'm+D}-\frac{N'}{D'}=\frac{N'D'm+D'N-N'D'm-DN'}{D'(D'm+D)}$$

$$=\frac{D'N-DN'}{D'(D'm+D)}=\frac{D'N-DN'}{D'D''}\ldots (2)$$

The numerators of (1) and (2) are the same, with contrary signs; and, to find its value, we have only to go back to the first two convergents $\frac{a}{1}$ and $\frac{ab+1}{b}$, the difference of which is $\frac{1}{b}$.

430. Since the denominators of the convergents increase to infinity if the series continue sufficiently far, it is possible to take two consecutive convergents whose difference shall be less than any assignable number; wherefore, as two consecutive convergents comprehend between them the value of the continued fraction, it follows that a convergent can be found whose value shall differ from that of the fraction by less than any assigned number.

For example, if the value of a continued fraction be required to within $\frac{1}{1000}$, the convergents must be continued till the product of the denominators of the last and last but one is at least 1000. The last convergent will then have the degree of approximation required.

The convergents are fractions in the lowest terms; for if a convergent $\frac{N}{D}$ admits of lower terms, some quantity q must be a common measure of N and D. Whence (Art. 29) q must be a measure of the multiples $N'D$ and ND', and of (Art. 29) $DN'-ND'$, or ± 1, which is impossible.

431. Each convergent is a nearer approximation to the true value of the con-

tinued fraction than that which precedes. For, let $\frac{N''}{D''}=\frac{N'm+N}{D'm+D}$ be a convergent in which m is the last quotient employed; then, if the complete quotient $m+\frac{1}{n}+$, &c., be denoted by y, and y be substituted for m in the expression of $\frac{N''}{D''}$, it is evident (employing x to denote the value of the continued fraction) that

$$x=\frac{N'y+N}{D'y+D}.$$

Subtracting each of the convergents $\frac{N}{D}$, $\frac{N'}{D'}$ from this value of x,

$$\frac{N'y+N}{D'y+D}-\frac{N}{D}=\frac{(DN'-ND')y}{D(D'y+D)}=\frac{\pm y}{D(D'y+D)},$$

and
$$\frac{N'y+N}{D'y+D}-\frac{N'}{D'}=\frac{ND'-DN'}{D'(D'y+D)}=\frac{\mp 1}{D'(D'y+D)}.$$

But $y>1$ and $D'>D$ $\therefore$ $D'(D'y+D)>D(D'y+D)$;

$$\therefore \frac{y}{D(D'y+D)}>\frac{1}{D'(D'y+D)}.$$

Whence $\frac{N'}{D'}$ is a nearer approximation to the value of x than $\frac{N}{D}$.

432. Among continued fractions those have been particularly distinguished in which the denominators, after a certain number of changes, are continually repeated in the same order, in which the continued fraction so formed is said to be *periodic*, and may then always be considered as the root of a quadratic equation or a surd.

To prove this, take a continued fraction entirely *periodic*.

$$x=\frac{1}{p+\frac{1}{p+\frac{1}{p+,\ \&c.}}}$$

Then, since the number of these fractions is unlimited, it follows that the sum of all after the first is also x; whence

$$x=\frac{1}{p+x} \therefore x^2+px=1$$
$$\therefore x=-\tfrac{1}{2}p\pm\tfrac{1}{2}\sqrt{p^2+4};$$

in which case the above continued fraction serves to determine the value of $\sqrt{p^2+4}$, since we have, by transposition,

$$\tfrac{1}{2}\sqrt{p^2+4}=\tfrac{1}{2}p+x=\frac{p}{2}+\frac{1}{p+\frac{1}{p+\frac{1}{p+,\ \&c.;}}}$$

and if p in this last expression be put equal to 2, we shall have

$$\sqrt{2}=1+\frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2+,\ \&c.}}}}$$

A continued fraction is also called periodic when the denominators occur periodically in pairs, threes, fours, &c.; thus,

$$\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{p+\cfrac{1}{q+, \&c.}}}} \qquad \text{or } \cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r+\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r.}}}}}}$$

Again, a fraction may be irregular in some of its first terms, and only become periodic at a certain distance from its commencement.

In either of these cases, as above, the value of x, the sum of all the terms, may be obtained by the resolution of an equation of the second degree. To prove this in a general manner, let

$a, b, \ldots.$ &c., be the quotients which form the non-periodic part,

$p, q, \ldots.$ &c., be the quotients which form the periodic part;

then
$$x=a+\cfrac{1}{b+\cfrac{\vdots}{\vdots+\cfrac{1}{p+\cfrac{1}{q+, \&c.;}}}}$$

and, representing by y the value of the periodic part,
$$p+\cfrac{1}{q+, \&c.,}$$

we have
$$x=a+\cfrac{1}{b+\cfrac{\vdots}{\vdots+\cfrac{1}{y}}} \qquad \text{and } y=p+\cfrac{1}{q+\cfrac{\vdots}{\vdots+\cfrac{1}{y}.}}$$

Consider these continued fractions as terminating with the partial fraction $\frac{1}{y}$, and deduce the convergents; we have (Art. 426) two equations of the following form:

$$x=\frac{P'y+P}{Q'y+Q},\ y=\frac{R'y+R}{S'y+S}.$$

The value of y, given by the first of these equations, is

$$y=\frac{P-Qx}{Q'x-P'},$$

which substituted in the second, gives, after reduction,

$$\frac{P-Qx}{Q'x-P'}=\frac{R'(P-Qx)+R(Q'x-P')}{S'(P-Qx)+S(Q'x-P')},$$

which is an equation of the second degree in x.

By way of illustration, take the following fraction:

$$x=a+\cfrac{p}{q+\cfrac{p}{q+\cfrac{p}{q}, \&c.}} \quad (1) \qquad \text{or } x-a=\cfrac{p}{q+\cfrac{p}{q+, \&c.}} \quad (2)$$

$\therefore x-a=\frac{p}{q+x-a}$; or, resolving the equation, $x=\frac{2a-q+\sqrt{q^2+4p}}{2}$.

If we transpose $\frac{2a}{2}$ or a, and substitute for $x-a$ its value (2), we have

$$\frac{\sqrt{q^2+4p}-q}{2}=\cfrac{p}{q+\cfrac{p}{q+\cfrac{p}{q}, \&c.;}}$$

or, making $q=2a$,

$$\sqrt{a^2+p}=a+\frac{p}{2a+}\frac{p}{2a+}\frac{p}{2a+}\frac{p}{2a}, \&c.$$

A similar mode of solution may be applied to continued surds or expressions of the form

$$\sqrt{a+\sqrt{a+\sqrt{a}}}, \&c.,$$

the value of which, though apparently infinite, is always determinable by a certain equation, and in some cases in a real integral or fractional quantity ; for, putting

$$x=\sqrt{a+\sqrt{a+\sqrt{a}}}, \&c.,$$

we shall have, by squaring both numbers,

$$x^2=a+\sqrt{a+\sqrt{a+}}, \&c.,$$

the latter part of which is evidently equal to the original surd; whence

$$x^2=a+x, \text{ or } x^2-x=a \therefore x=\tfrac{1}{2}\pm\sqrt{\tfrac{1}{4}+a},$$

where, if $a=2$, the expression becomes

$$\sqrt{2+\sqrt{2+\sqrt{2+}}}, \&c., =2 \text{ or } -1.$$

433. The process for developing any quantity, x, in a continued fraction, consists in making successively $x=a+\frac{1}{x'}$, $x'=b+\frac{1}{x''}$, $x''=c+\frac{1}{x'''}$, &c., a being the greatest whole number contained in x, b the greatest whole number contained in x', c the greatest whole number contained in x'', &c.

The numbers a, b, c, &c., being found, it is evident that if x', x'', &c., are replaced by their values, the required development is obtained, viz.,

$$x=a+\frac{1}{b+}\frac{1}{c+}\frac{1}{d+}, \&c.$$

EXAMPLE.

Let it be required to convert $\sqrt{19}$ into a continued fraction.

$$\sqrt{19}=4+\frac{1}{x^{\text{I}}} \therefore x^{\text{I}}=\frac{1}{\sqrt{19}-4}=\frac{\sqrt{19}+4}{3}.*$$

$$\frac{\sqrt{19}+4}{3}=x^{\text{I}}=2+\frac{1}{x^{\text{II}}} \therefore x^{\text{II}}=\frac{3}{\sqrt{19}-2}=\frac{\sqrt{19}+2}{5}.$$

By proceeding in this way we shall obtain the following :

$$x=\sqrt{19}=4+\frac{1}{x^{\text{I}}};$$

$$x^{\text{I}}=\frac{\sqrt{19}+4}{3}=2+\frac{1}{x^{\text{II}}};$$

$$x^{\text{II}}=\frac{\sqrt{19}+2}{5}=1+\frac{1}{x^{\text{III}}};$$

$$x^{\text{III}}=\frac{\sqrt{19}+3}{2}=3+\frac{1}{x^{\text{IV}}};$$

* Multiplying both numerator and denominator by $\sqrt{19}+4$.

$$x^{\text{IV}}=\frac{\sqrt{19}+3}{5}=1+\frac{1}{x^{\text{V}}};$$

$$x^{\text{V}}=\frac{\sqrt{19}+2}{3}=2+\frac{1}{x^{\text{VI}}};$$

$$x^{\text{VI}}=\sqrt{19}+4=8+\frac{1}{x^{\text{VII}}};$$

$$x^{\text{VII}}=\frac{\sqrt{19}+4}{3}=2+\frac{1}{x^{\text{VIII}}}.$$

Hence $\sqrt{19}=4+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{8}}}}}}.$

x^{VII} being the same as x^{I}, it is evident that, omitting the 4, the greatest integral part of $\sqrt{19}$, the quotients 2, 1, 3, 1, 2, 8, already found, will always return again in the same order to infinity.

Should it be required to convert the square root of 19 into a series of converging fractions without first reducing it to the continued form, they may be obtained, after the method before employed, from the integral parts of the above results only.

Quotients, 4, 2, 1, 3, 1, 2, 8, 2.

$$\frac{1}{0},\ \frac{4}{1},\ \frac{9}{2},\ \frac{13}{3},\ \frac{48}{11},\ \frac{61}{14},\ \frac{170}{39},\ \frac{1421}{326}.$$

EXAMPLES.

(1) $\frac{251}{764}$. Ans. Quotients, $\frac{1}{3}$, 22, 1, 4, 2.

Convergents, $\frac{1}{3}, \frac{22}{67}, \frac{23}{70}, \frac{114}{347}, \frac{251}{764}$.

(2) $\frac{1769}{5537}$. Ans. Quotients, $\frac{1}{3}$, 7, 1, 2, 4, 5, 1, 2.

Convergents, $\frac{1}{3}, \frac{7}{22}, \frac{8}{25}, \frac{23}{72}, \frac{100}{313}, \frac{523}{1637}, \frac{623}{1950}, \frac{1769}{5537}$.

(3) $\sqrt{31}$. Ans. The quotients are 5, 1, 1, 3, 5, 3, &c.

And the convergents, $\frac{5}{1}, \frac{6}{1}, \frac{11}{2}, \frac{39}{7}, \frac{206}{37}, \frac{657}{118}$, &c.

(4) $\sqrt{28}$. Ans. The quotients are 5, 3, 2, 3, 10, &c.

And the convergents, $\frac{5}{1}, \frac{16}{3}, \frac{37}{7}, \frac{127}{24}, \frac{1307}{247}$, &c.

(5) $\sqrt{45}$. Ans. Quotients, 6, 1, 2, 2, 2, 1, 12, &c.

Convergents, $\frac{6}{1}, \frac{7}{1}, \frac{20}{3}, \frac{47}{7}, \frac{114}{17}, \frac{161}{24}, \frac{2046}{305}$.

434. The converse of the proposition stated in Art. 432 is true, viz., that the root of an equation of the second degree may be expressed in functions of the coefficients of the equation by continued fractions.

The general form of the equation of the second degree may be written

$$ax^2-bx-c=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which b and c may be essentially negative. This may be put under the form

$$ax=b+\frac{c}{x}.$$

Multiplying the fraction $\frac{c}{x}$ above and below by a, it becomes

$$ax=b+\frac{ac}{ax}=b+\frac{ac}{b}+\frac{ac}{b}+\frac{ac}{b}+\frac{ac}{b}+, \&c.$$

$$\therefore x=\frac{1}{a}\left(b+\frac{ac}{b}+\frac{ac}{b}+\frac{ac}{b}+\frac{ac}{b}+, \&c.\right) \qquad \text{Q. E. D.}$$

If $a=1$, this becomes

$$x=b+\frac{c}{b}+\frac{c}{b}+\frac{c}{b}+, \&c.$$

If $b=0$,

$$x^2=c,$$

$$x=0+\frac{c}{0}+\frac{c}{0}+, \&c.,$$

which has no signification. But if we make

$$x^2=(z-a)^2,$$

a^2 being the greatest square contained in c, we have

$$x^2=z^2-2az+a^2=c;$$

$$\therefore z^2-2az=c-a^2;$$

or, putting $c-a^2=\gamma$,

$$z^2-2az=\gamma,$$

$$z-2a=\frac{\gamma}{z},$$

and

$$z=2a+\frac{\gamma}{2a}+, \&c.$$

But since $x=z-a$, $x=a+\frac{\gamma}{2a}+\frac{\gamma}{2a}+, \&c.$

To apply this, let the equation be

$$x^2=8 \therefore a=2, \gamma=4,$$

$$\therefore x=2+\frac{4}{4}+\frac{4}{4}+\frac{4}{4}+, \&c.,$$

or

$$x=2+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+, \&c.$$

The above result may be obtained in a more simple manner; thus, put

$$x^2=c=a^2+\beta \therefore x^2-a^2=\beta \therefore (x-a)(x+a)=\beta$$

$$\therefore x=a+\frac{\beta}{a+x}=a+\frac{\beta}{2a}+\frac{\beta}{2a}+, \&c.,$$

which shows that the square root of any number which is the sum of a square, and of another number, is a continued fraction.

Thus, if we have $x^2=7$ ∴ $\alpha=2$, $\beta=3$,

$$\sqrt{7}=2+\frac{3}{4+}\frac{3}{4+}, \&c.$$

435. Continued fractions furnish a method of resolving in whole numbers the indeterminate equation

$$ax+by=c \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

In this equation a, b, c are whole numbers, and the first two are supposed to have no common factor. Let us conceive that we have developed the relation $\frac{a}{b}$ into a continued fraction, and that we have calculated all the convergents; the last will be equal to this relation itself. Let us subtract from it the next to the last, which I represent by $\frac{a'}{b'}$. The numerator of the difference will be $ab'-ba'$, and by the property of Art. 430 we have

$$ab'-ba'=\pm 1 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Multiplied by $\pm c$, this equality becomes

$$a\times\pm b'c+b\times\mp a'c=c;$$

then equation (1) is satisfied by taking $x=\pm b'c$, $y=\pm a'c$.

This solution being known, we know (Art. 161) that all the others are given by the formulas

$$x=\pm b'c-bt,\; y=\mp a'c+at,$$

t designating any whole number whatever. We take the upper or lower sign according as we have $+$ or $-$ in the equality (2), or, what is the same thing, according as the convergent $\frac{a}{b}$ is of an even or uneven rank.

EXAMPLE.

Let there be the equation

$$261x-82y=117.$$

If we reduce $\frac{261}{82}$ to a continued fraction, we find

Quotients, 3, 5, 2, 7.

Convergents, $\frac{3}{1}, \frac{16}{5}, \frac{35}{11}, \frac{261}{82}$.

If we take the numerator of the difference $\frac{261}{82}-\frac{35}{11}$, and observe that $\frac{261}{82}$ is a convergent of an even rank, we have

$$261\times 11-82\times 35=+1.$$

Then, multiplying by 117,

$$261\times 11\times 117-82\times 35\times 117=117.$$

The equation, then, is satisfied by making $x=11\times 117=1287$ and $y=35\times 117=4095$; then, finally, the general values of x and y are

$$x=1287+82t,\; y=4095+261t.$$

If we divide 1287 by 82, and 4095 by 261, we find $1287=82\times 15+57$ and $4095=261\times 15+180$. Then, observing that t is any whole number whatever, we can write more simply

$$x=57+82t,\; y=180+261t.$$

436. The following theorem will be found useful in the resolution of indeterminate equations of the second degree.

Let $p^2-Aq^2=\pm D$ be an indeterminate equation, in which $D<\sqrt{A}$. I assert, that if this equation is resolvable, the fraction $\frac{p}{q}$ will be found among the fractions which converge toward $\sqrt{A}$.

From the above equation we derive $p-q\sqrt{A}=\frac{\pm D}{p+q\sqrt{A}}$, and, therefore, $\frac{p}{q}-\sqrt{A}$, which I represent by $\frac{\pm\delta}{q^2}=\frac{\pm D}{q(p+q\sqrt{A})}$; then $\delta=\frac{Dq}{p+q\sqrt{A}}$.

Let $\frac{p_0}{q_0}$ be the converging fraction which precedes $\frac{p}{q}$, and which is of such a nature that the sign of δ will be the same with that of D; it will remain to be proved that we have $\frac{Dq}{p+q\sqrt{A}}<\frac{q}{q+q_0}$, or $D(q+q_0)<p+q\sqrt{A}$.

In the second member, instead of p, I put its value, $q\sqrt{A}\pm\frac{\delta}{q}$; the inequality to be proved can then be written thus:

$$(q+q_0)(\sqrt{A}-D)+(q-q_0)\sqrt{A}\pm\frac{\delta}{q}>0.$$

But this inequality is manifest, since we have $\sqrt{A}>D$, $q>q_0$, and since the part $(q-q_0)\sqrt{A}$, which is at least equal to $\sqrt{A}$, by itself surpasses $\frac{\delta}{q}$, which is less than unity. $\frac{p}{q}$, then, will always be found in the fractions which converge toward $\sqrt{A}$, so that it will only be necessary to develop $\sqrt{A}$ in a continued fraction, and to calculate the converging fractions which result, in order to have all the solutions in entire numbers of the equation

$$x^2-Ay^2=\pm D,$$

D being $<\sqrt{A}$.

METHOD OF RESOLVING IN RATIONAL NUMBERS INDETERMINATE EQUATIONS OF THE SECOND DEGREE.

437. Let the proposed general equation be

$$ax^2+bxy+cy^2+dx+ey+f=0,$$

in which x and y are the indeterminates, and a, b, c, d, e, f the given entire numbers, positive or negative. We first derive from this equation the following:

$$2ax+by+d=\sqrt{[(by+d)^2-4a(cy^2+ey+f)]}.$$

If we make, to abridge, the radical $=t$, $b^2-4ac=A$, $bd-2ae=g$, $d^2-4af=h$, we shall have the two equations

$$2ax+by+d=t,$$
$$Ay^2+2gy+h=t^2.$$

If we multiply the last of these equations by A, and make, again, $Ay+g=v$, $g^2-Ah=B$, we shall have the transformed equation

$$v^2-At^2=B.$$

Reciprocally, if we can find values of v and t which satisfy the equation

$v^2-At^2=B$, we deduce from it the values of the indeterminates x and y in the proposed equation, viz.,

$$y=\frac{v-g}{A},\ x=\frac{t-by-d}{2a},$$

in which we should observe that both v and t may be taken with either sign, as we may desire.

If we find the solution of the proposed equation in rational numbers, it will suffice to resolve, by means of these numbers, the transformed $v^2-At^2=B$; but if we wish to resolve the proposed in entire numbers, it will not only be necessary that t and v be entire numbers, but that the values of t and v, substituted in those of x and y, give for these indeterminates entire numbers. At present we will only occupy ourselves with the resolution in rational numbers.

438. Every indeterminate equation of the second degree can be reduced, as we have just seen, to the form $v^2-At^2=B$; but, whatever may be the rational numbers t and v, we can suppose that they are reduced to a common denominator. Hence, making $v=\frac{x}{z}$, $t=\frac{y}{z}$, we shall have to resolve the equation

$$x^2-Ay^2=Bz^2,$$

in which now x, y, z are entire numbers.

We can suppose that these three numbers have not a same common divisor; for if they had had one, we could have made it disappear by division.

In the same manner, we can suppose that the numbers A and B have no square divisors; for if we had had, for example, $A=A'k^2$, $B=B'l^2$, we might have made $ky=y'$, $lz=z'$, and the equation to be resolved would have become

$$x^2-A'y'^2=B'z'^2,$$

in which A′ and B′ have no longer a square factor.

The equation $x^2-Ay^2=Bz^2$ being thus prepared, we shall observe that any two of the indeterminates x, y, z can not have a common divisor; for if θ^2, for example, should divide x^2 and y^2, it must necessarily divide also Bz^2; but it can not divide z^2, since the three numbers x, y, z have no common divisors; neither can θ^2 divide B, since B has no square factor. x and y, therefore, are prime with respect to each other; for the same reason, x and z are primes with respect to each other, as well as y and z.

I assert, moreover, that A and B can be supposed to be positive; for we can only have, as regards the signs of the terms of one equation, the following three suppositions:

$$x^2-Ay^2=+Bz^2,$$
$$x^2-Ay^2=-Bz^2,$$
$$x^2+Ay^2=+Bz^2.$$

(I omit the supposition $x^2+Ay^2=-Bz^2$, since it is evidently impossible.)

Of these three combinations the second coincides with the third by a simple transposition; but if we multiply the third by B, and make $Bz=z'$, $AB=A'$, we shall have

$$z'^2-A'y^2=Bx^2.$$

The equation to be resolved, therefore, can always be reduced to the form

$$x^2-By^2=Az^2,$$

in which A and B are positive numbers, and do not contain any square factor.

439. The method which we shall proceed to follow for the resolution of this equation is that given by Lagrange, in the *Mémoires de Berlin*, 1767. It consists in producing, by means of transformations, the successive diminution of the coefficients A and B until one of them becomes equal to zero, in which case the solution can be immediately deduced from known formulas.

The equation thus reduced is of the form $x^2-y^2=Az^2$, or $x^2-By^2=z^2$, but these two formulas do not differ, and it will suffice to give the solution of the first, $x^2-y^2=Az^2$. To do this, decompose A into two factors α, β (which will always be prime with regard to each other, since A has no square factor), and suppose that z also is decomposed into two factors p, q, such that we have $A=\alpha\beta$, $z=pq$, we shall have the equation $(x+y)(x-y)=\alpha\beta p^2q^2$, which we can, in general, satisfy by taking $x+y=\alpha p^2$, $x-y=\beta q^2$; this supposition gives

$$x=\frac{\alpha p^2+\beta q^2}{2},\ y=\frac{\alpha p^2-\beta q^2}{2},\ z=pq;$$

hence the three indeterminates x, y, z will be expressed by means of two arbitrary quantities p and q; if it should happen that the values of x and y contain the fraction $\frac{1}{2}$, x, y, z must each be multiplied by two.

Such is the general solution of the equation $x^2-y^2=Az^2$, a solution which will comprise as many particular formulas as there are ways of decomposing A into two factors.

For example, if $A=30$; there are four ways of decomposing 30 into two factors, viz., 1.30, 2.15, 3.10, 5.6; hence will result these four solutions of the equation $x^2-y^2=30z^2$,

1°. $x=p^2+30q^2$, $y=p^2-30q^2$, $z=2pq$,
2°. $x=2p^2+15q^2$, $y=2p^2-15q^2$, $z=2pq$,
3°. $x=3p^2+10q^2$, $y=3p^2-10q^2$, $z=2pq$,
4°. $x=5p^2+6q^2$, $y=5p^2-6q^2$, $z=2pq$.

440. Let us proceed to the general equation $x^2-By^2=Az^2$; observe that this equation, being the same with $x^2-Az^2=By^2$, we can, without diminishing the generality of the theorem, suppose that the coefficient of the second member is the greater of the two. In case of equality, the reduction that we shall indicate would always be employed.

Let, then, the proposed equation be $x^2-By^2=Az^2$, in which we suppose, at the same time, $A>B$, A and B positive, and free from any square factor.

We have already proved that x and y are primes as regards each other; y and A, therefore, are equally prime to one another; for if y^2 and A had a common divisor θ, x^2 also must, necessarily, be divisible by θ, and x and y would not then be primes to one another.

But since y and A are primes to one another, if we suppose that the proposed equation is resolvable, and that we can, therefore, find determinate values of x and y, $x=M$, $y=N$, we shall also be able to satisfy the equation of the first degree,

$$M=nN-y'A,$$

in which M, N, A will be given numbers prime to one another, and n, y' two indeterminates.

Hence, in general, without knowing the particular solutions $x=M$, $y=N$, we can suppose $x=ny-Ay'$, n and y' being two indeterminates; and, substituting this value of x in the proposed equation, we shall have, after having divided by A,

$$\left(\frac{n^2-B}{A}\right)y^2-2nyy'+Ay'^2=z^2.$$

But since y and A are prime to one another, this equation can not subsist unless $\frac{n^2-B}{A}$ be an entire number. Let this entire number $=A'k^2$, k^2 being the greatest square which can be a divisor of it, we shall have

$$n^2-B=AA'k^2,$$

and the equation to be resolved will become

$$A'k^2y^2-2nyy'+Ay'^2=z^2.$$

We perceive that if there be any value whatsoever of n which renders n^2B divisible by A, this value can be augmented or diminished by any multiple of A, without n^2-B ceasing to be divisible by A; hence, we can suppose that its value is comprised between the limits 0 and A, or even between the more extended limits $-\frac{1}{2}A$ and $+\frac{1}{2}A$.

We conclude from this, that in trying successively for n all the entire numbers from $-\frac{1}{2}A$ to $+\frac{1}{2}A$, we shall encounter, necessarily, one or more values which will render n^2-B divisible by A, provided, however, the equation is resolvable; and in case these values will not render n^2-B divisible by A, we can conclude with certainty that the proposed equation is not resolvable.

441. Suppose, then, that we have found one or more values of n which fulfill the required condition, it will be necessary with each of these values to continue the calculation in the following manner:

Resume the equation $A'k^2y^2-2nyy'+Ay'^2=z^2$; if we multiply it by $A'k^2$, and if we make, to abridge,

$$A'k^2y-ny'=x',\ kz=z',$$

the transformed will be

$$x'x'-By'y'=A'z'z'.$$

This transformed could be resolved, if we could determine the solution of the proposed equation, since the values of x', y', z' are easily deduced from those of x, y, z; reciprocally, the proposed will be resolved, if we find the solution of its transformed. For, from the known values of x', y', z', we can equally deduce those of x, y, z; and it matters little whether these last values be under an entire or fractional form, since we have regard only to the resolution in rational numbers, and since, after we have found any fractional values of x, y, z, we can reduce them to a common denominator and suppress it.

Since we can suppose the number $n<\frac{1}{2}A$, it is clear that $\frac{n^2-B}{Ak^2}$ or A' will be $<\frac{1}{4}A$, and, at the same time, positive; for n can not be $<\sqrt{B}$, since otherwise n^2-B would be $<B$, and could not be divisible by A. The proposed equation, therefore, will be reduced to an equation in every respect similar, in which the coefficient A', which takes the place of A, is less than $\frac{1}{4}A$.

442. If we have, again, $A'>B$, we can, in like manner, from the equation $x'^2-By'^2=A'z'^2$, deduce a second transformed,

$$x''^2-By''^2=A''z''^2,$$

in which A'' will be $<\frac{1}{4}A'$, and always positive. To obtain this second transformed, there will be no new condition to be fulfilled, for having already found

$\frac{n^2-B}{A'}=Ak^2$, if we make $n=\mu A'+n'$, and if we take the indeterminate μ in such a way that $n'<\frac{1}{2}A'$, it is easy to see that $\frac{n'^2-B}{A'}$ will be an entire positive number less than $\frac{1}{4}A'$; we have, consequently,

$$n'^2-B=A'A''k'^2,$$

A'' being less than $\frac{1}{4}A'$, and not containing any square factor.

If it should happen that A'', again, were greater than B, we should continue this system of transformed equations, in which B is constant, until we arrive at one of this form

$$x^2-By^2=Cz^2,$$

in which C will be positive and $<B$.

443. But after we have passed into the second member of this equation the term which has the greatest coefficient, which gives

$$x^2-Cz^2=By^2,$$

we can proceed in a similar manner to the reduction of the coefficient B by a second system of transformed equations

$$x'^2-Cz'^2=B'y'^2,$$
$$x''^2-Cz''^2=B''y''^2,$$
$$\&c. \qquad \&c.,$$

in which the coefficients B', B'', &c., will be positive, and will diminish in at least a quadruple ratio, and thus we shall soon arrive at a transformed

$$x^2-Cz^2=Dy^2,$$

in which the coefficient D will be less than C.

But the series of positive and decreasing numbers A, B, C, D will not go on *ad infinitum;* it will terminate necessarily at unity, and when we shall have arrived at this term, the resolution of the last transformed, which is given at once, will make known those of all the preceding equations, and, consequently, that of the proposed.

GAUSS'S METHOD OF SOLVING BINOMIAL EQUATIONS.

444. The solution of $x^n-1=0$, it has been proved (Art. 299), can always be reduced to the case where n is prime; and the case of n a prime number, by a method invented by Gauss, may be made to depend upon the solution of equations whose degrees do not exceed the greatest prime number which is a divisor of $n-1$. The leading feature of Gauss's method is to represent the imaginary roots by a series of powers of any one of them, whose indices form a geometrical instead of an arithmetical progression. Thus, if m be a number (and such, called primitive roots of n, can always be found) whose several powers from 1 to $n-1$, when divided by n, leave different remainders, and a be any imaginary root, then all the roots may manifestly be represented by

$$a^m,\ a^{m^2},\ a^{m^3},\ \ldots\ a^{m^{n-1}};$$

or, since $m^{n-1}=\mu n+1$, where μ is an integer, by a, a^m, a^{m^2}, &c., $a^{m^{n-2}}$.

445. The advantage of this mode of representing the roots is, (1) that they can be distributed into periods, each of which, when continued, will produce the roots of that period in the same order; and (2) that the product of any

number of such periods will be equal to the sum of a certain number of periods, the importance of which properties will be seen in the use made of them.

(1) Let $n-1=rs$, r being a prime factor of $n-1$, and let $m^r=h$; then the roots may be written in vertical columns, each consisting of r terms, as follows:

$$\begin{array}{lllll} a & a^{h} & a^{h^2} & \ldots & a^{h^{s-1}} \\ a^{m} & a^{mh} & a^{mh^2} & \ldots & a^{mh^{s-1}} \\ a^{m^2} & a^{m^2h} & a^{m^2h^2} & \ldots & a^{m^2h^{s-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a^{m^{r-1}} & a^{m^{r-1}h} & a^{m^{r-1}h^2} & \ldots & a^{m^{r-1}h^{s-1}}, \end{array}$$

and if any one of the periods formed by the horizontal rows be continued, the roots in that period will be produced in the same order; thus, if the first row were continued, the indices would be $h^s=m^{rs}=m^{n-1}=\mu n+1$, $h^{s+1}=m^{rs+r}=(\mu n+1)m^r=\mu nh+h$, &c., and the corresponding roots, a, a^h, &c.

(2) Let any two of the above periods be represented by

$$a^{a}+a^{ah}+a^{ah^2}+, \text{ \&c.}, +a^{ah^{s-1}}$$
$$a^{b}+a^{bh}+a^{bh^2}+, \text{ \&c.}, +a^{bh^{s-1}},$$

and let us multiply them together, using each term of the lower line in succession as a multiplier, and starting at that term of the upper line which stands over it, and producing the upper line so as to supply the terms neglected at the beginning, the result is

$$\begin{array}{lllll} a^{a+b} & +a^{ah+b} & +a^{ah^2+b} & +, \text{ \&c.}, & +a^{ah^{s-1}+b} \\ a^{(a+b)h} & +a^{(ah+b)h} & +a^{(ah^2+b)h} & +, \text{ \&c.}, & +a^{(ah^{s-1}+b)h} \\ a^{(a+b)h^2} & +a^{(ah+b)h^2} & +a^{(ah^2+b)h^2} & +, \text{ \&c.}, & +a^{(ah^{s-1}+b)h^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a^{(a+b)h^{s-1}} & +a^{(ah+b)h^{s-1}} & +a^{(ah^2+b)h^{s-1}} & +, \text{ \&c.}, & +a^{(ah^{s-1}+b)h^{s-1}}; \end{array}$$

and therefore, collecting the vertical columns into periods, we get

$$\Sigma(a^{a})\Sigma(a^{b})=\Sigma(a^{a+b})+\Sigma(a^{ah+b})+\Sigma(a^{ah^2+b})+\ldots+\Sigma(a^{ah^{s-1}+b}),$$

or the product of two periods is equal to the sum of s periods; and, consequently, the product of any number of periods will be equal to the aggregate of a certain number of periods.

EXAMPLE I.

$x^7-1=0$; $6=3.2$, $\therefore r=3$, $s=2$; also, 3, 3^2, 3^3, 3^4, 3^5, when divided by 7, leave different remainders, viz., 3, 2, 6, 4, 5; $\therefore m=3$, and the roots are

$$p_1=a+a^6$$
$$p_2=a^3+a^4$$
$$p_3=a^2+a^5,$$

and $$p_1+p_2+p_3=-1.$$

Also,
$$p_1p_2=a^4+a^2+a^5+a^3=p_2+p_3$$
$$p_2p_3=a^5+a^6+a+a^2=p_1+p_3$$
$$p_1p_3=a^3+a+a^6+a^4=p_1+p_2;$$
$$\therefore p_1p_2+p_2p_3+p_1p_3=-2,$$

and $$p_1p_2p_3=p_1^2+p_1+p_2=2+p_3+p_1+p_2=1.$$

Therefore the cubic which has p_1, p_2, p_3 for its roots, is $p^3+p^2-2p-1=0$.

EXAMPLE II.

$x^{17}-1=0$; $16=2.8$; also, the powers of 3 from 0 to 15, when divided by 17, leave remainders

$$1\ 3\ 9\ 10\ 13\ 5\ 15\ 11\ 16\ 14\ 8\ 7\ 4\ 12\ 2\ 6,$$

$$\therefore p=a+a^9+a^{13}+a^{15}+a^{16}+a^8+a^4+a^2$$
$$q=a^3+a^{10}+a^5+a^{11}+a^{14}+a^7+a^{12}+a^6;$$

then $p+q=-1$, and

$$\begin{aligned} pq=&a^4+a^{12}+a^{16}+a+a^2+a^{11}+a^7+a^5\\ &a^2+a^6+a^8+a^9+a+a^{14}+a^{12}+a^{11}\\ &\cdots\cdots\cdots\cdots\\ &a^8+a^7+a^{15}+a^2+a^4+a^5+a^{14}+a^{10}\\ =&p+q+p+p+p+q+q+q=-4; \end{aligned}$$

therefore, p and q are roots of $z^2+z-4=0$.

Next, the periods p, q may be resolved respectively into the periods

$$\left.\begin{aligned} r&=a+a^{13}+a^{16}+a^4\\ s&=a^9+a^{15}+a^8+a^2 \end{aligned}\right\},\quad \left.\begin{aligned} t&=a^3+a^5+a^{14}+a^{12}\\ u&=a^{10}+a^{11}+a^7+a^6 \end{aligned}\right\};$$

$$\therefore r+s=p,$$

and

$$\left.\begin{aligned} rs=&a^{10}+a^5+a^8+a^{13}\\ &a^{11}+a^{14}+a^2+a^{16}\\ &\cdots\cdots\cdots\\ &a^6+a^3+a^{15}+a \end{aligned}\right\}=p+q=-1;$$

therefore, r, s are roots of $z^2-pz-1=0$; and, similarly, t, u are roots of $z^2-qz-1=0$.

Lastly, the periods r, s, t, u may be resolved respectively into

$$\left.\begin{aligned} r_1&=a+a^{16}\\ r_2&=a^{13}+a^4 \end{aligned}\right\},\quad \left.\begin{aligned} s_1&=a^9+a^8\\ s_2&=a^{15}+a^2 \end{aligned}\right\},\quad \left.\begin{aligned} t_1&=a^3+a^{14}\\ t_2&=a^5+a^{12} \end{aligned}\right\},\quad \left.\begin{aligned} u_1&=a^{10}+a^7\\ u_2&=a^{11}+a^6 \end{aligned}\right\};$$

then

$$r_1+r_2=r,$$
$$r_1r_2=a^{14}+a^{12}+a^3+a^5=t,$$
$$\therefore r_1,\ r_2 \text{ are roots of } z^2-rz+t=0;$$

and r_1, the greatest root of this equation, $=a+\frac{1}{a}=2\cos\frac{\pi}{17}$.

For further information upon the theory of numbers, the student is referred to the *Théorie des Nombres* of Legendre, the *Disquisitiones Arithmeticæ* of Gauss, of which there is an excellent French translation (*Recherches Arithmétiques*) by Poullet-Delisle; to Barlow's Theory of Numbers; to the article of Ivory in the fourth volume of the supplement of the *Encyc. Britan.*; and to the Memoirs by Libri, in tome v., 1838 (*Étrangères*), and by Cauchy, in tome xvii., 1840, of the *Mémoires* of the French Academy of Sciences.

S. 4,685 5713 ; V. —0,51 : T. 5819 ; V. +1,00.

0^d	4^d	N.		0	1	2	3	4	5	6	7	8	9	dif. et p.
24	0′	1440	158.	3625	3927	4228	4530	4831	5133	5434	5736	6037	6338	6 181; 7 211; 8 242; 9 272
	10	41		6640	6941	7243	7544	7845	8146	8448	8749	9050	9351	
	20	42		9653	9954									
			159.			0255	0556	0857	1158	1459	1760	2061	2362	
	30	43		2663	2964	3265	3566	3867	4168	4469	4770	5070	5371	301: 1 30; 2 60; 3 90; 4 120; 5 151; 6 181; 7 211; 8 241; 9 271
	40	44		5672	5973	6273	6574	6875	7175	7476	7777	8077	8378	
5″	50	1445		8678	8979	9280	9580	9881						
			160.						0181	0481	0782	1082	1383	
	1	46		1683	1983	2284	2584	2884	3184	3485	3785	4085	4385	
	10	47		4685	4985	5286	5586	5886	6186	6486	6786	7086	7386	
	20	48		7686	7986	8285	8585	8885	9185	9485	9785			300: 1 30; 2 60; 3 90; 4 120; 5 150; 6 180; 7 210; 8 240; 9 270
			161.									0084	0384	
	30	49		0684	0984	1283	1583	1883	2182	2482	2781	3081	3380	
10	40	1450		3680	3980	4279	4578	4878	5177	5477	5776	6075	6375	
	50	51		6674	6973	7273	7572	7871	8170	8470	8769	9068	9367	
	2	52		9666	9965									
			162.			0264	0563	0862	1161	1460	1759	2058	2357	299: 1 30; 2 60; 3 90; 4 120; 5 150; 6 179; 7 209; 8 239; 9 269
	10	53		2656	2955	3254	3553	3852	4150	4449	4748	5047	5345	
	20	54		5644	5943	6241	6540	6839	7137	7436	7734	8033	8331	
15	30	1455		8630	8928	9227	9525	9824						
			163.						0122	0420	0719	1017	1315	
	40	56		1614	1912	2210	2508	2807	3105	3403	3701	3999	4297	
	50	57		4596	4894	5192	5490	5788	6086	6384	6682	6979	7277	
	3	58		7575	7873	8171	8469	8767	9064	9362	9660	9958		298: 1 30; 2 60; 3 89; 4 119; 5 149; 6 179; 7 209; 8 238; 9 268
			164.										0255	
	10	59		0553	0851	1148	1446	1743	2041	2339	2636	2934	3231	
20	20	1460		3529	3826	4123	4421	4718	5016	5313	5610	5908	6205	
	30	61		6502	6799	7097	7394	7691	7988	8285	8582	8880	9177	
	40	62		9474	9771									
			165.			0068	0365	0662	0959	1256	1553	1850	2146	297: 1 30; 2 59; 3 89; 4 119; 5 149; 6 178; 7 208; 8 238; 9 267
	50	63		2443	2740	3037	3334	3631	3927	4224	4521	4817	5114	
	4	64		5411	5707	6004	6301	6597	6894	7190	7487	7783	8080	
25	10	1465		8376	8673	8969	9265	9562	9858					
			166.							0155	0451	0747	1043	
	20	66		1340	1636	1932	2228	2525	2821	3117	3413	3709	4005	
	30	67		4301	4597	4893	5189	5485	5781	6077	6373	6669	6965	
	40	68		7261	7556	7852	8148	8444	8740	9035	9331	9627	9922	296: 1 30; 2 59; 3 89; 4 118; 5 148; 6 178; 7 207; 8 237; 9 266
	50	69	167.	0218	0514	0809	1105	1400	1696	1991	2287	2582	2878	
30	5	1470		3173	3469	3764	4060	4355	4650	4946	5241	5536	5831	
	10	71		6127	6422	6717	7012	7308	7603	7898	8193	8488	8783	
	20	72		9078	9373	9668	9963							
			168.					0258	0553	0848	1143	1438	1733	295: 1 30; 2 59; 3 89; 4 118; 5 148; 6 177; 7 207; 8 236; 9 266
	30	73		2027	2322	2617	2912	3207	3501	3796	4091	4386	4680	
	40	74		4975	5269	5564	5859	6153	6448	6742	7037	7331	7626	
35	50	1475		7920	8215	8509	8803	9098	9392	9686	9981			
			169.									0275	0569	
	6	76		0864	1158	1452	1746	2040	2335	2629	2923	3217	3511	
	10	77		3805	4099	4393	4687	4981	5275	5569	5863	6157	6450	
	20	78		6744	7038	7332	7626	7920	8213	8507	8801	9094	9388	294: 1 29; 2 59; 3 88; 4 118; 5 147; 6 176; 7 206; 8 235; 9 265
	30	79		9682	9975									
			170.			0269	0563	0856	1150	1443	1737	2030	2324	
40	40	1480		2617	2911	3204	3497	3791	4084	4377	4671	4964	5257	
	50	81		5551	5844	6137	6430	6723	7017	7310	7603	7896	8189	
	7	82		8482	8775	9068	9361	9654	9947					
			171.							0240	0533	0826	1119	293: 1 29; 2 59; 3 88; 4 117; 5 147; 6 176; 7 205; 8 234; 9 264
	10	83		1412	1704	1997	2290	2583	2876	3168	3461	3754	4046	
	20	84		4339	4632	4924	5217	5509	5802	6095	6387	6680	6972	
45	30	1485		7265	7557	7849	8142	8434	8727	9019	9311	9604	9896	
	40	86	172.	0188	0480	0773	1065	1357	1649	1941	2233	2526	2818	
	50	87		3110	3402	3694	3986	4278	4570	4862	5154	5446	5737	
	8	88		6029	6321	6613	6905	7197	7488	7780	8072	8364	8655	292: 1 29; 2 58; 3 88; 4 117; 5 146; 6 175; 7 204; 8 234; 9 263
	10	89		8947	9239	9530	9822							
			173.					0113	0405	0697	0988	1280	1571	
50	20	1490		1863	2154	2446	2737	3028	3320	3611	3903	4194	4485	
	30	91		4776	5068	5359	5650	5941	6233	6524	6815	7106	7397	
	40	92		7688	7979	8270	8561	8852	9143	9434	9725			
			174.									0016	0307	
	50	93		0598	0889	1180	1471	1761	2052	2343	2634	2925	3215	291: 1 29; 2 58; 3 87; 4 116; 5 146; 6 175; 7 204; 8 233; 9 262
	9	94		3506	3797	4087	4378	4669	4959	5250	5540	5831	6121	
55	10	1495		6412	6702	6993	7283	7574	7864	8155	8445	8735	9026	
	20	96		9316	9606	9897								
			175.				0187	0477	0767	1057	1348	1638	1928	290: 1 29; 2 58; 3 87; 4 116; 5 145
	30	97		2218	2508	2798	3088	3378	3668	3958	4248	4538	4828	
	40	98		5118	5408	5698	5988	6278	6567	6857	7147	7437	7727	
	50	99		8016	8306	8596	8885	9175	9465	9754				
			176.								0044	0333	0623	
″	″	N.		0	1	2	3	4	5	6	7	8	9	

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