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ELEMENTARY TREATISE

ON

ALGEBRA,

THEORETICAL AND PRACTICAL,

ADAPTED TO THE INSTRUCTION OF YOUTH IN SCHOOLS AND COLLEGES.

BY JAMES RYAN, A. M.,

AUTHOR OF THE DIFFERENTIAL AND INTEGRAL CALCULUS; THE NEW AMERICAN GRAMMAR OF ASTRONOMY, &c.

TO WHICH IS ADDED,

AN APPENDIX,

CONTAINING AN ALGEBRAIC METHOD OF DEMONSTRATING THE PROPOSITIONS IN THE FIFTH BOOK OF EUCLID'S ELEMENTS, ACCORDING TO THE TEXT AND ARRANGEMENT IN SIMSON'S EDITION,

By ROBERT ADRAIN, L.L.D. F.A.P.S. F.A.A.S., &c

AND PROFESSOR OF MATHEMATICS AND NATURAL PHILOSOPHY, IN COLUMBIA COLLEGE, NEW-YORK.

SIXTH EDITION,

GREATLY ENLARGED AND IMPROVED, BY THE AUTHOR.

NEW YORK:

HENRY G. LANGLEY, 8 ASTOR HOUSE.

1845.

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TO

THE FOURTH EDITION.

THE author has endeavoured to accommodate his Algebra to the present state of science in the United States. Considerable alterations and improvements have been made in the different sections of the original work. There are also introduced two new chapters, containing Figurate and Polygonal Numbers, Vanishing Fractions, Indeterminate Coefficients, Indeterminate and Diophantine Analysis. The chapters upon these subjects have chiefly been derived from Euler, Bonnycastle, Young, and Bourdon.

JAMES RYAN.

New York, July 4, 1838.

ADVERTISEMENT.

As Utility is the great object aimed at in this Publication, I have spared no pains to make a careful selection of materials, from the most approved sources, which may tend to elucidate, in a full and clear manner, the Elements of Algebra, both in theory and practice.

Those authors of whose labours I have principally availed myself, are *Euler*, *Clairaut*, *Lacroix*, *Garnier*, *Bezout*, *Lagrange*, *Newton*, *Simson*, *Emerson*, *Wood*, *Bonnycastle*, *Bridge*, and *Bland*.

To Bland's Algebraical Problems, (a work compiled for the use of Students in one of the first Universities in Europe), I am chiefly indebted for the problems in Simple, Pure, and Quadratic Equations.

By permission of the learned *Dr. Adrain*, I have added, as an Appendix, his method of demonstrating algebraically the propositions in the fifth book of Euclid's Elements.

JAMES RYAN.

New York, *July* 1, 1824.

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AN

ELEMENTARY TREATISE

ON

ALGEBRA.

INTRODUCTION.

EXPLANATION OF THE ALGEBRAIC METHOD OF NOTATION:— DEFINITIONS AND AXIOMS.

1. *Algebra* is a general method of computation, in which abstract quantities and their several relations are made the subject of calculation, by means of alphabetical letters and other signs.

2. The letters of the alphabet may be employed at pleasure for denoting any quantities, as algebraical symbols or abbreviations; but, in general, quantities whose values are *known* or *determined*, are expressed by the *first* letters, a, b, c, &c.; and *unknown* or *undetermined* quantities are denoted by the *last* or final ones, u, v, w, x, &c.

3. Quantities are equal when they are of the same magnitude. The abbreviation $a=b$ implies that the quantity denoted by a is equal to the quantity denoted by b, and is read *a equal to b*; $a>b$, or *a greater than b*, that the quantity a is greater than the quantity b; and $a<b$, or *a less than b*, that the quantity a is less than the quantity b.

4. Addition is the joining of magnitudes into one sum. The sign of addition is an erect cross; thus, $a+b$ implies the sum of a and b, and is called *a plus b*, if a represent 8 and b, 4; then, $a+b$ represents 12, or $a+b=8+4=12$.

5. Subtraction is the taking as much from one quantity as is equal to another. Subtraction is denoted by a single line; as $a-b$, or *a minus b*, which is the part of a remaining, when a part equal to b has been taken from it; if $a=9$, and $b=5$; $a-b$ expresses 9 diminished by 5, which is equal to 4, or $a-b=9-5=4$.

6. Also, the difference of two quantities a and b; when it is not known which of them is the greater, is represented by the sign $\sim$; thus, $a\sim b$ is $a-b$, or $b-a$; and $a\underset{\sim}{+}b$ signifies the sum or difference of a and b.

7. Multiplication is the adding together so many numbers or quantities equal to the multiplicand as there are *units* in the multiplier, into one sum called the product. Multiplication is expressed by an oblique cross, by a point, or by simple apposition; thus, $a\times b$, $a\,.\,b$, or ab, signifies the quantity denoted by a, is to be multiplied by the quantity denoted by b; if $a=5$ and $b=7$; then $a\times b=5\times 7=35$, or $a\,.\,b=5\,.\,7=35$, or $ab=5\times 7=35$.

Scholium. The multiplication of numbers cannot be expressed by simple apposition. A *unit* is a magnitude considered as a whole complete within itself. And a whole number is composed of units by continued additions; thus, one plus one composes two, $2+1=3$, $3+1=4$, &c.

8. Division is the subtraction of one quantity from another as often as it is contained in it; or the finding of that quotient, which, when multiplied by a given divisor, produces a given dividend.

Division is denoted by placing the dividend before the sign $\div$, and the divisor after it; thus $a\div b$, implies that the quantity a is to be divided by the quantity b. Also, it is frequently denoted by placing one of the two quantities over the other, in the form of a fraction; thus, $\frac{a}{b}=a\div b$; if $a=12$, $b=4$; then $a\div b=\frac{a}{b}=12\div 4=\frac{12}{4}=3$.

9. A *simple fraction* is a *number* which by continual addition composes a unit, and the number of such fractions contained in a unit, is denoted by the denominator, or the number below the line; thus, $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1$. A *number* composed of such simple fractions, by continual addition, may properly be termed a multiple fraction; the number of simple fractions composing it, is denoted by the upper figure or numerator. In this sense, $\frac{2}{3}$, $\frac{3}{3}$, $\frac{4}{3}$, are multiple fractions; and $\frac{3}{3}=1$, $\frac{4}{3}=\frac{3}{3}+\frac{1}{3}=1+\frac{1}{3}=1\frac{1}{3}$.

10. When any quantities are enclosed in a parentheses, or have a line drawn over them, they are considered as one quantity with respect to other symbols; thus $a-(b+c)$, or $a-\overline{b+c}$; implies the excess of a above the sum of b and c. Let $a=9$, $b=3$, and $c=2$; then $a-(b+c)=9-(3+2)=9-5=4$, or $a-\overline{b+c}=9-\overline{3+2}=9-5=4$. Also, $(a+b)\times$

$(c+d)$, or $\overline{a+b}\times\overline{c+d}$, denotes that the sum of a and b is to be multiplied by the sum of c and d; thus, let $a=4$, $b=2$, $c=3$, and $d=5$; then $(a+b)\times(c+d)=(4+2)\times(3+5)=6\times8=48$, or $\overline{a+b}\times\overline{c+d}=\overline{4+2}\times\overline{3+5}=6\times8=48$. And $(a-b)\div(c+d)$, or $\frac{a-b}{c+d}$; implies the excess of a above b, is to be divided by the sum of c and d; if $a=12$, $b=2$, $c=4$, and $d=1$; then, $(a-b)\div(c+d)=(12-2)\div(4+1)=10\div5=2$, or $\frac{a-b}{c+d}=\frac{12-2}{4+1}=\frac{10}{5}=2$.

The line drawn over the quantities is sometimes called a *vinculum.*

11. *Factors* are the numbers or quantities, from the multiplication of which, the proposed numbers or quantities are produced; thus, the factors of 35 are 7 and 5, because $7\times5=35$; also, a and b are the factors of ab; 3, a^2, b and c^2, are the factors of $3a^2bc^2$; and $a+b$ and $a-b$ are the factors of the product $(a+b)\times(a-b)$.

When a number or quantity is produced by the multiplication of two or more factors, it is called a composite number or quantity; thus, 35 is a composite number, being produced by the product of 7 and 5; also, $5acx$ is a composite quantity, the factors of which are 5, a, c, and x.

12. When the factors are all *equal* to each other, the product is called a *power* of one of the factors, and the factor is called the *root* of the product or the power. When there are two equal factors, the product is called the *second power* or *square* of either factor, and the factor is called the *second root* or *square root* of the power. When there are three equal factors, the product is called the *third power* or *cube* of either factor, and the factor is called the third root or *cube root* of the power. And so on for any number of equal factors.

13. Instead of setting down in the manner of other products, the equal factors which multiplied together constitute a power, it is evidently more convenient to set down only one of the equal factors, (or, in other words, the root of the power,) and to designate their number by small figures or letters placed near the root. These figures or letters are always placed at the upper and right side of the root, and are called the *indices* or *exponents* of the power.

For example:

$a\times a\times a\times a$ or $aaaa$ is denoted thus, a^4;
$y\times y\times y\times y\times y$ or $yyyyy$, thus, y^5;

where a^4 and y^5 are the powers ; a and y the roots, and 4 and 5 the indices or exponents of the powers. Again : $4ax^2 \times 4ax^2 \times 4ax^2$, is thus abridged, $(4ax^2)^3$; where $(4ax^2)^3$ is the power, $4ax^2$ the root, and 3 the index or exponent of the power. The same method is adopted, whatever be the form of the root : thus, $(a^2-x^2-y^2)\times(a^2-x^2-y^2)\times(a^2-x^2-y^2)$ is written briefly thus, $(a^2-x^2-y^2)^3$, where $(a^2-x^2-y^2)^3$ is the power, $a^2-x^2-y^2$ the root of the power, and 3 its index or exponent.

N. B. Care must always be taken, to embrace the root in parentheses, except where it is expressed by a single character.

14. The *coefficient* of a quantity is the number or letter prefixed to it ; being that which shows how often the quantity is to be taken ; thus, in the quantities $3b$ and $5x^2$, 3 and 5 are the coefficients of b and x^2. Also, in the quantities $3ay$ and $5a^2x$, $3a$ and $5a^2$ are the coefficients of y and x.

15. When a quantity has no number prefixed to it, the quantity has unity for its coefficient, or it is supposed to be taken only once ; thus, x is the same as $1x$; and when a quantity has no sign before it, the sign $+$ is always understood ; thus, $3a^2b$ is the same as $+3a^2b$, and $5a-3b$ is the same as $+5a-3b$.

16. Quantities which can be expressed in finite terms, or the roots of which can be accurately expressed, are *rational* quantities ; thus, $3a$, $\frac{2}{5}a$, and the square root of $4a^2$, are rational quantities ; for if $a=10$; then, $3a=3\times 10=30$; $\frac{2}{5}a=\frac{2}{5}\times 10=\frac{20}{5}=4$; and the square root of $4a^2=$ the square root of $4\times 10^2=$ the square root of $4\times 10\times 10=$ the square root of $400=20$.

17. An *irrational* quantity, or *surd*, is that of which the value cannot be accurately expressed in numbers, as the square root of 3, 5, 7, &c. ; the cube root of 7, 9, &c.

18. The roots of quantities are expressed by means of the radical sign $\sqrt{}$, with the proper index annexed, or by fractional indices placed at the right-hand of the quantity ; thus, $\sqrt{a}$, or $a^{\frac{1}{2}}$, expresses the square root of a ; $\sqrt[3]{(a+x)}$, or $(a+x)^{\frac{1}{3}}$, the cube root of $(a+x)$; $\sqrt[4]{(a+x)}$, or $(a+x)^{\frac{1}{4}}$, the fourth root of $(a+x)$. When the roots of quantities are expressed by fractional indices ; thus, $a^{\frac{1}{2}}$, $(a+x)^{\frac{1}{3}}$, $(a+x)^{\frac{1}{4}}$; they are generally read a in the power $(\frac{1}{2})$, or a with $(\frac{1}{2})$ for an index ; $(a+x)$ in the power $(\frac{1}{3})$, or $(a+x)$ with $(\frac{1}{3})$ for an index ; and $(a+x)$ in the power $(\frac{1}{4})$, or $(a+x)$ with $(\frac{1}{4})$ for an index.

19. *Like* quantities are such as consist of the same letters or

the same combination of letters, or that differ only in their numeral coefficients; thus, $5a$ and $7a$; $4ax$ and $9ax$; $+2ac$ and $9ac$; $-5ca$; &c., are called like quantities; and *unlike* quantities are such as consist of different letters, or of different combination of letters; thus, $4a$, $3b$, $7ax$, $5ay^2$, &c. are unlike quantities.

20. Algebraic quantities have also different denominations, according to the sign $+$, or $-$.

Positive, or affirmative quantities, are those that are additive, or such as have the sign $+$ prefixed to them; as, $+a$, $+6ab$, or $9ax$.

21. *Negative* quantities are those that are subtractive, or such as have the sign $-$ prefixed to them; as, $-x$, $-3a^2$, $-4ab$, &c. A negative quantity is of an opposite nature to a positive one, with respect to addition and subtraction: the condition of its determination being such, that it must be subtracted when a positive quantity would be added, and the reverse.

22. Also quantities have different denominations, according to the number of terms (connected by the signs $+$ or $-$) of which they consist; thus, a, $3b$, $-4ad$, &c., quantities consisting of *one* term, are called simple quantities, or monomials; $a+x$, a quantity consisting of *two* terms, a binomial; $a-x$ is sometimes called a residual quantity. A trinomial is a quantity consisting of *three* terms; as, $a+2x-3y$; a quadrinomial of *four*; as, $a-b+3x-4y$; and a polynomial, or multinomial, consists of an indefinite number of terms. Quantities consisting of more than one term may be called compound quantities.

23. Quantities the signs of which are all positive or all negative, are said to have *like* signs; thus, $+3a$, $+4x$, $+5ab$, have *like* signs; also, $-4a$, $-3b$, $-4ac$. When some are positive, and others negative, they have *unlike* signs; thus, the quantities $+3a$ and $-5ab$ have *unlike* signs; also, the quantities $-3ax$, $+3a^2x$: and the quantities $-b$, $+b$.

24. If the quotients of two pairs of numbers are equal, the numbers are *proportional*, and the first is to the second, as the third to the fourth; and any quantities, expressed by such numbers, are also proportional; thus, if $\frac{a}{b}=\frac{c}{d}$; then a is to b as c to d. The abbreviation of the proportion; $a : b :: c : d$; and it is sometimes written $a : b = c : d$; if $a=8$, $b=4$, $c=12$, and $d=6$; then, $\frac{8}{4}=\frac{12}{6}=2$, and $8 : 4 :: 12 : 6$.

25. A *term*, is any part or member of a compound quantity, which is separated from the rest by the signs + and —; thus, a and b are the terms of $a+b$; and $3a$, $-2b$, and $+5ad$, are the terms of the compound quantity $3a-2b+5ad$. In like manner, the terms of a product, fraction, or proportion, are the several parts or quantities of which they are composed; thus, a and b are the terms of ab, or of $\frac{a}{b}$; and a, b, c, d, are the terms of the proportion $a : b :: c : d$.

26. A *measure*, or *divisor*, of any quantity, is that which is contained in it some exact number of times; thus, 4 is a measure of 12, and 7 is a measure of $35a$, because $\frac{35a}{7}=5a$.

27. A *prime* number, is that which has no exact divisor, except itself, or unity; 2, 3, 5, 7, 11, &c. and the intervening numbers; 4, 6, 8, &c. are composite numbers. (Art. 11.)

28. *Commensurable* numbers, or quantities, are such as have a common measure; thus, 6 and 8, $8ax$, and $4b$, are commensurable quantities; the common divisors being 2 and 4; also, $4ax^2$ and $5ax$ are commensurable, the common divisor being ax.

29. Also, two or more numbers are said to be *prime* to each other, when they have no common measure or divisor, except unity; as 3 and 5, 7 and 9, 11 and 13, &c.

30. A *multiple* of any quantity, is that which is some exact number of times that quantity; thus, 12 is a multiple of 4; and $15a$ is a multiple of $3a$, because $\frac{15a}{3a}=5$.

31. The *reciprocal* of a quantity is that quantity inverted or unity divided by it. Thus, the reciprocal of a, or of $\frac{a}{1}$ is $\frac{1}{a}$, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$ and the reciprocal of $\frac{a-b}{a+b}$ is $\frac{a+b}{a-b}$.

32. The *reciprocal* of the powers and roots of quantities, is frequently written with a negative index or exponent; thus, the reciprocal of $a^2=\frac{1}{a^2}$, may be written a^{-2}; the reciprocal of $(a+x)^3=\frac{1}{(a+x)^3}$, may be written, $(a+x)^{-3}$; but this method of notation requires some farther explanation, which will be given in a subsequent part of the work.

33. A *function* of one or more quantities, is an expression into which those quantities enter in any manner whatever,

either combined, or not, with known quantities; thus, $a+2x$, $ax+3ax^2$, $5ax^{\frac{1}{2}}-3a^2$, &c. are functions of x; and $3ax^2+xy^2$, $2(x^2+5xy)^2$, &c. are functions of x and y.

34. When quantities are connected by the sign of equality, the expression itself is called an *equation*; thus, $a+b=c+d$, means that the quantities a and b, are equal to the quantities c and d; and this is called an *equation;* it is divided into two members by the sign of equality, $a+b$ is the *first*, and $c+d$ the second member of the *equation*.

35. In algebraical operations the word *therefore*, or *consequently*, often occurs. To express this word, the sign $\therefore$ is generally made use of: thus, $a=b$, *therefore*, $a+c=b+c$; is expressed $\therefore a+c=b+c$.

Also ∞ is the sign of infinity; signifying that the quantity standing before it is of an unlimited value, or greater than any quantity that can be assigned.

36. The signs + and —, give a kind of *quality* or *affection* to the quantities to which they are annexed. As all those terms which have the sign + prefixed to them, are to be *added* (Art. 4), and those quantities which have the sign — prefixed to them, are to be *subtracted*, (Art. 5), from the terms which precede them; the former has a tendency to *increase*, and the latter to *diminish*, the quantities with which they are combined; thus, the compound quantity, $a-x$, will therefore be positive or negative, according to the effect which it produces upon some third quantity b; if a be greater than x, then, (since a is *added*, and b *subtracted*) $b+a-x$ is $>b$; but if a be less than x; then, $b+a-x$ is $<b$.

In the first place, let $a=10$, $x=6$, and $b=8$; then $b+a-x=8+10-6$, which is >8; since $10-6=4$, a positive quantity; therefore, $a-x$ is positive. Next, let $a=12$, $x=14$, and $b=20$; then $b+a-x=20+12-14$, which is <20; since $12-14=-2$, a negative quantity; therefore $a-x$ is negative. In like manner, it may be shown that the expression $a-b+c-d$ is positive or negative according as $a+c$ is $>$ or $<b+d$; and so of all compound quantities whatever.

37. The use of these several *signs*, *symbols*, and *abbreviations*, may be exemplified in the following manner:

EXAMPLES.

Example. 1. In the algebraic expression $a+b+c-d$, let $a=8$, $b=7$, $c=4$, and $d=6$; then

$$a+b+c-d=8+7+4-6=19-6=13.$$

Ex. 2. In the expression $ab+ax-by$, let $a=5$, $b=4$

$x=8$, and $y=12$; then, to find its value, we have $ab+ax-by=5\times4+5\times8-4\times12$

$$=20+40-48$$
$$=60-48=12.$$

Ex. 3. What is the value of $\frac{3ax+2y}{a+b}$, where $a=4$, $x=5$, $y=10$, and $b=6$?

Here $3ax+2y=3\times4\times5+2\times10=60+20=80$, and $a+b=4+6=10$;

$$\therefore \frac{3ax+2y}{a+b}=\frac{80}{10}=8.$$

Ex. 4. What is the value of $a^2+2ab-c+d$, when $a=6$, $b=5$, $c=4$, and $d=1$? Ans. 93.

Ex. 5. What is the value of $ab+ce-bd$, when $a=8$, $b=7$, $c=6$, $d=5$, and $e=1$? Ans. 27.

Ex. 6. In the expression $\frac{ax+by}{b+x}$, let $a=5$, $b=3$, $x=7$, and $y=5$; what is its numerical value? Ans. 5.

Ex. 7. In the expression $\frac{ax^2+b^2}{bx-a^2-c}$, let $a=3$, $b=5$, $c=2$, $x=6$; What is its numerical value? Ans. 7.

Ex. 8. What is the value of $a^2\times(a+b)-2abc$, where $a=6$, $b=5$, and $c=4$? Ans. 156.

Ex. 9. There is a certain algebraic expression consisting of three *terms* connected together by the sign *plus*; the first term of it arises from *multiplying* three times the square of a by the quantity b; the second is the product of a, b and c; and the *third* is *two thirds* of the product of a and b. Required the expression in algebraic writing, and its numerical value, where $a=4$, $b=3$, and $c=2$? Ans. 176.

DEFINITIONS.

38. A *proposition*, is some truth advanced, which is to be demonstrated, or proved; or something proposed to be done or performed; and is either a problem or theorem.

39. A *problem*, is a proposition or question, stated, in order to the investigation of some unknown truth; and which requires the truth of the discovery to be demonstrated.

40. A *theorem*, is a proposition, wherein something is advanced or asserted, the truth of which is proposed to be demonstrated or proved.

41. A *corollary*, or *consectary*, is a truth derived from some

proposition already demonstrated, without the aid of any other proposition.

42. A *lemma*, signifies a proposition previously laid down, in order to render more easy the demonstration of some theorem, or the solution of some problem that is to follow.

43. A *scholium*, is a note, or remark, occasionally made on some preceding proposition, either to show how it might be otherwise effected; or to point out its application and use.

44. An *axiom*, is a self-evident truth, or proposition universally assented to, or which requires no formal proof.

45. As axioms are the first principles upon which all mathematical demonstrations are founded, I will point out those that are necessary to be observed in the study of Algebra, as there will be frequent occasion to advert to them.

AXIOMS.

46. When no difference can be shown or imagined between two quantities, they are equal.

47. Quantities equal to the same quantity, are equal to each other.

48. If to equal quantities equal quantities be added, the wholes will be equal. Thus, if $a=b$, then $a+c=b+c$; if $a-b=c$, then adding b, $a-b+b=c+b$, or $a=c+b$.

49. If from equal quantities equal quantities be subtracted, the remainders will be equal.

If $a=b$, then, $a-2=b-2$; if $b+c=a+c$, then $b=a$.

50. If equal quantities be multiplied by equal numbers or quantities, the products will be equal.

Thus, if $a=b$, $3a=3b$; if $a=\frac{b}{3}$, $3a=b$; if $a=b$, $ca=cb$; and if $a=b$, $a\times a=b\times b$, or $a^2=b^2$.

51. If equal quantities be divided by equal numbers or quantities, the quotients will be equal.

Thus, if $5a=10b$, $\frac{5a}{5}=\frac{10b}{5}$, or $a=2b$; if $ca=cb$, $\frac{ca}{c}=\frac{cb}{c}$, or $a=b$; and if $a^2=ba$, then $\frac{a^2}{a}=\frac{ba}{a}$, or $a=b$.

Scholium. Articles (49), (50), (51), might have been deduced from Art. (48); but they are all easily admitted as axioms.

52. If the same quantity be added to and subtracted from another, the value of the latter will not be altered. Thus, if $a=c$, then $a+b=c+b$, and $a+b-b=c+b-b$, or $a=c$.

This might be inferred from Art. (48).

53. If a quantity be both multiplied and divided by another, its value will not be altered. Thus, if $a=b$; then $3a=3b$, and dividing by 3, $\frac{3a}{3}=\frac{3b}{3}$, or $a=b$.

CHAPTER I.

ON THE

ADDITION, SUBTRACTION, MULTIPLICATION,

AND

DIVISION OF ALGEBRAIC QUANTITIES.

---

§ I. *Addition of Algebraic Quantities.*

54. The *addition* of algebraic quantities is performed by connecting those that are *unlike* with their proper signs, and collecting those that are *like* into one sum; for the more ready effecting of which, it may not be improper to premise a few *propositions*, from which all the necessary rules may be derived.

55. *If two or more quantities are like, and have like signs, the sum of their coefficents prefixed to the same letter, or letters, with the same sign, will express the sum of these quantities.*

Thus, $5a$ added to $7a$ is $=12a$;

And $-5a$ added to $-3a$ is $=-8a$.

For, if the symbol a be made to represent any quantity or thing, which is the object of calculation, $5a$ will represent five times that thing, and $7a$ seven times the same thing, whatever may be the denomination or numeral value of a; and consequently, if the quantities $5a$ and $7a$ are to be incorporated, or added together, their sum will be twelve times the thing denoted by a, or $12a$.

Moreover, since a negative quantity is denoted by the sign of subtraction: thus, if $a+b=a-c$, $b=-c$, and $c=-b$. A debt is a negative kind of property, a loss a negative gain, and a gain a negative loss.

Therefore it is plain that the quantities, $-5a$ and $-3a$ will produce, in any mixed operation, a contrary effect to that of the positive quantities with which they are connected; and consequently, after incorporating them in the same manner as the latter, the sign $-$ must be prefixed to the result; so that if A be greater than a, it is evident that $5(\text{A}-a)+3(\text{A}-a)$, or $(5\text{A}-5a)+(3\text{A}-3a)=8\text{A}-8a$; and therefore the sum of the quantities $-5a$ and $-3a$, when taken in their isolated state, will, by a necessary extension of the proposition be $=-8a$.

56. *If two quantities are like, but have unlike signs, the difference of their coefficients, prefixed to the same letter, or letters, with the sign of that which hath the greater coefficient, will express the sum of those quantities.*

Thus $+6a$ added to $-4a$ is $=+2a$;
And $-6a$ added to $+4a$ is $=-2a$.

Since, Art. (36), the compound quantity $a-b+c-d$, &c. is positive or negative, according as the sum of the positive terms is greater or less than the sum of the negative ones, the aggregate or sum of the quantities $4a-2a+2a-2a$, or $6a-4a$, will be $+2a$: since the sum of the positive terms is greater than the sum of the negative ones. And the sum of the quantities $a-4a+3a-2a$, or $4a-6a$, will be $-2a$; since the sum of the negative terms is greater than the sum of the positive ones.

Corollary. Hence it appears, that if the sum of the positive terms be equal to the sum of the negative ones, their aggregate or sum will be nothing. Thus $5a-5a=0$; and $5a-3a+4a-6a=9a-9a=0$.

57. The preceding proposition is demonstrated in the following manner by Bonnycastle *in his Algebra.* Vol. II. 8vo.

Where the quantities are supposed to be like, but to have unlike signs, the reason of the operation will readily appear, from considering, that the addition of algebraic quantities, taken in a general sense, or without any regard to their particular values, means only the uniting of them together, by means of the arithmetical operations denoted by the signs + and — ; and as these are of contrary, or opposite natures, the less quantity must be taken from the greater, in order to obtain the incorporated mass, and the sign of the greater prefixed to the result. So that if $6a$ is to be added to $4\,(\text{A}-a)$, or to $4\text{A}-4a$, the result will evidently be $4\text{A}+6a-4a$, or $4\text{A}+2a$; and if $4a$ is to be added to $6\,(\text{A}-a)$, or to $6\text{A}-6a$, the result will be $6\text{A}+4a-6a$, or $6\text{A}-2a$; whence, by making this proposition general, as in the *last*, the sum of the isolated quantities $6a$ and $-4a$ will be $+2a$, and that of $4a$ and $-6a$ will be $-2a$.

58. *If two or more quantities be unlike, their sum can only be expressed by writing them after each other, with their proper signs.*

Thus, the sum of $2a$ and $2b$, can only be expressed, with the sign + between them, which denotes that the operation of addition is to be performed when we assign values to a and b.

For, if $a=10$, and $b=5$; then the sum of $2a$ and $2b$ can be neither $4a$ nor $4b$, that is, neither $4\times10=40$, nor $4\times5=20$; but $2\times10+2\times5=20+10=30$. In like manner, the sum of $3a$, $-5b$, $2c$, and $-8d$, can no otherwise be incorporated, or added together, than by means of the signs $+$ and $-$; thus, $3a-5b+2c-8d$.

These propositions being well understood, the following practical rules, for performing the addition of algebraic quantities, which is generally divided into three cases, are readily deduced from them.

CASE I.

When the quantities are like, and have like signs.

RULE.

59. Add all the *numeral* coefficients together, to their sum prefix the common sign when necessary, and subjoin the common quantities, or letters.

EXAMPLE 1.

$$\begin{array}{r} 2x+3a-4b \\ 3x+4a-\ b \\ 7x+\ a-7b \\ x+9a-9b \\ 9x+\ a-\ b \\ x+8a-3b \\ \hline 23x+26a-25b \end{array}$$

In this example, in adding up the first column, we say, $1+9+1+7+3+2=23$, to which the common letter x is subjoined. It is not necessary to prefix the sign $+$ to the result, since the sign of the leading term of any compound algebraic expression, when it is positive, is seldom expressed; for (14) when a quantity has no sign before it, the sign $+$ is always understood. And it may be observed when it has no numeral coefficient, unity or 1 is always understood.

Also, the sum of the second column is found thus, $8+1+9+1+4+3=26$, to which the sign $+$ is prefixed, and the common letter a annexed.

Again, the sum of the third column is found thus; $3+1+9+7+1+4=25$, to which the sign $-$ is prefixed, and the common letter b subjoined. So that the sum of all the quantities is expressed by 23 times x plus 26 times a minus 25 times b.

Ex. 2.	Ex. 3.
$9xy-4bc+7x^2$	$5a^3-3x^2+3y-19$
$4xy-\ bc+3x^2$	$4a^3-\ x^2+4y-17$
$xy-7bc+4x^2$	$a^3-7x^2+7y-14$
$8xy-4bc+\ x^2$	$7a^3-\ x^2+\ y-\ 1$
$7xy-\ bc+9x^2$	$8a^3-9x^2+9y-20$
$xy-3bc+\ x^2$	$7a^3-11x^2+\ y-\ 8$
$30xy-20bc+25x^2$	$32a^3-32x^2+25y-79$

Ex. 4. Add together $2x+3a$, $4x+a$, $5x+8a$, $7x+2a$, and $x+a$. Ans. $19x+15a$.

Ex. 5. Add together $7x^2-5bc$, $3x^2-bc$, x^2-4bc, $5x^2-bc$, and $4x^2-4bc$. Ans. $20x^2-15bc$.

Ex 6. Required the sum of $3x^3+4x^2-x$, $2x^3+x^2-3x$, $7x^3+2x^2-2x$, and $4x^3+2x^2-3x$. Ans. $16x^3+9x^2-9x$.

Ex. 7. What is the sum of $7a^3-3a^2b+2ab^2-3b^3$, $ab^2-a^2b-b^3+4a^3$, $-5b^3+5ab^2-4a^2b+6a^3$, and $-a^2b+4ab^2-4b^3+a^3$? Ans. $18a^3-9a^2b+12ab^2-13b^3$.

Ex. 8. Add together $2x^2y-x+2$, $x^2y-4x+3$, $4x^2y-3x+1$, and $5x^2y-7x+7$. Ans. $12x^2y-15x+13$.

Ex. 9. Required the sum of $30-13x^{\frac{1}{2}}-3xy$, $23-10x^{\frac{1}{2}}-4xy$, $-14x^{\frac{1}{2}}-7xy+14$, $-5xy+10-16x^{\frac{1}{2}}$, and $1-2x^{\frac{1}{2}}-xy$.

Ans. $78-55x^{\frac{1}{2}}-20xy$.

Ex. 10. Add $3(x+y)^2-4(a-b)^3$, $(x+y)^2-(a-b)^3$, $-7(a-b)^3+5(x+y)^2$, and $2(x+y)^2-(a-b)^3$ together. Ans. $11(x+y)^2-13(a-b)^3$.

CASE II.

When the quantities are like, but have unlike signs.

RULE.

60. Add all the positive coefficients into one sum, and those that are negative into another; subtract the *lesser* of these sums from the *greater*; to this *difference*, annex the common letter or letters, prefixing the sign of the *greater*, and the result will be the sum required.

EXAMPLE 1.

$$\begin{array}{r} 7x^3-3x^2+3x \\ -4x^3+\ x^2-4x \\ -\ x^3-2x^2+7x \\ 9x^3+6x^2-9x \\ 3x^3-5x^2+6x \\ -5x^3+3x^2-6x \\ \hline 9x^3\quad *\quad -3x \end{array}$$

In adding up the first column, we say $3+9+7=+19$, and $-(5+1+4)=-10$; then, $+19-10=+9=$ the aggregate sum of the coefficients, to which the common quantity x^3 is annexed.

In the second column, the sum of the positive coefficients is $3+6+1=10$, and the sum of the negative ones is $-(5+2+3)=-10$; then, $10-10=0$; consequently, (by Cor. Art. 56), the aggregate sum of the second column is nothing. And in the third column, the sum of the positive coefficients is $6+7+3=16$, and the sum of the negative one is $-(6+9+4)=-19$; then $+16-19=-3$; to which the common letter is annexed.

Ex. 2

$$\begin{array}{r} 5x^2-6a+4x-3 \\ -2x^2+\ a-9x+7 \\ 7x^2+7a+7x-1 \\ -\ x^2-3a-2x+3 \\ +3x^2+\ a-4x+4 \\ -7x^2-4a+3x-5 \\ \hline 5x^2-4a-\ x+5 \end{array}$$

Ex. 3.

$$\begin{array}{r} 4ab+3xy-2ax+\ c \\ -\ ab-\ xy+\ ax-5c \\ 5ab-2xy-7ax+7c \\ -4ab+\ xy+\ ax+\ c \\ 7ab-3xy+4ax-\ c \\ -\ ab-\ xy-\ ax+4c \\ \hline 10ab-3xy-4ax+7c \end{array}$$

Ex. 4.

$$\begin{array}{r} 3(a+b)^{\frac{1}{2}}-5(x^2+y^2)^2+3(a^3+c^2)^3+9xy \\ -\ (a+b)^{\frac{1}{2}}+\ (x^2+y^2)^2-5(a^3+c^2)^3-4xy \\ +8(a+b)^{\frac{1}{2}}-6(x^2+y^2)^2+8(a^3+c^2)^3+\ xy \\ -2(a+b)^{\frac{1}{2}}-\ (x^2+y^2)^2-7(a^3+c^2)^3-3xy \\ +5(a+b)^{\frac{1}{2}}-7(x^2+y^2)^2-\ (a^3+c^2)^3-\ xy \\ \hline 13(a+b)^{\frac{1}{2}}-18(x^2+y^2)^2-2(a^3+c^2)^3+2xy \end{array}$$

Ex. 5. Required the sum of $4a^2$, $-5a^2$, a^2, $-6a^2$, $9a^2$, and $-a^2$. Ans. $2a^2$.

Ex. 6. Required the sum of $4x^2-3x+4$, $x-2x^2-5$, $1+3x^2-5x$, $2x-4+7x^2$, $13-x^2-4x$. Ans. $11x^2-9x+9$.

Ex. 7. Required the sum of $4x^3-2x+y$, $4x-y-x^3$, $9y+7x^3-x$, $21x-2y+9x^3$. Ans. $19x^3+22x+7y$.

Ex. 8. Required the sum of $5a^3-2ab+b^2$, $ab-2b^2-a^3$, $b^2-3ab+4a^3$, $4ab+2a^3-4b^2$. Ans. $10a^3-4b^2$.

Ex. 9. What is the sum of $2a-3x^2$, $5x^2-7a$, $-3a+x^2$, and $a-3x^2$? Ans. $-7a$.

Ex. 10. What is the sum of $4-3x$, $x-5$, $2x-4$, $-4x+13$, and $-5x+1$? Ans. $9-9x$.

CASE III.

When the quantities are unlike, or when like and unlike are mixed together.

RULE.

61. When the quantities are unlike, write them down, one after another, with their signs and coefficients prefixed; but when some are like, and others unlike, collect all the like quantities together, by taking their sums or differences, as in the foregoing cases, and set down those that are unlike as before.

Example 1. Add together the quantities $7a^2$, $-5b$, $+4d$, $-9a$, and $8c^2$.

Here, the quantities are all unlike; ∴ (Art. 58), their sum must be written thus;

$$7a^2-5b+4d-9a+8c^2.$$

When several quantities are to be added together, in whatever order they are placed, their values remain the same. Thus, $7a^2-5b+4d-9a+8c^2$, $8c^2-5b+4d-9a+7a^2$, or $4d-5b-9a+8c^2+7a^2$, are equivalent expressions: though it is usual, in such cases, to take them so that the leading term shall be positive.

Ex. 2.

$$\begin{array}{l}
3x-y+d \\
4a-x-3y \\
5xy+7ax+y^2 \\
3ax-2xy+4x^2 \\
5y+2d+5x \\
\hline
7x+y+3d+3xy+10ax+4a+y^2+4x^2.
\end{array}$$

Ex. 3.

$$\begin{array}{l} 4x^3-3xy+3y\ -3\ \ -3x^2 \\ 5y^2+5x^3-3x^2+3xy+5y \\ 30\ +6x^2+2x\ -3y^2-2x^3 \\ 2x^2-8\ \ -5xy-7y\ -2y^2 \\ \hline 7x^3-5xy+y+19+2x^2+2x. \end{array}$$

In Ex. 2. Collecting together like quantities, and beginning with $3x$, we have $3x+5x-x=8x-x=(8-1)\,x=7x$; $5y-y-3y=(5-1-3)\,y=(5-4)\,y=y$; $d+2d=(1+2)\,d=3d$; $5xy-2xy=(5-2)xy=3xy$; and $3ax+7ax=(3+7)ax=10ax$; besides which there are three quantities $+4a$, $+y^2$, $+4x^2$; which are unlike, and do not coalesce with any of the others; the sum required therefore is,

$$7x+y+3d+3xy+10ax+4a+y^2+4x^2.$$

In Ex. 3. Beginning with $4x^3$, we have,
$4x^3+5x^3-2x^3=(4+5-2)x^3=(9-2)x^3=7x^3$;
$-3xy+3xy-5xy=(3-5-3)xy=(3-8)xy=-5xy$;
$+3y+5y-7y=(3+5)y-7y=(8-7)y=+y$;
$-3+30-8=30-(8+3)=30-11=+19$;
$2x^2-3x^2-3x^2+6x^2=8x^2-(3+3)x^2=(8-6)x^2=+2x^2$;
$5y^2-3y^2-2y^2=5y^2-(3+2)y^2=(5-5)y^2=0\times y^2=0$; $+2x=2x$.

When quantities with literal coefficients are to be added together; such as mx, ny, px^2, qy^2, &c. (where m, n, p, q, &c., may be considered as the coefficients of x, y, x^2, y^2, &c.) it may be done by placing the coefficients of *like* quantities one after another (with their proper signs), under a vinculum, or in a parentheses, and then annexing the common quantity to the sum or difference.

Ex. 4.

$$\begin{array}{r} ax+by+\ b \\ bx+dy+2b \\ \hline (a+b)x+(b+d)y+3b \end{array}$$

Ex. 5.

$$\begin{array}{r} ax^3+bx^2+cx \\ ex^3-dx^2-fx \\ \hline (a+e)x^3+(b-d)x^2+(c-f)x \end{array}$$

In Ex. 4. The sum of ax and bx, or $ax+bx$, is expressed by $(a+b)x$; the sum of $+by$ and $+dy$, or $+by+dy$, is $=+(b+d)y$.

In Ex. 5. The sum of ax^3 and ex^3, or ax^2+ex^2 is $=(a+e)x^3$; the sum of $+bx^2$ and $-dx^2$, or $+bx^2-dx^2$, is $=(b-d)x^2$; and the sum of $+ex$ and $-fx$, or $+ex-fx$, is $=+(c-f)x$. Any multinomial may be expressed in like manner, thus; the multinomial $mx^2+nx^3-px^2-qx^2$ may be expressed by $(m+n-p-q)x^2$; and the mixed multinomial $pxy+qy^2-rxy+my^2-nxy$, by $(p-r-n)xy+(q+m)y^2$; &c.

Ex. 6. Add $2x^2+y^2+9$, $7xy-3ab-x^2$, $4xy-y-9$, and $x^2y-xy+3x^2$ together.

Ans. $4x^2+y^2+10xy-3ab-y+x^2y$.

Ex. 7. Add together $72a^2$, $24bc$, $70xy$, $-18a^2$, and $-12bc$.

Ans. $54a^2+12bc+70xy$.

Ex. 8. What is the sum of $43xy$, $7x^2$, $-12ay$, $-4ab$, $-3x^2$, and $-4ay$? Ans. $43xy+4x^2-16ay-4ab$.

Ex. 9. What is the sum of $7xy$, $-16bc$, $-12xy$; $18bc$, and $5xy$? Ans. $2bc$.

Ex. 10. Add together $5ax$, $-60bc$, $7ax$, $-4xy$, $-6ax$, and $-12bc$. Ans. $6ax-72bc-4xy$.

Ex. 11. Add $8a^2x^2-3ax$, $7ax-5xy$, $9xy-5ax$, and $xy+2a^2x^2$ together. Ans. $10a^2x^2-ax+5xy$.

Ex. 12. Add $2x^2-3y^2+6$, $9xy-3ax-x^2$, $4y^2-y-6$, and $x^2y-3xy+3x^2$ together. Ans. $4x^2+y^2+6xy-3ax-y+x^2y$.

Ex. 13. Add $2x^{\frac{1}{2}}-4x^{\frac{1}{3}}+x^2$, $5x^2y-ab+x^{\frac{1}{3}}$, $4x^2-x^3$, and $2x^{\frac{1}{3}}-3+2x^{\frac{1}{2}}$ together.

Ans. $4x^{\frac{1}{2}}-x^{\frac{1}{3}}+5x^2+5x^2y-ab-x^3-3$.

Ex. 14. Required the sum of $4x^2+7(a+b)^2$, $4y^2-5(a+b)^2$, and $a^3-4x^2-3y^2-(a+b)^2$. Ans. $a^3+y^2+(a+b)^2$.

Ex. 15. Required the sum of $ax^4-bx^3+cx^2$, $bcx^2-acx^3-c^2x$, and ax^2+c-bx.

Ans. $ax^4-(b+ac)x^3+(c+bc+a)x^2-(c^2+b)x+c$.

Ex. 16. Required the sum of $5a+3b-4c$, $2a-5b+6c+2d$, $a-4b-2c+3e$, and $7a+4b-3c-6e$.

Ans. $15a-2b-3c+2d-3e$.

§ II. *Subtraction of Algebraic Quantities.*

62. Subtraction in Algebra, is finding the difference between two algebraic quantities, and connecting those quantities together with their proper signs; the practical rule for performing the operation is deduced from the following *proposition.*

63. *To subtract one quantity from another, is the same thing as to add it with a contrary sign. Or, that to subtract a posi-*

tive quantity, is the same as to add a negative; and to subtract a negative, is the same as to add a positive.

Thus, if $3a$ is to be subtracted from $8a$, the result will be $8a-3a$, which is $5a$; and if $b-c$ is to be subtracted from a, the result will be $a-(b-c)$, which is equal to $a-b+c$: For since, in this case, it is the difference between b and c that is to be taken from a, it is plain, from the quantity $b-c$, which is to be subtracted, being less than b by c, that if b be only taken away, too much will have been deducted by the quantity c; and therefore c must be added to the result to make it correct.

This will appear more evident from the following consideration; Thus, if it were required to substract 6 from 9, the difference is properly $9-6$, which is 3; and if $6-2$ were subtracted from 9, it is plain that the remainder would be greater by 2, than if 6 only were subtracted; that is, $9-(6-2)=9-6+2=3+2=5$, or $9-6+2=9-4=5$.

Also, if in the above demonstration, $b-c$ were supposed negative, or $b-c=-d$; then, because c is greater than b by d, reciprocally $c-b=d$, so that to subtract $-d$ from a, it is necessary to write $a+d$.

64. The preceding proposition demonstrated after the manner of *Garnier*.

Thus, if $b-c$ is to be subtracted from the quantity a; we will determine the remainder in quantity and sign, according to the condition which every remainder must fulfil; that is, if one quantity be subtracted from another, the remainder added to the quantity that is subtracted, the sum will be the other quantity. Therefore, the result will be $a-b+c$, because $a-b+c+b-c=a$.

This method of reasoning applies with equal facility to compound quantities: in order to give an example;
suppose that from $6a-3b+4c$,
we are to subtract, $5a-5b+6c$;
designating the remainder by R, we have the equality,

$$R+5a-5b+6c=6a-3b+4c:$$

which will not be altered (Art. 49.) by subtracting $5a$, adding $5b$, and subtracting $6c$, from each member of the equality; therefore the result will be,

$$R=6a-3b+4c-5a+5b-6c,$$

or, by making the proper reductions,

$$R=a+2b-2c.$$

65. Another demonstration of the same proposition in *La place's* manner.

Thus we can write,

$$a=a+b-b \quad \ldots\ldots (1),$$

$$a-c=a-c+b-b \quad \ldots\ldots (2);$$

so that if from a we are to subtract $+b$ or $-b$, or, which is the same, if in a we suppress $+b$, or $-b$, the remainder, from transformation (1), must be $a-b$ in the first case, and $a+b$ in the second. Also, if from $a-c$ we take away $+b$ or $-b$, the remainder, from (2), will be $a-c-b$, or $a-c+b$.

66. Hence, we have the following general rule for the subtraction of algebraic quantities.

RULE.

Change the signs of all the quantities to be *subtracted* into the contrary signs, or conceive them to be so changed, and then add, or connect them together, as in the several cases of addition.

EXAMPLE 1. From $18ab$ subtract $14ab$.

Here, changing the sign of $14ab$, it becomes $-14ab$, which being connected to $18ab$ with its proper sign, we have $18ab -14ab=(18-14)ab=4ab$. Ans.

Ex. 2. From $15x^2$ subtract $-10x^2$.

Changing the sign of $-10x^2$, it becomes $+10x^2$, which being connected to $15x^2$ with its proper sign, we have $15x^2+10x^2=25x^2$. Ans.

Ex. 3. From $24ab+7cd$ subtract $18ab+7cd$.

Changing the signs of $18ab+7cd$, we have $-18ab-7cd$, therefore, $24ab+7cd-18ab-7cd=6ab$. Ans.

Or,

$$\begin{array}{r} 24ab+7cd \\ -18ab-7cd \\ \hline 6ab \end{array} \quad \text{Ans.}$$

Ex. 4. Subtract $7a-5b+3ax$ from $12a+10b+13ax-3ab$,

Changing the signs of all the terms of $7a-5b+3ax$; it becomes,

$$\left.\begin{array}{l} 12a+10b+13ax-3ab \\ -7a+5b-3ax \end{array}\right\}$$

$\therefore$ by addition, $5a+15b+10ax-3ab$.

Ex. 5. From $3ab-7ax+7ab+3ax$, take $4ab-3ax-4xy$,

Changing the signs of all the terms of $4ab-3ax-4xy$,

$$\left.\begin{array}{l} 3ab-7ax \\ 7ab+3ax \\ -4ab+3ax+4xy \end{array}\right\}$$

$\therefore$ by addition, $6ab-ax+4xy$. Ans.

Ex. 6.

From $36a-12b+7c$
Take $14a-\ \ 4+7c-8$

Rem. $22a-\ \ 8b+8$ Ans.

In the above example, one row is set under the other, that is, the quantities to be subtracted in the lower line; then, beginning with $14a$, and conceiving its sign to be changed, it becomes $-14a$, which being added to $36a$, we have $36a-14a=22a$; also, $-4b$, with its sign changed, added to $-12b$, will give $4b-12b=(4-12)b=-8b$; in like manner, $7c-7c=0$, and -8, with its sign changed, $=+8$. The following examples are performed in the same manner as the last.

Ex. 7.

From $3x-4a+\ b$
Take $2x+3a-7b$

Rem. $x-7a+8b$

Ex. 8.

$a+\ b$
$a-\ b$

$*+2b$

Ex. 9.

From $3ab-4cx+\ y$
Take $4ax+2x^2-3y^2$

Rem. $3ab-4ax+y-4cx-2x^2+3y^2$

Ex. 10.

$7x^3+3x^2-x$
$6x^3-2x^2+8x$

x^3+5x^2-9x

Ex. 11.

From $5x^2-4xy+5$
Take $4x^2-4xy+9$

Rem. $x^2\ \ *\ \ -4$

Ex. 12.

$7x^2-8$
$9x^2+5ab-3x^3$

$3x^3-2x^2-5ab-8$

Ex. 13.

From $ax^3-bx^2+\ x$
Take px^3-cx^2+ex

$(a-p)x^3-(b-c)x^2+(1-e)x$

Ex. 14.

From $bx^3+qx^2-rx+py^2$
Take $ax^3-cx^2+mx-sy^2$

$(b-a)x^3+(q+c)x^2-(r+m)x+(p+s)y^2$

67. As quantities in a parentheses, or under a vinculum, are

considered as one quantity with respect to other symbols (Art. 10,) the sign prefixed to quantities in a parentheses affects them *all*; when this sign is *negative*, the signs of all those quantities must be changed in putting them into the parentheses.

Thus, in (Ex. 13), when $-cx^2$ is subtracted from $-bx^2$, the result is $-bx^2+cx^2$, or $-(b-c)x^2$: because the sign $-$ prefixed to $(b-c)$ changes the signs of b and c; or it may be written $+(c-b)x^2$.

Again, in (Ex. 14), when $+mx$ is subtracted from $-rx$, the result is $-rx-mx$; and, as this means that the sum of rx and mx is to be *subtracted*, that *negative* sum is to be expressed by $-(rx+mx)=-(r+m)x$. For the same reason, the *multinomial* quantity $-my^2+n^2y^2-aby^2-ry^2+6y^2$, when put into a parentheses, with a *negative* sign prefixed, becomes

$$-(m-n^2+ab+r-6)y^2.$$

Ex. 15. From $a-b$, subtract $a+b$. Ans. $-2b$.

Ex. 16. From $7xy-5y+3x$, subtract $3xy+3y+3x$.

Ans. $4xy-8y$

Ex. 17. What is the difference between $7ax^2+5xy-12ay+5bc$, and $4ax^2+5xy-8ay-4cd$.

Ans. $3ax^2-4ay+5bc+4cd$.

Ex. 18. From $8x^2-3ax+5$, take $5x^2+2ax+5$.

Ans. $3x^2-5ax$.

Ex. 19. From $a+b+c$, take $-a-b-c$.

Ans. $2a+2b+2c$.

Ex. 20. From the sum of $3x^3-4ax+3y^2$, $4y^2+5ax-x^3$, $y^2-ax+5x^3$, and $3ax-2x^2-y^2$; take the *sum* of $5y^2-x^2+x^3$, $ax-x^3+4x^2$, $3x^3-ax-3y^2$, and $7y^2-ax+7$.

Ans. $4x^3+4ax-2y^2-5x^2-7$.

Ex. 21. From the sum of $x^2y^2-x^2y-3xy^2$, $9xy^2-15-3x^2y^2$, and $70+2x^2y^2-3x^2y$, subtract the sum of $5x^2y^2-20+xy^2$, $3x^2y-x^2y^2+ax$, and $3xy^2-4x^2y^2-9+a^2x^2$.

Ans. $2xy^2-7x^2y-ax-a^2x^2+84$.

Ex. 22. From $a^3x^2y^2-m^2x^3+3cx-4x^2-9$: take $a^2x^2y^2-n^2x^3+c^2x+bx^2+3$.

Ans. $(a^3-a^2)x^2y^2-(m^2-n^2)x^3+(3c-c^2)x-(4+b.)x^2-12$.

§ III. *Multiplication of Algebraic Quantities.*

In the multiplication of algebraic quantities, the following propositions are necessary to be observed.

68. *When several quantities are multiplied continually together, the product will be the same, in whatever order they are multiplied.*

Thus, $a\times b=b\times a=ab$.

For it is evident, from the nature of multiplication, that the product contains either of the factors as many times as the other contains an unit. Therefore, the product ab contains a as many times as b contains an unit, that is, b times.

And the same quantity ab, contains b as many times as a contains an unit, that is, a times. Consequently, $a\times b=ba=ab$; so that, for instance, if the numeral value of a be 12, and of b, 8, the product ab will be 12×8, or 8×12, which, in either case, is 96.

In like manner it will appear that $abc=cab=bca$, &c.

69. *If any number of quantities be multiplied continually together, and any other number of quantities be also multiplied continually together, and then those two products be multiplied together; the whole product thence arising will be equal to that arising from the continual multiplication of all the single quantities.*

Thus, $ab\times cd=a\times b\times c\times d=abcd$.

For $ab=a\times b$, and $cd=c\times d$; if x be put $=cd$, then $ab\times cd=ab\times x=a\times b\times x$; but x is $=cd=c\times d$, $\therefore ab\times x=ab\times c\times d=a\times b\times cd=abcd$.

70. *If two quantities be multiplied together, the product will be expressed by the product of their numeral coefficients with the several letters subjoined.*

Thus, $7a\times5b=35ab$.

For $7a$ is $=7\times a$, and $5b=5\times b$, $\therefore 7a\times5b=7\times a\times5\times b=7\times5\times a\times b=35\times ab=35ab$.

71. *The powers of the same quantity are multiplied together by adding the indices.*

Thus, to multiply a^2 by a^3, it is necessary to write the letter a only once, and to give it for an exponent the sum $2+3$, the exponents of the factors; that is, $a^2\times a^3=a^{2+3}=a^5$; because $a^2=a\times a$, and $a^3=a\times a\times a$; therefore $a^2\times a^3=a\times a\times a\times a\times a=a^5$. In general, the product of a^m by a^n, m and n being always entire positive numbers, is a^{m+n}. In fact, a^m is the abbreviation of $a\times a\times a$, &c., continued to m factors, and a^n is $a\times a\times a$, &c., continued to n factors; therefore $a^m\times a^n=a\times a\times a\times a\times a$, &c., continued to $m+n$ factors; which (Art. 12) is a^{m+n}.

Reciprocally a^{m+n} can be replaced by $a^m\times a^n$. The quantity a^m is sometimes called an *exponential.*

72. *If two quantities having like signs are multiplied together, the sign of the product will be* $+$; *if their signs are unlike, the sign of the product will be*$-$.

1. A positive quantity being multiplied by a positive one, the product is positive; thus $+a\times+b=+ab$, because $+a$ is to be added to itself as often as there are units in b, and consequently the product will be $+ab$.

2. A negative quantity being multiplied by a positive one, the product is negative; thus, $-a\times+b=-ab$; because $-a$ is to be added to itself as often as there are units in b, and therefore the product is $-ab$. Or, since adding a negative quantity is equivalent to subtracting a positive one, the more of such quantities that are added, the greater will the whole diminution be, and the sum of the whole, or the product, must be negative.

3. A positive quantity being multiplied by a negative one, the product is negative; thus, $+a\times-b=-ab$; because $+a$ is to be subtracted as often as there are units in b, and consequently the product is $-ab$.

4. A negative quantity being multiplied by a negative one, the product is positive; thus, $-a\times-b=+ab$. For, $a\times-b=-ab$, that is, when the positive quantity a is multiplied by the negative quantity b, the product indicates that a must be subtracted as often as there are units in b; but when a is negative, its subtraction is equivalent to the addition of an equal positive quantity; therefore, in this case, an equal positive quantity must be added as often as there are units in b.

73. *If all the terms of a compound quantity be multiplied separately by a simple one, the sum of all the products taken together, will be equal to the product of the whole compound quantity by the simple one.*

For, in the first place, if $a+b$ be multiplied by c, the product will be $ca+bc$: Since $a+b$ is to be repeated as many times as there are units in b; the product of a by c, that is, ca, is too little by the product of b by c, that is, cb; it is necessary then to augment ca by cb, which will give for the product sought $ca+cb$, where the term $+cb$ arises from multiplying $+b$ by c. It would be found by reasoning in like manner, that the product of c by $a+b$ must be $ca+cb$, where $+cb$ is $c\times+b$. If, in the second place, $a-b$ be multiplied (where a is greater than b) by c, the product will be $ca-cb$. Since $a-b$ is to be repeated as many times as there are units in c; the product of a by c will give too great a result by the product cb; it is necessary then to diminish the product ca by cb, so that the true product is $ca-cb$.

Let, for example, $7-2$ be multiplied by 4; the product will be $28-8$, or 20;

For, 7×4, or 28, is too great by 2×4, or by 8; therefore, the true product will be the first diminished by the second, or $28-8$, that is 20. In fact, $7-2$, or $5\times4=20$. *The term* $-cb$ *of the product, is the product* of $-b$ by c.

It would be found, by reasoning in like manner, that the product of c by $a-b$, must be $ac-bc$, the same as in the preceding, and in which the term $-bc$ is the product of c by $-b$.

If, in the third place, $a+b+d$ be multiplied by c, the product will be $ca+cb+cd$.

For, let $a+b$ be designated by e; then, $e+d$ multiplied by c is equal to $ce+cd$; but ce is equal to $c\times(a+b)=ca+cb$, because e is equal to $a+b$; therefore $(a+b+d)\times c=ca+cb+cd$. Also, if $(a+b)-d$ be multiplied by c, the product will be $ca+cb-cd$; for let $(a+b)=e$, then $(e-d)\times c=ce-cd=c(a+b)-cd=ca+cb-cd$.

Finally, it may be demonstrated in like manner, that if any polynomial, $a+b-d+e-f$, &c., be multiplied by c, the product will be $ca+cb-cd+ce-cf$, &c. Also, if a quantity c be multiplied by any polynomial $a+b-d+e$, &c., the product will be $ac+bc-dc+ec$, &c.

75. *If a compound quantity be multiplied by a compound quantity, the product will be equal to every term of one factor, multiplied by every term of the other factor, and the products added together.*

Let, in the first place, $a+b$ be multiplied by $c+d$: $a+b$ taken c times is $ca+cb$, as we have already proved; but this product is too little by the binomial $a+b$ repeated d times, it is necessary then to add to it $da+db$, and we will have $ca+cb+da+db$ for the product sought; in which the term $+db$ arises from the multiplication of $+b$ by $+d$.

Suppose, in the second place, that $a+b$ is multiplied by $c-d$, the product will be $ca+cb-da-db$.

Because the product of $a+b$ by c, that is, $ca+cb$, is too great by that of $a+b$ by d, which is $da+db$; we will have therefore the true product equal to $ca+cb-da-db$, where *the term* $-db$ *is the product of* $+b$ by $-d$; in multiplying $c-d$ by $a+b$, we will find that $-bd$ is *the product* of $-d$ by $+b$.

Let, in the third place, $a-b$ be multiplied by $c-d$; the product will be $ca-cb-da+db$.

For, the product of $a-b$ by c, that is, $ca-cb$, is too little by that of $a-b$ by d, which is $da-db$; because the multiplier c is too great by d; it is necessary then to subtract the second product from the first, and the difference will be (66) $ca-cb-da+db$.

Here *the term* $+bd$ *results from* $-b$ by $-d$.

Finally, if $a+b+e$ be multiplied by $c+d$ the product will be $ca+cb+ce+ad+bd+de$.

For, in designating $a+b$ by h; then, $(h+e)\times(c+d)=hc+ec+dh+ed$, which is equal to $h\times(c+d)+ec+ed=(a+b)\times(c+d)+ec+ed=ca+cb+ce+ad+bd+de$.

The same mode of reasoning may be extended to compound quantities composed of any number of terms whatever.

76. Cor. Hence, in general, if any two terms which are multiplied have different signs, their product must be preceded by the sign $-$, and if they have the same sign, the product is affected with the sign $+$; agreeably to what has been demonstrated (Art. 72.) where simple quantities, or isolated factors, such as, $+a$, $+b$, $-a$, $-b$, were only considered.

From the division of algebraic quantities into *simple* and *compound*, there arises three cases of Multiplication: the practical rules for performing the operation are easily deduced from the preceding propositions.

CASE I.

When the factors are both simple quantities.

RULE.

77. Multiply the coefficients together, to the product subjoin the letters belonging to both the factors, and the result, with the proper sign prefixed, will be the product required.

	Ex. 1.	Ex. 2.	Ex. 3.	Ex. 4.
Multiply	$3ab$	$5x$	$-6y$	$-4a^2$
By	$4c$	$-3a$	$+3x$	$-6x^2$
Product	$12abc$	$-15ax$	$-18xy$	$+24a^2x^2$

	Ex. 5.	Ex. 6.	Ex. 7.	Ex. 8.
Mul.	$2ax$	$-3a^2c$	x^2y^2	$-5a^2b^2c$
By	$-8ax$	$+5ac^2$	$-7xy$	$-4a^2b^2x$
Pro.	$-16a^2x^2$	$-15a^3c^3$	$-7x^3y^3$	$+20a^4b^4cx$

Ex. 9. Required the product of $4abc$ and $3a^2c$.

Ans. $12a^3bc^2$.

Ex. 10. Required the product of $-7axy$ and $-2acx$.

Ans. $+14a^2cx^2y$.

Ex. 11. Required the product of $7x^2y^3$ and $-3y^2x^3$.

Ans. $-21x^5y^5$.

Ex. 12. Required the product of a^3 and $-a^5$. Ans. $-a^8$.

Ex. 13. Required the product of axz and bx^2z.

Ans. abx^3z^2.

Ex. 14. Required the product of $-xyz$ and abc.

Ans. $-abcxyz$.

Ex. 15. Required the product of $-4b^2cd^2$ and $-2a^3bc^2d$.

Ans. $8a^3b^3c^3d^3$.

Ex. 16. Required the product of $-3a^3$ and $4a$.

Ans. $-12a^4$.

Ex. 17. Required the product of a^2b^3c by a^3bc^2d.

Ans. $a^5b^4c^3d$.

CASE II.

When one factor is Compound and the other Simple.

RULE.

78. Multiply *each term* of the compound factor by the simple factor, as in the last case; then these products placed one after another with their proper signs, will be the product required.

Ex. 1.

Multiply $4xy-3ax+2y$

by $4ax$

Product $16ax^2y-12a^2x^2+8axy$

Ex. 2.

Mul. $4x^3-3x^2-8$

by $-2ax$

Pro. $-8ax^4+6ax^3+16ax$

Ex. 3.

Mul. $8a^3-7b^2+3a-1$

by $2b$

Pro. $16a^3b-14a^2b+6ab-2b$

Ex. 4.

Mul. $3x^2yz^2-xy^2z-2a^2y$

by $-x^2yz$

Pro. $-3x^4y^2z^3+x^3y^3z^2+2a^2x^2y^2z$

Ex. 5. Multiply $8a^2x^2-3b+c$ by $2ac$.
Ans. $16a^3cx^2-6abc+2ac^2$.

Ex. 6. Multiply $-3x^2-4a^2+5$ by $-4ax$.
Ans. $12ax^3+16a^3x-20ax$.

Ex. 7. Multiply a^2+ax+x^2 by ax.
Ans. $a^3x+a^2x^2+ax^3$.

Ex. 8. Multiply x^2-xy+y^2 by $-x^2y$.
Ans. $-x^4y+x^3y^2+x^2y^3$.

Ex. 9. Multiply $3a^2-2ab+3b^2$ by a^2b^2.
Ans. $3a^4b^2-2a^3b^3+3a^2b^4$.

Ex. 10. Multiply a^2x^2-ax+9 by 5. Ans. $5a^2x^2-5ax+45$.

Ex. 11. Multiply $2cd-3ab-3$ by $4ac$.
Ans. $8ac^2d-12a^2bc-12ac$.

Ex. 12. Multiply $7xz+3ab-5y^2$ by $-xy$.
Ans. $-7x^2yz-3abxy+5xy^3$.

Ex. 13. Multiply $a+b-c-d$ by $abcd$.
Ans. $a^2bcd+ab^2cd-abc^2d-abcd^2$.

CASE III.

When both factors are compound quantities.

RULE.

79. Multiply every term of the multiplicand by each term of the multiplier successively, as in the last case; then, add or connect all the partial products together, and the sum will be the product required.

Note. It is necessary to observe that *like* quantities are generally placed *under each other*, in order to facilitate their addition. And if several compound quantities are to be multiplied continually together; thus,

$$(a+b)\times(a-b)\times(a^2+ab+b^2)\times(a^2-ab+b^2).$$

Multiply the first factor by the second, and then that product by the third, and so on to the last factor; but it is sometimes more concise not to observe the order in which the compound quantities, or factors, are placed, as can be readily seen from the following examples.

EXAMPLE 1.

Multiplicand	$2a^4-3ba^3-5b^2a^2$
Multiplier	$a^3-2ba^2+3b^2a$
1st partial pro.	$2a^7-3ba^6-5b^2a^5$
second	$-4ba^6+6b^2a^5+10b^3a^4$
third	$+6b^2a^5-9b^3a^4-15b^4a^3$
Total prod.	$=2a^7-7ba^6+7b^2a^5+\ b^3a^4-15b^4a^3$

Ex. 2.

Multiply	$a+b$
by	$a-b$
1st partial prod.	a^2+ab
second	$-ab-b^2$
Total product	a^2 * $-b^2$

Ex. 3.

Multiply a^2+ab+b^2

by a^2-b^2

1st partial product	$a^4+a^3b+a^2b^2$
second	$-a^2b^2-ab^3-b^4$
Total prod.	a^4+a^3b * $-ab^3-b^4$

Ex. 4.

Multiply $a^4+a^3b-ab^3-b^4$

by a^2-ab+b^2

1st partial prod.	$a^6+a^5b-a^3b^3-a^2b^4$
second	$-a^5b-a^4b^2+a^2b^4+ab^5$
third	$+a^4b^2+a^3b^3-ab^5-b^6$
Total product	a^6 * * * * $-b^6$

Ex. 5.

Multiply a^2+ab+b^2

by a^2-ab+b^3

1st partial prod.	$a^4+a^3b+a^2b^2$
second	$-a^3b-a^2b^2-ab^2$
third	$+a^2b^2+ab^3+b^4$
Total prod.	$a^4 \quad * \quad +a^2b^2 \quad * \quad +b^4$

Ex. 6.

Multiply $a^4+a^2b^2+b^4$
by a^2-b^2

1st partial product	$a^6+a^4b^2+a^2b^4$
second	$-a^4b^2-a^2b^4-b^6$
Total product	$a^6 \quad * \quad * \quad -b^6$

Ex. 7.

Multiply a^2+ab+b^2
by $a-b$

1st partial prod.	$a^3+a^2b+ab^2$
second	$-a^2b-ab^2-b^3$
Total product	$a^3 \quad * \quad * \quad -b^3$

Ex. 8.

Multiply a^2-ab+b^2
by $a+b$

first	$a^3-a^2b+ab^2$
second	$+a^2b-ab^2+b^3$
Product	$a^3 \quad * \quad * \quad +b^3$

Ex. 9.

Mul.	a^3-b^3
by	a^3+b^3
1st.	$a^6-a^3b^3$
2nd.	$+a^3b^3-b^6$
Prod.	$a^6 \quad * \quad -b^6$

Ex. 10.

a^2-ab+b^2
$a-b$

$a^3-a^2b+ab^2$
$-a^2b+ab^2-b^3$

$a^3-2a^2b+2ab^2-b^3$

Ex. 11.

$$\begin{array}{ll} \text{Multiply} & a^2+ab+b^2 \\ \text{by} & a\ +b \\ \hline \text{first} & a^3+a^2b+ab^2 \\ \text{second} & \quad\ \ +a^2b+ab^2+b^3 \\ \hline \text{Product} & a^3+2a^2b+2ab^2+b^3 \\ \hline \end{array}$$

Ex. 12.

$$\begin{array}{lllllll} \text{Mult.} & a^3+2a^2b+2ab^2+b^3 \\ \text{by} & a^3-2a^2b+2ab^2-b^3 \\ \hline \text{1st.} & a^6+2a^5b+2a^4b^2+a^3b^3 \\ \text{2nd.} & \quad -2a^5b-4a^4b^2-4a^3b^3-2a^2b^4 \\ \text{3d.} & \qquad\qquad +2a^4b^2+4a^3b^3+4a^2b^4+2ab^5 \\ \text{4th.} & \qquad\qquad\qquad\quad -a^3b^3-2a^2b^4-2ab^5-b^6 \\ \hline \text{prod.} & a^6 \quad * \quad * \quad * \quad * \quad * \quad -b^6 \\ \hline \end{array}$$

When the quantities to be multiplied together have literal coefficients, proceed as before, putting the sum or difference of the coefficients of the resulting terms into a parentheses, or under a vinculum, as in Addition.

Ex. 13.

$$\begin{array}{ll} \text{Mult.} & x^2-ax+p \\ \text{by} & x^2+bx+3 \\ \hline \text{1st.} & x^4-ax^3+px^2 \\ \text{2nd.} & \quad +bx^3-abx^2+bpx \\ \text{3d.} & \qquad\qquad +3x^2-3ax+3p \\ \hline \text{prod.} & x^4-(a-b)x^3+(p-ab+3)x^2+(bp-3a)x+3p \\ \hline \end{array}$$

Ex. 14.

$$\begin{array}{ll} \text{Mult.} & ax^2-\ bx\ +c \\ \text{by} & \ x^2-\ cx\ +1 \\ \hline \text{1st.} & ax^4-\ bx^3+\ cx^2 \\ \text{2nd.} & \qquad -acx^3+bcx^2-c^2x \\ \text{3d.} & \qquad\qquad\quad +\ ax^2-bx+c \\ \hline \text{prod.} & ax^4-(b+ac)x^3+(c+bc+a)x^2-(c^2+b)+c \\ \hline \end{array}$$

Ex. 15. Required the continual product of $a+2x$, $a-2x$, and a^2+4x^2.

$$\begin{array}{rl} \text{Multiply} & a+2x \\ \text{by} & a-2x \\ \hline & a^2+2ax \\ & \quad -2ax-4x^2 \\ \hline \text{Multiply} & a^2-4x^2 \\ \text{by} & a^2+4x^2 \\ \hline & a^4-4a^2x^2 \\ & \quad +4a^2x^2-16x^4 \\ \hline \text{Total product} & a^4 \quad * \quad -16x^4 \\ \hline \end{array}$$

It may be necessary to observe, that it is usual, in some cases, to write down the quantities that are to be multiplied together, in a parentheses, or under a vinculum, without performing the whole operation; thus, $(a+2x)\times(a-2x)\times(a^2+4x^2)$. This method of representing the multiplication of compound quantities by barely indicating the operation that is to be performed on them, is preferable to that of executing the entire process; particularly when the product of two or more factors is to be divided by some other quantity; because, in this case, any term that is common to both the divisor and dividend may be more readily suppressed; as will be evident, from various instances, in the following part of the work.

Ex. 16. Required the product of $a+b+c$ by $a-b+c$. Ans. $a^2+2ac-b^2+c^2$.

Ex. 17. Required the product of $x+y+z$ by $x-y-z$. Ans. $x^2-y^2-2yz-z^2$.

Ex. 18. Required the product of $1-x+x^2-x^3$ by $1+x$. Ans. $1-x^4$.

Ex. 19. Multiply $a^3+3a^2b+3ab^2+b^3$ by $a^2+2ab+b^2$. Ans. $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$.

Ex. 20. Multiply $4x^2y+3xy-1$ by $2x^2-x$. Ans. $8x^4y+2x^3y-2x^2-3x^2y+x$.

Ex. 21. Multiply $x^3+x^2y+xy^2+y^3$ by $x-y$. Ans. x^4-y^4.

Ex. 22. Multiply $3x^3-2a^2x^2+3a^3$ by $2x^3-3a^2x^2+5a^3$. Ans. $6x^6-13a^2x^5+6a^4x^4+21a^3x^3-19a^5x^2+15a^6$.

Ex. 23. Multiply $2a^2-3ax+4x^2$ by $5a^2-6ax-2x^2$. Ans. $10a^4-27a^3x+34a^2x^2-18ax^3-8x^4$.

Ex. 24. Required the continual product of $a+x$, $a-x$, $a^2+2ax+x^2$, and $a^2-2ax+x^2$. Ans. $a^6-3a^4x^2+3a^2x^4-x^6$.

Ex. 25. Required the product of x^3-ax^2+bx-c, and x^2-2x+3.

Ans. $x^5-(a+2)x^4+(b+2a+3)x^3-(c+2b+3a)x^2+(2c+3b)x-3c$.

Ex. 26. Required the product of mx^2-nx-r and $nx-r$.
Ans. $mnx^3-(n^2+mr)x^2+r^2$.

Ex. 27. Required the product of px^2-rx+q and x^2-rx-q.
Ans. $px^4-(r+pr)x^3+(q+r^2-pq)x^2-q^2$.

Ex. 28. Multiply $3x^2-2xy+5$ by $x^2+2xy-3$.
Ans. $3x^4+4x^3y-4x^2\times(1+y^2)+16xy-15$.

Ex. 29. Multiply $a^3+3a^2b+3ab^2+b^3$ by $a^3-3a^2b+3ab^2-b^3$.
Ans. $a^6-3a^4b^2+3a^2b^4-b^6$.

Ex. 30. Multiply $5a^3-4a^2b+5ab^2-3b^3$ by $4a^2-5ab+2b^2$.
Ans. $20a^5-41a^4b+50a^3b^2-45a^2b^3+25ab^4-6b^5$.

Ex. 31. Required the continual product of $a+x$, $a^2+2ax+x^2$, and $a^3+3a^2x+3ax^2+x^3$.
Ans. $a^6+6a^5x+15a^4x^2+20a^3x^3+15a^2x^4+6ax^5+x^6$.

Ex. 32. Required the continual product of $a-x$, $a^2-2ax+x^2$, and $a^3-3a^2x+3ax^2-x^3$.
Ans. $a^6-6a^5x+15a^4x^2-20a^3x^3+15a^4x^2-6ax^5+x^6$.

§ IV. *Division of Algebraic Quantities.*

80. In the *Division* of algebraic quantities, the same circumstances are to be taken into consideration as in their multiplication, and consequently the following *propositions* must be observed.

81. *If the sign of the divisor and dividend be like, the sign of the quotient will be + ; if unlike, the sign of the quotient will be—.*

The reason of this proposition follows immediately from multiplication.

Thus, if $+a\times+b=+ab$; therefore $\frac{+ab}{+a}=+b$:

$+a\times-b=-ab$; $\therefore$ $\frac{-ab}{+a}=-b$:

$-a\times+b=-ab$; $\therefore$ $\frac{-ab}{-a}=+b$:

$-a\times-b=+ab$; $\therefore$ $\frac{+ab}{-a}=-b$:

82. *If the given quantities have coefficients, the coefficient of the quotient will be equal to the coefficient of the dividend divided by that of the divisor.*

Thus, $4ab\div2b$, or $\frac{4ab}{2b}=2a$.

For, by the nature of division, the product of the quotient, multiplied by the divisor, is equal to the dividend; but the coefficient of a product is equal to the product of the coefficients of the factors (Art. 70). Therefore, $4ab \div 2b = \frac{4}{2} \times \frac{ba}{b} = 2a$.

83. *That the letters of the quotient are those of the dividend not common to the divisor, when all the letters of the divisor are common to be dividend:* for example, the product abc, divided by ab, gives c for the quotient, because the product of ab by c is abc.

84. *But when the divisor comprehends other letters, not common to the dividend, then the division can only be indicated and the quotient written in the form of a fraction, of which the numerator is the product of all the letters of the dividend, not common to the divisor, and the denominator all those of the divisor not common to the dividend:* thus, abc divided by amb, gives for the quotient $\frac{c}{m}$, in observing that we suppress the common factor ab, in the divisor and dividend without altering the quotient, and the division is reduced to that of $\frac{c}{m}$, which admits of no farther reduction without assigning numeral values to c and m.

85. *If all the terms of a compound quantity be divided by a simple one, the sum of the quotients will be equal to the quotient of the whole compound quantity.*

Thus, $\frac{ab}{a} + \frac{ac}{a} + \frac{ad}{a} = \frac{ab+ac+ad}{a} = b+c+d$.

For, $(b+c+d) \times a = ab+ac+ad$.

86. *If any power of a quantity be divided by any other power of the same quantity, the exponent of the quotient will be that of the dividend, diminished by the exponent of the divisor.*

Let us occupy ourselves, in the first place, with the division of two exponentials of the same letter; for instance, $\frac{a^m}{a^n}$, m and n being any positive whole numbers, so that we can have,

$$m > n,\ m = n,\ m < n.$$

It may be necessary to observe that, according to what has been demonstrated (71), with regard to exponentials of the same letter, the letter of the quotient must also be a, and if the unknown exponent of a be designated by x, then a^x will be the quotient, and from the nature of division,

$$a^m = a^n \times a^x = a^{n+x};$$

from which there necessarily results the following equality between the exponents,

$$m=n+x;$$

And as, subtracting n from each of these equal quantities, the two remainders are equal (Art. 49), we shall have,

$$m-n=x \ldots . (1).$$

Therefore, in the first case, where m is $>n$, the exponent of the quotient is $m-n$; thus,

$$a^5 \div a^3=a^{5-3}=a^3, \text{ and } a^3 \div a=a^{3-1}=a^2.$$

Also, it may be demonstrated in like manner, that $(a+x)^5 \div (a+x)^2=(a+x)^{5-2}=(a+x)^3$; and $\frac{(2x+y)^7}{(2x+y)^5}=(2x+y)^{7-5}=(2x+y)^2$.

In the second case, where $m=n$, we shall have,

$$a^m=a^n \times a^x=a^m \times a^x=a^{m+x};$$

From which there results between the exponents the equality,

$$m=m+x,$$

and subtracting m from each of these equals (Art. 49),

$$m-m=x, \text{ or } x=0 \ldots . (2);$$

therefore, the exponent of the quotient will be equal to 0, or $a^x=a^0$, a result which it is necessary to explain. For this purpose, let us resume the division of a^m by a^m, which gives unity for the quotient, or $\frac{a^m}{a^m}=1$; and as two quotients, arising from the same division, are necessarily equal; therefore,

$$a^0=1.$$

Hence, as a may be any quantity whatever, we may conclude that; *any quantity raised to the power zero, must be equal to unity, or* 1, *and that reciprocally unity can be translated into* a^0. This conclusion takes place whatever may be the value of a; which may also be demonstrated in the following manner.

Thus, let $a^0=y$; then, by squaring each member, $a^0 \times a^0=y \times y$, or $a^0=y^2$;

therefore, (47), $y^2=y$,

and (51), $\frac{y^2}{y}=\frac{y}{y}$,

or $y=1$;

but $a^0=y$; consequently $a^0=1$.

In the third case, where m is less than n; let $n=m+d$, d being the excess of n above m; we shall always have,

$$a^m=a^{m+d} \times a^x=a^{m+d+x},$$

and equalising the exponents, because the preceding equality cannot have place, but under this consideration,

$$m=m+d+x,$$

subtracting $m+d$ from both sides, the final result will be

$$x = -d \ldots\ldots (3);$$

then the quotient is a^{-d}.

In order to explain this, let us resume the division of a^m by a^n, or by $a^{m+d} = a^m \times a^d$; by suppressing the factor a^m, which is common to the dividend and divisor, according to what has been demonstrated with regard to the division of letters (Art. 84), we have for the quotient $\frac{1}{a^d}$: therefore,

$$a^{-d} = \frac{1}{a^d} \ldots\ldots (4);$$

This transformation is very useful in various analytical operations; in order to see more clearly the meaning of it, we may recollect that a^{+d} is the same as $a \times a \times a$, &c., continued to d factors; therefore, according to the acceptation and opposition of the signs, a^{-d} must represent $a \times a \times a$ &c., continued to d factors in the divisor.

Hence, according to the results (1), (2), and (3), the proposition is general, when m and n are any whole numbers whatever; thus, $a^3 \div a^5 = a^{3-5} = a^{-2}$, or $\frac{1}{a^2}$; because the divisor multiplied by the quotient is equal to the dividend, $a^5 \times a^{-2} = a^{5-2} = a^3 =$ the dividend, and $\frac{1}{a^2} \times a^5 = \frac{a^5}{a^2} = a^{5-2} = a^3 =$ the dividend, therefore, $a^{-2} = \frac{1}{a^2}$. In like manner it may be shown that, $\frac{1}{a^3} = a^{-3}$, $\frac{1}{a^4} = a^{-4}$, &c. But, according to the result (4), in general, $\frac{1}{a^d} = a^{-d}$, where d may be any whole number whatever; hence the method of notation pointed out, (Art. 32), is evident.

87. If a compound quantity is to be divided by a compound quantity, it frequently occurs that the division cannot be performed, in which case, the division can be only indicated, in representing the quotient by a fraction, in the manner that has been already described (Art. 8).

88. *But if any of the terms of the dividend can be produced by multiplying the divisor by any simple quantity, that simple quantity will be the quotient of all those terms. Then the remaining terms of the dividend may be divided in the same manner, if they can be produced by multiplying the divisor by any other simple quantity; and by continuing the same*

method, until the whole dividend is exhausted; the sum of all those simple quantities will be the quotient of the whole compound quantity.

The reason of this is, that as the whole dividend is made up of all its parts, the divisor is contained in the whole as often as it is contained in all its parts. Thus, $(ab+cb+ad+cd)\div(a+c)$ is equal to $b+d$:

For $b\times(a+c)=ab+cb$; and $d\times(a+c)=ad+cd$; but the sum of $ab+cb$ and $ad+cd$ is equal to $ab+cb+ad+cd$, which is equal to the dividend; therefore $b+d$ is the quotient required.

Also, $(a^2+3ab+2b^2)\div(a+b)$ is equal to $a+2b$.

For, it is evident in the first place, that the quotient will include the term a, since otherwise we should not obtain a^2. Now, from the multiplication of the divisor $a+b$ by a, arises a^2+ab; which quantity being subtracted from the dividend, leaves a remainder $2ab+2b^2$; and this remainder must also be divided by $a+b$, where it is evident that the quotient of this division must contain the term $2b$: again, $2b$, multiplied by $a+b$, produces $2ab+2b^2$; consequently $a+2b$ is the quotient required; which, multiplied by the divisor $a+b$, ought to produce the dividend $a^2+3ab+2b^2$. See the operation at length:

$$
\begin{array}{r}
a+b)a^2+3ab+2b^2(a+2b \\
a^2+\ ab\qquad\quad \\
\hline
2ab+2b^2 \\
2ab+2b^2 \\
\hline
*\qquad
\end{array}
$$

89. Scholium. If the divisor be not exactly contained in the dividend; that is, if by continuing the operation as above, there be a remainder which cannot be produced by the multiplication of the divisor by any simple quantity whatever; then place this remainder over the divisor, in the form of a *fraction*, and annex *it* to the part of the quotient already determined; the result will be the complete quotient.

But in those cases where the operation will not terminate without a remainder; it is commonly most convenient to express the quotient, as in (Art. 87).

90. *Division* being the converse of *multiplication*, it also admits of *three* cases.

CASE I.

When the divisor and dividend are both simple quantities.

RULE.

91. Divide, at first, the coefficient of the dividend by that of the divisor; next, to the quotient annex those letters or factors of the dividend that are not found in the divisor; finally, prefix the proper sign to the result, and it will be the quotient required.

Note. Those letters in the dividend, that are common to it with the divisor, are expunged, when they have the same exponent; but when the exponents are not the same, the exponent of the divisor is subtracted from the exponent of the dividend, and the remainder is the exponent of that letter in the quotient.

EXAMPLE 1. Divide $18ax^2$ by $3ax$.

$$\frac{18ax^2}{3ax}=\frac{18}{3}\times\frac{a}{a}\times\frac{x^2}{x}=6\times1\times x^{2-1}=6x.$$

Or, $\dfrac{18ax^2}{3ax}=\dfrac{18}{3}\times a^{1-1}\times x^{2-1}=6\times a^0\times x=6x.$ See (Art 86.)

Ex. 2. Divide $-48a^2b^2c^2$ by $16abc$.

In the first place, $48\div16=3=$ the coefficient of the quotient, next, $a^2b^2c^2\div abc=a^{2-1}\times b^{2-1}\times c^{2-1}=abc$; now, annexing abc to 3, we have $3abc$, and, prefixing the sign $-$; because the signs of the dividend and divisor are unlike; the result is $-3abc$, which is the quotient required.

Or, the operation may be performed thus,

$$\frac{-48a^2b^2c^2}{16abc}=-\frac{48}{16}\times\frac{a^2}{a}\times\frac{b^2}{b}\times\frac{c^2}{c}=-3\times a\times b\times c=-3abc.$$

Ex. 3. Divide $-21x^3y^3z^4$ by $-7x^2y^2z^3$.

$$\frac{-21x^3y^3z^4}{-7x^2y^2z^3}=+\frac{21}{7}x^{3-2}\times y^{3-2}\times z^{4-3}=+3xyz.$$

Ex. 4. Divide $28a^4b^5c^7$ by $-7a^2b^2c^5$.

$$28a^4b^5c^7\div-7a^2b^2c^5=-\frac{28}{7}\times\frac{a^4}{a^2}\times\frac{b^5}{b^2}\times\frac{c^7}{c^5}=-4\times a^{4-2}\times b^{5-2}\times c^{7-5}=-4\times a^2\times b^3\times c^2=-4a^2b^3c^2.$$

In order that the division could be effected according to the above rule; it is necessary, in the first place, that the divisor contains no letter that is not to be found in the dividend: in the second place, that the exponent of the letters, in the divisor, do not surpass at all that which they have in the dividend; finally, that the coefficient of the divisor, divides exactly that of the dividend.

When these conditions do not take place, then, after cancelling the letters, or factors, that are common to the dividend and divisor; the quotient is expressed in the manner of a fraction, as in (Art. 84).

Ex. 5. Divide $48a^3b^5c^2d$ by $64a^3b^3c^4e$.

The quotient can be only indicated under a fractional form, thus,

$$\frac{48a^3b^5c^2d}{64a^3b^3c^4e}.$$

But the coefficients 48 and 64 are both divisible by 16, suppressing this common factor, the coefficient of the numerator will become 3, and that of the denominator 4. The letter a having the same exponent 3 in both terms of the fraction, it follows that a^3 is a common factor to the dividend and divisor, and that we can also suppress it. The exponent of the letter b is greater in the dividend than in the divisor; it is necessary to divide b^5 by b^3, and the quotient will be b^2, or $\frac{b^5}{b^3}=b^{5-3}=b^2$, which factor will remain in the numerator.

With respect to the letter c, the greater power of it is in the denominator; dividing c^4 by c^2, we have c^2, or $\frac{c^4}{c^2}=c^{4-2}=c^2$, therefore the factor c^2 will remain in the denominator.

Finally, the letters d and e remain in their respective places; because, in the present state, they cannot indicate any factor that is common to either of them.

By these different operations, the quotient, in its most simple form, is $\frac{3b^2d}{4c^2e}$.

Note. The division of such quantities belongs, properly speaking, to the reduction of algebraic fractions.

Ex. 6. Divide $36x^2y^2$ by $9xy$. Ans. $4xy$.

Ex. 7. Divide $30a^2by^2$ by $-6aby$. Ans. $-5ay$.

Ex. 8. Divide $-42c^3x^3y$ by $7c^2x^2$. Ans. $-6cxy$.

Ex. 9. Divide $-4ax^2y^3$ by $-axy^2$. Ans. $+4xy$.

Ex. 10. Divide $16a^5b^3cx$ by $-4a^3bdy$. Ans. $-\frac{4a^2b^2cx}{dy}$.

Ex. 11. Divide $-18a^3b^2c^2$ by $12a^5b^3x$. Ans. $-\frac{3c^2}{2a^2bx}$.

Ex. 12. Divide $17xyzw^2$ by $xzyw$. Ans. $17w$.

Ex. 13. Divide $-12a^3b^3c^3$ by $-6abc$. Ans. $2a^2b^2c^2$.

Ex. 14. Divide $-9x^2y^2z^3$ by $x^4y^4z^4$. Ans. $-\frac{9}{x^2y^2z}$.

Ex. 15. Divide $39a^9$ by $13a^5$. Ans. $3a^4$.

CASE II.

When the divisor is a simple quantity, and the dividend a compound one.

RULE.

92. Divide each term of the dividend separately by the simple divisor, as in the preceding case ; and the sum of the resulting quantities will be the quotient required.

EXAMPLE 1. Divide $18a^3+3a^2b+6ab^2$ by $3a$.

Here, $\frac{18a^3}{3a}=6a^2$, $\frac{3a^2b}{3a}=ab$, and $\frac{6ab^2}{3a}=2b^2$;

therefore, $\frac{18a^3+3a^2b+6ab^2}{3a}=6a^2+ab+2b^2$.

Ex. 2. Divide $20a^2x^3-12a^2x^2+8a^3x^2-2a^4x^2$ by $2ax^2$.

Here, $\frac{20a^2x^3}{2ax^2}=10ax$, $-12a^2x^2\div 2ax^2=-6a$, $8a^3x^2\div 2ax^2 =+4a^2$, and $-2a^4x^2\div 2ax^2=-a^3$;

hence $\frac{20a^2x^3-12a^2x^2+8a^3x^2-2a^4x^2}{2ax^2}=10ax-6a+4a^2-a^3$.

Ex. 3. Divide $20a^2x-15ax^2+30axy^2-5ax$ by $5ax$.

Here $20a^2x\div 5ax=4a$, $-15ax^2\div 5ax=-3x$, $30axy^2\div 5ax=6y^2$, and $-5ax\div 5ax=-1$;

therefore, $\frac{20a^2x-15ax^2+30axy^2-5ax}{5ax}=4a-3x+6y^2-1$.

Ex. 4. Divide $5a^6x-25a^5x^2+50a^4x^3-50a^3x^4+25a^2x^5-5ax^6$ by $5ax$.

Here $\frac{5a^6x}{5ax}=a^5$, $\frac{-25a^5x^2}{5ax}=-5a^4x$, $\frac{+50a^4x^3}{5ax}=+10a^3x^2$, $\frac{-50a^3x^4}{5ax}=-10a^2x^3$, $\frac{+25a^2x^5}{5ax}=+5ax^4$, and $\frac{-5ax^6}{5ax}=-x^5$;

therefore, $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ is the quotient required.

Ex. 5. Divide $3a^4x^2-3a^2x^4$ by $-3a^2x^2$. Ans. x^2-a^2.

Ex. 6 Divide $21a^3x^3-7a^2x^2-14ax$ by $7ax$.

Ans. $3a^2x^2-ax-2$.

Ex. 7. Divide $12abc - 48ax^2y^2 + 64a^2b^2c^2 - 16a^2b^2$ by $-16ab$. Ans. $ab - \frac{3c}{4} + \frac{3x^2y^2}{b} - 4abc^2$.

Ex. 8. Divide $72x^2y^2z^2 - 12axyz + 24bcxyz$ by $12xyz$.
Ans. $6xyz - a + 2bc$.

Ex. 9. Divide $4x^2y^2 - x^4y^4 + 3ax^3y^3$ by x^3y^3.
Ans. $\frac{4}{xy} - xy + 3a$.

Ex. 10. Divide $5a - 7b + 6c - 3ac^2 + 9c^3$ by $3c$.
Ans. $\frac{5a}{3c} - \frac{7b}{3c} + 2 - ac + 3c^2$.

Ex. 11. Divide $-60x^7y + 50x^6y^2 - 40x^5y^3 + 30x^4y^4 - 20x^3y^5 + 10x^2y^6 - 5xy^7$ by $-5xy$.
Ans. $y^6 - 2xy^5 + 4x^2y^4 - 6x^3y^3 + 8x^4y^2 - 10x^5y + 12x^6$.

CASE III.

When the dividend and divisor are both compound quantities.

RULE.

93. Arrange both the dividend and divisor according to the exponents of the same letter, beginning with the *highest*, and place the divisor at the right hand of the dividend; then divide the first term of the dividend by the first term of the divisor, as in Case I., and place the result under the divisor.

Multiply the whole divisor by this partial quotient, and subtract the product from the dividend, and the remainder will be a new dividend.

Again, divide that term of the new dividend, which has the highest exponent, by the first term of the divisor, and the result will be the second term of the quotient. Proceed in the same manner as before, repeating the operation till the dividend is exhausted, and nothing remains, as in common arithmetic. *This rule is evident from* (*Art.* 88).

Example 1. Divide $12a^5b^2 - 6a^4b^3 + 8a^2b^5 - 4a^3b^4 - 22a^6b + 5a^7$ by $4a^2b^2 - 2a^3b + 5a^4$.

It can be readily perceived that the letter a is the one to be chosen, in order to arrange the terms of the dividend and divisor according to *its* powers, beginning with the dividend, $5a^7$ is the term which contains the highest power of a; placing $5a^7$ for the first term, $-22a^6b$, for the second, and so on; the terms of the dividend, arranged according to the powers of a, are written thus;

$$5a^7-22a^6b+12a^5b^2-6a^4b^3-4x^3b^4+8a^2b^5.$$

And the terms of the divisor, arranged according to the powers of a, are written thus;

$$5a^4-2a^3b+4a^2b^2.$$

	Dividend.	*Divisor.*
	$5a^7-22a^6b+12a^5b^2-6a^4b^3-4a^3b^4+8a^2b^5$	$5a^4-2a^3b+4a^2b^2$
	$5a^7-\ 2a^6b+4a^5b^2$	*Quotient.* $a^3-4a^2b+2b^3$
Rem.	$-20a^6b+8a^5b^2-6a^4b^3-4a^3b^4+8a^2b^5$	
	$-20a^6b+8a^5b^2-16a^4b^3$	
Rem.	$+10a^4b^3-4a^3b^4+8a^2b^5$	
	$+10a^4b^3-4a^3b^4+8a^2b^5$	
Rem.	0	

The sign of the first term $5a^7$ of the dividend being the same as that of $5a^4$, the first term of the divisor, the sign of the first term of the quotient is $+$, which is omitted (Art. 14). Dividing $5a^7$ by $5a^4$, the quotient is a^3, which is written under the divisor. Multiplying successively the three terms of the divisor by the first term a^3 of the quotient, and writing the product under the corresponding terms of the dividend; subtracting $5a^7-2a^6b+4a^5b^2$ from the dividend, the remainder is

$$-20a^6b+8a^5b^2-6a^4b^3-4a^3b^4+8a^2b^5.$$

Dividing $-20a^6b$ the first term of this new dividend by $5a^4$, the result will be $-4a^2b$, this quotient having the sign $-$, because the dividend and divisor have different signs. Multiplying all the terms of the divisor by $-4a^2b$; we have $-20a^6b+8a^5b^2-16a^4b^3$; subtracting this result from the partial dividend, the remainder will be $10a^4b^3-4a^2b^4+8a^2b^5$, dividing the first term of this new partial dividend, $10a^4b^2$, by the first term $5a^4$ of the divisor, multiplying all the divisor by the result $+2b^3$, and subtracting the product from the last partial dividend, nothing remains; therefore the last term of the quotient sought is $+2b^3$, and the entire quotient is $a^3-4a^2b+2b^3$.

94. It is very proper to observe that in division, the multiplications of different terms of the quotient by the divisor, produce frequently terms which are not found in the dividend, and which it is necessary to divide afterward by the first term of the divisor. These terms are such as are destroyed when the dividend is formed by the multiplication of the quotient and divisor.

See a remarkable example of these reductions:

Ex. 2. Divide a^3-b^3 by $a-b$.

Division.		Multiplication.
Dividend.	*Divisor.*	Mul. $a-b$
a^3-b^3	$a-b$	by a^2+ab+b^2
a^3-a^2b		
	Quotient.	a^3-a^2b
a^2b-b^3	a^2+ab+b^2	$+a^2b-ab^2$
a^2b-ab^2		$+ab^2-b^3$
ab^2-b^3		a^3 * * $-b^3$
ab^2-b^3		
* *		

The first term a^3 of the dividend divided by the first term a of the divisor, gives a^2 for the first term of the quotient; multiplying the divisor $a-b$ by a^2, the first term of the quotient, the result is a^3-a^2b; subtracting a^3-a^2b from the dividend, the term a^3 destroys the first term of the dividend; but there remains the term $-a^2b$, which is not found at first in the dividend; therefore the remainder is a^2b-b^3. Because the term a^2b contains the letter a, we can divide it by the first term of the divisor, and we obtain $+ab$, which is the second term of the quotient. Multiplying the divisor by $+ab$, the product is a^2b-ab^2, which being subtracted from a^2b-b^3; the first term a^2b destroys the term a^2b which arose from the preceding operation; but there remains the term $-ab^2$, which being not yet in the dividend; the remainder is therefore ab^2-b^3. Dividing ab^2 by a, the result is b^2, which is the third term of the quotient; multiplying the divisor by b^2, we have ab^2-b^3; and subtracting this result from the last remainder, the terms of both destroy one another; so that nothing remains.

In order to comprehend well the mechanism of the division, it is only necessary to take a glance at the multiplication of the quotient a^2+ab+b^2 by the divisor $a-b$, and it will be readily seen that all the terms reproduced in the partial divisions are those which destroy one another in the result of the multiplication.

Ex. 3. Divide y^3-1 by $y-1$.

Dividend.	*Divisor.*
y^3-1	$y-1$
y^3-y^2	
y^2-1	*Quotient.* y^2+y+1
y^2-y	
$y-1$	
$y-1$	
*	

Ex. 4. Divide a^6-x^6 by $a-x$.

Dividend.	*Divisor.*
a^6-x^6	$a-x$
a^6-a^5x	
a^5x-x^6	*Quotient.* $a^5+a^4x+a^3x^2+a^2x^3+ax^4+x^5$
$a^5x-a^4x^2$	
$a^4x^2-x^6$	
$a^4x^2-a^3x^3$	
$a^3x^3-x^6$	
$a^3x^3-a^2x^4$	
$a^2x^4-x^6$	
$a^2x^4-ax^5$	
ax^5-x^6	
ax^5-x^6	
* *	

Ex. 5 Divide x^5+a^5 by $x+a$.

Dividend.	*Divisor.*
x^5+a^5	$x+a$
x^5+ax^4	*Quotient.*
$-ax^4+a^5$	$x^4-ax^3+a^2x^2-a^3x+a^4$
$-ax^4-a^2x^3$	
$a^2x^3+a^5$	
$a^2x^3+a^3x^2$	
$-a^3x^2+a^5$	
$-a^3x^2-a^4x$	
a^4x+a^5	
a^4x+a^5	
* *	

95. When we apply the rule, (Art. 93), to the division of algebraic quantities of which one is not a factor of the other, we know it is impossible to effect the division; because that we arrive, in the course of the operation, at a remainder, of which the first term cannot be divided by that of the divisor. In this case, the remainder is made the numerator of a fraction whose denominator is the divisor; and the fraction thus arising, with its proper sign, is annexed to the other part of the quotient, in order to render its value complete.

Ex. 6. Divide $a^3+a^2b+2b^3$ by a^2+b^2.

	Dividend.	*Divisor.*
	$a^3+a^2b+2b^3$	a^2+b^2
	a^3+ab^2	*Quotient.*
1st rem.	$a^2b-ab^2+2b^3$	$a+b+\dfrac{b^3-ab^2}{a^2+b^2}$
	a^2b+b^3	
2d rem.	$-ab^2+b^3$	

The first term $-ab^2$ of the remainder, cannot be divided by a^2, the first term of the divisor; thus the division terminates at this point. The fraction $\dfrac{-ab^2+b^3}{a^2+b^2}$, having the remainder for its numerator, and the divisor for its denominator, is annexed to the partial quotient $a+b$; and the complete quotient is $a+b+\dfrac{b^3-ab^2}{a^2+b^2}$.

96. It is necessary to remark, that the operation of divi-

sion may be considered as terminated, when the highest power of the letter, in the first or leading term of the remainder, by which the process is regulated, is less than the first term of the divisor; as the succeeding part of the quotient, after this, would necessarily become fractional; and which may be carried on, *ad infinitum*, like a decimal fraction.

This subject belongs to algebraic fractions, and as it is of considerable importance in analysis, we will treat of it with a near attention in the next Chapter.

97. In the preceding examples, the product of the first term of the quotient by the divisor, is placed under the dividend; then the reduction is made by subtraction; and every succeeding product is managed in like manner. In the following examples, the signs of all the terms of the product are changed in placing it under the dividend; and then the reduction is performed by the rules of addition; which is the method adopted by some of the most refined *Analysts.*

Ex. 7. Divide $a^4+2a^2b^2+b^4-c^4$ by $a^2+b^2+c^2$.

	Dividend.	*Divisor.*
	$a^4+2a^2b^2+b^4-c^4$	$a^2+b^2+c^2$
	$-a^4-a^2b^2-a^2c^2$	*Quotient.*
1st. rem.	$a^2b^2-a^2c^2+b^4-c^4$	$a^2+b^2-c^2$
	$-a^2b^2-b^2c^2-b^4$	
2d. rem.	$-a^2c^2-b^2c^2-c^4$	
	$+a^2c^2+b^2c^2+c^4$	
	* * *	

Ex. 8. Divide $6x^4-96$ by $3x-6$.

Dividend.	*Divisor.*
$6x^4-96$	$3x-6$
$-6x^4+12x^3$	*Quotient.*
$+12x^3-96$	$2x^3+4x^2+8x+16$
$-12x^3+24x^2$	
$+24x^2-96$	
$-24x^2+48x$	
$48x-96$	
$48x-96$	
* *	

Ex. 9. Divide $8a^6-4a^3b^2+4a^3+2a^3-b^2+1$ by $2a^3-b^2+1$.

Dividend.	*Divisor.*
$8a^6-4a^3b^2+4a^3+2a^3-b^2+1$	$2a^3-b^2+1$
$8a^6+4a^3b^2-4a^3$	*Quotient.*
$2a^3-b^2+1$	$4a^3+1$
$-2a^3+b^2-1$	
* * *	

98. The division of algebraic quantities can be sometimes facilitated by decomposing, at sight, a quantity into its factors ; thus, in the above example, the divisor forms the last three terms of the dividend, it is only necessary to seek if it be a factor of the first three ; but those have visibly for a common factor $4a^3$, for $8a^6-4a^3b^2+4a^3=4a^3\times(2a^3-b^2+1)$.
By this observation, the dividend will become

$$4a^3(2a^3-b^2+1)+2a^3-b^2+1,$$

or $$(2a^3-b^2+1)\times(4a^3+1):$$

therefore the division is immediately effected, by suppressing the factor $2a^3-b^2+1$ equal to the divisor, and the quotient will be $4a^3+1$.

Experience, in algebraic calculations, will suggest a great many remarks of this kind, by which the operations can be frequently abridged.

99. It sometimes happens that, in arranging the dividend and the divisor according to the same letter, there occur several terms in which this letter has the same exponent : In this case, it is necessary to range in the same column those terms, observing to order them according to another letter, common to the two quantities.

Ex. 10. Divide $-a^4b^2+b^2c^4-a^2c^4-a^6+2a^4c^2+b^6+2b^4c^2+a^2b^4$ by $a^2-b^2-c^2$.

Ordering the dividend according to the letter a, we will place in the same column the terms $-a^4b^2$ and $+2a^4c^2$, in another the terms $+a^2b^4$ and $-a^2c^4$; finally, in the last column the three terms $+b^6$, $+2b^4c^2$, $+b^2c^4$, ordering them according to the exponents of the letter b ; then the quantities, so arranged, will stand thus :

Dividend.	*Divisor.*
$-a^6-a^4b^2+a^2b^4+b^6$	$a^2-b^2-c^2$
$+2a^4c^2-a^2c^4+2b^4c^2$	
$+b^2c^4$	*Quotient.*
$+a^6-a^4b^2$	$-a^4-2a^2b^2-b^4$
$-a^4c^2$	$+a^2c^2-b^2c^2$

$$\begin{array}{ll}
\text{1st rem.} & -2a^4b^2+a^2b^4+b^6 \\
 & +a^4c^2-a^2c^4+2b^4c^2 \\
 & \qquad\qquad +b^2c^4 \\
 & +2a^4b^2-2a^2b^4 \\
 & \qquad -2a^2b^2c^2 \\ \hline
\text{2d rem.} & +a^4c^2-a^2b^4+b^6 \\
 & \qquad -2a^2b^2c^2+2b^4c^2 \\
 & \qquad -a^2c^4+b^2c^4 \\
 & -a^4c^2+a^2b^2c^2 \\
 & \qquad +a^2c^4 \\ \hline
\text{3d rem.} & -a^2b^4+b^6 \\
 & -a^2b^2c^2+2b^4c^2 \\
 & \qquad +b^2c^4 \\
 & +a^2b^4-b^6 \\
 & \qquad -b^4c^2 \\ \hline
\text{4th rem.} & -a^2b^2c^2+b^4c^2 \\
 & \qquad +b^2c^4 \\
 & +a^2b^2c^2-b^4c^2 \\
 & \qquad -b^2c^4 \\ \hline
 & * \quad *
\end{array}$$

Ex. 11. Divide $ax^4-(b+ac)x^3+(c+bc+a)x^2-(c^2+b)x+c$ by ax^2-bx+c.

Dividend.	*Divisor.*
$ax^4-(b+ac)x^3+(c+bc+a)x^2-(c^2+b)x+c$	ax^2-bx+c
$-ax^4 \quad +bx^3 \quad -cx^2$	
$-acx^3+(bc+a)x^2-(c^2+b)x+c$	*Quotient.*
$+acx^3 \quad -bcx^2 \quad +c^2x$	x^2-cx+1

$$\begin{array}{c}
ax^2-bx+c \\
-ax^2+bx-c \\ \hline
* \quad * \quad *
\end{array}$$

100. The following practical examples may be wrought according to either of the methods pointed out, (Art. 93, 97); but in complicated cases, the latter should be preferred. See Example 10.

Ex. 12. Divide $x^6-x^4+x^3-x^2+2x-1$ by x^2+x-1.
Ans. $x^4-x^3+x^2-x+1$.

Ex. 13. Divide $a^5+5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^3-3a^2x+3ax^2-x^3$.
Ans. $a^2-2ax+x^2$.

Ex. 14. Divide $2x^3-19x^2+26x-16$ by $x-8$.
Ans. $2x^2-3x+2$.

Ex. 15. Divide $48y^3-76ay^2-64a^2y+105a^3$ by $2y-3a$.
Ans. $24y^2-2ay-35a^2$.

Ex. 16. Divide a^2-b^2 by $a-b$. Ans. $a+b$.

Ex. 17. Divide a^4-x^4 by a^2-x^2. Ans. a^2+x^2.

Ex. 18. Divide a^6-b^6 by $a^3+2a^2b+2ab^2+b^3$.
Ans. $a^3-2a^2b+2ab^2-b^3$.

Ex. 19. Divide $a^4+a^2b^2+b^4$ by a^2-ab+b^2.
Ans. a^2+ab+b^2.

Ex. 20. Divide $25x^6-x^4-2x^3-8x^2$ by $5x^3-4x^2$.
Ans. $5x^3+4x^2+3x+2$.

Ex. 21. Divide $a^2+4ab+4b^2+c^2$ by $a+2b$.

Ans. $a+2b+\frac{c^2}{a+2b}$.

Ex. 22. Divide $8a^4-2a^3b-13a^2b^2-3ab^3$ by $4a^2+5ab+b^2$.
Ans. $2a^2-3ab$.

Ex. 23. Divide $20a^5-41a^4b+50a^3b^2-45a^2b^3+25ab^4-6b^5$ by $4a^2-5ab+2b^2$.
Ans. $5a^3-4a^2b+5ab^2-3b^3$.

Ex. 24. Divide $a^4+8a^3x+24a^2x^2+32ax^3+16x^4$ by $a+2x$.
Ans. $a^3+6a^2x+12ax^2+8x^3$.

Ex. 25. Divide $x^4-(a-b)x^3+(p-ab+3)x^2+(bp-3a)x+3p$ by x^2-ax+p. Ans. x^2+bx+3.

Ex. 26. Divide $ax^3-(a^2+b)x^2+b^2$ by $ax-b$.
Ans. x^2-ax-b.

Ex. 27. Divide $y^6+a^2y^4+b^4y^2-a^6-2b^2y^4-a^4y^2-2a^4b^2-a^2b^4$ by $y^4+2a^2y^2+a^4-b^2y^2+a^2b^2$.
Ans. $y^2-a^2-b^2$.

Ex. 28. Divide $9x^6-46x^5+95x^2+150x$ by x^2-4x-5.
Ans. $9x^4-10x^3+5x^2-30x$.

Ex. 29. Divide $6a^4+9a^2-15a$ by $3a^2-3a$.
Ans. $2a^2+2a+5$.

Ex. 30. Divide $2a^4-16a^3b+31a^2b^2-38ab^3+24b^4$ by $2a^2-3ab+4b^2$. Ans. $a^2-5ab+6b^2$.

Ex. 31. Divide $a^8+8a^7x+28a^6x^2+56a^5x^3+70a^4x^4+56a^3x^5+28a^2x^6+8ax^7+x^8$ by $a^4+4a^3x+6a^2x^2+4ax^3+x^4$.
Ans. $a^4+4a^3x+6a^2x^2+4ax^3+x^4$.

Ex. 32. Divide $a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6$ by $a^3-3a^2x+3ax^2-x^3$. Ans. $a^3-3a^2x+3ax^2-x^3$

§ V. *Some General Theorems, Observations, &c.*

101. NEWTON calls Algebra *Universal Arithmetic.* This denomination, says LAGRANGE, in his *Traité de la Résolution des Equations numériques*, is exact in some respects; but it does not make sufficiently known the real difference between Arithmetic and Algebra.

Algebra differs from Arithmetic chiefly in this; that in the latter, every figure has a determinate and individual value peculiar to itself; whereas the algebraic characters being general, or independent of any particular or partial signification, represent all sorts of numbers, or quantities according to the nature of the question to which they are applied.

Hence, when any of the operations of addition, subtraction, &c., are to be made upon numbers, or other magnitudes, which are represented by the letters, a, b, c, &c., it is obvious that the results so obtained will be general; and that any particular case, of a similar kind, may be readily derived from them, by barely substituting for every letter its real numeral value, and then computing the amount accordingly.

Another advantage, also, which arises from this general mode of notation, is, that while the figures employed in Arithmetic disappear in the course of the operation, the characters used in Algebra always retain their original form, so as to show the dependence they have upon each other in every part of the process; which circumstance, together with that of representing the operations of addition, subtraction, &c., by means of certain signs, renders both the language and algorithm of this science extremely simple and commodious.

Besides the advantages which the algebraic method of notation possesses over that of numbers, it may be observed, that even in this early part of the science we are furnished with the means of obtaining several general theorems that could not be well established by the principles of Arithmetic.

102. *The greater of any two numbers is equal to half their sum added to half their difference, and the less is equal to half their sum minus half their difference.*

Let a and b be any two numbers, of which a is the greater; let their sum be represented by s; and their difference by d. Then,

$$\left.\begin{array}{l} a+b=s \\ a-b=d \end{array}\right\}$$

$$\left.\begin{array}{lll} \therefore \text{ by addition, } & 2a=s+d & \text{(Art. 48);} \\ \text{and} & a=\frac{s}{2}+\frac{d}{2} & \text{(Art. 51);} \end{array}\right\}$$

$$\left.\begin{array}{lll} \text{By subtraction, } & 2b=s-d & \text{(Art. 49);} \\ \text{and } \therefore & b=\frac{s}{2}-\frac{d}{2} & \text{(Art. 51);} \end{array}\right\}$$

Cor. 1. Hence if the sum and difference of any two numbers be given, we can readily find each of the numbers; thus, if s be equal to the sum of two numbers, and d equal to the difference; then the general expression for the first, is $\frac{s+d}{2}$, and for the second $\frac{s-d}{2}$.

Whatever may be the numeral values that we assign to s and d, or whatever values these letters must represent in a particular question, we have but to substitute them in the above expressions, in order to ascertain the numbers required: For example,

Given the sum of two numbers equal to 36, and the difference equal to 8:

Then, by substituting 36 for s, and 8 for d, in $\frac{s+d}{2}$ and $\frac{s-d}{2}$, we have $\frac{s+d}{2}=\frac{36+8}{2}=\frac{44}{2}=22$, and $\frac{s-d}{2}=\frac{36-8}{2}=\frac{28}{2}=14$. So that, 22 and 14 are the numbers required.

Cor. 2. Also, if it were required *to divide the number s into two such parts, that the first will exceed the second by d.* It appears evident, that the general expression for the first part is $\frac{s+d}{2}$, and for the second $\frac{s-d}{2}$; s and d representing any numbers whatever.

103. The general expression $\frac{s+d}{2}$ may be found after the manner of *Garnier*. Thus, let x represent the first part; then according to the enunciation of the question, $x-d$ will be the second; and, as any quantity is equal to the sum of all its parts, we have therefore,

$$x+x-d=s, \text{ or } 2x-d=s.$$

This equality will not be altered, by adding the number d to each member, and then it becomes,

$$2x-d+d=s+d, \text{ or } 2x=s+d;$$

dividing each member by 2, we have the equality, $x=\frac{s+d}{2}$; in which we read that the number sought is equal to half the sum of the two numbers s and d; thus the relation between the unknown and known numbers remaining the same, the question is resolved in general for all numbers s and d.

104. We have not here the numerical value of the unknown quantity; but the system of operations that is to be performed upon the given quantities; in order to deduce from them, according to the conditions of the problem, the value of the quantity sought; and the expression that indicates these operations, is called *a formula.*

It is thus, for example, that if we denote by a the tens of a number, and the units by b, we have this constant composition of a square, or this *formula*,

$$a^2+2ab+b^2;$$

this algebraic expression is a brief enunciation of the rules to be pursued in order to pass from a number to its square.

105. From whence we infer that, *if a number be divided into any two parts, the square of the number is equal to the square of the two parts, together with twice the product of those parts.*

Which may be demonstrated thus; let the number n be divided into any two parts a and b;

$$\text{Then } n=a+b,$$
$$\text{and } n=a+b;$$

$\therefore$ by Multiplication, $n^2=a^2+2ab+b^2$ (Art. 50).

106. *If the sum and difference of any two numbers or quantities be multiplied together, their product gives the difference of their squares, observing to take with the sign — that of the two squares whose root is subtracted.*

Let M and N represent any two quantities, or polynomials whatever, of which M is the greater; then $(M+N)\times(M-N)$ is equal to M^2-N^2; for the operation stands thus;

$$(M+N)\times(M-N)=\left.\begin{matrix}M^2+MN\\ -MN-N^2\end{matrix}\right\}=M^2-N^2.$$

107. When we put $M=a^3$, and $N=b^3$; then,

$(a^3+b^3)\times(a^3-b^3)=a^6-b^6$; (See Ex. 9. page 30).

Where a^6 is the square of a^3, and b^6 that of b^3, and this last square is subtracted from the first.

Reciprocally, *the difference of two squares* M^2-N^2, *can be put under the form* $(M+N)\times(M-N)$.

This result is a *formula* that should be remembered.

108. *The difference of any two equal powers of different quantities is always divisible by the difference of their roots, whether the exponent of the power be even or odd.* For since

$$\frac{x^2-a^2}{x-a}=x+a;$$

$$\frac{x^3-a^3}{x-a}=x^2+ax+a^2;$$

$$\frac{x^4-a^4}{x-a}=x^3+ax^2+a^2x+a^3;$$

$$\frac{x^5-a^5}{x-a}=x^4+ax^3+a^2x^2+a^3x+a^4;$$

$$\frac{x^6-a^6}{x-a}=x^5+ax^4+a^2x^3+a^3x^2+a^4x+a^5;$$

We may conclude that in general, x^m-a^m is divisible by $x-a$, m being an entire positive number; that is,

$$\frac{x^m-a^m}{x-a}=x^{m-1}+ax^{m-2}+\ldots+a^{m-2}x+a^{m-1}\ \ldots\ (1).$$

109. *The difference of any two equal powers of different quantities, is also divisible by the sum of their roots, when the exponent of the power is an even number.* For since

$$\frac{x^2-a^2}{x+a}=x-a;$$

$$\frac{x^4-a^4}{x-a}=x^3-ax^2+a^2x-a^3;$$

&c. &c . . . .

Hence we may conclude that, in general,

$$\frac{x^{2m}-a^{2m}}{x+a}=x^{2m-1}-ax^{2m-2}+\ldots+a^{2m-2}x-a^{2m-1}\ .\ (2).$$

110. *And the sum of any two equal powers of different quantities, is also divisible by the sum of their roots, when the exponent of the power is an odd number.* For since

$$\frac{x^3+a^3}{x+a}=x^2-ax+x^2;$$

$$\frac{x^5+a^5}{x+a}=x^4-ax^3+a^2x^2+a^3x+a^4;$$

Hence we may conclude that, in general,

$$\frac{x^{2m+1}+a^{2m+1}}{x+a}=x^{2m}-ax^{2m-1}+\ldots-a^{2m-1}x+a^{2m}.\ (3).$$

6*

111. In the formulæ (1), (2), (3), as well as in all others of a similar kind, it is to be observed, that if m be any whole number whatever, $2m$ will always be an even number, and $2m+1$ an odd number; so that $2m$ is a general formula for even numbers, and $2m+1$ for odd numbers.

112. Also, if a in each of the above formulæ, be taken $=1$, and x being always considered greater than a; they will stand as follows:

$$\frac{x^m-1}{x-1}=x^{m-1}+x^{m-2}+x^{m-3}+\ \ldots\ +x+1\ \ldots\ (4).$$

$$\frac{x^{2m}-1}{x+1}=x^{2m-1}-x^{2m-2}+x^{2m-3}-\ \ldots\ +x-1\ \ldots\ (5).$$

$$\frac{x^{2m+1}+1}{x+1}=x^{2m}-x^{2m-1}+x^{2m-2}-\ldots-x+1\ \ldots\ (6).$$

113. And if any two unequal powers of the same root be taken, it is plain, from what is here shown, that

$$x^m-x^n,\text{ or }x^n(x^{m-n}-1)\ \ldots\ldots\ (7),$$

is divisible by $x-1$, whether $m-n$ be even or odd; and that

$$x^m-x^n,\text{ or }x^n(x^{m-n}-1)\ \ldots\ldots\ (8),$$

is divisible by $x+1$, where $m-n$ is an even number; as also that

$$x^m+x^n,\text{ or }x^n(x^{m-n}+1)\ \ldots\ldots\ (9),$$

is divisible by $x+1$, when $m-n$ is an odd number.

114. It is very proper to remark, that the number of all the factors, both equal and unequal, which enter in the formation of any product whatever, is called the degree of that product. The product a^2b^3c, for example, which comprehends six simple factors, is of the *sixth degree;* this, a^7b^2c is of the *tenth degree;* and so on.

Also, that if all the terms of a polynomial, or compound quantity, be of the same degree, it is said to be *homogeneous.* And it is evident from the rules established in Multiplication, *that if two polynomials be homogeneous; their product will be also homogeneous; and of the degree marked by the sum of the numbers which designate the degree of those factors.*

Thus, in Ex. 1, page 29, the multiplicand is of the fourth degree, the multiplier of the third, and the product of the degree $4+3$, or of the seventh degree.

In Ex, 12, page 31, the multiplicand is of the third degree, the multiplier of the third, and the product of the degree $3+3$, or of the sixth degree.

Hence, we can readily discover, by inspection only, the errors of a product, which might be committed by forgetting some one of the factors in the partial multiplications.

CHAPTER II.

ON

ALGEBRAIC FRACTIONS.

115. We have seen in the division of two simple quantities (Art. 84,) that when certain letters, factors in the divisor, are not common to the dividend, and reciprocally, the division can only be indicated, and then the quotient is represented by a fraction whose numerator is the product of all the letters of the dividend, not common to the divisor, and denominator, all those letters of the divisor, not common to the dividend.

Let, for example, *abmn* be divided by *cdmn*; then,

$$\frac{abmn}{cdmn}=\frac{ab}{cd}.$$

It may be observed, that the fraction $\frac{ab}{cd}$ may be a whole number for certain numeral values of the letters a, b, c, and d; thus, if we had $a=4$, $b=6$, $c=2$, $d=3$; but that, generally speaking, it will be a numerical fraction which can be reduced to a more simple expression.

§ I. *Theory of Algebraic Fractions.*

116. It is evident (Art. 103,) that *if we perform the same operation on each of the two members of an equality*, that is, *upon two equivalent quantities or numbers, the results shall always be equal.*

It is by passing thus from the fractional notation to the algorithm of equality, that the process to be pursued in the researches of properties and rules, becomes simple and uniform.

117. Let therefore the equality be

$$a=b\times v \;\ldots\ldots\; (1).$$

when we divide both sides by b which has no factor common with a, we shall have

$$\frac{a}{b}=v \;\ldots\ldots\; (2).$$

Thus v will represent the value of the fraction $\frac{a}{b}$, or the quotient of the division of a by b.

118. *If the numerator and denominator of a fraction be both multiplied, or both divided by the same quantity, its value will not be altered.*

For, if we multiply by m the two members of the equality (1), we will have these equivalent results,

$$ma = mb \times v \ldots\ldots (3);$$

dividing both by mb, we shall have

$$\frac{ma}{mb} = v;$$

but $\frac{a}{b} = v$; therefore

$$\frac{ma}{mb} = v = \frac{a}{b} \ldots\ldots (4),$$

m being any whole or fractional number whatever.

119. *If the fraction is to be multiplied by m, it is the same whether the numerator be multiplied by it, or the denominator divided by it.*

For, if we divide by b, the two members of the equality (3), we obtain the following,

$$\frac{ma}{b} = m \times v \ldots\ldots (5).$$

The equality (1) may also be put under the form

$$a = \frac{1}{m} b \times mv \ldots\ldots (6),$$

whence we derive, dividing each side by $\frac{1}{m}b$,

$$\frac{a}{\frac{1}{m}b} = m \times v \ldots\ldots (7).$$

120. *If a fraction is to be divided by m, it is the same whether the numerator be divided by m, or the denominator multiplied by it.*

For, from the equality (1), we deduce these

$$(8) \ldots . \frac{a}{m} = b \times \frac{v}{m},\ a = mb \times \frac{v}{m} \ldots . (9),$$

dividing the first by b and the second by mb, in order to have $\frac{v}{m}$, they become

$$(10) \ldots . \frac{\frac{a}{m}}{b} = \frac{v}{m};\ \frac{a}{mb} = \frac{v}{m} \ldots . (11).$$

It is to be observed, that in $\frac{\frac{a}{m}}{b}$, the numerator is $\frac{a}{m}$ and the denominator b, and that we employ the greater line for separating the numerator from the denominator.

121. *If two fractions have a common denominator, their sum will be equal to the sum of their numerators divided by the common denominator.*

For, let now the two equalities be

$$(12) \ldots . a=b\times v;\ a'=b\times v' \ldots . (13),$$

corresponding to the fractions

$$\frac{a}{b}=v,\ \frac{a'}{b'}=v',$$

which have the same denominator; adding the two equalities (12) and (13), we shall have

$$a+a'=bv+bv'=b(v+v');$$

and dividing both members by b, in order to have the sum sought $v+v'$, it becomes

$$\frac{a+a'}{b}=v+v' \ldots . (14).$$

Note. In adding the above equalitiés, the corresponding members are added; that is, the two members on the left-hand side of the sign $=$, are added together, and likewise those on the right. The same thing is to be understood when two equalities are subtracted, multiplied, &c.

122. *If two fractions have a common denominator, their difference is equal to the difference of their numerators divided by the common denominator.*

For, if we subtract the equality (13) from (12), we shall have

$$a-a'=bv-bv'=b(v-v');$$

dividing each side by b, and we will obtain

$$\frac{a-a'}{b}=v-v' \ldots\ldots (15).$$

123. Let us suppose that the fractions have different denominators, or that we have the equalities

$$a=b\ .\ v,\ a'=b'\ .\ v';$$

we will multiply the two members of the first by b', and those of the second by b, an operation which will give

$$ab'=bb'v,\ a'b=bb'v';$$

then adding and subtracting, we have

$$ab'\pm a'b=bb'(v\pm v'),$$

the double sign $\pm$ which we read *plus* or *minus*, indicating at the same time both addition and subtraction; dividing each side by bb', in order to find the sum and difference sought $v\pm v'$, we will have

$$\frac{ab' \pm a'b}{bb'} = v \pm v' \ldots\ldots (16);$$

from whence we might readily derive the rule for the addition and subtraction of fractions not reduced to the same denominator.

124. It would be without doubt more simple to have recourse to property (4) in order to reduce to the same denominator the fractions

$$\frac{a}{b} = v, \frac{a'}{b'} = v';$$

but our object is to show, that the principle of equality is sufficient to establish all the doctrine of fractions.

125. We have given the rule for multiplying a fraction by a whole number, which will also answer for the multiplication of a whole number by a fraction.

Now, let us suppose that two fractions are to be multiplied by one another.

Let the two equalities be

$$a = b \, . \, v, \; a' = b' \, . \, v';$$

multiplying one by the other, the two products will be equal; thus,

$$aa' = bb' \, . \, vv',$$

and dividing each side by bb', in order to have the product sought vv', we will obtain

$$\frac{aa'}{bb'} = vv' \ldots\ldots (17).$$

Therefore *the product of two fractions, is a fraction having for its numerator the product of the numerators, and for its denominator that of the denominators.*

126. It now remains to show how a whole number is to be divided by a fraction; and also, how one fraction is to be divided by another.

Let, in the first case, the two equalities be

$$m = m; \; a = b \, . \, v;$$

if we divide one by the other, the two quotients will be equal, that is,

$$\frac{m}{a} = \frac{m}{bv};$$

and multiplying both sides by b, in order to have the expression $\frac{m}{v}$, we shall find

$$\frac{mb}{a} = \frac{m}{v} \ldots\ldots (18).$$

Therefore, *to divide a whole number by a fraction, we must multiply the whole number by the reciprocal of the fraction, or which is the same, by the fraction inverted*

Let, in the second case, the two equalities be

$$a=b\,.\,v,\ a'=b'\,.\,v';$$

if the first equality be divided by the second, we shall have

$$\frac{a}{a'}=\frac{b\,.\,v}{b'.v'};$$

multiplying each side by b' and dividing by b, for the purpose of obtaining the expression $\frac{v}{v'}$, we will arrive at

$$\frac{ab'}{a'b}=\frac{v}{v'}=\frac{a}{b}\times\frac{b'}{a'}\ldots\ldots (19).$$

Therefore, *to divide one fraction by another, we must multiply the fractional dividend by the reciprocal of the fractional divisor, or which is the same, by the fractional divisor inverted.*

127. These properties and rules should still take place in case that a and b would represent any polynomials whatever.

According to the transformation $a^{-d}=\frac{1}{a^d}$, demonstrated (Art. 86), we can change a quantity from a fractional form to that of an integral one, and reciprocally. So that, we have $\frac{b}{a}=b\times\frac{1}{a}=b\times a^{-1}=ba^{-1}$, $\frac{b}{a^d}=b\times\frac{1}{a^d}=b\times a^{-d}=ba^{-d}$, and $a^{-2}b^{-2}d^{-2}=\frac{1}{a^2}\times\frac{1}{b^2}\times\frac{1}{d^2}=\frac{1}{a^2b^2d^2}$. In like manner any quantity may be transferred from the numerator to the denominator, and reciprocally, by changing the sign of its index: Thus, $\frac{a^2b}{c^2}=\frac{b}{a^{-2}c^2}=\frac{bc^{-2}}{a^{-2}}=\frac{c^{-2}}{a^{-2}b^{-1}}$, and $\frac{a^{-3}x^{-2}z^{-1}}{c^{-m}b^2y^{-n}}=\frac{c^m y^n}{a^3b^2x^2z}$.

128. *If the signs of both the numerator and denominator of a fraction be changed, its value will not be altered.*

$$\text{Thus, } \frac{-a}{-b}=\frac{+a}{+b}=+\frac{a}{b}=\frac{a}{b};\ \frac{a-b}{c-d}=\frac{b-a}{d-c}.$$

Which appears evident from the *Division* of algebraic quantities having like or unlike signs. Also, *if a fraction have the negative sign before it, the value of the fraction will not be altered by making the numerator only negative, or by changing the signs of all its terms,* Thus, $-\frac{a}{b}=+\frac{-a}{b}$, and $-\frac{a-b}{c+d}=+\frac{b-a}{c+d}=\frac{b-a}{c+d}$.

And, in like manner, *the value of a fraction having a negative sign before it*, will not be altered by making the denominator only negative: Thus,

$$-\frac{a-b}{c-d}=+\frac{a-b}{d-c}=\frac{a-b}{d-c}.$$

129. *Note.* It may be observed, that if the numerator be equal to the denominator, the fraction is equal to unity; thus, if $a=b$, then $\frac{a}{b}=\frac{a}{a}=1$: Also, if a is $>b$, the fraction is greater than unity; and in each of those two cases it is called *an improper fraction:* But if a is $<b$, then the fraction is less than unity, and in this case, it is called *a proper fraction.*

§ II. *Method of finding the Greatest Common Divisor of two or more Quantities.*

130. The greatest common *divisor* of two or more quantities, is the greatest quantity which divides each of them exactly. Thus, the greatest common divisor of the quantities $16a^2b^2$, $12a^2bc$ and $4abc^2$, is $4ab$.

131. If one quantity measure two others, it will also measure their sum or difference. Let c measure a by the units in m, and b by the units in n, then $a=mc$, and $b=nc$; therefore $a+b=mc+nc=(m+n)c$; and $a-b=mc-nc=(m-n)c$; or $a\pm b=(m\pm n)c$; consequently c measures $a+b$ (their sum) by the units in $m+n$, and $a-b$ (their difference) by the units in $m-n$.

132. Let a and b be any two numbers or quantities, whereof a is the greater; and let $p=$ quotient of a divided by b, and $c=$ remainder; $q=$ quotient of b divided by c, and $d=$ remainder; $r=$ quotient of c divided by d, and the remainder $=0$; thus,

$b)\ a(p$
$\quad pb$
$\quad$ —
$\quad c)\ b\ (q$
$\qquad qc$
$\qquad$ —
$\qquad d)\ c\ (r$
$\qquad\quad rd$
$\qquad\quad$ —
$\qquad\quad 0$

Then, since in each case the divisor multiplied by the quotient *plus* the remainder is equal to be dividend; we have

$c=rd$, hence $qc=qrd$ (Art. 50);

$b=qc+d=qrd+d=(qr+1)d$; and $pb=pqrd+pd=(pqr+p)d$ (Art. 61.),

$\therefore\ a=pb+c=pqrd+pd+rd=(pqr+p+r)d$.

Hence, since p, q, and r, are *whole numbers* or *integral quantities*, d is contained in b as many times as there are units in $qr+1$, and in a as many times as there are units in $pqr+p+r$; consequently the last divisor d is a common measure of a and b; and this is evidently the case, whatever be the length of the operation, provided that it be carried on till the remainder is nothing.

This last divisor d is also the *greatest* common measure of a and b. For let x be a common measure of a and b; such that $a=mx$, and $b=nx$, then $pb=pnx$; and $c=a-pb=mx-pnx=(m-pn)x$, also $d=b-qc=nx-(qmx-qpnx)=na-qmx+pqnx=(n-qm+pqn)x$; (because $qc=qmx-qpnx$) therefore x measures d by the units in $n-qm+pqn$, and as it also measures a, and b, the numbers, or quantities a, b, and d have a common measure. Now the greatest common measure of d is *itself;* consequently d is the greatest common measure of a and b.

133. To find the greatest common measure of three numbers, or quantities, a, b, c; let d be the greatest common measure of a and b, and x the greatest common measure of d and c; then x is the greatest common measure of a, b, and c. For, as a, b, and d have a common measure; if d and c have also a common measure, that same number or quantity will measure a, b, and c; and if x be the greatest common measure of d and c, it will also be the greatest common measure of a, b, and c.

And, in like manner, if there be any number of quantities; a, b, c, d, &c.; and that x is the greatest common measure of a and b; y the greatest common measure of x and c; z the greatest common measure of y and d; &c. &c.; then will y be the greatest common measure of a, b, and c; z the greatest common measure of a, b, c, and d; &c. &c.

134. The preceding method of demonstration is similar to that given by BRIDGE *in his Treatise on the Elements of Algebra.* The following is according to the manner of GARNIER. Thus, to find the greatest common divisor of any number of quantities A, B, C, &c., it is sufficient to know the method of finding the greatest common divisor of two numbers or quantities. For this purpose, we will at first seek the greatest common divisor D of the quantities A and B, then the greatest common divisor D′ of D and C, and so on, and finally the last greatest common divisor will be that which was required.

Let, in order to demonstrate it, the three quantities be A, B, C; we will have

$$\begin{array}{ll} \text{1st} \ldots \left\{ \begin{array}{l} A = mD, \\ B = nD, \end{array} \right. \\ \text{2d} \ldots \left\{ \begin{array}{l} D = rD', \\ C = qD', \end{array} \right. \end{array} \quad \text{whence} \quad \left\{ \begin{array}{l} A = mrD', \\ B = nrD', \\ \\ C = qD'; \end{array} \right.$$

m and n are necessarily prime to one another, otherwise D would not be the greatest common divisor of A and B ; r and q are also prime to one another, in order that D′ may be the greatest common divisor of D and C. Now rD', the greatest common divisor of A and B, cannot be the greatest common divisor of A, B, and C, unless that r be equal to q, or a factor of q ; but r and q being prime to one another ; D′ remains the greatest common divisor of A, B, and C.

135. As the problem of finding the greatest common divisor of any two quantities A and B, is the same as to reduce a fraction $\frac{A}{B}$ to its most simple expression ; because that in dividing A and B by their greatest common divisor, we have the two least quotients possible ; admitting this enunciation, and supposing $A > B$.

The greatest common divisor of A and B, cannot exceed B ; it could be B itself, which we can readily know, if we perform the division of A by B, which gives

$$\frac{A}{B} = q + \frac{R}{B} \ldots\ldots (1),$$

q being the integral quotient, and R the remainder, if A is not exactly divisible by B. The fraction $\frac{A}{B}$ being changed into $q + \frac{R}{B}$, cannot be reduced unless that $\frac{R}{B}$ or its reciprocal $\frac{B}{R}$ is reducible, because q is an integral quantity which is always irreducible ; or B being $>$R, the quantity which ought to reduce $\frac{B}{R}$, cannot exceed R, it might be R itself, which we will know in performing the division of B by R, which gives

$$\frac{B}{R} = q' + \frac{R'}{R} \ldots\ldots (2),$$

q' being the integral part of the quotient, and R′ the remainder $<$R ; we say still that the reduction of $\frac{R}{B}$ depends on that of $\frac{R'}{R}$, or its reciprocal, because that q' is an irreducible quantity ; so that by continuing in this manner we shall have the following decompositions :

$$\frac{R}{R'}=q''+\frac{R''}{R'} \dots (3),$$
$$\frac{R'}{R''}=q'''+\frac{R'''}{R''} \dots (4).$$

We see very clearly that the quantity which ought to reduce $\frac{A}{B}$ is that which must reduce $\frac{R}{B}$ or $\frac{B}{R}$, which must reduce $\frac{R'}{R}$ or $\frac{R}{R'}$, which must reduce $\frac{R''}{R'}$ or $\frac{R'}{R''}$.

If, for example, $R'''=0$, this quantity cannot be greater than R''; R'' is therefore the greatest quantity which can reduce the fraction $\frac{A}{B}$; consequently it is the greatest common divisor of A and B.

136. Let $R''=0$ and $R'=1$: unity will be, according to what has been above demonstrated, the greatest common divisor of A and B; the fraction $\frac{A}{B}$ will therefore itself be the most simple expression, that is, it will be irreducible. *Reciprocally, the last divisor being unity, we may conclude that the fraction proposed is irreducible, or in its lowest terms.*

137. It may also be shown, that the greatest common measure of two quantities will, in no respect, be altered, by multiplying or dividing either of them by any quantity which is not a divisor of the other, or that contains no factor which is common to both of them; thus, let the quantities ab and ac be taken, of which the common measure is a; then, if ab be multiplied by d, they will become abd, and ac; where it is evident that a is the common measure, as before. And, conversely, if the first of the two quantities abd, ac, be divided by d, they will become ab, ac, where a is still the common measure.

138. But it will not be the same if one or two of the quantities be multiplied or divided by a quantity which is a divisor of the other, or has a common factor with it; for if the first of the two quantities ab, ac, be multiplied by c, they will become abc, ac, of which the common divisor is ac, instead of a; and, conversely, if the first of the two quantities abc and ac, be divided by c, they will become ab and ac; of which the common divisor is a, instead of ac.

139. Hence, if the numbers or quantities be $mncN$, $pqcN'$; the common factor c, to simplify the operation, may be suppressed, observing, in the meantime, after having found the

greatest common divisor a, of the two quotients N and N′, to multiply it by this factor c, and the product will be the greatest common divisor sought. Also, if a factor d is introduced into the two quantities, it is necessary to divide the greatest common divisor by this factor.

140. As the foregoing demonstration may be extended to any algebraic quantities whatever, we are therefore conducted to this practical rule.

To find the greatest common divisor of two or more compound algebraic quantities.

RULE.

141. Arrange the two quantities according to the order of their powers, and divide that which is of the highest dimensions by the other, having first expunged any factor that may be contained in all the terms of the divisor without being common to those of the dividend; then divide this divisor by the remainder, simplified, if necessary, as before; and so on, for each remainder and its preceding divisor, till nothing remains: then the divisor last used will be the greatest common divisor required. And the greatest common divisor, of *more than two* compound quantities, is found in like manner; by finding in the first place the greatest common divisor of two of them, as above, and then of that common divisor and the third, and so on. The last divisor, thus found, will be the greatest common divisor of all the quantities.

EXAMPLE 1. The greatest common divisor of the compound quantities $3a^3-3a^2b+ab^2-b^3$ and $4a^2b-5ab^2+b^3$, is required.

Dividend.	*Divisor.*
$3a^3-3a^2b+ab^2-b^3$	$(4a^2b-5ab^2+b^3)\div b=$
4	$4a^2-5ab+b^2$
$12a^3-12a^2b+4ab^2-4b^3$	
$12a^3-15a^2b+3ab^2$	*Partial quot.* $3a$
$(3a^2b+ab^2-4b^3)\div b=$	
$3a^2+ab-4b^2$	
4	*Divisor.*
$12a^2+4ab-16b^2$	$4a^2-5ab+b^2$
$12a^2-15ab+3b^2$	*Partial quot.* 3.
$19ab-19b^2$	

Dividend.	*Divisor.*
$4a^2-5ab+b^2$	$(19ab-19b^2)\div 19b=$
$4a^2-4ab$	$a-b$
$-ab+b^2$	
$-ab+b^2$	*Quot.* $4a-b$
$*\quad *$	

Here the quantities are already arranged according to the powers of the letter a; the first is taken for a dividend, and the second for a divisor. In the first place, the factor b is found in every term of the divisor, and not in every term of the dividend; therefore, the divisor is divided by the factor b, and the result is $4a^2-5ab+b^2$; but the first term of this result will not divide exactly that of the dividend, on account of the factor 4, which is not in the dividend; the dividend is therefore multiplied by 4 in order to render the division of their first terms complete. Now, the dividend $12a^3-12a^2b+4ab^2-4b^3$ is divided by the divisor $4a^2-5ab+b^2$, and the partial quotient is $3a$. Multiplying the divisor by this quotient, and subtracting the product from the dividend, the remainder is $3a^2b+ab^2-4b^3$, a quantity which, according to (Art. 135), must still have with $4a^2-5ab+b^2$ the same greatest common divisor as the first.

Suppressing the factor b, common to all the terms of the remainder, or, which is the same, dividing the remainder by b, and multiplying the result by 4, to render possible the division of its first term by that of the divisor, we have then for the dividend the quantity

$$12a^2+4ab-16b^2,$$

and for the divisor the quantity

$$4a^2-5ab+b^2;$$

the partial quotient is 3.

Multiplying the divisor by the quotient, and subtracting the product from the dividend, the remainder is

$$19ab-19b^2,$$

and the question is now reduced to finding the greatest common divisor of $19ab-19b^2$ and $4a^2-5ab+b^2$.

But the letter a, according to which the division has been performed, being of the second degree in the divisor, and only of the first in the remainder; it is necessary therefore to take the last divisor for a new dividend, and the remainder for a new divisor.

Having, at the commencement of this new division, divided the divisor $19ab-19b^2$ by the factor $19b$, common to all its terms, and which is not at all common to those of the dividend; therefore the dividend is $4a^2-5ab+b^2$, the divisor $a-b$, and the quotient $4a-b$;

The operation is completed, because nothing remains; and consequently, (Art. 135), $a-b$ is the greatest common divisor sought.

If we divide the two proposed quantities by $a-b$, the quotients will be

$$3a^2+b^2 \text{ and } 4ab-b^2:$$

Whence, the two given quantities are thus decomposed as follows:

$$(3a^2+b^2)\times(a-b),\ (4ab-b^2)\times(a-b).$$

Ex. 2. Required the greatest common divisor of $3a^2-2a-1$ and $4a^3-2a^2-3a+1$.

Dividend.	*Divisor.*
$4a^3-2a^2-3a+1$	$3a^2-2a-1$
3	
$12a^3-6a^2-9a+3$	
$12a^3-8a^2-4a$	*Partial quot.* $4a$

	Divisor.
$2a^2-5a+3$	$3a^2-2a-1$
3	
$6a^2-15a+9$	
$6a^2-4a-2$	*Partial quot.* 2

$$(-11a+11)\div-11=$$

Dividend.	
$3a^2-2a-1$	$a-1$
$3a^2-3a$	
$a-1$	
$a-1$	*Complete quot.* $3a+1$
* *	

In the above operation, the remainder $-11a+11$ is divided by -11, (its greatest simple divisor with a negative sign), so as to make the leading term positive: or, which is the same, if any of the divisors, in the course of the operation, become negative, they may have their signs changed, or be taken affirmatively, without altering the truth of the result; thus, in the above operation, changing the signs of $-11a+11$, it becomes $11a-11$, and dividing $11a-11$ by its greatest simple divisor 11, we have $a-1$, as before.

Therefore $a-1$ is the greatest common divisor sought, and the two given quantities may be readily decomposed, thus;

$$(3a+1)\times(a-1),\ (4a^2+2a-1)\times(a-1).$$

Ex. 3. Required the greatest common divisor of a^3-b^3, $a^3+2a^2b+2ab^2+b^3$, and $a^4+a^2b^2+b^4$.

In the first place, the greatest common divisor of a^3-b^3 and $a^3+2a^2b+2ab^2+b^3$, is a^2+ab+b^2, which is found thus;

$$\begin{array}{l|l}
\textit{Dividend.} & \textit{Divisor.} \\
a^3+2a^2b+2ab^2+b^3 & a^3-b^3 \\
a^3 \qquad\qquad\quad -b^3 & \overline{} \\
\hline
& \textit{Partial quot. } 1 \\
(2a^2b+2ab^2+2b^3)\div 2b= & \\
\textit{Dividend.} & \\
a^3-b^3 & a^2+ab+b^2 \\
a^3+a^2b+ab^2 & \\
\hline
-a^2b-ab^2-b^3 & \\
-a^2b-ab^2-b^3 & \textit{Complete quot. } a-b \\
\hline
* \quad * \quad * &
\end{array}$$

Hence, the greatest common divisor of a^3-b^3 and $a^3+2a^2b+2ab^2+b^3$, is a^2+ab+b^2; and the greatest common divisor of a^2+ab+b^2 and $a^4+a^2b^2+b^4$, is found to be a^2-ab+b^2, thus;

$$\begin{array}{l|l}
\textit{Dividend.} & \textit{Divisor.} \\
a^4+a^2b^2+b^4 & a^2+ab+b^2 \\
a^4+a^3b+a^2b^2 & \\
\hline
-a^3b+b^4 & \textit{Quotient.} \\
-a^3b-a^2b^2-ab^3 & a^2-ab+b^2 \\
\hline
a^2b^2+ab^3+b^4 & \\
a^2b^2+ab^3+b^4 & \\
\hline
* \quad * \quad * &
\end{array}$$

Consequently a^2+ab+b^2 is the greatest common divisor which was required; and dividing each of the given quantities by this divisor, we will thus decompose them as follows: $(a-b)\ (a^2+ab+b^2)$, $(a+b)\ (a^2+ab+b^2)$, $(a^2-ab+b^2)\ (a^2+ab+b^2)$.

142. It has been remarked (Art. 136), that if the last divisor be unity, and the remainder nothing; then the fraction is

already in its lowest terms; this observation is applicable to numbers, and as in algebraic quantities, the greatest simple divisor may be readily found by inspection.

Now, it only remains to discover, if compound algebraic quantities can admit of a compound divisor.

If, by proceeding according to the Rule (Art. 141), no compound divisor can be found, that is, if the last remainder be only a simple quantity; we may conclude the case proposed does not admit of any, but is already in its lowest terms.

Ex. 4. Required the greatest common divisor of a^2+ax+x^2 and $a^3+2a^2x+3ax^2+4x^3$. It is plain by inspection that they do not admit of any simple divisor; then the operation according to the rule will stand thus;

Dividend.	*Divisor.*
$a^3+2a^2x+3ax^2+4x^3$	a^2+ax+x^2
$a^3+a^2x+ax^2$	
$a^2x+2ax^2+4x^3$	
$a^2x+ax^2+x^3$	*Partial quot.* $a+x$

$(ax^2+3x^3)\div x^2=$

Dividend.	
a^2+ax+x^2	$a+3x$
a^2+3ax	
$-2ax+x^2$	
$-2ax-6x^2$	*Partial quot.* $a-2x$
$* \quad +7x^2$	

Here, the last remainder is found to be the simple quantity $7x^2$; we may therefore conclude that the given quantities do not admit of any divisor whatever.

143. When the quantity which is taken for the divisor contains many terms where the letter, according to which we have arranged, has the same exponent; then every successive remainder becomes more complicated than the preceding one; in this case, *Analysts* make use of various artifices, which can only be learned by experience.

Ex. 5. Required the greatest common divisor of $a^2b+ac^2-d^3$ and $ab-ac+d^2$.

	Dividend.	*Divisor.*
	$a^2b+ac^2-d^3$	$ab-ac+d^2$
	$a^2b-a^2c+ad^2$	
rem.	$a^2c+ac^2-ad^2-d^3$	*Partial quot.* a

Dividing at first a^2b by ab, we find for the quotient, a;

multiplying the divisor by this quotient, and subtracting the product from the dividend, the remainder contains a new term, a^2c, arising from the product of $-ac$ by a.

By proceeding after this manner there will be no progress made in the operation; for, taking $a^2c+ac^2-ad^2-d^3$ for a dividend, and multiplying it by b, to render possible the divisor by ab, we will have

$$\begin{array}{ll|l} & \textit{Dividend.} & \textit{Divisor.} \\ & a^2bc+abc^2-abd^2-bd^3 & ab-ac+d^2 \\ & a^2bc-a^2c^2+acd^2 & \\ \cline{2-2} & & \textit{Partial quot.} \\ \text{rem.} & a^2c^2+abc^2-acd^2-abd^2-bd^3 & ac \end{array}$$

and the term $-ac$ will still reproduce a term a^2c^2, in which the exponent of a is 2.

To avoid this inconveniency, we must observe that the divisor $ab-ac+d^2=a(b-c)+d^2$, reuniting the terms $ab-ac$ into one, and putting, to abridge the calculations, $b-c=m$; we will have for the divisor $am+d^2$; it is necessary to multiply all the dividend $a^2b+ac^2-d^3$ by the factor m, for the purpose of finding a new dividend whose first term would be divisible by the quantity am forming the first term of the divisor; the operation will become,

$$\begin{array}{ll|l} & \textit{Dividend.} & \textit{Divisor.} \\ & a^2bm+ac^2m-d^3m & am+d^2 \\ & a^2bm+abd^2 & \\ \cline{2-2} & & \textit{Partial quot.} \\ \text{1st rem.} & +ac^2m-abd^2-d^3m & ab+c^2 \\ & +ac^2m+c^2d^2 & \\ \cline{2-2} \text{2d rem.} & -abd^2-c^2d^2-d^3m & \end{array}$$

By the first operation, the terms involving a^2 are taken away from the dividend, and there remain no terms involving a except in the first power. In order to make them disappear, we will at first divide the term ac^2m by am, and it gives for the quotient c^2; multiplying the divisor by the quotient, and subtracting the product from the dividend, we will have the second remainder; taking this second remainder for a new dividend, and cancelling in it the factor d^2, which is not a factor of the divisor, it will become

$$-ab-c^2-dm;$$

multiplying by m, we shall have

$$\begin{array}{ll|l} & \textit{Dividend.} & \textit{Divisor.} \\ & -abm-c^2m-dm^2 & am+d^2 \\ & -abm-bd^2 & \\ \cline{2-2} & & \textit{Partial quot.}\ -b. \\ \text{rem.} & +bd^2-c^2m-dm^2 & \end{array}$$

The remainder, $bd^2-c^2m-dm^2$, of this last division does not contain the letter a; it follows, then, that if there exist between the proposed quantities a common divisor, it must be independent of the letter a.

Having arrived at this point, we cannot continue the division with respect to the letter a; but observing that if there be a common divisor, independent of a, of the two quantities $bd^2-c^2m-dm^2$ and $am+d^2$, it may divide separately the two parts am and d^2 of the divisor; for, in general, if a quantity be arranged according to the powers of the letter a, every term of this quantity, independent of a, must divide separately the quantities by which the different powers of this letter are multiplied.

In order to be convinced of what has just been said, it is sufficient to observe, that in this case each of the proposed quantities should be the product of a quantity dependent on a, and of a common divisor which does not at all depend on it. Now, if we have, for example, the expression

$$Aa^4+Ba^3+Ca^2+Da+E,$$

in which the letters A, B, C, D, E, designate any quantities whatever, independent of a, and if we multiply it by a quantity M, also independent of a, the product,

$$MAa^4+MBa^3+MCa^2+MDa+ME,$$

arranged according to a, will still contain the same powers of a as before; but the coefficient of each of these powers will be a multiple of M.

This being admitted, if we substitute for m the quantity $(b-c)$, which this letter represents, we shall have the quantities

$$bd^2-c^2(b-c)-c(b-c)^2,$$
$$a(b-c)+d^2;$$

now it is plain that $b-c$ and d^2 have no common factor whatever: therefore the two proposed quantities have not a common divisor.

144. The greatest common divisor of two quantities may sometimes be obtained without having recourse to the general Rule. Some of the methods that are used by *Analysts* for this purpose, will be exemplified by the following Examples.

Ex. 6. Required the greatest common divisor of $a^4b^2+a^3b^3+b^4c^2-a^4c^2-a^3bc^2-b^2c^4$, and $a^2b+ab^2+b^3-a^2c-abc-b^2c$.

After having arranged these quantities according to the powers of the letter a, we shall have

$$(b^2-c^2)a^4+(b^3-bc^2)a^3+b^4c^2-b^2c^4$$
$$(b-c)a^2+(b^2-bc)a+b^3-b^2c;$$

it may at first be observed, that if they admit of a common divisor, which should be independent of the letter a, it must divide separately each of the quantities by which the different powers of a are multiplied, (Art. 143), as well as the quantities $b^4c^2-b^2c^4$ and b^3-b^2c, which comprehend not at all this letter.

The question is therefore reduced to finding the common divisors of the quantities b^2-c^2 and $b-c$, and, to verify afterward, if, among these divisors, there be found some that would also divide b^3-bc^2 and b^2-bc, $b^4c^2-b^2c^4$ and b^3-b^2c.

Dividing b^2-c^2 by $b-c$, we find an exact quotient $b+c$: $b-c$ is therefore a common divisor of the quantities b^2-c^2 and $b-c$, and it appears that they cannot have any other divisor, because the quantity $b-c$ is divisible but by itself and unity. We must therefore try if it would divide the other quantities referred to above, or, which is equally as well, if it would divide the two proposed quantities; but it will be found to succeed, the quotients coming out exactly,

$$(b+c)a^4+(b^2+bc)a^3+b^3c^2+b^2c^3;$$
$$\text{and} \qquad a^2+ba+b^2.$$

In order to bring these last expressions to the greatest possible degree of simplicity, it is expedient to try if the first be not divisible by $b+c$; this division being effected, it succeeds, and we have now only to seek the greatest common divisor of these very simple quantities;

$$a^4+ba^3+b^2c^2, \text{ and } a^2+ba+b^2.$$

Operating on these, according to the Rule, (Art. 141), we will arrive, after the second division, at a remainder containing the letter a in the first power only; and as this remainder is not the common divisor, hence we may conclude that the letter a does not make a part of the common divisor sought, which is consequently composed but of the factor $b-c$.

Ex. 7. Required the greatest common divisor of $(d^2-c^2)\times a^2+c^4-d^2c^2$ and $4da^2-(2c^2+4cd)a+2c^3$.

Arranging these quantities according to d, we have

$$(a^2-c^2)d^2+c^4-a^2c^2, \text{ or } (a^2-c^2)d^2-(a^2-c^2)c^2,$$
$$\text{and} \qquad (4a^2-4ac)\times d-(a-c)\times 2c^2;$$

it is evident, by inspection only, that a^2-c^2 is a divisor of the first, and $a-c$ of the second. But a^2-c^2 is divisible by $a-c$; therefore $a-c$ is a divisor of the two proposed quantities: Dividing both the one and the other by $a-c$, the quotients will be

$$(a+c)\times(d^2-c^2), \text{ and } 4ad-2c^2;$$

which, by inspection, are found to have no common divisor, consequently $a-c$ is the greatest common divisor of the proposed quantities.

Ex. 8. Required the greatest common divisor of y^4-x^4 and $y^3-y^2x-yx^2+x^3$. Ans. y^2-x^2.

Ex. 9. Required the greatest common divisor of a^4-b^4 and a^6-b^6. Ans. a^2-b^2.

Ex. 10. Required the greatest common divisor of $a^4+a^3b-ab^3-b^4$ and $a^4+a^2b^2+b^4$. Ans. a^2+ab+b^2.

Ex. 11. Required the greatest common divisor of $a^2-2ax+x^2$ and $a^3-a^2x-ax^2+x^3$. Ans. $a^2-2ax+x^2$.

Ex. 12. Find the greatest common divisor of $6x^3-8yx^2+2y^2x$ and $12x^2-15yx+3y^2$. Ans. $x-y$.

Ex. 13. Find the greatest common divisor of $36b^2a^6-18b^2a^5-27b^2a^4+9b^2a^3$ and $27b^2a^5-18b^2a^4-9b^2a^3$.

Ans. $9b^2a^4-9b^2a^3$.

Ex. 14. Find the greatest common divisor of $(c-d)a^2+(2bc-2bd)a+(b^2c-b^2d)$ and $(bc-bd+c^2-cd)a+(b^2d+bc^2-b^2c-bcd)$. Ans. $c-d$.

Ex. 15. Find the greatest common divisor of $x^3+9x^2+27x-98$ and $x^2+12x-28$. Ans. $x-2$.

§ III. METHOD OF FINDING THE LEAST COMMON MULTIPLE OF TWO OR MORE QUANTITIES.

145. The least common multiple of two or more quantities is the least quantity in which each of them is contained without a remainder. Thus, $20abc$ is the least common multiple of $5a$, $4ac$, and $2b$.

146. The least common multiple of any number of quantities, literal or numeral, monomial or polynomial, may be easily found thus :

Resolve each quantity into its simplest factors, putting the product of equal factors when there are any in the form of powers, then multiply all together the highest powers of every root concerned, and the product will be the least common multiple required.

Ex. 1. Required the least common multiple of a^3b^2x, $acbx^2$, abc^2d.

Here the quantities are already exhibited in the form required. Therefore the least common multiple is $a^3b^2c^2dx^2$.

Ex. 2. Required the least common multiple of $2a^2x$, $4ax^2$, and $6x^3$.

Here the literal quantities are already in the form required. The coefficients resolved into their simplest factors become 2, 2^2, 2×3. The least common multiple is therefore $2^2\times3\times a^2x^3=12a^2x^3$.

Ex. 3. Required the least common multiple of $12a^2y(a+b)$, $6a^3y^2+12a^2\,by^2+6ab^2y^2$, and $4a^2y^2$.

These quantities resolved into their simplest factors become

$$2^2\times3\times a^2y(a+b)$$
$$2\ \times3\times ay^2(a+b)^2$$
$$2^2\times a^2y^2$$

Hence the least common multiple required is $2^2\times3\times a^2y^2(a+b)^2=12a^2y^2(a+b)^2$.

Ex. 3. Required the least common multiple of $8a$, $4a^2$, and $12ab$. Ans. $24a^2b$.

Ex. 4. Required the least common multiple of a^2-b^2, $a+b$, and a^2+b^2. Ans. a^4-b^4.

Ex. 5. Required the least common multiple of $72a$, $15b$, $9ab$, and $3a^2$. Ans. $135a^2b$.

Ex. 6. Required the least common multiple of $a^3+3a^2b+3ab^2+b^3$, $a^2+2ab+b^2$, a^2-b^2. Ans. $a^4+2a^3b-2ab^3-b^4$.

Ex. 7. Required the least common multiple of $a+b$, $a-b$, a^2+ab+b^2, and a^2-ab+b^2. Ans. a^6-b^6

§ IV. REDUCTION OF ALGEBRAIC FRACTIONS.

CASE I.

To reduce a mixed quantity to an improper fraction.

RULE.

147. Multiply the integral part by the denominator of the fraction, and to the product annex the numerator with its proper sign: under this sum place the former denominator, and the result is the improper fraction required.

Ex. 1. Reduce $3x+\frac{2b}{5a}$ to an improper fraction.

The integral part $3x$, multiplied by the *denominator* $5a$ of the fraction plus the *numerator* $(2b)$, is equal to $3x\times5a+2b=15ax+2b$;

Hence, $\frac{15ax+2b}{5a}$ is the fraction required.

Ex. 2. Reduce $5a-\frac{3x}{y}$ to an improper fraction.

Here $5a\times y=5ay$; to this add the numerator with its proper sign, viz. $-3x$; and we shall have $5ay-3x$.

Hence, $\frac{5ay-3x}{y}$ is the fraction required.

Ex. 3. Reduce $x^2-\frac{a^2-y^2}{x}$ to an improper fraction.

Here, $x^2\times x=x^3$; adding the numerator a^2-y^2 *with its proper sign:* It is to be recollected that the sign $-$ affixed to the fraction $\frac{a^2-y^2}{x}$ *means* that the whole of that fraction is to be *subtracted*, and consequently that the sign of each term of the numerator must be *changed*, when it is combined with x^3, hence the improper fraction required is $\frac{x^3-a^2+y^2}{x}$. Or, as $-\frac{a^2-y^2}{x}=\frac{-a^2+y^2}{x}=\frac{y^2-a^2}{x}$; (Art. 67), the proposed mixed quantity $x^2-\frac{a^2-y^2}{x}$, may be put under the from $x^2+\frac{y^2-a^2}{x}$, which is reduced as Ex. 1. Thus, $x^2\times x+y^2-a^2=x^3+y^2-a^2$; hence, $x^2+\frac{y^2-a^2}{x}=\frac{x^3+y^2-a^2}{x}$.

Ex. 4. Reduce $5a^2+\frac{3x^2-a+7}{2ax}$ to an improper fraction.

Here, $5a^2\times 2ax=10a^3x$; adding the numerator $3x^2-a+7$ to this, and we have $10a^3x+3x^2-a+7$.

Hence, $\frac{10a^3x+3x^2-a+7}{2ax}$ is the fraction required.

Ex. 5. Reduce $4x^2-\frac{3ab+c}{2ac}$ to an improper fraction.

Here, $4x^2\times 2ac=8acx^2$, in adding the numerator with its proper sign; the sign $-$ prefixed to the fraction $\frac{3ab+c}{2ac}$ signifies that it is to be taken negatively, or that the whole of that fraction is to be subtracted; and consequently that the sign of each term of the numerator must be changed when it is combined with $8acx^2$; hence, $\frac{8acx^2-3ab-c}{2ac}$ is the fraction required. Or, as $-\frac{3ab+c}{2ac}=+\frac{-3ab-c}{2ac}=\frac{-3ab-c}{2ac}$ (Art. 108); hence the reason of changing the signs of the numerator is evident.

Ex. 6. Reduce $x-\frac{a^2-x^2}{x}$ to an improper fraction.

Ans. $\frac{2x^2-a^2}{x}$.

Ex. 7. Reduce $ab-\frac{a^2+c}{5x}$ to an improper fraction.

Ans. $\frac{5abx-a^2-c}{5x}$.

Ex. 8. Reduce $ax^2-\frac{3b}{a}$ to an improper fraction.

Ans. $\frac{a^2x^2-3b}{a}$.

Ex. 9. Reduce $a-x+\frac{a^2-ax}{x}$ to an improper fraction.

Ans. $\frac{a^2-x^2}{x}$.

Ex. 10. Reduce $3x^2-\frac{4x-9}{7a}$ to an improper fraction.

Ans. $\frac{21ax^2-4x+9}{7a}$.

Ex. 11. Reduce $5x-\frac{2x-5}{3}$ to an improper fraction.

Ans. $\frac{13x+5}{3}$.

Ex. 12. Reduce $1+2x-\frac{4x-4}{5x}$ to an improper fraction.

Ans. $\frac{x+10x^2+4}{5x}$.

CASE II.

To reduce an improper fraction to a whole or mixed quantity.

RULE.

148. Observe which terms of the *numerator* are divisible by the *denominator* without a remainder, the quotient will give the *integral part*; and put the remaining terms of the numerator, if any, over the denominator for the fractional part; then the two joined together with the proper sign between them, will give the mixed quantity required.

Ex. 1. Reduce $\frac{x^3+2ax^2+b}{x^2}$ to a mixed quantity.

Here, $\frac{x^3+2ax^2}{x^2}=x+2a$ is the integral part, and $\frac{b}{x^2}$ is the fractional part;

therefore $x+2a+\frac{b}{x^2}$ is the mixed quantity required.

Ex. 2. Reduce $\frac{x^8+x^4y^4+y^8}{x^4+x^2y^2+y^4}$ to a whole quantity.

Dividend.	*Divisor.*
$x^8+x^4y^4+y^8$	$x^4+x^2y^2+y^4$
$x^8+x^6y^2+x^4y^4$	
$-x^6y^2+y^8$	*Quotient.*
$-x^6y^2-x^4y^4-x^2y^6$	$x^4-x^2y^2+y^4$
$x^4y^4+x^2y^6+y^8$	
$x^4y^4+x^2y^6+y^8$	
*	

Here the operation is performed according to the rule (Art. 93), and the quotient $x^4-x^2y^2+y^4$ is the whole quantity required.

Ex. 3. Reduce $\frac{ax-2b^2}{x}$ to a mixed quantity.

Here, $\frac{ax}{x}=a$ is the integral, and $\frac{2b^2}{x}$ the fractional part;

therefore $a-\frac{2b^2}{x}$ is the mixed quantity required.

Ex. 4. Reduce $\frac{x^2-a^2+b}{x+a}$ to a mixed quantity.

$x+a)x^2-a^2+b(x-a+\frac{b}{x+a}$ the mixed quantity required.

$$\begin{array}{l} x^2+ax \\ \hline -ax-a^2 \\ -ax-a^2 \\ \hline \quad * \quad +b \end{array}$$

Here the remainder b is placed over the denominator $x+a$, and annexed to the quotient as in (Art. 89).

Ex. 5. Reduce $\frac{3a^2b^2+6ab-2x+2c}{3ab}$ to a mixed quantity.

Here $\frac{3a^2b^2+6ab}{3ab}=ab+2$ is the integral part,

and $\frac{-2x+2c}{3ab}=-\frac{2x-2c}{3ab}=+\frac{2c-2x}{3ab}$ (Art. 128), is the fractional part;

$\therefore ab+2-\frac{2x-2c}{3ab}$, or $ab+2+\frac{2c-2x}{3ab}$ is the mixed quantity required.

Ex. 6. Reduce $\frac{21ax^2-4x+9}{7a}$ to a mixed quantity.

Ans. $3x^2-\frac{4x-9}{7a}$.

Ex. 7. Reduce $\frac{8x^2y^2-3ax-6b}{4x^2}$ to a mixed quantity.

Ans. $2y^2-\frac{3ax+6b}{4x^2}$.

Ex. 8. Reduce $\frac{x^4-a^4}{x^2+a^2}$ to a whole quantity.

Ans. x^2-a^2.

Ex. 9. Reduce $\frac{27a^3+3b^2-4x-9a^2}{9a^2}$ to a mixed quantity.

Ans. $3a-1+\frac{3b^2-4x}{9a^2}$.

Ex. 10. Reduce $\frac{x^4-3x^2y^2+4ax}{x^2-3y^2}$ to a mixed quantity.

Ans. $x^2+\frac{4ax}{x^2-3y^2}$.

Ex. 11. Reduce $\frac{x^6+3ax^2-a^6-b}{x^3+a^3}$ to a mixed quantity.

Ans. $x^3-a^3+\frac{3ax^2-b}{x^3+a^3}$.

Ex. 12. Reduce $\frac{3x^2-12ax+y-9x}{3x}$ to a mixed quantity.

Ans. $x-4a-3+\frac{y}{3x}$.

CASE III.

To reduce a fraction to its lowest terms, or most simple expression.

RULE.

149. Observe what quantity will divide all the terms both of the numerator and denominator *without a remainder:* Divide them by this quantity, and the fraction is reduced to its lowest terms. Or, find their greatest common divisor, according to the method laid down in (Art. 141); by which divide both the numerator and denominator, and it will give the fraction required.

EXAMPLE 1.

Reduce $\frac{14x^3+7ax^2+28x}{21x^2}$ to its lowest terms.

The coefficient of every term of the numerator and denominator of the fraction is divisible by 7, and the letter x also enters into every term; therefore $7x$ will divide both the numerator and denominator without a remainder.

Now $\frac{14x^3+7ax^2+28x}{7x}=2x^2+ax+4$, and $\frac{21x^2}{7x}=3x$; hence the fraction in its lowest term is $\frac{2x^2+ax+4}{3x}$.

Ex. 2. Reduce $\frac{30a^2b^2c-6abc^2-12a^2c^2b}{36abcx}$ to its lowest terms.

Here the quantity which divides both the numerator and denominator without a remainder is evidently $6abc$; then $\frac{30a^2b^2c-6abc^2-12a^2c^2b}{6abc}=5ab-c-2ac$; and $\frac{36abcx}{6abc}=6x$;

Hence $\frac{5ab-c-2bc}{6x}$ is the fraction in its lowest terms.

Ex. 3. Reduce $\frac{a^2-b^2}{a^4-b^4}$ to its lowest terms.

Here, $a^4-b^4=(a^2+b^2)\times(a^2-b^2)$, (Art. 107.); and, consequently, a^2-b^2 will divide both the numerator and denominator without a remainder; that is, $\frac{a^2-b^2}{a^2-b^2}=1=$ new numerator, and $\frac{(a^2+b^2)\times(a^2-b^2)}{a^2-b^2}=a^2+b^2=$ new denominator; hence, $\frac{1}{a^2+b^2}$ is the fraction in its lowest terms.

Ex. 4. Reduce $\frac{x^4-3ax^3-8a^2x^2+18a^3x-8a^4}{x^3-ax^2-8a^2x+6a^3}$ to its lowest terms.

Here, by proceeding according to the method of (Art. 141), we find the greatest common measure of the numerator and denominator to be $x^2+2ax-2a^2$; thus,

$$\begin{array}{l|l}
x^4-3ax^3-8a^2x^2+18a^3x-8a^4 & x^3-ax^2-8a^2x+6a^3 \\
x^4-\ \ ax^3-8a^2x^2+\ \ 6a^3x & \overline{\textit{Partial quot. } x-2a} \\
\hline
\quad -2ax^3+12a^3x-\ \ 8a^4 & \\
\quad -2ax^3+\ \ 2a^2x^2+16a^3x-12a^4 & \\
\hline
\end{array}$$

remaind. . . . $-2a^2x^2-4a^3x+4a^4$;

then, $\frac{-2a^2x^2-4a^3x+4a^4}{-2a^2}=x^2+2ax-2a^2=$ the next divisor;

$$\begin{array}{l}
x^2+2ax-2a^2)x^3-\ \ ax^2-8a^2x+6a^3(x-3a \\
\qquad\qquad\qquad\ x^3+2ax^2-2a^2x \\
\qquad\qquad\qquad\ \overline{\quad -3ax^2-6a^2x+6a^3} \\
\qquad\qquad\qquad\quad -3ax^2-6a^2x+6a^3 \\
\qquad\qquad\qquad\quad \overline{\quad * \qquad * \qquad *\quad}
\end{array}$$

And, dividing both terms by the greatest common measure, thus found, we have the fraction in its lowest terms; but the numerator, divided by the greatest common measure, gives $x-3a$, as above, equal to the new numerator; and the denominator, divided by the same, gives $x^2-5ax+4a^2$; thus,

$$\begin{array}{l|l}
x^4-3ax^3-\ \ 8a^2x^2+18a^3x-8a^4 & x^2+2ax-2a^2 \\
x^4+2ax^3-\ \ 2a^2x^2 & \overline{\qquad\qquad\qquad} \\
\hline
\quad -5ax^3-\ \ 6a^2x^2+18a^3x & \quad \textit{Quotient.} \\
\quad -5ax^3-10a^2x^2+10a^3x & x^2-5ax+4a^2 \\
\hline
\qquad\qquad 4a^2x^2+\ \ 8a^3x-8a^4 & \\
\qquad\qquad 4a^2x^2+\ \ 8a^3x-8a^4 & \\
\hline
\qquad\qquad * \qquad * \qquad * &
\end{array}$$

Hence, the fraction in its lowest terms is

$$\frac{x-3a}{x^2-5ax+4a^2}.$$

150. In addition to the methods pointed out in (Art. 144), for finding the greatest common divisor of two algebraic quantities, it may not be improper to take notice here of another method, given by SIMPSON, in his *Algebra*, which may be used to great advantage, and is very expeditious in reducing fractions, which become laborious by ordinary methods, to the lowest expression possible. Thus, fractions that have in them more than two different letters, and one of the letters rises only to a single dimension, either in the numerator or in the denominator, it will be best to divide the numerator or denominator (whichever it is) into two parts, so that the said letter may be found in every term of the one part, and be totally excluded out of the other: this being done, let the greatest common divisor of these two parts be found, which will evidently be a divisor to the whole, and by which the division of the other quantity is to be tried; as in the following example.

Ex. 5. Reduce $\frac{x^3+ax^2+bx^2-2a^2x+bax-2ba^2}{x^2-bx+2ax-2ab}$ to its lowest terms.

Here the denominator being the least compounded, and b rising therein to a single dimension only; I divide the same into the parts x^2+2ax, and $-bx-2ab$; which, by inspection, appear to be equal to $(x+2a)x$, and $(x+2a)\times -b$. Therefore $x+2a$ is a divisor to both the parts, and likewise to the whole, expressed by $(x+2a)\times(x-b)$; so that one of these two factors, if the fraction given can be reduced to lower terms, must also measure the numerator: but the former is found to succeed, the quotient coming out $x^2-ax+bx-ab$, exactly: whence the fraction is reduced to $\frac{x^2-ax+bx-ab}{x-b}$, which is not reducible farther by $x-b$, since the division does not terminate without a remainder, as upon trial will be found.

Ex. 6. Reduce $\frac{5a^5b+10a^4b^2+5a^3b^3}{a^5b+2a^4b^2+2a^3b^3+a^2b^4}$ to its lowest terms.

Here, the greatest simple divisor of the numerator and denominator is evidently, a^2b; Now, $\frac{5a^5b+10a^4b^2+5a^3b^3}{a^2b}=5a^3$ $+10a^2b+5ab^2$; and $\frac{a^5b+2a^4b^2+2a^3b^3+a^2b^4}{a^2b}=a^3+2a^2b-$

ab^2+b^3. Hence the result is $\frac{5a^3+10a^2b+5ab^2}{a^2+2a^2b+2ab^2+b^2}$; and the greatest common measure of this result is $a+b$, which is found thus;

$$a^3+2a^2b+2ab^2+b^3)\ 5a^3+10a^2b+5ab^2(5$$
$$5a^3+10a^2b+10ab^2+5b^3$$

remainder . . . . $-5ab^2-5b^3$

And $\frac{-5ab^2-5b^3}{-5b^2}=a+b$, which by another operation is found to divide the numerator without a remainder; and consequently dividing both the numerator and denominator of the fraction $\frac{5a^3+10a^2b+5ab^2}{a^3+2a^2b+2ab^2+b^3}$ by $a+b$, we have the fraction in its lowest terms; that is, $\frac{5a^3+10a^2b+5ab^2}{a+b}=5a^2+5ab$; and $\frac{a^2+2a^2b+2ab^2+b^3}{a+b}=a^2+ab+b^2$:

Hence $\frac{5a^2+5ab}{a^2+ab+b^2}$ is the fraction in its lowest terms.

Ex. 7. Reduce $\frac{14x^2y^2-21x^3y^2}{7x^3y}$ to its lowest terms.

Ans. $\frac{2y-3xy}{x}$

Ex. 8. Reduce $\frac{51x^3-17x^2+34x}{17x}$ to its lowest terms.

Ans. $\frac{3x^2-x+2}{x^4}$.

Ex. 9. Reduce $\frac{a-b}{a^3+b^3}$ to its lowest terms.

Ans. $\frac{1}{a^2+ab+b^2}$.

Ex. 10. Reduce $\frac{x^4+a^2x^2+a^4}{x^4+ax^3-a^3x-a^4}$ to its lowest terms.

Ans. $\frac{x^2-ax+a^2}{x^2-a^2}$.

Ex. 11. Reduce $\frac{7a^2-23ab+6b^2}{5a^3-18a^2b+11ab^2-6b^3}$ to its lowest terms.

Ans. $\frac{7a-2b}{5a^2-3ab+2b^2}$.

Ex. 12. Reduce $\frac{a^4-b^4}{a^6-b^6}$ to its lowest terms.

Ans. $\frac{a^2+b^2}{a^4+a^2b^2+b^4}$.

Ex. 13. Reduce $\frac{y^4-x^4}{y^3-y^2x-yx^2+x^3}$ to its lowest terms.

Ans. $\frac{y^2+x^2}{y-x}$.

Ex. 14. Reduce $\frac{a^3-2a^2b+2ab^2-b^3}{a^4+a^2b^2+b^4}$ to its lowest terms.

Ans. $\frac{a-b}{a^2+ab+b^2}$.

Ex. 15. Reduce $\frac{a^3-3a^2x+3ax^2-x^3}{a^2-x^2}$ to its lowest terms.

Ans. $\frac{a^2-2ax+x^2}{a+x}$.

Ex. 16. Reduce $\frac{a^3+2ba^2+3b^2a^2}{2a^4-3ba^3-5b^2a^2}$ to its lowest terms.

Ans. $\frac{a+2b+3b^2}{2a^2-3ba-5b^2}$.

Ex. 17. Reduce $\frac{a^2-2ax+x^2}{a^3-a^2x-ax^2+x^3}$ to its lowest terms.

Ans. $\frac{1}{a+x}$.

Ex. 18. Reduce $\frac{a^3-b^2a}{a^2+2ab+b^2}$ to its lowest terms.

Ans. $\frac{a^2-ab}{a+b}$.

CASE IV.

To reduce fractions to other equivalent ones, that shall have a common denominator.

RULE I.

151. Multiply each of the numerators separately, into all the denominators, *except its own*, for the new numerators, and *all the denominators together* for the common denominator.

It is necessary to remark, that, if there are whole or mixed quantities, they must be reduced to improper fractions, and then proceed according to the rule.

Ex. 1. Reduce $\frac{3a}{4}$, $\frac{5b}{c}$ and $\frac{x}{a}$ to a common denominator.

$$\left.\begin{array}{r} 3a\times c\times a=3a^2c \\ 5b\times 4\times a=20ab \\ x\times c\times 4=4cx \end{array}\right\}\text{new numerators;}$$

$$4\times c\times a=4ac \text{ common denominator;}$$

Hence the fractions required are $\frac{3a^2c}{4ac}$, $\frac{20ab}{4ac}$, and $\frac{4cx}{4ac}$.

Ex. 2. Reduce $\frac{2x+1}{3b}$, and $\frac{2a^2}{x}$ to a common denominator.

$$\left.\begin{array}{r} (2x+1)\times x=2x^2+x \\ 2a^2\times 3b=6a^2b \end{array}\right\}\text{new numerators;}$$

$$3b\times x=3bx \text{ common denominator;}$$

Hence the fractions required are $\frac{2x^2+x}{3bx}$, and $\frac{6a^2b}{3bx}$.

Ex. 3. Reduce $\frac{3}{4}$, $\frac{5x}{3}$, and $a+\frac{3x^2}{5}$ to a common denominator.

Here $a+\frac{3x^2}{5}=\frac{5a+3x^2}{5}$.

$$\left.\begin{array}{r} 3\times 3\times 5=45 \\ 5x\times 4\times 5=100x \\ (5a+3x^2)\times 4\times 3=60a+36x^2 \end{array}\right\}\text{new numerator;}$$

$$4\times 3\times 5=60 \text{ common denominator;}$$

Hence the fractions required are $\frac{45}{60}$, $\frac{100x}{60}$, and $\frac{60a+36x^2}{60}$.

RULE II.

152. Find the least common multiple of all the denominators of the given fractions, (Art. 147), and it will be the common denominator required.

Divide the common denominator by the denominator of each fraction, separately, and multiply the quotient by the respective numerators, and the products will be the numerators of the fractions required.

Ex. 4. Reduce $\frac{3a^2b}{x^2}$ and $\frac{5ab}{4ax^2}$ to the least common denominator.

Here $4ax^2$ is the least common multiple of x^2 and $4ax^2$;

$$\left.\begin{array}{l}\text{then } \dfrac{4ax^2}{x^2}\times 3a^2b=4a\times 3a^2b=12a^3b \\ \text{and } \dfrac{4ax^2}{4ax^2}\times 5ab=5ab\end{array}\right\}\text{new numerators.}$$

Hence $\dfrac{12a^3b}{4ax^2}$ and $\dfrac{5ab}{4ax^2}$ are the fractions required.

Or, as $4ax^2$ (the least common multiple) is the denominator of one of the fractions, it is only necessary to reduce the fraction $\dfrac{3a^2b}{x^2}$ to an equivalent one, whose denominator shall be $4ax^2$; hence, $\dfrac{4ax^2}{x^2}=4a$, and $\dfrac{3a^2b}{x^2}\times\dfrac{4a}{4a}=\dfrac{3a^2b\times 4a}{x^2\times 4a}=\dfrac{12a^2b}{4ax^2}$ is the fraction required.

These rules appear evident from (Art. 118). For, let $\dfrac{a}{b}, \dfrac{c}{d}, \dfrac{e}{f}$ be the proposed fractions; then $\dfrac{adf}{bdf}, \dfrac{cbf}{bdf}, \dfrac{edb}{bdf}$, are fractions of the same value with the former, having the common denominator bdf. Since $\dfrac{adf}{bdf}=\dfrac{a}{b}$; $\dfrac{cbf}{bdf}=\dfrac{c}{d}$; and $\dfrac{edb}{bdf}=\dfrac{e}{f}$.

Ex. 5. Reduce $\dfrac{3a^2b}{4cx^2}$, $\dfrac{y}{2x}$, and $\dfrac{5x^2}{8ac^2}$ to the least common denominator.

Here, the least common multiple of $4cx^2$, $2x$, and $8ac^2$, (Art. 147), is $8ac^2x^2$; then,

$$\left.\begin{array}{l}\dfrac{8ac^2x^2}{4cx^2}\times 3a^2b=2ac\times 3a^2b=6a^3bc \\ \dfrac{8ac^2x^2}{2x}\times y=4ac^2x\times y=4ac^2xy \\ \dfrac{8ac^2x^2}{8ac^2}\times 5x^2=x^2\times 5x^2=5x^4\end{array}\right\}\text{new numerators;}$$

Hence $\dfrac{6a^3bc}{8ac^2x^2}$, $\dfrac{4ac^2xy}{8ac^2x^2}$, and $\dfrac{5x^4}{8ac^2x^2}$ are the fractions required.

Ex. 6. Reduce $\dfrac{5x}{a+x}$, $\dfrac{a-x}{3}$, and $\dfrac{1}{2x}$ to a common denominator.

Ans. $\dfrac{30x^2}{6ax+6x^2}$, $\dfrac{2a^2x-2x^3}{6ax+6x^2}$, and $\dfrac{3a+3x}{6ax+6x^2}$.

Ex. 7. Reduce $\frac{2x+3}{x}$, and $\frac{5x+2}{3ab}$ to a common denominator.

Ans. $\frac{6abx+9ab}{3abx}$, and $\frac{5x^2+2x}{3abx}$.

Ex. 8. Reduce $\frac{x}{3}$, $\frac{x+1}{4}$, and $\frac{x-1}{1+x}$ to a common denominator.

Ans. $\frac{4x^2+4x}{12x+12}$, $\frac{3x^2+6x+3}{12x+12}$, and $\frac{12x-12}{12x+12}$.

Ex. 9. Reduce $\frac{a}{b}$, $\frac{2c^2}{d}$, and $x+\frac{3a-x^2}{x}$ to a common denominator.

Ans. $\frac{adx}{bdx}$, $\frac{2bc^2x}{bdx}$, and $\frac{3abd}{bdx}$.

Ex. 10. Reduce $\frac{x^2}{5y}$, $a-\frac{x^2-5}{3x}$, and $7+\frac{4a-15}{2}$ to a common denominator.

Ans. $\frac{6x^3}{30xy}$, $\frac{30axy-10x^2y+50y}{30xy}$, and $\frac{60axy-15xy}{30xy}$.

Ex. 11. Reduce $\frac{a}{a^2-x^2}$, $\frac{3b}{4a-4x}$, and $\frac{5x}{a+x}$ to other equivalent fractions having the least common denominator.

Ans. $\frac{4a}{4a^2-4x^2}$, $\frac{3ab+3bx}{4a^2-4x^2}$, and $\frac{20ax-20x^2}{4a^2-4x^2}$.

Ex. 12. Reduce $\frac{1}{a^2+2ax+x^2}$, $\frac{1}{a^2-x^2}$, and $\frac{5y}{a^4-x^4}$ to the least common denominator.

Ans. $\frac{a^3+ax^2-a^2x-x^3}{a^5-ax^4+a^4x-x^5}$, $\frac{a^3+ax^2+a^2x+x^3}{a^5-ax^4+a^4x-x^5}$,

and $\frac{5ay+5xy}{a^5-ax^4+a^4x-x^5}$.

§ V. ADDITION AND SUBTRACTION OF ALGEBRAIC FRACTIONS.

To add fractional quantities together.

RULE.

153. Reduce the fractions, if necessary, to a common denominator, by the rules in the last case, then add all the nume-

rators together, and under their sum put the common denominator; bring the resulting fraction to its lowest terms, and it will be the sum required.

Ex. 1. Add $\frac{2x}{3}$, $\frac{5x}{7}$, and $\frac{x}{9}$ together.

$$\left.\begin{array}{r} 2x\times7\times9=126x \\ 5x\times3\times9=135x \\ x\times7\times3=\ \ 21x \\ \hline 3\times7\times9=189 \end{array}\right\} \therefore \frac{126x+135x+21x}{189}=\frac{282x}{189}=x+\frac{93x}{189} \text{ is the sum required.}$$

Ex. 2. Add $\frac{a}{b}$, $\frac{2a}{3b}$, and $\frac{5b}{4a}$ together.

$$\left.\begin{array}{r} a\times3b\times4a=12a^2b \\ 2a\times b\times4a=\ \ 8a^2b \\ 5b\times3b\times b=15b^3 \\ \hline b\times3b\times4a=12ab^2 \end{array}\right\} \therefore \frac{12a^2b+8a^2b+15b^3}{12ab^2}=\frac{20a^2b+15b^3}{12ab^2}=\text{(dividing by } b) \ \frac{20a^2+15b^2}{12ab} \text{ is the sum required.}$$

Or, the least common multiple of the denominators may be found, and then proceed, as in (Art. 152).

It is generally understood that mixed quantities are reduced to improper fractions, before we perform any of the operations of Addition and Subtraction. But it is best to bring the fractional parts only to a common denominator, and to affix their sum or difference to the sum or difference of the integral parts, interposing the proper sign.

Ex. 3. It is required to find the sum of $a-\frac{3x^2}{b}$, and $b+\frac{2ax}{c}$.

Here, $a-\frac{3x^2}{b}=\frac{ab-3x^2}{b}$, and $b+\frac{2ax}{c}=\frac{bc+2ax}{c}$.

$$\text{Then, } \left.\begin{array}{l} (ab-3x^2)\times c=abc-3cx^2 \\ (bc+2ax)\times b=b^2c+2abx \end{array}\right\} \text{numerators.}$$

$$b\times c=bc=\text{ denominator.}$$

$$\therefore \frac{abc-3cx^2+b^2c+2abx}{bc}=\frac{abc+b^2c}{bc}+\frac{2abx-3cx^2}{bc}=a+b+\frac{2abx-3cx^2}{bc}$$

is the sum required.

Or, bringing the fractional parts only to a common denominator,

Thus, $\left.\begin{array}{l} 3x^2 \times c = 3cx^2 \\ 2ax \times b = 2abx \end{array}\right\}$ numerators,

And $b \times c = bc$ common denominator.

Whence $a - \frac{3cx^2}{bc} + b + \frac{2abx}{bc} = a + b + \frac{2abx - 3cx^2}{bc}$ the sum.

Ex. 4. It is required to find the sum of $5x + \frac{x-2}{3}$ and $4x - \frac{2x-3}{5x}$.

Here, $\left.\begin{array}{l} (x-2) \times 5x = 5x^2 - 10x \\ (2x-3) \times 3 = 6x - 9 \end{array}\right\}$ numerators,

And $3 \times 5x = 15x$ common denominator.

Whence $5x + \frac{5x^2-10x}{15x} + 4x - \frac{6x-9}{15x} = 9x + \frac{5x^2 - 10x}{15x} + \frac{9-6x}{15x} = 9x + \frac{5x^2-16x+9}{15x}$ the sum required.

Here, $-\frac{6x-9}{15x}$ is evidently $= \frac{9-6x}{15x}$ (Art. 128); but we might change the fractions into other equivalent forms before we begin to add or subtract; thus, the fractional part of the proposed quantity $4x - \frac{2x-3}{5x}$ may be transformed by changing the signs of the numerator, (Art. 128), and the quantity itself can be written thus, $4x + \frac{3-2x}{5x}$: It is well to keep this transformation in mind, as it is often necessary to make use of it in performing several algebraical operations.

Ex. 5. Add $\frac{3a^2}{2b}$, $\frac{2a}{5}$ and $\frac{b}{7}$ together.

Ans. $\frac{105a^2+28ab+10b^2}{70b}$.

Ex. 6. Add $\frac{x}{x-3}$ and $\frac{x}{x+3}$ together. Ans. $\frac{2x^2}{x^2-9}$.

Ex. 7. Add $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$ together.

Ans. $\frac{2a^2+2b^2}{a^2-b^2}$.

Ex. 8. Add $\frac{a+x}{a-x}$ and $-\frac{a-x}{a+x}$ together.

Ans. $\frac{4ax}{a^2-x^2}$.

Ex. 9. Add $2x+\frac{x-2}{3}$ and $3x+\frac{2x-3}{4}$ together.

Ans. $5x+\frac{10x-17}{12}$.

Ex. 10. Add $4x$, $\frac{7x}{9}$ and $2+\frac{x}{5}$ together.

Ans. $4x+2+\frac{44x}{45}$.

Ex. 11. Add $5x-\frac{2x}{7}$ and $\frac{5x}{9}-4x$ together.

Ans. $x+\frac{17x}{63}$.

Ex. 12. It is required to find the sum of $2a$, $\frac{a}{a-x}$, and $\frac{a-x}{a}$.

Ans. $2a+2+\frac{x^3}{a^2-ax}$.

To subtract one fractional quantity from another.

RULE.

154. Reduce the fractions to a common denominator, if necessary, and then subtract the numerators from each other, and under the difference write the common denominator, and it will give the difference of the fractions required.

Or, enclose the fractional quantity to be subtracted in a parentheses; then, prefixing the negative sign, and performing the operation, observing the same remarks and rules as in addition, the result will be the difference required.

The reason of this is evident; because, adding a negative quantity is equivalent to subtracting a positive one (Art. 63); thus, prefixing the negative sign to the fractional quantity $\frac{a-b}{c}$, it becomes $-\left(\frac{a-b}{c}\right)=-\frac{a-b}{c}=\frac{b-a}{c}$; to the fractional quantity $-\frac{x^2+a}{y}$, it becomes $-\left(-\frac{x^2+a}{y}\right)=+\frac{x^2+a}{y}$ (Art. 128); to the fractional quantity $-\frac{ax-b}{5}$, it becomes $-\left(-\frac{ax-b}{5}\right)=\frac{ax-b}{5}$; to the mixed quantity $5x-\frac{3a+b}{y}$, it becomes $-\left(5x-\frac{3a+b}{y}\right)=-5x+\frac{3a+b}{y}$; and to the mixed quantity $-3a+\frac{2-x}{c}$, it becomes $-\left(-3a+\frac{2-x}{c}\right)=3a-\frac{2-x}{c}=3a+\frac{x-2}{c}$.

Ex. 1. Subtract $\frac{3x}{5}$, from $\frac{5x}{7}$.

Here $\left.\begin{array}{l}3x\times7=21x\\5x\times5=25x\end{array}\right\}$ numerators, $\left\{\begin{array}{l}\therefore\ \frac{25x-21x}{35}=\frac{4x}{35}\text{ is the}\\\text{difference required.}\end{array}\right.$

$5\times7=35$ com. denom.

Ex. 2. Subtract $\frac{2a-4x}{5c}$ from $\frac{x-y}{3b}$.

Here $\left.\begin{array}{l}(2a-4x)\times3b=6ab-12bx\\(x-y)\times5c=\quad5cx-5cy\end{array}\right\}$ numerators.

$5c\times3b=15bc$ common denominator.

Whence, $\frac{5cx-5cy}{15bc}-\frac{6ab-12bx}{15bc}=\frac{5cx-5cy}{15bc}+\frac{12bx-6ab}{15bc}=\frac{5cx-5cy+12bx-6ab}{15bc}$ is the difference required.

Or, by prefixing the negative sign to the quantity $\frac{2a-4x}{5c}$, it becomes $-\frac{2a-4x}{5c}=\frac{4x-2a}{5c}$; then it only remains to add $\frac{4x-2a}{5c}$ and $\frac{x-y}{3b}$ together, as in addition, and the result will be the same as above.

Ex. 3. From $2ab+\frac{a-x}{a+x}$ subtract $2ab-\frac{a-x}{a+x}$.

Here prefixing the negative sign to the quantity $2ab-\frac{a-x}{a+x}$, we have $-\left(2ab-\frac{a-x}{a+x}\right)=-2ab+\frac{a-x}{a+x}$; hence the difference of the proposed fractions is equivalent to the sum of $2ab+\frac{a-x}{a+x}$, and $-2ab+\frac{a+x}{a-x}$; but the sum of the fractional parts $\frac{a-x}{a+x}$ and $\frac{a+x}{a-x}$, is $\frac{2a^2+2x^2}{a^2-x^2}$: Therefore the difference required is $2ab-2ab+\frac{2a^2+2x^2}{a^2-x^2}=\frac{2a^2+2x^2}{a^2-x^2}$.

Ex. 4. From $\frac{10x-9}{15}$ subtract $\frac{3x-5}{7}$.

Here $\left.\begin{array}{l}(10x-9)\times 7=70x-63\\(3x-5)\times 15=45x-75\end{array}\right\}$ numerators.

$15\times 7=105$ common denominator.

Therefore, $\frac{70x-63}{105}-\frac{45x-75}{105}=$

$\frac{70x-63-45x+75}{105}=\frac{25x+12}{105}$ is the fraction required.

Ex. 5. From $\frac{a+b}{a-b}$ subtract $\frac{a-b}{a+b}$. Ans. $\frac{4ab}{a^2-b^2}$

Ex. 6. From $\frac{1}{a-x}$ subtract $\frac{1}{a+x}$ Ans. $\frac{2x}{a^2-x^2}$.

Ex. 7. From $\frac{4x+2}{3}$ subtract $\frac{2x-3}{3x}$. Ans. $\frac{4x^2+3}{3x}$.

Ex. 8. From $3x+\frac{x}{b}$ subtract $x-\frac{x-a}{c}$

Ans. $2x+\frac{cx+bx-ab}{bc}$.

Ex. 9. Subtract $\frac{2x+7}{8}$ from $\frac{3x^2+a^2}{3b}$.

Ans. $\frac{24x^2+8a^2-6bx-21b}{24b}$.

Ex. 10. Subtract $4x-\frac{2x-3}{5}$ from $5x+\frac{x-2}{3}$

Ans. $x+\frac{11x-19}{15}$.

Ex. 11. Subtract $\frac{a+x}{a(a-x)}$ from $a+\frac{a-x}{a(a+x)}$.

Ans. $a-\frac{4x}{a^2-x^2}$.

Ex. 12. Required the difference of $3x$ and $\frac{3a+12x}{5}$

Ans. $\frac{3x-3a}{5}$

Ex. 13. From $2x+\frac{5x-2}{7}$ subtract $3x-\frac{4x+5}{6}$.

Ans. $\frac{16x+23}{42}$

§ VI. MULTIPLICATION AND DIVISION OF ALGEBRAIC FRACTIONS.

To multiply fractional quantities together.

RULE.

155. Multiply their numerators together for a new numerator, and their denominators together for a new denominator; reduce the resulting fraction to its lowest terms, and it will be the product of the fractions required.

It has been already observed, (Art. 119), that when a fraction is to be multiplied by a whole quantity, the numerator is multiplied by that quantity, and the denominator is retained:

Thus, $\frac{a}{b}\times c=\frac{ac}{b}$, and $\frac{2x}{b}\times 5=\frac{10x}{b}$; or, which is the same, making an improper fraction of the integral quantity, and then proceeding according to the rule, we have $\frac{a}{b}\times\frac{c}{1}=\frac{ac}{b}$, and $\frac{2x}{b}\times\frac{5}{1}=\frac{10x}{b}$.

Hence, if a fraction be multiplied by its denominator, the product is the numerator; thus, $\frac{a}{b}\times b=\frac{ab}{b}=b$. In like manner, the result being the same, whether the numerator be multiplied by a whole quantity, or the denominator divided by it, the latter method is to be preferred, when the denominator is some multiple of the multiplier: Thus, let $\frac{ad}{bc}$ be the fraction, and c the multiplier; then $\frac{ad}{bc}\times c=\frac{adc}{bc}=\frac{ad}{b}$; and $\frac{ad}{bc}\times c=\frac{ad}{bc\div c}=\frac{ad}{b}$, as before.

Also, when the numerator of one of the fractions to be multiplied, and the denominator of the other, can be divided by some quantity which is common to each of them, the quotients may be used instead of the fractions themselves; thus,

$\frac{a+b}{a-b}\times\frac{x}{a+b}=\frac{x}{a-b}$; cancelling $x+b$ in the numerator of the one, and denominator of the other.

Ex. 1. Multiply $\frac{3a}{5}$ by $\frac{4a}{7}$.

$\left.\begin{array}{l}3a\times 4a=12a^2=\text{ numerator,}\\ 5\times 7=35=\text{ denominator ;}\end{array}\right\}$ ∴ the fraction required is $\frac{12a^2}{35}$.

Ex. 2. Multiply $\frac{3x+2}{4}$ by $\frac{8x}{7}$.

Here, $(3x+2)\times 8x=24x^2+16x=$ numerator,
and $4\times 7=28=$ denominator ;

Therefore, $\frac{24x^2+16x}{28}=$ (dividing the numerator and denominator by 4) $\frac{6x^2+4x}{7}$, the product required.

Ex. 3. Multiply $\frac{a^2-x^2}{3a}$ by $\frac{7x^2}{a-x}$.

Here, $(a^2-x^2)\times 7x^2=(a+x)\times(a-x)\times 7x^2=$ numerator (Art. 106), and $3a\times(a-x)=$ denominator ; see Ex. 15, (Art. 79.)

Hence, the product is $\frac{(a+x)\times(a-x)\times 7x^2}{3a\times(a-x)}=$ (dividing the numerator and denominator by $a-x$,) $\frac{7x^2(a+x)}{3a}=\frac{7ax^2+7x^3}{3a}$.

Ex. 4. Multiply $a+\frac{x}{5}$ by $a-\frac{x}{3}$.

Here, $a+\frac{x}{5}=\frac{5a+x}{5}$, and $a-\frac{x}{3}=\frac{3a-x}{3}$:

Then, $(5a+x)\times(3a-x)=15a^2-2ax-x^2=$ new numerator, and $5\times 3=15=$ denominator : Therefore, $\frac{15a^2-2ax-x^2}{15}=a^2-\frac{2ax+x^2}{15}$ is the product required.

156. But, when mixed quantities are to be multiplied together, it is sometimes more convenient to proceed, as in the multiplication of integral quantities, without reducing them to improper fractions.

Ex. 5. Multiply $x^2-\frac{1}{2}x+\frac{2}{3}$ by $\frac{1}{3}x+2$.

$$\begin{array}{l} x^2-\frac{1}{2}x+\frac{2}{3} \\ \frac{1}{3}x+2 \\ \hline \frac{1}{3}x^3-\frac{1}{6}x^2+\frac{2}{9}x \\ \qquad +2x^2-x+\frac{4}{3} \\ \hline \frac{1}{3}x^3+\frac{11}{6}x^2-\frac{7}{9}x+\frac{4}{3} \\ \hline \end{array}$$

Ex. 6. Multiply $\frac{3x^2-5x}{14}$ by $\frac{7a}{3x^3-3x}$.

Ans. $\frac{3ax-5a}{6x^2-6}$.

Ex. 7. Multiply $\frac{3x^2}{5x-10}$ by $\frac{15x-30}{2x}$. Ans. $\frac{9x}{2}$.

Ex. 8. Multiply $\frac{2a-2x}{3ab}$ by $\frac{3ax}{5a-5x}$. Ans. $\frac{2x}{5b}$.

Ex. 9. It is required to find the continual product of $\frac{3a}{5}$, $\frac{2x^2}{3}$, and $\frac{a+b}{ax}$. Ans. $\frac{2ax+2bx}{5}$.

Ex. 10. It is required to find the continued product of $\frac{a^4-x^4}{a^2-y^2}$, $\frac{a+y}{a^2+x^2}$, and $\frac{a-y}{a-x}$. Ans. $a+x$.

Ex. 11. It is required to find the continued product of $\frac{a^2-x^2}{a+b}$, $\frac{a^2-b^2}{a+x}$, and $\frac{a}{ax-x^2}$. Ans. $\frac{a^2-ab}{x}$.

Ex. 12. Multiply $x^2-\frac{3}{4}x+1$ by $x^2-\frac{1}{2}x$.

Ans. $x^4-\frac{5}{4}x^3+\frac{11}{8}x^2-\frac{1}{2}x$.

To divide one fractional quantity by another.

RULE.

157. Multiply the dividend by the *reciprocal* of the divisor, or which is the same, *invert* the divisor, and proceed, in every respect, as in multiplication of algebraic fractions; and the product thus found will be the quotient required.

When a fraction is to be divided by an integral quantity; the process is the reverse of that in multiplication; or, which is the same, multiply the denominator by the integral, (Art. 120), or divide the numerator by it. The latter mode is to be preferred, when the numerator is a multiple of the divisor.

Ex. 1. Divide $\frac{5x}{a}$ by $\frac{b}{c}$.

The divisor $\frac{b}{c}$ inverted, becomes $\frac{c}{b}$, hence $\frac{5x}{a}\times\frac{c}{b}=\frac{5cx}{ab}$ is the fraction required.

Ex. 2. Divide $\frac{3a-3x}{a+b}$ by $\frac{5a-5x}{a+b}$.

The divisor $\left(\frac{5a-5x}{a+b}\right)$ *inverted*, becomes $\frac{a+b}{5a-5x}$; hence $\frac{3a-3x}{a+b}\times\frac{a+b}{5a-5x}=\frac{3a-3x}{5a-5x}=\frac{3(a-x)}{5(a-x)}=\frac{3}{5}$ is the quotient required.

Ex. 3. Divide $\frac{a^2-b^2}{x}$ by $a+b$.

The *reciprocal* of the divisor is $\frac{1}{a+b}$; hence $\frac{a^2-b^2}{x}\times\frac{1}{a+b}$ $=\frac{(a+b)(a-b)}{x\times(a+b)}=\frac{a-b}{x}$ is the quotient required.

Or, $\frac{a^2-b^2}{a+b}=a-b$; hence $\frac{a-b}{x}$ is the fraction required.

Ex. 4. Divide $\frac{x^2-a^2}{a+c}$ by $a+\frac{x^2-a^2}{a}$.

Here, $a+\frac{x^2-a^2}{a}=\frac{a^2+x^2-a^2}{a}=\frac{x^2}{a}$; then, the fraction $\frac{x^2-a^2}{a+c}$ divided by $\frac{x^2}{a}$ becomes $\frac{x^2-a^2}{a+c}\times\frac{a}{x^2}=\frac{ax^2-a^3}{ax^2+cx^2}=$ the quotient required.

158. But it is, however, frequently more simple in practice to divide mixed quantities by one another, without reducing them to improper fractions, as in division of integral quantities, especially when the division would terminate.

Ex. 5. Divide $x^4-\frac{5}{4}x^3+\frac{11}{8}x^2-\frac{1}{2}x$ by $x^2-\frac{1}{2}x$.

$$
\begin{array}{l}
x^2-\tfrac{1}{2}x)\,x^4-\tfrac{5}{4}x^3+\tfrac{11}{8}x^2-\tfrac{1}{2}x\,(x^2-\tfrac{3}{4}x+1 \\
\qquad\quad x^4-\tfrac{1}{2}x^3 \\
\hline
\qquad\qquad -\tfrac{3}{4}x^3+\tfrac{11}{8}x^2-\tfrac{1}{2}x \\
\qquad\qquad -\tfrac{3}{4}x^3+\tfrac{3}{8}x^2 \\
\hline
\qquad\qquad\qquad\quad x^2-\tfrac{1}{2}x \\
\qquad\qquad\qquad\quad x^2-\tfrac{1}{2}x \\
\hline
\qquad\qquad\qquad\quad * \quad *
\end{array}
$$

Ex. 6. Divide $\frac{4a}{3}$ by $\frac{3a}{5}$. Ans. $\frac{20}{9}$.

Ex. 7. Divide $\frac{4x+2}{5}$ by $\frac{2x+1}{5x}$. Ans. $2x$.

Ex. 8. Divide $\frac{9x^2-3x}{5}$ by $\frac{x^2}{5}$. Ans. $\frac{9x-3}{x}$.

Ex. 9. Divide $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x^2+bx}{x-b}$. Ans. $x+\frac{b^2}{x}$.

Ex. 10. Divide $\frac{2x^2}{a^3+x^3}$ by $\frac{x}{x+a}$. Ans. $\frac{2x}{x^2-ax+a^2}$.

Ex. 11. Divide $\frac{a^3-x^3}{a+x}$ by $\frac{a-x}{a^2+2ax+x^2}$.

Ans. $a^3+2a^2x+2ax^2+x^3$.

Ex. 12. Divide $x^4-\frac{13}{6}x^3+x^2+\frac{4}{3}x-2$ by $\frac{4}{3}x-2$.

Ans. $\frac{3}{4}x^3-\frac{1}{2}x^2+1$

CHAPTER III.

ON

SIMPLE EQUATIONS,

INVOLVING ONLY ONE UNKNOWN QUANTITY.

159. In addition to what has been already said, (Art. 34), it may be here observed, that the expression, in algebraic symbols, of two equivalent phrases contained in the enunciation of a question, is called an *equation*, which, as has been remarked by GARNIER, differs from an *equality*, in this, that the first comprehends an unknown quantity combined with certain known quantities; whereas the second takes place but between quantities that are known. Thus, the expression $a=\frac{s}{2}+\frac{d}{2}$, (Art. 102), according to the above remark, is called an equality; because the quantities a, s, and d, are supposed to be known. And the expression $x+x-d=s$, (Art. 103), is called an equation, because the unknown quantity x, is combined with the given quantities d and s. Also, $x-a=0$ is an

equation which asserts that $x-a$ is equal to nothing, and therefore, that the positive part of the expression is equal to the negative part.

160. A *simple* equation is that which contains only the first power of the unknown quantity, or the unknown quantity merely in its simplest form, after the terms of the equation have been properly arranged:

Thus, $x+a=b$; $ax+bx=c$; or, $\frac{x}{4}+\frac{x}{3}=d$, &c. where x denotes the unknown quantity, and the other letters, or numbers, the known quantities.

§ I. REDUCTION OF SIMPLE EQUATIONS.

161. *Any quantity may be transposed from one side of an equation to the other, by changing its sign.*

Because, in this transposition, the same quantity is merely added to or subtracted from each side of the equation; and, (Art. 48, 49,) if equals be added to or subtracted from equal quantities, the sums or remainders will be equal. Thus, if $x+5=12$; by subtracting 5 from each side, we shall have

$$x+5-5=12-5;$$

but $5-5=0$, and $12-5=7$; hence $x=7$.

Also, if $x+a=b-2x$; by subtracting a from each side, we shall have

$$x+a-a=b-2x-a;$$

and by adding $2x$ to each side, we shall have

$$x+a-a+2x=b-2x-a+2x;$$

but $a-a=0$, and $-2x+2x=0$: therefore

$$x+2x=b-a, \text{ or } 3x=b-a.$$

Again, if $ax-c=d$, and c be added to each side, $ax-c+c=d+c$, or $ax=d+c$.

Also, if $5x-7=2x+12$; by subtracting $2x$ from each side, we shall have

$$5x-7-2x=2x+12-2x, \text{ or } 3x-7=12;$$

subtracting -7, or, which is the same thing, adding $+7$ to each side of this last equation, and we shall have

$$3x-7+7=12+7;$$

but $7-7=0$, $\therefore 3x=19$.

Finally, if $x-a+b=c-2x+d$, then, by subtracting b from each side, we shall have

$$x-a+b-b=c-2x+d-b;$$

and adding $a+2x$ to each side, it becomes

$$x-a+b-b+a+2x=c-2x+d-b+a+2x;$$

but $a-a=0$, $b-b=0$, and $-2x+2x=0$;

therefore, $x+2x=c+a-b+d$, or $3x=c+a-b+d$.

Cor. 1. Hence, if the signs of the terms on each side of an equation be changed, the two sides still remain equal; because in this change every term is transposed: Thus, if $-x+b-c=a-9+x$; then, $x-b+c=9-a-x$; or, which is the same thing, by transposing the right-hand side to the left, and the reverse, we shall have $9-a-x=x-b+c$.

Cor. 2. Hence, when the known and unknown quantities are connected in an equation by the signs + or —, they may be separated by transposing the known quantities to one side, and the unknown to the other.

Thus, if $3x-9-a=12+b-4x^2$; then, $4x^2+3x=a+b+21$.

Also, if $3x^2-2+x=b-4x^3-3x^4$; then, $3x^4+4x^3+3x^2+x=b+2$.

Hence also, if any quantity be found on both sides of an equation, it may be taken away from each; thus, if $x+a=a+5$, then $x=5$; if $x-b=c+d-b$, then $x=c+d$; because, by adding b to each side, we shall have $x-b+b=c+d-b+b$; but $b-b=0$, $\therefore x=c+d$.

162. *If every term on each side of an equation be multiplied by the same quantity, the results will be equal*: because, in multiplying every term on each side by any quantity, the value of the whole side is multiplied by that quantity; and, (Art. 50), if equals be multiplied by the same quantity, the products will be equal.

Thus, if $x=5+a$, then $6x=30+6a$, by multiplying every term by 6. And, if $\frac{x}{2}=4$, then, multiplying each side by 2, we have $\frac{x}{2}\times2=4\times2$, or $x=8$, because, (Art. 155), $\frac{x}{2}\times2=x$.

Also, if $\frac{x}{4}-3=a-b$, then, by multiplying every term by 4, we shall have $x-12=4a-4b$.

Again, if $2x-\frac{3}{2}+1=x$; then, $4x-3+2=2x$; and $4x-2x=3-2$, or $2x=1$.

Cor. 1. Hence, an equation of which any part is fractional, may be reduced to an equation expressed in integers, by multiplying every term by the denominator of the fraction; but if there be more fractions than one in the given equation, it may be so reduced by multiplying every term by the product of the denominators, or by the least common multiple of them; and it will be of more advantage, to multiply by the least common multiple, as then the equation will be in its lowest terms.

Let $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=11$; then, if every term be multiplied by 24, which is the product of all the denominators; we have $\frac{x}{2}\times 24+\frac{x}{3}\times 24+\frac{x}{4}\times 24=11\times 24$; and $12x+8x+6x=264$; or, if every term of the proposed equation be multiplied by 12, which is the least common multiple of 2, 3, 4, (Art. 146); we shall have $6x+4x+3x=132$, an equation in its lowest terms.

Cor. 2. Hence also, if every term on both sides have a common divisor, that common divisor may be taken away; thus, if $\frac{3x}{5}+\frac{a+6}{5}=\frac{2x+7}{5}$, then, multiplying every term by 5, we shall have $3x+a+6=2x+7$, or $x=1-a$.

Also, if $\frac{ax}{c}-\frac{b}{c}+\frac{3}{c}=\frac{7-x}{c}$, then multiplying by c, we shall have $ax-b+3=7-x$, or $ax+x=b+4$.

163. *If every term on each side of an equation be divided by the same quantity, the results will be equal:* Because, by dividing every term on each side by any quantity, the value of the whole side is divided by that quantity; and, (Art. 51), if equals be divided by the same quantity, the products will be equal.

Thus, if $6a^2+3x=9$; then, dividing by 3, $2a^2+x=3$.

Also, if $ax^2+bx=acx$; then, dividing every term by the common multiplier x, we shall have $\frac{ax^2}{x}+\frac{bx}{x}=\frac{acx}{x}$, or $ax+b=ac$.

Cor. 1. Hence, if every term on both sides have a common multiplier, that common multiplier may be taken away.

Thus, if $ax+ad=ab$, then, dividing every term by the common multiplier a, we shall have $x+d=b$.

Also, if $\frac{ax}{c}+\frac{ab}{c}=\frac{4a^2x}{c}$; then dividing by the common multiplier $\frac{a}{c}$, or (which is the same thing) multiplying by $\frac{c}{a}$, we shall have $x+b=4ax$.

Cor. 2. Also, if each member of the equation have a common divisor, the equation may be reduced by dividing both sides, by that common divisor.

Thus, if $ax^2-a^2x=abx-a^2b$, or $(ax-a^2)x=(ax-a^2)b$; then it is evident that each side is divisible by $ax-a^2$, whence $x=b$.

Again, if $x^2-a^2=x+a$; then, because $x^2-a^2=(x+a)$. $(x-a)$, it is evident that each side is divisible by $x+a$; and hence we have $\frac{x^2-a^2}{x+a}=\frac{x+a}{x+a}$, or $x-a=1$, and $x=a+1$.

164. *The unknown quantity may be disengaged from a divisor or a coefficient, by multiplying or dividing all the terms of the equation by that divisor or coefficient.*

Thus, if $2x+4=b$, then $x+2=\frac{b}{2}$ and $x=\frac{b}{2}-2$.

Also, let $\frac{x}{2}+9=17$; then, multiplying by 2, we shall have $\frac{x}{2}\times2+18=17\times2$,

$$\text{or } x+18=34, \therefore x=34-18.$$

Again, let $ax+bx=c-d$, or, which is the same, let $(a+b)x=c-d$; then, dividing both sides by $a+b$, the coefficient of x, and we shall have

$$x=\frac{c-d}{a+b}.$$

Finally, let $\frac{x}{a}-\frac{x}{b}=c+d$; then, the equation may be put under this form,

$$\left(\frac{1}{a}-\frac{1}{b}\right)x=c+d\ ;$$

and dividing each side by $\frac{1}{a}-\frac{1}{b}$, we shall have $x=(c+d)\div\left(\frac{1}{a}-\frac{1}{b}\right)$; which may be still farther reduced, because $\frac{1}{a}-\frac{1}{b}=\frac{b-a}{ab}$; therefore

$$x=(c+d)\div\frac{b-a}{ab},$$

$$\text{or } x=(c+d)\times\frac{ab}{b-a},$$

$$\therefore x=\frac{abc+abd}{b-a}.$$

165. *Any proportion may be converted into an equation; for the product of the extremes is equal to the product of the means.*

Because, if $a : b :: x : d$; then $\frac{a}{b}=\frac{x}{d}$, (Art. 24), and $\therefore$ (Art. 162), $ad=bx$, by clearing of fractions.

Let $3x : 5x :: 2x : 7$; then $7\times 3x=2x\times 5x$,
or $21x=10x^2$: and $\therefore 21=10x$.

Again, let $5x+20 : 4x+4 :: 5 : x+1$; then,
$(5x+20)\times(x+1)=5\times(4x+4)$;
or, $5x^2+25x+20=20x+20$;
and (Art. 161), $5x^2+25x=20x$;
$\therefore$ (Art. 163), $5x+25=20$.

166. When an *unknown* quantity enters into, or forms a part of an *equation;* and if the equation can be so ordered, that the unknown quantity may stand by *itself* on one side, with its simple or first power, and only known quantities on the other, the quantity that was before unknown, will then become known.

Thus, suppose $3x+18=5x-2$; then, by transposing $3x$ and -2, we shall have

$$18+2=5x-3x, \text{ or } 20=2x;$$

$$\text{therefore, } x=\frac{20}{2}=10.$$

Here, in the above equation, the value of the unknown quantity x, becomes known, and 10 is the value of x that fulfils the condition required, which we can readily see verified, by substituting this value of x in the given equation; thus,

$$3x=3\times 10=30, \text{ and } 5x=5\times 10=50;$$

hence, $3x+18=30+18=48$, and $5x-2=50-2=48$; therefore 10 is the true value of x, which answers the condition required, and this value of x is called *the root of the equation.*

167. Hence the root of an equation is such a number or quantity, as, being substituted for the unknown quantity, will make both sides of the equation vanish or equal to each other: Thus, in the simple equation

$$3x-9+6=0;$$

the value of x must be such, that if substituted for it, both sides must vanish, because the right-hand side is 0; but this value is found to be 1, for by transposition

$$3x=9-6=3,$$

and dividing by 3, we shall have

$$\frac{3x}{3}=\frac{3}{3}, \text{ or } x=1;$$

therefore 1 is the root of the given equation, which can be easily verified by substituting it for x; thus,

$$3x-9+6=3\times1-9+6=3-9+6=9-9=0.$$

Hence, the value of the unknown quantity being substituted in the equation, will always reduce it to $0=0$.

§ II. RESOLUTION OF SIMPLE EQUATIONS,

Involving only one unknown Quantity.

168. The resolution of simple equations is the disengaging of the unknown quantity, in all such expressions, from the other quantities with which it is connected; and making it stand alone, on one side of the equation, so as to be equal to such as are known on the other side, or, which is the same thing, the value of the unknown quantity cannot be ascertained till we transform the given equation, by the addition, subtraction, multiplication, or division of equal quantities, so that we may fully arrive at the conclusion,

$$x=n,$$

n being a number, or a formula, which indicates the operations to be performed upon known numbers. This number n being substituted for x in the primitive equation, has the property of rendering the first member equal to the second. And this value of the unknown quantity, as has been already observed, is called the *root* of the equation, this word has not here the same acceptation as in (Art. 15.)

169. In the resolution of simple equations, involving only one unknown quantity, the following rules, which are deduced from the Articles in the preceding Section, are to be observed.

RULE I.

When the unknown quantity is only connected with known quantities by the signs plus or minus.

170. Transpose the known quantities to one side of the equation, so that the unknown may stand by *itself* on the other; and then the unknown quantity becomes known.

Ex. 1. Given $x+8=9$, to find the value of x.

By transposition, $x=9-8$, $\therefore x=1$.

Ex. 2. Given $3x-4=2x+5$, to find the value of x.

By transposition, $3x-2x=5+4$, $\therefore x=9$.

Ex. 3. Given $x+a=a+5$, to find the value of x.

By taking a from both sides, we have

$x=5$; or by transposition,

$x=a-a+5$; but $a-a=0$ $\therefore x=5$.

Ex. 4. Given $9-x=2$, to find the value of x.

By changing the signs of all the terms, we have

$$-9+x=-2,$$

by transposition, $x=9-2$, $\therefore x=7$.

It may be remarked, that it is the general practice of Analysts, to make the unknown quantity appear on the left-hand side of the equation, which is principally the reason for changing the signs.

Ex. 5. Given $-b-x=a-c$ to find x in terms of a, b, and c.

(161. Cor. 1), by changing the signs of all the terms, we have $b+x=c-a$; $\therefore$ by transposition, $x=c-b-a$.

Ex. 6. Given $2x-4+7=3x-2$, to find the value of x.

(161.) by transposition, $2x-3x=4-7-2$, and (161. Cor. 1), by changing the signs, $3x-2x=7+2-4$; but $3x-2x=x$, and $7+2-4=5$; $\therefore x=5$.

Ex. 7. Given $7x+3-5=6x-2+7$, to find the value of x.
Ans. $x=7$.

Ex. 8. Given $3x+5-2-2x-7=0$, to find the value of x.
Ans. $x=4$.

Ex. 9. Given $x-3+4-6=0$, to find the value of x.
Ans. $x=5$.

Ex. 10. Given $7+x=2x+12$, to find the value of x.
Ans. $x=-5$.

Ex. 11. Given $12-3x=9-2x$, to find the value of x.
Ans. $x=3$.

Ex. 12. Given $x-a+b-c=0$, to find the value of x in terms of a, b, and c.
Ans. $x=a-b+c$.

Ex. 13. Given $x-a+b=2x-2a+b$, to find the value of x in terms of a and b.
Ans. $x=a$.

Ex. 14. Given $2x+a=x+b$, to find x in terms of a and b.
Ans. $x=b-a$.

RULE II.

171. Transpose the known quantities to one side of the equation, and the unknown to the other, as in the last Rule; then, if the unknown quantity has a coefficient, its value may be found by dividing each side of the equation by the coefficient, or by the sum of the coefficients.

Ex. 1. Given $3x+9=18$, to find the value of x.

By transposition, $3x=18-9$, or $3x=9$; dividing both sides of the equation by 3, the coefficient of x, we have $\frac{3x}{3}=\frac{9}{3}$, $\therefore x=3$.

Ex. 2. Given $2x-3=9-x$, to find the value of x.

By transposition, $2x+x=9+3$,

by collecting the terms, $3x=12$,

by division, $\frac{3x}{3}=\frac{12}{3}$; $\therefore x=4$.

Ex. 3. Given $7-4x=3x-7$, to find the value of x.

By transposition, $-4x-3x=-7-7$,

by collecting the terms, $-7x=-14$,

by changing the signs, $7x=14$,

by division, $\frac{7x}{7}=\frac{14}{7}$; $\therefore x=2$.

Ex. 4. Given $6x+10=3x+22$, to find the value of x.

By transposition, $6x-3x=22-10$,

by collecting the terms, $3x=12$,

by division, $\frac{3x}{3}=\frac{12}{3}$; $\therefore x=4$.

Ex. 5. Given $ax+b=c$ to find the value of x in terms of a, b, and c.

By transposition, $ax=c-b$,

by division, $\frac{ax}{a}=\frac{c-b}{a}$; $\therefore x=\frac{c-b}{a}$.

The value of x is equal to $c-b$ divided by a, which may be positive or negative, according as c is greater or less than b; thus, if $c=9$, $b=5$, $a=2$, then $x=\frac{9-5}{2}=2$; if $c=12$, $b=16$, and $a=2$, then, $\frac{12-16}{2}=\frac{-4}{2}=-2$.

Ex. 6. Given $3x-4=7x-16$, to find the value of x.

Ans. $x=3$.

Ex. 7. Given $9-2x=3x-6$, to find the value of x.

Ans. $x=3$.

Ex. 8. Given $ax^2+bx=9x^2+cx$, to find the value of x in terms of a, b, &c. Ans. $x=\frac{c-b}{a-9}$.

Ex. 9. Given $x-9=4x$, to find the value of x.

Ans. $x=-3$.

Ex. 10. Given $5ax-c=b-3ax$, to find the value of x in terms of a, b, and c. Ans. $x=\frac{b+c}{8a}$.

Ex. 11. Given $3x-1+9-5x=0$, to find the value of x.

Ans. $x=4$.

Ex. 12. Given $ax=ab-ac$, to find the value of x.

Ans. $x=b-c$.

Ex. 13. Given $x^2+2x=(x+a)^2$, to find the value of x.

Ans. $x=\frac{a^2}{2-2a}$

Ex. 14. Given $(x-1)^2=x+1$, to find the value of x.

Ans. $x=3$.

Ex. 15. Given $x^3+2x^2+x=(x^2+3x)\times(x-1)+16$, to find the value of x. Ans. $x=4$.

RULE III.

172. If in the equation there be any irreducible fractions, in which the unknown quantity is concerned, multiply every term of the equation by the denominators of the fractions in succession, or by their least common multiple; and then proceed according to Rules I, and II.

Ex. 1. Given $\frac{2x}{4}+1=x-9$, to find the value of x.

Multiplying by 4, $2x+4=4x-36$,

by transposition, $2x-4x=-36-4$,

by collecting the terms, $-2x=-40$,

by changing the signs, $2x=40$,

by division, $\frac{2x}{2}=\frac{40}{2}$; $\therefore x=20$.

Ex. 2. Given $\frac{x}{2}-\frac{x}{3}+3=5-\frac{x}{4}$, to find the value of x.

Multiplying by 2, $x-\frac{2x}{3}+6=10-\frac{2x}{4}$,

. . . . . by 3, $3x-2x+18=30-\frac{6x}{4}$,

. . . . by 4, $12x-8x+72=120-6x$.

by transposing, and collecting, $10x=48$,

by division, $\frac{10x}{10}=\frac{48}{10}$; $\therefore x=4\frac{4}{5}$.

Or, it is more concise and simple to multiply the equation by the least common multiple of the denominators; because, then the equation is reduced to its lowest terms; thus,

Multiplying by 12, the least common multiple of 2, 3, and 4, we have, $6x-4x+36=60-3x$,

by transposition, $5x=24$,

by division, $\frac{5x}{5}=\frac{24}{5}$; $\therefore x=4\frac{4}{5}$.

Ex. 3. Given $x-\frac{x}{3}-1=\frac{x}{5}+\frac{x}{6}$, to find the value of x.

Here 30 is the least common multiple of 3, 5, and 6;

Multiplying by 30, $30x-\frac{30x}{3}-30=\frac{30x}{5}+\frac{30x}{6}$,

$\therefore 30x-10x-30=6x+5x$,

by transposition, $9x=30$,

by division, $\frac{9x}{9}=\frac{30}{9}=\frac{10}{3}$; $\therefore x=3\frac{1}{3}$.

Ex. 4. Given $\frac{x}{4}-a=\frac{x}{5}-3$, to find the value of x.

Here 20, the product of 4 and 5, being their least common multiple,

Multiplying by 20, $\frac{20x}{4}-20a=\frac{20x}{5}-60$,

$\therefore 5x-20a=4x-60$,

by transposition, $5x-4x=20a-60$,

$\therefore x=20a-60$.

Ex. 5. Given $\frac{ax}{5}-\frac{bx}{5}=\frac{2a}{5}$, to find the value of x.

Multiplying by 5, $\frac{5ax}{5}-\frac{5bx}{5}=\frac{5\times 2a}{5}$,

$\therefore ax-bx=2a$,

by collecting the coefficients, $(a-b)x=2a$,

$\therefore$ by division, $x=\frac{2a}{a-b}$.

Ex. 6. Given $\frac{2ax}{c}+\frac{3bx}{2}=\frac{5x}{a}+3$, to find the value of x.

Here $2ac$, the product of 2, a, and c, being the least common multiple,

Multiplying by $2ac$, $4a^2x+3abcx=10cx+6ac$,

by transposition, and collecting the coefficients, we shall have

$(4a^2+3abc-10c)x=6ac$,

$\therefore$ by division, $x=\frac{6ac}{4a^2+3abc-10c}$.

Ex. 7. Given $3x-\frac{x-4}{4}-4=\frac{5x+14}{3}-\frac{1}{12}$, to find the value of x.

Multiplying by 12, the least common multiple,

we have $36x-3x+12-48=20x+56-1$,

by transposition, $36x-3x-20x=56-1+48-12$,

or $13x=91$,

by division, $\frac{13x}{13}=\frac{91}{13}$; $\therefore x=7$.

Ex. 8. Given $\frac{56}{5x+3}=\frac{63}{14x-5}$, to find the value of x.

Ans. $x=1$.

Ex. 9. Given $\frac{x+1}{2}+\frac{x+2}{3}=16-\frac{x+3}{4}$, to find the value of x.

Ans. $x=13$.

Ex. 10. Given $\frac{x-3}{2}+\frac{x}{3}=20-\frac{x-19}{2}$, to find the value of x.

Ans. $x=23\frac{1}{4}$.

Ex. 11. Given $x+\frac{11-x}{3}=\frac{19-x}{2}$, to find the value of x.

Ans. $x=5$.

Ex. 12. Given $\frac{x-5}{4}+6x=\frac{284-x}{5}$, to find the value of x.

Ans. $x=9$.

Ex. 13. Given $3x+\frac{2x+6}{5}=5+\frac{11x-37}{2}$, to find the value of x.

Ans. $x=7$.

Ex. 14. Given $\frac{6x-4}{3}-2=\frac{18-4x}{3}+x$, to find the value of x.

Ans. $x=4$.

Ex. 15. Given $\frac{ax-3}{5}-\frac{bx+2}{3}=\frac{2x-9}{2}-\frac{x-1}{3}$, to find the value of x.

Ans. $x=\frac{87}{10b+20-6a}$.

Ex. 16. Given $\frac{x-1}{7}-\frac{x+3}{2}=\frac{2x+1}{14}-\frac{x-3}{4}$, to find the value of x.

Ans. $x=-9\frac{6}{7}$.

RULE IV.

173. If the unknown quantity be involved in a proportion, the proportion must be converted into an equation (Art 165); and then proceed to resolve this equation according to the foregoing Rules.

Ex. 1. Given $3x-2:4::5x-9:2$, to find the value of x.

Multiplying extremes and means, we have

$$2(3x-2)=4(5x-9),$$

$$\text{or } 6x-4=20x-36,$$

by transposition, $6x-20x=-36+4$;

$$\text{or } -14x=-32,$$

by changing the signs, $14x=32$,

$$\text{by division, } \frac{14x}{14}=\frac{32}{14};\ \therefore x=2\tfrac{2}{7}.$$

Ex. 2. Given $3a : x :: b+5 : x-9$, to find the value of x.

Multiplying extremes and means, we have

$$3a\,.\,(x-9)=x\,.\,(b+5),$$
$$\text{or } 3ax-27a=bx+5x,$$
$$\text{by transposition, } 3ax-bx-5x=27a,$$
$$\text{collecting the coeff's, } (3a-b-5)x=27a,$$
$$\therefore \text{by division, } x=\frac{27a}{3a-b-5}.$$

Ex. 3. Given $\frac{x-5}{4} : x-5 :: \frac{2}{3} : \frac{3}{4}$, to find the value of x.

Multiplying extremes and means, we have

$$\frac{3}{4}\,.\left(\frac{x-5}{4}\right)=\frac{2}{3}\,.\,(x-5),$$
$$\text{or } \frac{3x-15}{16}=\frac{2x-10}{3},$$
$$\text{by clearing of fractions, } 9x-45=32x-160,$$
$$\text{by transposition, } 9x-32x=45-160,$$
$$\text{collecting and changing signs, } 23x=115,$$
$$\text{by division, } \frac{23x}{23}=\frac{115}{23};\ \therefore x=5.$$

Ex. 4. Given $2x-3 : x-1 :: 4x : 2x+2$, to find the value of x.

Multiplying extremes and means, we shall have

$$(2x-3)\,.\,(2x+2)=4x(x-1),$$
$$\text{or } 4x^2-2x-6=4x^2-4x,$$
$$\text{by transposition, \&c., } 2x=6,$$
$$\therefore \text{by division, } x=3.$$

Ex. 5. Given $a+x : b :: c-x : d$, to find the value of x in terms of a, b, c, and d.

Multiplying extremes and means, $ad+dx=bc-bx$,

$$\text{by transposition, } bx+dx=bc-ad,$$
$$\text{or } (b+d)x=bc-ad,$$
$$\therefore \text{by division, } x=\frac{bc-ad}{b+d}.$$

Ex. 6. Given $\frac{x-1}{3} : x+2 :: \frac{3}{4} : 1$, to find the value of x.

Multiplying extremes, &c., $\frac{x-1}{3}=\frac{3x+6}{4}$,

$$\text{clearing of fractions, } 4x-4=9x+18,$$
$$\text{by transposition, } 4x-9x=18+4,$$

changing the signs, &c., $5x=-22$,

$\therefore$ by division, $x=-\dfrac{22}{5}=-4\frac{2}{5}$.

Ex. 7. Given $2x-1 : x+1 :: \dfrac{3x}{2} : \dfrac{x}{4}$, to find the value of x.

Ans. $x=-1\frac{3}{4}$.

Ex. 8. Given $x+3 : a :: b : \dfrac{1}{x}$ to find the value of x.

Ans. $x=\dfrac{3}{ab-1}$.

Ex. 9. Given $\dfrac{1}{2} : \dfrac{3x}{4} :: 5 : 2x-2$, to find the value of x.

Ans. $x=-\frac{4}{11}$.

Ex. 10. Given $\dfrac{4}{7} : \dfrac{3}{4} :: x-1 : \dfrac{2x-1}{4}$, to find the value of x.

Ans. $x=1\frac{4}{13}$.

Ex. 11. Given $\dfrac{x^2-1}{3a} : \dfrac{x+1}{3a} :: 6 : 3$, to find the value of x.

Ans. $x=3$.

§ III. EXAMPLES IN SIMPLE EQUATIONS,

Involving only one unknown Quantity.

174. It is necessary to observe that an equation expressing but a relation between abstract numbers or quantities, may agree with many questions whose enunciations would differ from that of the one proposed: but the principles of the resolution of equations being independent of any hypothesis upon the nature and magnitude of quantities; it follows, therefore, that the value of the unknown quantity substituted in the equation, will always reduce it to $0=0$, although it may not agree with the particular question. This is what will happen, when the value of the unknown quantity shall be negative; for it is evident that when a concrete question is the subject of inquiry, it is not a negative quantity which ought to be the value of the unknown, or which could satisfy the question in the direct sense of the enunciation.

The negative root can only verify the primitive equation of a problem, by changing in it the sign of the unknown; this equation will therefore agree then with a question in which the relation of the unknown to the known quantities shall be different from that which we had supposed in the first enunciation. We see therefore that the negative roots indicate not an absolute impossibility, but only relative to the actual enunciation of the question.

The rules of Algebra, therefore, make not only known certain contradictions, which may be found in enunciations of problems of the first degree; but they still indicate their rectification, in rendering subtractive certain quantities which we had regarded as additive, or additive certain quantities which we had regarded as subtractive, or in giving for the unknown quantities, values affected with the sign —.

Hence, it follows, that we may regard as forming, properly speaking, but one question, those whose enunciations are not connected to one another in such a manner, that the solution which satisfies one of the enunciations, can, by a simple change of the sign, satisfy the other.

We must nevertheless observe that we can make upon the signs and values of the terms of an equation, hypotheses which do not agree with the enunciation of a concrete question, whereas the change which we will make in this enunciation might be always represented by the equation.

These principles, which will be illustrated by examples, are applicable to equations of all degrees, and to determinate equations containing many unknown quantities.

The question which conducts to the equation,

$$ax+b=cx+d,$$

is not well enunciated for $a>c$, and $b>d$, since the first member is greater than the second.

Thus the formula

$$x=\frac{d-b}{a-c},$$

gives for x a negative value; but by rendering the unknown x negative, the equation is changed into the following,

$$b-ax=d-cx,$$

which is possible under the above relations between a and c, b and d, and which gives then for x an absolute value.

If we have $b>d$ and $c>a$, the two subtractions become impossible in the formula

$$x=\frac{d-b}{a-c};$$

but in order to resolve the equation, let us subtract $cx+b$ from both members, which would be impossible, because that $cx+b$ is greater than each of the two members: we must therefore, on the contrary, take away $ax+d$ from both sides, and it becomes

$$b-d=cx-ax;$$

from whence we deduce

$$x=\frac{b-d}{c-a}.$$

This formula compared to the preceding, differs from it in this, that the signs of both terms of the fraction are changed.

We may therefore conclude, *that we can operate on negative isolated quantities, as we would do if they had been positive.*

These principles will be clearly elucidated, when we come to treat of the solutions of Problems producing simple Equations: we shall now proceed to illustrate the Rules in the preceding Section, by a variety of practical examples.

Ex. 1. Given $21+\frac{3x-11}{16}=\frac{5x-5}{8}+\frac{97-7x}{2}$, to find the value of x.

Multiplying both sides of the equation by 16, the least common multiple of 16, 8, and 2, we shall have

$$336+3x-11=10x-10+776-56x;$$

$\therefore$ by transposition,

$$3x-10x+56x=11-10+776-336,$$

$$\text{or } 49x=441;$$

$$\text{by division, } x=\frac{441}{49}, \therefore x=9.$$

Ex. 2. Given $x+\frac{3x-5}{2}=12-\frac{2x-4}{3}$, to find the value of x.

Multiplying both sides of the equation by 6, the product of 2 and 3, which is the least common multiple, we have

$$6x+9x-15=72-4x+8;$$

$$\therefore \text{ by transposition, } 6x+9x+4x=72+8+15,$$

$$\text{or } 19x=95;$$

$$\text{by division, } x=\frac{95}{19}, \therefore x=5.$$

In this example, when the fraction $-\frac{2x-4}{3}$, is multiplied by 6, the result is $-\frac{12x-24}{3}=-(4x-8)=-4x+8$, or, which is the same thing, when the sign $-$ stands before a fraction, it may be transformed, so that the sign $+$ may stand before it, by changing the sign of every term in the numerator; therefore, we make the above step $-4x+8$, and not $4x-8$.

Ex. 3. Given $4x-\frac{x-1}{2}=x+\frac{2x-2}{5}+24$, to find the value of x.

Multiplying by 10, the least common multiple, and we have,

$$40x-5x+5=10x+4x-4+240,$$

by transposition, $40x-5x-10x-4x=240-4-5$,

$$\text{or, } 40x-19x=231;$$

$$\text{and } 21x=231,$$

$$\text{by division, } x=\frac{231}{21}, \therefore x=11.$$

Ex. 4. Given $2x-\frac{x}{2}+1=5x-2$, to find the value of x.

Multiplying by 2, we have,

$$4x-x+2=10x-4,$$

$\therefore$ by transposition, $4x-x-10x=-4-2$,

$$\text{or } -7x=-6,$$

$$\text{by changing the signs, } 7x=6,$$

$$\therefore \text{by division, } x=\frac{6}{7}.$$

Ex. 5. Given $3ax-2bx=3b-a$, to find the value of x.

Here, $3ax-2bx=(3a-2b)x$, by collecting the coefficients of x. Therefore,

$$(3a-2b)x-3b-a,$$

$$\text{by division, } x=\frac{3b=a}{3a-2b}.$$

Ex. 6. Given $bx+x=2x+3a$, to find the value of x.

$$\text{by transposition, } bx+x-2x=3a,$$

$$\text{or } (b-1)x=3a,$$

$$\therefore \text{by division, } x=\frac{3a}{b-1}.$$

Ex. 7. Given $\frac{3x}{a}-c+\frac{x}{b}=4x+\frac{2x}{d}$, to find the value of x.

Multiplying by abd, we have,

$$3bdx-abcd+adx=4abdx+2abx,$$

by transposition, $3bdx+adx-4abdx-2abx=abcd$,

$$\text{or } (3bd+ad-4abd-2ab)\,x=abcd,$$

$$\therefore \text{by division, } x=\frac{abcd}{3bd+ad-4abd-2ab}.$$

Ex. 8. Given $\frac{x}{5}-\frac{x}{6}+\frac{a}{6}=b+c$, to find the value of x.

Multiplying by 30, the product of 5 and 6, the product becomes

$$6x-5x+5a=30b+30c;$$

by transposition, $6x-5x=30b+30c-5a$,

and $\therefore x=30b+30c-5a$.

Ex. 9. Given $\frac{12-x}{9}:5x-\frac{14+x}{3}::1:8$, to find the value of x.

Multiplying extremes and means, we have

$$\frac{96-8x}{9}=5x-\frac{14+x}{3},$$

Multiplying by 9, the least common multiple,

$$96-8x=45x-42-3x,$$

by transposition, $-45x-8x+3x=-96-42$,

by changing the signs, $45x+8x-3x=96+42$,

or $50x=138$,

$\therefore$ by division, $x=\frac{138}{50}=2\frac{19}{25}$.

Ex. 10. Given $\frac{ax-b}{4}+\frac{a}{3}=\frac{bx}{2}-\frac{bx-a}{3}$, to find the value of x.

Multiplying by 12, the least common multiple of the denominators, and the equation will become,

$$3ax-3b+4a=6bx-4bx+4a, \quad . \quad . \quad (1),$$

by taking away $4a$ from each member, we shall have

$$3ax-3b=6bx-4bx=2bx,$$

by transposing $-3b$ and $2bx$, it becomes

$$3ax-2bx=3b,$$

by collecting the coefficients of x, we shall have

$$(3a-2b)x=3b,$$

by division, $x=\frac{3b}{3a-2b}$.

Ex. 11. Given $2ax+b=3cx+4a$, to find the value of x.

by transposition, $2ax-3cx=4a-b$,

by collecting the coefficients, $(2a-3c)x=4a-b$,

$\therefore$ by division, $x=\frac{4a-b}{2a-3c}$.

Ex. 12. Given $19x+13=59-4x$, to find the value of x.

by transposition, $19x+4x=59-13$,

or, $23x=46$;

$\therefore$ by division, $x=2$.

Ex. 13. Given $3x+4-\frac{x}{3}=46-2x$, to find the value of x.

Multiplying both sides by 3,

$$9x+12-x=138-6x,$$

by transposition, $9x+6x-x=138-12$,

or $14x=126$;

by division, $x=\frac{126}{14}$, $\therefore x=9$.

Ex. 14. Given $x^2+15x=35x-3x^2$, to find the value of x.

Dividing every term by x,

$$x+15=35-3x,$$

by transposition, $x+3x=35-15$,

or $4x=20$;

$\therefore x=5$.

Ex. 15. Given $\frac{x}{6}-\frac{x}{4}+10=\frac{x}{3}-\frac{x}{2}+11$, to find the value of x.

Here 12 is the least common multiple of 6, 4, 3, and 2;

$\therefore$ multiplying both sides of the equation by 12,

$$2x-3x+120=4x-6x+132;$$

by transposition, $2x-3x-4x+6x=132-120$,

or $8x-7x=12$;

$\therefore x=12$.

Ex. 16. Given $\frac{x-1}{7}+\frac{23-x}{5}=7-\frac{4+x}{4}$, to find the value of x. Ans. $x=8$.

Ex. 17. Given $\frac{7x+5}{3}-\frac{16+4x}{5}+6=\frac{3x+9}{2}$, to find the value of x. Ans. $x=1$.

Ex. 18. Given $\frac{17-3x}{5}-\frac{4x+2}{3}=5-6x+\frac{7x+14}{3}$, to find the value of x. Ans. $x=4$.

Ex. 19. Given $x-\frac{3x-3}{5}+4=\frac{20-x}{2}-\frac{6x-8}{7}+\frac{4x-4}{5}$, to find the value of x. Ans. $x=6$.

Ex. 20. Given $\frac{4x-21}{9}+3\frac{3}{4}+\frac{57-3x}{4}=241-\frac{5x-96}{12}-11x$, to find the value of x. Ans. $x=21$.

Ex. 21. Given $\frac{6x+18}{13}-4\frac{5}{6}-\frac{11-3x}{36}=5x-48-\frac{13-x}{12}-\frac{21-2x}{18}$, to find the value of x. Ans. $x=10$.

Ex. 22. Given $ax-\frac{a^2-3bx}{a}-ab^2=bx+\frac{6bx-5a^2}{2a}-\frac{bx+4a}{4}$, to find the value of x. Ans. $x=\frac{4ab^2-10a}{4a-3b}$.

Ex. 23. Given $\frac{7x+16}{21}-\frac{x+8}{4x-11}=\frac{x}{3}$, to find the value of x.

Ans. $x=8$.

Ex. 24. Given $\frac{6x+7}{9}+\frac{7x-13}{6x+3}=\frac{2x+4}{3}$, to find the value of x.

Ans. $x=4$.

Ex. 25. Given $\frac{4x+3}{9}+\frac{7x-29}{5x-12}=\frac{8x+19}{18}$, to find the value of x.

Ans. $x=6$.

Ex. 26. Given $12-x : \frac{x}{2} :: 4 : 1$, to find the value of x.

Ans. $x=4$.

Ex. 27. Given $\frac{5x+4}{2} : \frac{18-x}{4} :: 7 : 4$, to find the value of x.

Ans. $x=2$.

Ex. 28. Given $(2x+8)^2=4x^2+14x+172$, to find the value of x.

Ans. $x=6$.

Ex. 29. Given $\frac{3x+4}{5}+2x=\frac{22-x}{5}+16$, to find the value of x.

Ans. $x=7$.

Ex. 30. Given $\frac{7-x}{2}+4=\frac{3x-11}{4}+\frac{8x+15}{6}$, to find the value of x.

Ans. $x=3$.

Ex. 31. Given $\frac{x^2}{2}+\frac{x}{2}=\frac{3ax^2}{2}$, to find the value of x.

Ans. $x=\frac{1}{3a-1}$.

Ex. 32. Given $2x-\frac{x+3}{3}+15=\frac{12x+26}{5}$, to find the value of x.

Ans. $x=12$.

Ex. 33. Given $5ax-2b+4bx=2x+5c$, to find the value of x.

Ans. $x=\frac{5c+2b}{5a+4b-2}$.

Ex. 34. Given $\frac{2x-5}{18}+\frac{19-x}{3}=\frac{10x-7}{9}-\frac{5}{2}$, to find the value of x.

Ans. $x=7$.

Ex. 35. Given $x-\frac{2x+1}{3}=\frac{x+3}{4}$, to find the value of x.

Ans. $x=13$.

Ex. 36. Given $\frac{3x+5}{8}-\frac{21+x}{3}=39-5x$, to find the value of x.

Ans. $x=9$.

Ex. 37. Given $4x-\frac{19+2x}{5}=15-\frac{7x+11}{4}$, to find the value of x. Ans. $x=3$.

Ex. 38. Given $\frac{21-3x}{3}-\frac{4x+6}{6}=6-\frac{5x+1}{4}$, to find the value of x. Ans. $x=3$.

Ex. 39. Given $7\frac{5}{8}+\frac{3x-1}{4}-\frac{7x+3}{16}=\frac{8x+19}{8}$, to find the value of x. Ans. $x=7$.

Ex. 40. Given $\frac{6x+8}{11}-\frac{5x+3}{2}=\frac{27-4x}{3}-\frac{3x+9}{2}$, to find the value of x. Ans. $x=6$.

Ex. 41. Given $x+\frac{27-9x}{4}-\frac{5x+2}{6}=\frac{61}{12}-\frac{2x+5}{3}-\frac{29+4x}{12}$, to find the value of x. Ans. $x=5$.

Ex. 42. Given $\frac{7x-8}{11}+\frac{15x+8}{13}=3x-\frac{31-x}{2}$, to find the value of x. Ans. $x=9$.

Ex. 43. Given $\frac{5x-1}{2}-\frac{7x-2}{10}=6\frac{3}{5}-\frac{x}{2}$, to find the value of x. Ans. $x=3$.

Ex. 44. Given $\frac{10+x}{5}:\frac{4x-9}{7}::14:5$, to find the value of x. Ans. $x=4$.

Ex. 45. Given $\frac{17-4x}{4}:\frac{15+2x}{3}-2x::5:4$, to find the value of x. Ans. $x=3$.

Ex. 46. Given $16x+5:\frac{4x+14}{9x+31}::36x+10:1$, to find the value of x. Ans. $x=5$.

Ex. 47. Given $\frac{4x+3}{6x-43}:1::2x+19:3x-19$, to find the value of x. Ans. $x=8$.

Ex. 48. Given $5x+\frac{7x+9}{4x+3}=9+\frac{10x^2-18}{2x+3}$, to find the value of x. Ans. $x=3$.

Ex. 49. Given $\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}$, to find the value of x. Ans. $x=8$.

Ex. 50. Given $\frac{20x+36}{25}+\frac{5x+20}{9x-16}=\frac{4x}{5}+\frac{86}{25}$, to find the value of x. Ans. $x=4$.

Ex. 51. Given $\frac{10x+17}{18}-\frac{12x+2}{13x-16}=\frac{5x-4}{9}$, to find the value of x. Ans. $x=4$.

Ex. 52. Given $\frac{18x-19}{28}+\frac{11x+21}{6x+14}=\frac{9x+15}{14}$, to find the value of x. Ans. $x=7$.

Ex. 53. Given $\frac{a.(b^2+x^2)}{bx}=ac+\frac{ax}{b}$, to find the value of x.

Ans. $x=\frac{b}{c}$.

Ex. 54. Given $\frac{cx^m}{a+bx}=\frac{dx^m}{e+fx}$, to find the value of x.

Ans. $x=\frac{ad-ce}{cf-bd}$.

Ex. 55. Given $\frac{a}{bx}+\frac{c}{dx}+\frac{e}{fx}+\frac{g}{hx}=k$, to find the value of x.

Ans. $x=\frac{adfh+bcfh+bdeh+bdfg}{bdfhk}$.

Ex. 56. Given $(b+x).(b+x)-a.(b+c)=\frac{a^2c}{b}+x^2$, to find the value of x. Ans. $x=\frac{ac}{b}$.

Ex. 57. Given $\frac{3x-3}{4}-\frac{3x-4}{3}=5\frac{1}{3}-\frac{27+4x}{9}$, to find the value of x. Ans. $x=9$.

Ex. 58. Given $\frac{4x-34}{17}-\frac{258-5x}{3}=\frac{69-x}{2}$, to find the value of x. Ans. $x=51$.

Ex. 59. Given $2x-\frac{4x-2}{13}=\frac{2x+11}{5}-\frac{7-8x}{7}$, to find the value of x. Ans. $x=7$.

Ex. 60. Given $\frac{2x+1}{29}-\frac{402-3x}{12}=9-\frac{471-6x}{2}$, to find the value of x. Ans. $x=72$.

Ex. 61. Given $\frac{3a+x}{x}-5=\frac{6}{x}$, to find the value of x.

Ans. $x=\frac{3a-6}{4}$.

CHAPTER IV.

ON

THE SOLUTION OF PROBLEMS,

PRODUCING SIMPLE EQUATIONS.

175. The solution of a problem is the method of discovering, by analysis, quantities which will answer its several conditions; for this purpose, there are four things to be distinguished:

I. The given, that is to say, the known quantities, enunciated in the problem, and the quantities that are to be found.

II. The translation of the problem into algebraic language, which is composed of the translation of every distinct condition that it contains into an algebraic equation.

III. The resolution of the equations, that is, the series of transformations which the immediate translation must undergo, in order to arrive at an equation containing in the first member one unknown quantity alone in its simple state, and in the other a formula of operations to be performed upon the representations of given numbers.

IV. Finally, the numerical valuation, or the geometrical construction of this formula.

176. Algebraic problems and their solutions may be considered as of two kinds, that is, numerical and literal, or particular and general. In the numerical, or particular method of solution, unknown quantities are represented by letters, and the known ones by numbers, as in arithmetic. In the literal, or general solution, all quantities, known and unknown, are represented by letters, and the answers given in general terms. A problem solved in this way, furnishes a theorem, which may be applied to the solution of all questions of the same kind.

§ I. SOLUTION OF PROBLEMS PRODUCING SIMPLE EQUATIONS,

Involving only one unknown Quantity.

177. If from certain quantities which are known, another quantity be required which has a given relation to them, let the unknown quantity be represented by x; then, the condition enunciated in the problem being clearly understood, it can be easily translated into an algebraic equation, by means of the

signs pointed out in the Introduction. Having now brought the question into an algebraic form, the value of the unknown quantity can be readily found by the application of the rules delivered Chap. III.

Or, if there be more than one unknown quantity required, and that they bear given relations to one another, instead of assuming a symbol to represent each of them, it is more convenient to assume one only, and from the conditions of the problem to deduce expressions for the others in terms of that one and known quantities. And as the number of conditions ought to be one more than the number of quantities thus expressed, there will remain one to be translated into an equation; from which the value of the unknown quantity may be determined as above; and this being substituted in the other expressions, their values also may be discovered.

Problem I.

What number is that, to which 17 being added, the sum will be 48?

Let the required number be represented by x:

Then by the problem, $x+17=48$;

by transposition, $x=48-17$:

$\therefore x=31$.

Prob. 2. What number is that, from which a being subtracted, the remainder is b?

Let x represent the number required.

Then by the problem $x-a=b$;

by transposition, $x=a+b$.

Here, if $a=16$, and $b=14$; then $x=16+14=30$; that is, 30 is a number, from which 16 being subtracted, the remainder is 14.

Prob. 3. To find a number which, being subtracted from a, leaves b for a remainder.

Designating the unknown number by x, we shall have this translation,

$$a-x=b, \therefore x=a-b.$$

178. If we suppose $a=10$, $b=4$, we shall have $x=6$; then the subtraction is arithmetically performed. But if we had $a=10$, $b=14$, we must subtract 14 from 10, which cannot be done except in part, or that with respect to the portion of 14 equal to 10.

The excess, in as much as it exists subtractively, will indicate that the number x of which it is the representation must enter negatively in the enunciation where it is already subtracted from the number a, so that the enunciation of the problem is corrected and brought to these terms: *to find a number which being added to* 10, *the sum will be* 14; a problem whose translation is, designating the unknown quantity by x,

$$10+x=14;\ \therefore x=14-10=4;$$

whereas, the translation in the former case would be

$$10-x=14;\ \therefore x=10-14, \text{ or } x=-4.$$

The negative root -4, satisfies the equation of the problem, besides it announces a rectification in the enunciation; this is what appears evident, since the subtraction of a negative quantity is equivalent to the addition of a positive, (Art. 63). In fact, as has been already observed, (Art. 174), it makes known that the enunciation ought to be taken in an opposite sense to that which we first proposed in the problem.

Prob. 4. A person lends at interest for one year a certain capital at 5 per cent; at the end of the year, according to agreement, he is to receive a sum b, besides the principal and interest, and the whole sum he receives must be equal to the capital. I demand what is the capital?

Let the capital be designated by x:

Since 100 dollars becomes at the end of the year 105 dollars, we shall have the capital at the same time by this proportion,

$$100 : 105 :: x : \frac{105x}{100} = \text{the capital.}$$

The sum $\frac{105x}{100}+b$, by the problem, must be equal to x, we have therefore the equation

$$\frac{105x}{100}+b=x;\ \therefore 105x+100b=100x;$$

$$\text{by transposition, } 5x=-100b;$$

$$\therefore \text{ by division, } x=-20b.$$

179. Thus the capital shall be $-20b$. This answer does not agree with the problem, and still if this value $-20b$, be substituted for x in the equation found, we obtain

$$-\frac{105\times 20b}{100}+b=-20b,$$

and, performing the operations indicated in the first member, it becomes

$$-20b=-20b,$$

which is true. This value of x, although it is negative, satisfies the equation of the problem, as has been already observed (Art. 174), since its two members become *identically* equal by making the proper substitution.

If we return again to the enunciation, we discover that it is impossible that a capital augmented by the interest would remain equal to itself, and that much more this impossibility takes place, if, besides the interest, we add to it a sum b; it is necessary therefore that one of these two parts, namely, the interest at 5 per cent, and b, be subtracted.

In fact, if we carry into the first equation this circumstance $-x$, which is but $x=-$ a number, we find

$$-\frac{105}{100}x+b=-x;\ \therefore \frac{105x}{100}-b=x,$$

a translation of the enunciation, by supposing the interest additive to the capital, in which case, the sum b ought to be subtracted.

This equation, treated as the preceding, shall give

$$x=20b,$$

If the interest at 5 per cent be subtracted from 100, in which case 100 reduces itself to 95, we have the capital x at the end of the year, by the proportion

$$100 : 95 :: x : \frac{95x}{100}= \text{ the capital,}$$

$$\text{consequently, } \frac{95x}{100}+b=x;$$

multiplying by 100, and transposing, we shall have

$$100b=5x,\ \therefore x=20b.$$

The negative isolated result, that is, the negative value of x, would announce a rectification or a correction in the terms of the enunciation, and the problem proposed could be re-established in two ways.

PROB. 5. What number is that, the double of which exceeds its half by 6?

Let $x=$ the number;

Then by the problem, $2x-\frac{x}{2}=6$,

$\therefore$ multiplying by 2, $4x-x=12$,

or $3x=12$,

$\therefore$ by division, $x=4$.

PROB. 6. From two towns which are 187 miles distant, two travellers set out at the same time, with an intention of meeting. One of them goes 8 miles, and the other 9 miles a day. In how many days will they meet?

Let $x=$ the number of days required;
then $8x=$ the number of miles *one* travelled,
and $9x=$ the number the *other* travelled;

and since they meet, they must have travelled together the whole distance,

consequently, $8x+9x=187$,
or $17x=187$,
$\therefore$ by division, $x=11$.

PROB. 7. What number is that, from which 6 being subtracted, and the remainder multiplied by 11, the product will be 121?

Let $x=$ the number required;
Then by the problem $(x-6)\times 11=121$,
by transposition, $11x=121+66$,
or $11x=187$,
$\therefore$ by division, $x=17$.

PROB. 8. A Gentleman meeting 4 poor persons, distributed five shillings amongst them: to the second he gave twice, the third thrice, and to the fourth four times as much as to the first. What did he give to each?

Let $x=$ the pence he gave to the first,
$\therefore 2x=$ the pence given to the second,
and $3x=$ . . . . . . to the third,
$4x=$ . . . . . . to the fourth.
$\therefore$ by the problem, $x+2x+3x+4x=5\times 12=60$,
or $10x=60$,
by division, $x=6$,
and therefore he gave 6, 12, 18, 24 pence respectively to them.

PROB. 9. A Bookseller sold 10 books at a certain price; and afterwards 15 more at the same rate. Now at the latter time he received 35 shillings more than at the former. What did he receive for each book?

Let $x=$ the price of a book.
then $10x=$ the price of the first set,
and $15x=$ the price of the second set;
but by the problem, $15x=10x+35$;
$\therefore$ by transposition, $5x=35$;
and by division, $x=7$.

PROB. 10. A Gentleman dying bequeathed a legacy of 1400 dollars to three servants. A was to have twice as much as B; and B three times as much as C. What were their respective shares?

Let $x=$C's share,
$\therefore 3x=$B's share,
and $6x=$A's share.

them by the problem, $x+3x+6x=1400$,

$$\text{or } 10x=1400,$$

$$\therefore \text{by division, } x=140=\text{C's share.}$$

$\therefore$ A received 840 dollars; B, 420 dollars; and C, 140 dollars.

PROB. 11. There are two numbers whose difference is 15, and their sum 59. What are the numbers?

As their *difference* is 15, it is evident that the greater number must exceed the lesser by 15.

Let, therefore, $x=$ the lesser number;
then will $x+15=$ the greater;

$$\therefore \text{by the problem, } x+x+15=59,$$

$$\text{or } 2x+15=59,$$

$$\text{by transposition, } 2x=59-15=44,$$

$$\therefore \text{by division, } x=22 \text{ the lesser number,}$$

$$\text{and } x+15=22+15=37 \text{ the greater.}$$

PROB. 12. What two numbers are those whose difference is 9; and if three times the greater be added to five times the lesser, the sum shall be 35?

Let $x=$ the *lesser* number;
then $x+9=$ the *greater* number.
And 3 *times* the greater $=3(x+9)=3x+27$,
5 *times* the lesser $=5x$.

$$\therefore \text{by the problem, } (3x+27)+5x=35;$$

$$\text{by transposition, } 3x+5x=35-27,$$

$$\text{or } 8x=8;$$

$$\therefore \text{by division, } x=1 \text{ the lesser number,}$$

$$\text{and } x+9=1+9=10 \text{ the greater number.}$$

PROB. 13. What number is that, to which 10 being added, $\frac{3}{5}$ths of the sum will be 66?

Let $x=$ the number required;
then $x+10=$the number, with 10 added to it.

Now $\frac{3}{5}$ths of $(x+10)=\frac{3}{5}(x+10)=\dfrac{3(x+10)}{5}=\dfrac{3x+30}{5}$.

But, by the problem, $\frac{3}{5}$ths of $(x+10)=66$;

$$\therefore \frac{3x+30}{5}=66;$$

$$\text{by multiplication, } 3x+30=330;$$

$$\text{by transposition, } 3x=300;$$

$$\therefore \text{by division, } x=100.$$

PROB. 14. What number is that, which being multiplied by

6, the product increased by 18, and that sum divided by 9, the product shall be 20?

Let $x=$ the number required;
then $6x=$ the number multiplied by 6;
$6x+18=$ the product increased by 18;
and $\frac{6x+18}{9}=$ that sum divided by 9,

$\therefore$ by the problem, $\frac{6x+18}{9}=20$;
by multiplication, $6x+18=20\times 9$;
by transposition, $6x=180-18$,
or $6x=162$;
$\therefore$ by division, $x=27$.

PROB. 15. A post is $\frac{1}{5}$th in the earth, $\frac{3}{7}$ths in the water, and 13 feet out of the water. What is the length of the post?

Let $x=$ the length of the post;

then $\frac{x}{5}=$ the part of it in the earth,

$\frac{3x}{7}=$ the part of it in the water,

and $13=$ the part of it out of the water.

But by the problem, part in the earth + part in water + part out of water = whole part;

$$\therefore \left(\frac{x}{5}\right)+\left(\frac{3x}{7}\right)+13=x.$$

and $\frac{x}{5}\times 35+\frac{3x}{7}\times 35+13\times 35=35x$;
or $7x+15x+455=35x$;
by transposition, $455=35x-7x-15x=13x$,
or $13x=455$;
$\therefore$ by division, $x=35$, length of the post.

PROB. 16. After paying away $\frac{1}{4}$th and $\frac{1}{7}$th of my money, I had 850 dollars left. What money had I at first?

Let $x=$ the money in my purse at first;

then $\frac{x}{5}+\frac{x}{7}=$ money paid away.

But money at first − money paid away = money remaining;

$\therefore$ by the problem $x-\left(\frac{x}{4}+\frac{x}{7}\right)=850$,

or $x-\frac{x}{4}-\frac{x}{7}=850$.

Multiplying by 28, the product of 4 and 7, which is the least common multiple,

$$\text{and } 28x-\frac{x}{4}\times 28-\frac{x}{7}\times 28=850\times 28,$$

$$\text{or } 28x-7x-4x=23800,$$

$\therefore$ $17x=23800$; and by division, $x=1400$ dollars.

Prob. 17. What number is that, whose one half and one third, plus 12, shall be equal to itself?

Let $x=$ the number required;

then, by the problem, $x=\frac{x}{2}+\frac{x}{3}+12$;

Now to clear this of fractions, multiply by 6,

and $6x=3x+2x+72$;

by transposition, $6x-5x=72$;

$\therefore x=72$.

It can be readily proved that 72 is the number required;

thus, $\frac{72}{2}+\frac{72}{3}+12=36+24+12=72$.

All other problems in this Section may be proved in like manner.

Prob. 18. To find a number, whose half, minus 6, shall be equal to its third part, plus 10.

Let $x=$ the number required;

then by the problem, $\frac{x}{2}-6=\frac{x}{3}+10$,

$\therefore$ clearing of fractions, $3x-36=2x+60$,

by transposition, $3x-2x=60+36$,

$\therefore x=96$.

Prob. 19. Two persons, A and B, set out from one place, and both go the same road, but A goes a hours before B, and travels n miles an hour; B follows, and travels m miles an hour. In how many hours, and in how many miles travel, will B overtake A?

Let $x=$ the hours that B travelled;

then $x+a=$ the hours that A travelled.

Also $mx=$ the number of miles travelled by B;

and $n(x+a)=nx+na=$ the miles travelled by A;

$\therefore$ by the problem, $mx=nx+na$;

by transposition, $mx-nx=na$,

or $(m-n)x=na$;

$\therefore$ by division, $\frac{(m-n)x}{m-n}=\frac{na}{m-n}$,

$\therefore x=\frac{na}{m-n}$, the hours that B travelled.

Then $x+a=\frac{na}{m-n}+a=\frac{na+ma-na}{m-n}=\frac{ma}{m-n}$, the hours that A travelled; and $mx=\frac{mna}{m-n}=$ the miles travelled.

180. This is a general or literal solution, because m, n, a, may be any numbers or quantities taken at pleasure; for example,

Let $a=9$, $n=5$, and $m=7$;

Then, A travels 9 hours at the rate of 5 miles an hour, before B sets out; and B follows after at the rate of 7 miles an hour.

Now, by putting these values of a, n, and m, in the formula, found above; we have,

$x=\frac{na}{m-n}=\frac{9\times5}{7-5}=\frac{45}{2}=22\frac{1}{2}$, the hours that B travelled;

and $x=\frac{ma}{m-n}=\frac{9\times7}{7-5}=\frac{63}{2}=31\frac{1}{2}$, the hours travelled by A.

And $mx=7\times22\frac{1}{2}=157\frac{1}{2}$, the miles travelled by each.

Prob. 20. Four merchants entered into a speculation, for which they subscribed 4755 dollars; of which B paid three times as much as A; C paid as much as A and B; and D paid as much as C and B. What did each pay?

Here, if we knew how much A paid, the sum paid by each of the rest could be easily ascertained;

Let, therefore, $x=$ number of dollars A paid;
$3x=$ number B paid;
$4x=$ number C paid;
and $7x=$ number D paid;

$\therefore (x+3x+4x+7x=)15x=4755$,
and $x=317$.

$\therefore$ they contributed 317, 951, 1268, and 2219 dollars respectively.

Prob. 21. Let it be required to divide 890 dollars between three persons, in such a manner, that the first may have 180 more than the second, and the second 115 more than the third.

Here, it is manifest that if the least or third part were known, the remaining parts could be easily ascertained; therefore,

Let the *least* or *third* part . . $=x$.
Then the *second* part . . . $=x+115$.
$\therefore$ the *greatest* or *first* part . . $=x+115+180$.
But the sum of the three parts . $=890$.

$$\therefore 3x+115+115+180=890,$$
$$\text{or } 3x+410=890;$$
$$\therefore \text{by transposition, } 3x=890-410,$$
$$\text{or } 3x=480,$$
$$\therefore x=160= \textit{least} \text{ part.}$$
$$\therefore x+115=160+115=275= \textit{second} \text{ part.}$$
$$\text{and } x+115+180=160+115+180=455= \textit{greatest} \text{ part.}$$

Prob. 22. A prize of 2329 dollars was divided between two persons A and B, whose shares therein were in proportion of 5 to 12. What was the share of each?

$$\text{Let } 5x=\text{A's share};$$
$$\text{then } 12x=\text{B's share};$$
$$\therefore 5x+12x=2329, \text{ or } 17x=2329;$$
$$\text{and } x=137;$$
$\therefore$ their shares were 685 and 1644 dollars respectively.

Prob. 23. A fish was caught, whose tail weighed 9lbs.; his head weighed as much as his tail, and half his body; and his body weighed as much as his head and tail. What did the fish weigh?

Let $2x=$ the number of lbs. the body weighed; then $9+x$ $=$ the weight of the tail;

$$\therefore 9+9+x=2x;$$
$$\text{by transposition, } x=18;$$
$\therefore$ the fish weighed $36+27+9=72$lbs.

Prob. 24. A hare, 50 of her leaps before a greyhound, takes 4 leaps to the greyhound's three; but two of the greyhound's leaps are as much as three of the hare's. How many leaps must the greyhound take to catch the hare?

Let $3x=$ the number of leaps the greyhound must take;
$\therefore 4x=$ the number the hare takes in the same time,
$\therefore 4x+50=$ the whole number she takes,

$$\text{and } 2:3::3x:4x+50;$$
$$\therefore 9x=8x+100;$$
$$\text{by transposition, } x=100,$$
and the greyhound must take 300 leaps.

Prob. 25. The number of soldiers of an army is such, that its triple diminished by 1000, is equal to its quadruple augmented by 2000. What is this number?

Let x designate the number required ;
then, we are conducted to this equation,

$$3x-1000=4x+2000, \text{ whence } x=-3000,$$

which gives an absurd answer with respect to the terms of the question, since that a number of soldiers cannot be negative.

181. We shall render this impossibility very plain, by observing that the triple of a number being less than the quadruple of the same number, the triple diminished by 1000 is much less than the quadruple augmented by 2000. But by writing $-x$ in the place of $+x$ in the equation of the problem, then changing the signs of both sides, we find

$$3x+1000=4x-2000\ ;\ \therefore x=3000.$$

We can from the equation

$$3x+1000=4x-2000,$$

re-establish the enunciation of the problem in such a manner that there results from the solution an absolute number, that is,

$$x=3000.$$

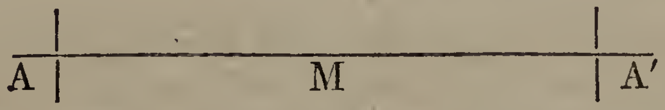
If in place of taking x for the representation of the unknown number, we had taken

$$3x-6000, \text{ or } x=x'-6000$$

we should find for the equation

$$x'-19000=4x'-22000\ ;$$

$\therefore$ by transposition, $22000-19000=4x'-3x'$,
and $\therefore x'=3000$ as before.



Thus the value $x=-3000$ being represented, on a line, by the length A′M, counted from A′ towards M, or to the left of A′, we pass by the substitution $x=x'-6000$ from the origin A′ to the origin A, to the left of A′, and distant from A′ by $6000=2\text{A}'\text{M}$; then the length $\text{AM}=x'$ is positive.

Prob. 26. A Courier sets out from *Trenton* for *Washington*, and travels at the rate of 8 miles an hour ; two hours after his departure another Courier sets out after him from *New-York*, supposed to be 68 miles distant from *Trenton*, and travels at the rate of 12 miles an hour. How far must the second Courier travel before he overtakes the first ?

Let x represent the number of miles which the second Courier travels before he overtakes the first ; then, by a little attention, we discover that this distance should be equal to the distance from *New-York* to *Trenton*, or $\text{NT}=68$ miles, plus the distance travelled by the first Courier in two hours which his departure preceded that of the second, together with the number of miles which the first travels whilst the second Courier is on route; that is, NM, or $x=\text{NT}+\text{TR}+\text{RM}$.

Let us translate the two last distances, that is, TR and RM; in the first place, $2\times 8=16=\text{TR}=$ the number of miles which the first Courier travels before the second sets out; then, in order to find an expression for MR; we shall say, since the distances passed over in an hour are as 8 : 12, or 2 : 3; as, 2 : 3 : : MR : x; and consequently $\text{MR}=\frac{2x}{3}$. So that we obtain for a translation of the enunciation,

$$x=68+16+\frac{2x}{3}=84+\frac{2x}{3};$$

by multiplication, $3x=252+2x$; $\therefore x=252$, that is to say, the two Couriers would meet when the second shall have travelled 252 miles. In fact, while the second travelled 252 miles, the first travelled 168 miles; since $\frac{2x}{3}$ is the expression for the number of miles which the first travelled while the second was on route; that is, substituting 252 for x, $\frac{2x}{3}=\frac{2\times 252}{3}=\frac{504}{3}=168$ miles.

Now, the place from whence the first Courier departed, being 68 miles distant from New-York, besides he has the advantage of having travelled 16 miles before the other set out. Consequently $68+16+168$ must be equal to the number of miles which the second Courier travels before they meet; that is, $68+16+168=252$.

We see here an example of verification of the value of the unknown; it is a proof which the student can, and should always make.

182. In order to have a general solution of this problem, let us therefore represent in general, by a the distance between the two places of departure, which was 68 miles in the preceding question, by b the number of hours which the departure of the first precedes that of the second, by c the number of miles that the first Courier travels per hour, and by d the number which the second travels in the same time. Let $x=$ the distance which the second Courier must travel before they meet; then, we shall have the distance travelled by the first Courier during the time that the second has been travelling, by calculating the fourth term of a proportion that commences thus;

$$d : c :: x : \frac{c \times x}{d} \text{ or } \frac{cx}{d}.$$

The first Courier travelling c miles an hour, he will have travelled $c \times b$ miles before the second set out.

Therefore by the condition of the problem, we shall have

$$x = \frac{cx}{d} + bc + a\text{; whence } x = \frac{d(cb+a)}{d-c},$$

which gives the solution of all questions of the same kind.

In order to show the use of this formula, let us resume again the preceding enunciation, and by recollecting that we must replace a by 68, b by 2, c by 8, and d by 12.

Then the value of x becomes

$$x = \frac{12(16+68)}{12-6} = 252 \text{ miles as before.}$$

Prob. 27. What two numbers are those, whose difference is 10, and if 15 be added to their sum, the whole will be 43?
Ans 9 and 19.

Prob. 28. What two numbers are those, whose difference is 14, and if 9 times the lesser be subtracted from six times the greater, the remainder will be 33? Ans. 17 and 31.

Prob. 29. What number is that, which being divided by 6, and 2 subtracted from the quotient, the remainder will be 2?
Ans. 24.

Prob. 30 What two numbers are those, whose difference is 14, and the quotient of the greater divided by the lesser 3?
Ans. 21 and 7.

Prob. 31. What two numbers are those, whose sum is 60, and the greater is to the lesser as 9 to 3? Ans. 45 and 15.

Prob. 32. What number is that, which being added to 5, and also multiplied by 5, the product shall be 4 times the sum?
Ans. 20.

Prob. 33. What number is that, which being multiplied by 12, and 48 added to the product, the sum shall be 18 times the number required? Ans. 8.

Prob. 34 What number is that, whose $\frac{1}{4}$ part exceeds its $\frac{1}{5}$ part by 32? Ans. 640.

Prob. 35. A Captain sends out $\frac{1}{3}$ of his men, plus 10; and there remained $\frac{1}{2}$, minus 15; how many had he? Ans. 150.

Prob. 36. What number is that, from which if 8 be subtracted, three-fourths of the remainder will be 60? Ans. 88.

Prob. 37. What number is that, the treble of which is as much above 40, as its half is below 51? Ans 26.

Prob. 38. What number is that, the double of which exceeds four-fifths of its half by 40? Ans 25.

PROB. 39. At a certain election, 946 men voted, and the candidate chosen had a majority of 558. How many men voted for each? Ans. 194 for one, and 752 for the other.

PROB. 40. After paying away $\frac{1}{9}$ of my money, and then $\frac{1}{8}$ of the remainder, I had 140 dollars left: what had I at first?

Ans. 180 dollars.

PROB. 41. One being asked how old he was, answered, that the product of $\frac{1}{25}$ of the years he had lived, being multiplied by $\frac{5}{6}$ of the same, would be his age. What was his age?

Ans. 30.

PROB. 42. After A had lent 10 dollars to B, he wanted 8 dollars in order to have as much money as B; and together they had 60 dollars. What money had each at first?

Ans. A 36 and B 24.

PROB. 43. Upon measuring the corn produced by a field, being 48 bushels; it appeared that it yielded only one third part more than was sown. How much was that?

Ans. 36 bushels.

PROB. 44. A Farmer sold 96 loads of hay to two persons. To the first one half, and to the second one fourth of what his stack contained. How many loads did that stack contain?

Ans. 128 loads.

PROB. 45. A Draper bought three pieces of cloth, which together measured 159 yards. The second piece was 15 yards longer than the first, and the third 24 yards longer than the second. What was the length of each?

Ans. 35, 50 and 74 yards respectively.

PROB. 46. A cask which held 146 gallons, was filled with a mixture of brandy, wine, and water. In it there were 15 gallons of wine more than there were of brandy, and as much water as both wine and brandy. What quantity was there of each? Ans. 29, 44, and 73 gallons respectively.

PROB. 47. A person employed 4 workmen, to the first of whom he gave 2 shillings more than to the second; to the second 3 shillings more than to the third; and to the third 4 shillings more than to the fourth. Their wages amounted to 32 shillings. What did each receive?

Ans. 12, 10, 7, and 3 shillings respectively.

PROB. 48. A Father taking his four sons to school, divided a certain sum among them. Now the third had 9 shillings more than the youngest; the second 12 shillings more than the third; and the eldest 18 shillings more than the second; and the whole sum was 6 shillings more than 7 times the sum which the youngest received. How much had each?

Ans. 21, 30, 42, and 60 shillings respectively.

PROB. 49. It is required to divide the number 99 into five such parts, that the first may exceed the second by 3; be less than the third by 10; greater than the fourth by 9; and less than the fifth by 16. Ans. 17, 14, 27, 8, and 33.

PROB. 50. Two persons began to play with equal sums of money; the first lost 14 shillings, the other won 24 shillings, and then the second had twice as many shillings as the first. What sum had each at first? Ans. 52 shillings.

PROB. 51. A Mercer having cut 19 yards from each of three equal pieces of silk, and 17 from another of the same length, found that the remnants together were 142 yards. What was the length of each piece? Ans. 54 yards.

PROB. 52. A Farmer had two flocks of sheep, each containing the same number. From one of these he sells 39, and from the other 93; and finds just twice as many remaining in one as in the other. How many did each flock originally contain? Ans. 147

PROB. 53. A Courier, who travels 60 miles a day, has been dispatched five days, when a second is sent to overtake him, in order to do which he must travel 75 miles a day. In what time will he overtake the former? Ans. 20 days.

PROB. 54. A and B trade with equal stocks. In the first year A tripled his stock, and had \$27 to spare; B doubled his stock, and had \$153 to spare. Now the amount of both their gains was five times the stock of either. What was that? Ans. 90 dollars.

PROB. 55. A and B began to trade with equal sums of money. In the first year A gained 40 dollars, and B lost 40; but in the second A lost one-third of what he then had, and B gained a sum less by 40 dollars, than twice the sum that A had lost; when it appeared that B had twice as much money as A. What money did each begin with? Ans. 320 dollars.

PROB. 56. A and B being at play, severally cut packs of cards, so as to take off more than they left. Now it happened that A cut off twice as many as B left, and B cut off seven times as many as A left. How were the cards cut by each?

Ans. A cut off 48, and B cut off 28 cards.

PROB. 57. What two numbers are as 2 to 3; to each of which if 4 be added, the sums will be as 5 to 7?

Ans. 16 and 24.

PROB. 58. A sum of money was divided between two persons, A and B, so that the share of A was to that of B as 5 to 3; and exceeded five-ninths of the whole sum by 50 dollars. What was the share of each person?

Ans. 450, and 270 dollars.

PROB. 59. The joint stock of two partners, whose particular shares differed by 40 dollars, was to the share of the lesser as 14 to 5. Required the shares.

Ans. the shares are 90 and 50 dollars respectively.

PROB. 60. A Bankrupt owed to two creditors 1400 dollars; the difference of the debts was to the greater as 4 to 9. What were the debts? Ans. 900, and 500 dollars.

PROB. 61. Four places are situated in the order of the four letters A, B, C, D. The distance from A to D is 34 miles, the distance from A to B : distance from C to D : : 2 : 3, and one-fourth of the distance from A to B added to half the distance from C to D, is three times the distance from B to C. What are the respective distances?

Ans. AB=12, BC=4, and CD=18 miles.

PROB. 62. A General having lost a battle, found that he had only half his army plus 3600 men left, fit for action; one-eighth of his men plus 600 being wounded, and the rest, which were one-fifth of the whole army, either slain, taken prisoners, or missing. Of how many men did his army consist?

Ans. 24000.

PROB. 63. It is required to divide the number 91 into two such parts that the greater being divided by their difference, the quotient may be 7. Ans. 49 and 42.

PROB. 64. A person being asked the hour, answered that it was between five and six; and the hour and minute hands were together. What was the time?

Ans. 5 hours 27 minutes $16\frac{4}{11}$ seconds.

PROB. 65. Divide the number 49 into two such parts, that the greater increased by 6 may be to the less diminished by 11 as 9 to 2. Ans. 30 and 19.

PROB. 66. It is required to divide the number 34 into two such parts that the difference between the greater and 18, shall be to the difference between 18 and the less : : 2 : 3.

Ans. 22 and 12.

PROB. 67. What number is that to which if 1, 5, and 13, be severally added, the first sum shall be to the second, as the second is to the third. Ans. 3.

PROB. 68. It is required to divide the number 36 into three such parts, that one-half of the first, one-third of the second, and one-fourth of the third, shall be equal to each other.

Ans. 8, 12, and 16.

PROB. 69. Divide the number 116 into four such parts, that if the first be increased by 5, the second diminished by 4, the third multiplied by 3, and the fourth divided by 2, the result in each case shall be the same. Ans. 22, 31, 9, and 54.

Prob. 70. A Shepherd, in time of war, was plundered by a party of soldiers who took $\frac{1}{4}$ of his flock, and $\frac{1}{4}$ of a sheep; another party took from him $\frac{1}{3}$ of what he had left, and $\frac{1}{3}$ of a sheep; then a third party took $\frac{1}{2}$ of what now remained, and $\frac{1}{2}$ of a sheep. After which he had but 25 sheep left. How many had he at first? Ans. 103.

Prob. 71. A Trader maintained himself for 3 years at the expense of 50*l.* a year; and in each of those years augmented that part of his stock which was not so expended by $\frac{1}{3}$ thereof. At the end of the third year his original stock was doubled. What was that stock? Ans. 740*l.*

Prob. 72. In a naval engagement, the number of ships taken was 7 more, and the number burnt 2 fewer, than the number sunk. Fifteen escaped, and the fleet consisted of 8 times the number sunk. Of how many did the fleet consist?

Ans. 32.

Prob. 73. A cistern is filled in twenty minutes by three pipes, one of which conveys 10 gallons more, and the other 5 gallons less, than the third, *per* minute. The cistern holds 820 gallons. How much flows through each pipe in a minute?

Ans. 22, 7, and 12 gallons.

Prob. 74. A sets out from a certain place, and travels at the rate of 7 miles in five hours; and eight hours afterwards B sets out from the same place, and travels the same road at the rate of 5 miles in three hours. How long, and how far, must A travel before he is overtaken by B?

Ans. 50 hours, and 70 miles.

Prob. 75. There are two places, 154 miles distant, from which two persons set out at the same time to meet, one travelling at the rate of 3 miles in two hours, and the other at the rate of 5 miles in four hours. How long, and how far, did each travel before they met?

Ans. 56 hours; and 84, and 70 miles.

CHAPTER V.

ON

SIMPLE EQUATIONS,

INVOLVING TWO OR MORE UNKNOWN QUANTITIES.

183. It has been observed (Art. 159), that an equation was the translation into algebraic language of two equivalent phrases comprised in the enunciation of a question; but this question may comprehend in it a greater number, and if they are well distinguished two by two, and independent of one another, they furnish a certain number of equations.

Thus, for example, *let us propose to find two numbers, such that double the first added to the second, gives* 24, *and that five times the first, plus three times the second, make* 65. We find here two phrases, which express the same thing in different terms; 1st, *the double of an unknown number, plus another unknown number, then the equivalent* 24; 2d, *five times the first unknown number, plus three times the second, then the equivalent* 65.

The translation is easy, and it gives these two *determinate* equations:

$$2x+y=24;\ 5x+3y=65.$$

When two or more equations, involving as many unknown quantities, are independent of one another, they are called *determinate.* But if for the second of these two conditions we had substituted this: *and such that six times the first number, plus three times the second, make* 72; these two phrases express nothing more than the first two, since that we have only tripled two equal results; we should have but one translation, and consequently a single equation. It can therefore happen that we may have less equations than unknown quantities, and then the question is said to be *indeterminate;* because the number of conditions would be insufficient for the determination of the unknown quantities, as we shall see clearly illustrated in the following section.

§ I. ELIMINATION OF UNKNOWN QUANTITIES FROM ANY NUMBER OF SIMPLE EQUATIONS.

184. Elimination is the method of exterminating all the unknown quantities, except one, from two, three, or more given

equations, in order to reduce them to a single, or final equation, which shall contain only the remaining unknown, and certain known quantities.

185. In order to simplify the calculations, by avoiding fractions, we shall here make use of literal equations, which will modify the process of elimination : And also, to avoid the inconvenience arising from the multitude of letters which must be employed in order to represent the given quantities, when the number of equations involving as many unknown quantities surpasses two, we shall represent by the same letter all the coefficients of the same unknown quantity ; but we shall affect them with one or more accents, in order to distinguish them, according to the number of equations.

186. In the first place, any two simple equations, each involving the same two unknown quantities, may, in general, be written thus :

$$ax+by=c \quad . \quad . \quad . \quad . \quad . \quad . \quad (A),$$
$$a'x+b'y=c' \quad . \quad . \quad . \quad . \quad . \quad . \quad (B).$$

The coefficients of the unknown quantity x are represented both by a ; those of y by b ; but the accent, by which the letters of the second equation are affected, shows that we do not regard them as having the same value as their correspondents in the first. Thus a' is a quantity different from a, b' a quantity different from b.

187. We can readily see, by a few examples, how any two simple equations, each involving the same two unknown quantities, may be reduced to the above form.

Ex. 1. Let the two simple equations,

$$5x+3y-5=y-2x+7,$$
$$9x-2y+3=x-7y+16,$$

be reduced to the form of equations (A) and (B).

By transposition, these equations become

$$5x+3y-y+2x=7+5,$$
$$9x-2y-x+7y=16-3\,;$$

by reduction, we shall have

$$7x+2y=12,$$
$$8x+5y=13\,;$$

equations which are reduced to the form of (A) and (B), and which may be expressed under the form of the same literal equations, by substituting a, b, and c, for 7, 2, and 12 ; and a', b', and c', for 8, 5, and 13.

Ex. 2. Let the two simple equations,

$$mx+6y-7=px-2y+3,$$
$$rx-9y+6=3y-3x+12,$$

be reduced to the form of equations (A) and (B).

By transposition, these equations become

$$mx+6y-px+2y=3+7,$$
$$rx-9y-3y+3x=12-6;$$

by reduction, we shall have

$$(m-p)x+\ 8y=10,$$
$$(r+3)x-12y=6;$$

which are reduced to the form required, and which may be expressed under the form of the same literal equations, by substituting a for $m-p$, b for 8, c for 10, a' for $r+3$, b' for -12, and c' for 6.

In like manner any two simple equations may be reduced to the form of equations (A) and (B); hence we may conclude that a, b, c, a,′ b', and c', may be any given numbers or quantities whatever, *positive* or *negative*, *integral* or *fractional.*

It is to be always understood, that when we make use of the same letters, marked with different accents, they express different quantities. Thus, in the following equations, a, a', a'', are three different quantities; and the same of others.

188. Any three simple equations, each involving the same three unknown quantities, may be expressed thus;

$$ax+by+cz=d \quad . \quad . \quad . \quad . \quad (C),$$
$$a'x+b'y+c'z=d' \quad . \quad . \quad . \quad (D),$$
$$a''x+b''y+c''z=d'' \quad . \quad . \quad (E);$$

where a, b, c, d, a', b', c' d', a'', b'', c'', d'', are *known* quantities; and x, y, z, *unknown* quantities whose values may be found in terms of the known quantities.

In like manner, any four simple equations may be expressed thus;

$$ax+by+cz+du=e \quad . \quad . \quad . \quad . \quad (F),$$
$$a'x+b'y+c'z+d'u=e' \quad . \quad . \quad . \quad (G),$$
$$a''x+b''y+c''x+d''u=e'' \quad . \quad . \quad (H),$$
$$a'''x+b'''y+c'''z+d'''u=e''' \quad . \quad . \quad (I);$$

And so on for five, or more simple equations.

189. Analysts make use of various methods of eliminating unknown quantities from any number of equations, so as to have a final equation containing only one of the unknown quantities; some of which are only applicable in particular cases; but the most general methods of exterminating unknown quantities in simple equations, are the following.

FIRST METHOD.

190. Let us consider, in the first place, the equations,

$$ax+by=c \quad . \quad . \quad . \quad (A),$$
$$a'x+b'y=c' \quad . \quad . \quad . \quad (B).$$

It is evident that if one of the unknown quantities, x, for example, had the same coefficient in the two equations, it would be sufficient to subtract one from the other, in order to exterminate this unknown: Let, for example, the equations be

$$10x+11y=27,$$
$$10x+\ 9y=15;$$

if the second be subtracted from the first, we shall have

$$11y-9y=27-15, \text{ or } 2y=12.$$

It is very plain, that we can immediately render the coefficients of x equal, in the equations (A) and (B);

By multiplying the two members of the first by a', the coefficient of x in the second; and the two members of the second by a, the coefficient of x in the first; we shall thus obtain,

$$a'ax+a'by=a'c;$$
$$aa'x+ab'y=ac'.$$

Subtracting the first of these from the second, the unknown x will disappear, we shall have only

$$(ab'-a'b)y=ac'-a'c,$$

an equation which contains no more than the unknown quantity y, and we will deduce from it

$$y=\frac{ac'-a'c}{ab'-a'b} \quad . \quad . \quad . \quad (a).$$

By eliminating in the same manner the unknown quantity y, from the proposed equations; we would arrive at the equation

$$(ab'-a'b)x=b'c-bc';$$

from which we will deduce

$$x=\frac{b'c-bc'}{ab'-a'b} \quad . \quad . \quad . \quad (b).$$

191. The process which we have just employed, may be applied to all simple equations, for exterminating any number whatever of unknown quantities.

If we apply this process to three equations, involving x, y, and z, we will at first eliminate x between the first and second; then between the second and third; and we shall thus arrive at two equations, which involve only y and z, and between which we will afterward eliminate y, as in the preceding article.

If we effect the equation in z, at which we will arrive, we shall have a factor too much in all its terms; and consequently it will not be the most simple which might be obtained.

SECOND METHOD.

192. Let us resume again the equations,

(A) . . $ax+by=c$; $a'x+b'y=c'$ . . . (B):

If we find the value of x in terms of y and the known quantities in each of these equations, we shall have

$$x=\frac{c-by}{a},\quad x=\frac{c'-b'y}{a'};$$

the equality of the second members, furnishes the equation

$$\frac{c-by}{a}=\frac{c'-b'y}{a'},$$

which, by making proper reductions, gives

$$y=\frac{ac'-a'c}{ab'-a'b};$$

by substituting this value for y, in one of the values of x, we shall, after the reductions, have

$$x=\frac{b'c-bc'}{ab'-a'b};$$

These values of x and y are the same as before.

Now, it is evident, that by proceeding in the same manner, with three equations containing x, y, and z, we will find the value of x in each of them, then by comparing these values, we shall arrive at two equations, involving only y and z, from which we can eliminate y, as in equations (A) and (B). And, we can proceed, in a similar manner, when there are four equations with four unknown quantities; and so on, for five, or more equations.

THIRD METHOD.

193. Now, if in the equation (A), we find the value of x, in terms of y and the given quantities, we shall have

$$x=\frac{c-by}{a};$$

by substituting this value in equation (B), we shall have

$$a'\times\frac{c-by}{a}+b'y=c',$$

which, by reduction, becomes

$$(ab'-a'b)y=ac'-a'c,\ \therefore y=\frac{ac'-a'c}{ab'-a'b};$$

this value being substituted for y in the above value of x, after making the proper reductions, we shall obtain

$$x=\frac{b'c-bc'}{ab'-a'b}.$$

These values of x and y are the same as in the two former instances.

194. We might eliminate in like manner, when any number of simple equations are concerned; thus, for example: Let it be required to deduce from the three equations, (C), (D), and (E), (Art. 188), a single equation involving only the unknown quantity z.

By finding the value of x in each of these equations, in terms of y, z, and the given quantities, we shall have

$$x=\frac{d-by-cz}{a} \quad . \quad . \quad . \quad (1);$$

$$x=\frac{d'-b'y-c'z}{a'} \quad . \quad . \quad . \quad (2);$$

$$x=\frac{d''-b''y-c''z}{a''} \quad . \quad . \quad (3);$$

Putting the first value of x equal to the second, and also equal to the third, we shall have these two equations,

$$\frac{d-by-cz}{a}=\frac{d'-b'y-c'z}{a'};$$

$$\frac{d-by-cz}{a}=\frac{d''-b''y-c''z}{a''};$$

From which we deduce, by reduction and proceeding as in equations (A) and (B),

$$y=\frac{(a'c-ac')z+ad'-a'd}{ab'-a'b} \quad . \quad . \quad . \quad (4);$$

$$y=\frac{(a''c-ac')z+ad''-a''d}{ab''-a''b} \quad . \quad . \quad (5).$$

The equality of the second members furnishes the equation

$$\frac{(a'c-ac')z+ad'-a'd}{ab'-a'b}=\frac{(a''c-ac'')z+ad''-a''d}{ab''-u''b}$$

which, by proper reductions, will give the value of z: having obtained the value of z, substitute it in equation (4) or (5), and the value of y can be readily found.

Now, the values of y and z being known, by substituting them in the equation (1), (2), or (3); we shall easily obtain the value of x.

FOURTH METHOD.

195. Let, as before, the two equations be

$$(A) \quad . \quad . \quad . \quad ax+by=c; \quad a'x+b'y=c' \quad . \quad . \quad . \quad (B).$$

Multiplying equation (A) by some indeterminate quantity m, it will become

$$amx+bmy=mc;$$

and subtracting from this result equation (B), we shall have

$$(am-a')x+(bm-b')y=cm-c' \quad . \quad . \quad . \quad (6).$$

And since the value of m, in this equation, is indeterminate, we can take $bm-b'=0$, or $m=\frac{b'}{b}$; in which case the second term will disappear, we shall have

$$x=\frac{cm-c'}{am-a'}=\frac{c\times\frac{b'}{b}-c'}{a\times\frac{b'}{b}-a'}=\frac{cb'-bc'}{ab'-a'b};$$

which is the same value of x, as before.

Also, as the value of x, thus found, is independent of that of m, we can now take $am=a'$ or $m=\frac{a'}{a}$; according to which supposition the term involving x will vanish, and the result will give

$$y=\frac{ca'-ac'}{ba'-ab'}.$$

By changing the signs of the numerator and denominator (Art. 128) of this value, its denominator will be the same as that of x, since we shall have,

$$y=\frac{ac'-a'c}{ab'-ba'};$$

which is the same value of y as in each of the preceding methods.

This method, given by Bezout, is very simple for eliminating all the unknown quantities, except one; besides, it has the advantage of greater brevity above the preceding methods, as we can deduce the values of each of the unknown quantities from the same equation.

§ II. RESOLUTION OF SIMPLE EQUATIONS,

Involving two unknown Quantities.

196. When there are two *independent* simple equations, involving two unknown quantities, the value of each of them may be found by any of the following practical rules, which are easily deduced from the Articles in the preceding Section.

RULE I.

197. Multiply the first equation by the coefficient of one of the unknown quantities in the second equation, and the se-

cond equation by the coefficient of the same unknown quantity in the first. If the signs of the term involving the unknown quantity be alike in both, subtract one equation from the other; if unlike, add them together, and an equation arises in which only one unknown quantity is found.

Having obtained the value of the unknown quantity from this equation, the other may be determined by substituting in either equation the value of the quantity found, and thus reducing the equation to one which contains only the other unknown quantity.

Or, multiply or divide the given equations by such numbers, or quantities, as will make the term that contains one of the unknown quantities the same in each equation, and then proceed as before.

Ex. 1. Given $\left\{\begin{array}{l} 2x+3y=23, \\ 5x-2y=10, \end{array}\right\}$ to find the values of x and y.

Multiply the 1st equation by 5, then $10x+15y=115$;
2nd . . . . 2, . . $10x-\ 4y=\ 20$;

$\therefore$ by subtraction, $19y=95$,

by division, $y=\frac{95}{19}$, $\therefore y=5$.

Now, from the first of the preceding equations, we shall have $x=\frac{23-3y}{2}=(\text{since } y=5)\ \frac{23-15}{2}=\frac{8}{2}=4.$

The values of x and y might be found in a similar manner, thus:

Multiply the 1st equation by 2, then $4x+6y=46$;
2nd . . . . . 3, . $15x-6y=30$;

$\therefore$ by addition, $19x=76$,

by division, $x=\frac{76}{19}=4$.

Now, from the first of the preceding equations, we shall have $y=\frac{23-2x}{3}=(\text{since } x=4)\ \frac{23-8}{3}=\frac{15}{3}=5.$

Ex. 2. Given $\left\{\begin{array}{l} 4x+\ 9y=35, \\ 6x+12y=48, \end{array}\right\}$ to find the values of x and y.

Multiply the 1st equation by 6, then $24x+54y=210$;
2nd . . . . 4, . $24x+48y=192$;

$\therefore$ by subtraction, $6y=18$,

by division, $y=\frac{18}{6}=3$.

Now, from the first of the preceding equations, we shall have $x=\frac{35-9y}{4}=$(since $y=3$) $\frac{35-9\times3}{4}=\frac{35-27}{4}$,

$$\text{or } x=\frac{8}{4},\ \therefore x=2$$

The values of x and y may be found thus;

Multiply the 1st equation by 3, then $12x+27y=105$;
2nd . . . 2, . $12x+24y=96$;

$\therefore$ by subtraction, $3y=9$,

by division, $y=\frac{9}{3}=3$.

And $\therefore x=\frac{35-27}{4}=\frac{8}{2}=2$.

The numbers 3 and 2, by which we multiplied the given equations, are found thus;

The product of two numbers or quantities, divided by their greatest common measure, will give their least common multiple.

$\therefore \frac{6\times4}{2}=12$ the least common multiple,

Then $\frac{12}{4}=3$, the number by which the first equation is multiplied; and $\frac{12}{6}=2$, the number by which the second equation is multiplied.

By proceeding in a similar manner with other equations, the final equation will be always reduced to its lowest terms.

Ex. 3. Given $\left\{\begin{array}{l}5x+4y=58,\\3x+7y=67,\end{array}\right\}$ to find the values of x and y.

Multiply the 2nd equation by 5, then $15x+35y=335$;
1st . . . 3, . $15x+12y=174$;

$\therefore$ by subtraction, $23y=161$,

and $y=\frac{161}{23}=7$;

whence, $5x=58-4y=58-28=30$,

and $\therefore x=\frac{30}{5}=6$.

If the second equation had been multiplied by 4, and subtracted from the first when multiplied by 7, an equation would have arisen involving only x, the value of which might be determined, and thence, by substitution, the value of y.

Ex. 4. Given $\left\{\begin{array}{l} 6x-2y=14, \\ 5x-6y=-10, \end{array}\right\}$ to find the values of x and y.

Multiply the first equation by 3,

$$18x-6y=42;$$
$$\text{but } 5x-6y=-10;$$

$\therefore$ by subtraction, $13x=52$, and $x=4$,

$$\text{whence } y=\frac{5x+10}{6}=\frac{20+10}{6}=\frac{30}{6}=5.$$

198. These values being substituted in the place of x and y in each of the equations, shall render both members *identically* equal, or, what is the same thing, each of the equations will reduce to $0=0$.

Thus, by substituting 4 for x, and 5 for y, in the above equations, they become

$$\left.\begin{array}{l} 6\times4-2\times5=14, \\ 5\times4-6\times5=-10; \end{array}\right\} \text{ or } \left\{\begin{array}{l} 14=14; \\ -10=-10. \end{array}\right\}$$

Therefore, by transposition,

$$14-14=0, \text{ or } 0=0;$$
$$\text{and } -10+10=0, \text{ or } 0=0.$$

Since (Art. 56) $14-14=0$, and $10-10=0$.

If these conditions do not take place, it is evident that there must be an error in the calculation: therefore, the student, whenever he has any doubt respecting the answer, should always make similar substitutions.

Ex. 5. Given $\left\{\begin{array}{l} 11x+3y=100, \\ 4x-7y=4, \end{array}\right\}$ to find the values of x and y.

Mult. the 1st equation by 7, then $77x+21y=700$,

2d . . . 3, . $12x-21y=12$;

$\therefore$ by addition, $89x=712$,

by division, $x=\frac{712}{89}$;

and $\therefore x=8$;

whence $3y=100-11x=100-11\times8=100-88=12$;

$$\therefore y=\frac{12}{3}=4.$$

Ex. 6. Given $\left\{\begin{array}{l}\frac{x}{7}+7y=99,\\ \frac{y}{7}+7x=51,\end{array}\right\}$ to find the values of x and y.

Multiply each equation by 7,

$$\therefore x+49y=693,$$
$$\text{and } y+49x=357;$$

$$\therefore \text{by addition } 50x+50y=1050,$$
$$\text{and by division, } x+y=\frac{1050}{50}=21;$$
$$\text{but since } x+49y=693,$$

subtracting the upper equation from the lower,

$$\text{we have } 48y=693-21=672,$$
$$\therefore y=\frac{672}{48}=14,$$
$$\text{whence } x=21-y=21-14=7.$$

Ex. 7. Given $\left\{\begin{array}{l}\frac{x+2}{3}+\ \ 8y=\ \ 31,\\ \frac{y+5}{4}+10x=192,\end{array}\right\}$ to find the values of x and y.

Clearing the first equation of fractions,

$$x+2+24y=93;$$
$$\therefore \text{by transposition, } x+24y=91 \quad . \quad . \quad . \quad (1)$$

Clearing the second equation of fractions,

$$y+5+40x=768;$$
$$\therefore \text{by transposition, } 40x+y=763 \quad . \quad . \quad . \quad (2).$$

Multiplying equation (1) by 40, and subtracting equation (2) from it,

$$40x+960y=3640;$$
$$40x-\quad y=\ \ 763;$$

$$\therefore 959y=2877,$$
$$\text{and by division, } y=3;$$

From equation (1), $x=91-24y$,

$\therefore$ by substitution, $x=91-24\times 3$,

or $x=91-72$, $\therefore x=19$.

If from equation (2), multiplied by 24, equation (1) had been subtracted, an equation would have arisen involving only x, the value of which might be determined, and this being substituted in either of the equations, the value of y might also be found.

Ex. 8. Given $\left\{\begin{array}{l} x+y=a, \\ x-y=b, \end{array}\right\}$ to find the values of x and y.

By addition, $2x=a+b$; $\therefore x=\dfrac{a+b}{2}$.

By subtraction, $2y=a-b$, $\therefore y=\dfrac{a-b}{2}$.

Ex. 9. Given $\left\{\begin{array}{l} \frac{1}{2}x+2y=12, \\ \frac{1}{2}x-2y=\ \ 4, \end{array}\right\}$ to find the values of x and y

Multiply the 1st equation by 2, then $x+4y=24$;
2nd . . . . 2, . $x-4y=\ 8$;

by addition, $2x=32$,

$\therefore$ by division, $x=\dfrac{32}{2}=16$.

By subtraction, $8y=16$;
$\therefore$ by division, $y=\ 2$.

Or, the values of x and y may be found thus:
From the first equation subtract the second, and we have

$4y=8$, $\therefore y=2$.

Add the first equation to the second,

and $\therefore x=16$.

Ex. 10. Given $4x+3y=31$, and $3x+2y=22$; to find the values of x and y. Ans. $x=4$, $y=5$.

Ex. 11. Given $5x-4y=19$, and $4x+2y=36$, to find the values of x and y. Ans. $x=7$, $y=4$.

Ex. 12. Given $\dfrac{x+y}{3}-2y=2$, and $\dfrac{2x-4y}{5}+y=\dfrac{23}{5}$; to find the values of x and y. Ans. $x=11$, $y=1$.

Ex. 13. Given $\left.\begin{array}{r} \dfrac{2x-y}{2}+14=18, \\ \text{and } \dfrac{2y+x}{3}+16=19, \end{array}\right\}$ to find the values of x and y.

Ans. $x=5$, and $y=2$.

Ex. 14. Given $\left.\begin{array}{r} \dfrac{2x+3y}{6}+\dfrac{x}{3}=8, \\ \text{and } \dfrac{7y-3x}{2}-y=11, \end{array}\right\}$ to find the values of x and y.

Ans. $x=6$, and $y=8$.

Ex. 15. Given $\left.\begin{array}{l}3x+\dfrac{7y}{2}=22, \\ \text{and } 11y-\dfrac{2x}{5}=20,\end{array}\right\}$ to find the values of x and y.

Ans. $x=5$, and $y=2$.

Ex. 16. Given $\left.\begin{array}{l}x+1 : y :: 5 : 3, \\ \text{and } \dfrac{2x}{3}-\dfrac{5-y}{2}=\dfrac{41}{12}-\dfrac{2x-1}{4},\end{array}\right\}$ to find the values of x and y.

Ans. $x=4$, and $y=3$.

Ex. 17. Given $\left.\begin{array}{l}\dfrac{x-2}{5}-\dfrac{10-x}{3}=\dfrac{y-10}{4}, \\ \text{and } \dfrac{2y+4}{3}-\dfrac{2x+y}{8}=\dfrac{x+13}{4},\end{array}\right\}$ to find the values of x and y.

Ans. $x=7$, and $y=10$.

Ex. 18. Given $\left.\begin{array}{l}x+15y=53, \\ \text{and } y+\ 3x=27,\end{array}\right\}$ to find the values of x and y.

Ans. $x=8$, and $y=3$.

Ex. 19. Given $\left.\begin{array}{l}4x+\ 9y=51, \\ \text{and } 8x-13y=\ 9,\end{array}\right\}$ to find the values of x and y.

Ans. $x=6$, and $y=3$.

Ex. 20. Given $\left.\begin{array}{l}\dfrac{x}{6}+\dfrac{y}{4}=6, \\ \text{and } \dfrac{x}{4}+\dfrac{y}{6}=5\frac{2}{3},\end{array}\right\}$ to find the values of x and y.

Ans. $x=12$, and $y=16$.

RULE II.

199. Find the value of one of the unknown quantities in terms of the other and known quantities, in the more simple of the two equations; and substitute this value instead of the quantity itself in the other equation; thus an equation is obtained, in which there is only one unknown quantity; the value of which may be found as in the last Rule.

Ex. 1. Given $\left\{\begin{array}{l}x+2y=17, \\ 3x-\ y=\ 2,\end{array}\right\}$ to find the values of x and y.

From the first equation, $x=17-2y$;

Substituting therefore this value of x in the second equation,

$$3.(17-2y)-y=2,$$

$$\text{or } 51-6y-y=2;$$

by changing the signs, and transposing;

$$7y=51-2=49,$$
$$\therefore \text{ by division, } y=7;$$
$$\text{whence } x=17-2y=17-14=3.$$

Here a value of y might be determined from either equation, and substituted in the other; from which would arise an equation involving only x, the value of which might be found; and therefore the value of y also might be obtained by substitution, thus;

From the second equation, $y=3x-2$; substituting therefore this value of y in the first equation; we have,

$$x+2\,.\,(3x-2)=17,$$
$$\text{or } x+6x-4=17;$$
$$\therefore \text{ by transposition, } 7x=17+4=21$$
$$\text{by division, } x=\frac{21}{7},\ \therefore x=3;$$
$$\text{and } \therefore y=3x-2=3\times3-2=9-2=7.$$

Ex. 2. Given $\left\{\begin{array}{l}3x+y=60,\\ 5x+10=78+y,\end{array}\right\}$ to find the values of x and y.

From the first equation, $y=60-3x$;

Let the value of y be substituted in the second equation, and it becomes,

$$5x+10=78+(60-3x).$$

Then, by transposition, $8x=78+60-10$;

$$\text{and by division, } x=\frac{128}{8}=16.$$

Whence, $y=60-3x=60-3\times16=60-48$;

$$\therefore y=12.$$

Ex. 3. Given $\left\{\begin{array}{l}\dfrac{x+y}{3}=66-2y,\\ \dfrac{x-y}{3}=62-2x,\end{array}\right\}$ to find the values of x and y.

Mult. the 1st equation by 3, then

$$x+y=198-6y \quad \ldots \quad (1);$$
$$\text{2nd by 3, then } x-y=186-6x \quad \ldots \quad (2);$$
$$\text{From equation (1), we have } x=198-7y;$$
$$(2), \quad \ldots\ldots \quad 7x-y=186;$$

By substituting the above value of x, in the last equation, it becomes

$$7(198-7y)-y=186,$$
$$\text{or, } 1386-49y-y=186;$$
$$\text{by transposition, } -50y=186-1386=-1200,$$
$$\text{by changing the signs, } 50y=1200,$$

$$\therefore \text{ by division, } y=\frac{1200}{50}=24.$$

Whence, $x=198-7y=198-7\times24=198-168$,

$$\therefore x=30.$$

Ex. 4. Given $\left\{\begin{array}{l} x+2y=80, \\ x+\ \ y=60, \end{array}\right\}$ to find the values of x and y.

From the second equation, $x=60-y$:

By substituting this value of x in the 1st equation, we have,

$$60-y+2y=80,$$

by transposition, $y=80-60$,

$$\therefore y=20.$$

And $x=60-y=$ (by substitution) $60-20$,

$$\therefore x=40.$$

Ex. 5. Given $\left\{\begin{array}{l} x+2y=17, \\ 3x-\ y=\ 2, \end{array}\right\}$ to find the values of x and y.

From the 1st equation, $x=17-2y$.

And this value substituted in the second,

$$3(17-2y)-y=2,$$

or $51-6y-y=2$,

by transposition, &c., $7y=49$,

$\therefore$ by division, $y=7$,

whence, $x=17-2y=17-2\times7=17-14$,

$$\therefore y=3.$$

Ex. 6. Given $\left\{\begin{array}{l} x+y=5, \\ x^2-y^2=5, \end{array}\right\}$ to find the values of x and y.

From the first equation, $x=5-y$,

squaring both sides, $x^2=(5-y)^2$.

And by substituting this value for x^2 in the second equation, it becomes,

$$(5-y)^2-y^2=5,$$

by reduction, $25-10y=5$,

by transposition, $10y=20$,

$\therefore$ by division, $y=2$.

Whence, $x=5-y=5-2=3$.

Ex. 7. Given $\left\{\begin{array}{l} \frac{x}{8}+8y=194, \\ \frac{y}{8}+8x=131, \end{array}\right\}$ to find the values of x and y.

Multiplying the first equation by 8,

$$x+64y=1552,$$

$\therefore$ by transposition, $x=1552-64y$.

And substituting this value for x, in the second equation, it becomes,

$$\frac{y}{8}+8(1552-64y)=131,$$

by reduction, $y+99328-4096y=1048$,

by transposition, $4095y=98280$,

by division, $y=\frac{98280}{4095}$;

$\therefore y=24.$

Whence $x=1552-64y=1552-64\times24$,

or $x=1552-1536$;

$\therefore x=16.$

The value of y might be found from the second equation, in terms of x and the known quantities; which value of y substituted for it in the first, an equation would arise involving only x, the value of which might be found; and therefore the value of y also may be obtained by substitution.

Ex. 8. Given $\frac{5x+6y}{3}=27$, and $\frac{6x-5y}{4}=6$, to find the values of x and y.

Ans. $x=9$, and $y=6$.

Ex. 9. Given $15y+45x=300$, and $x+15y=36$, to find the values of x and y.

Ans. $x=6$, and $y=2$.

Ex. 10. Given $3x+y=60$, and $5x+10=78+y$, to find the values of x and y. Ans. $x=16$, and $y=12$.

Ex. 11. Given $10x-3y=38$, and $3x-y=11$, to find the the values of x and y. Ans. $x=5$, and $y=4$.

Ex. 12. Given $x+y=198-6y$, and $x-y=186-6x$, to find the values of x and y. Ans. $x=30$, and $y=24$.

Ex. 13. Given $\frac{x}{8}+y=26$, and $\frac{y}{2}+8x=131$, to find the values of x and y. Ans. $x=16$, and $y=24$.

Ex. 14. Given $\frac{x}{2}+\frac{y}{3}=7$, and $\frac{x}{3}+\frac{y}{2}=8$, to find the values of x and y. Ans. $x=6$, and $y=12$.

Ex. 15. Given $4x+y=34$, and $4y+x=16$, to find the values of x and y. Ans. $x=8$, and $y=2$.

Ex. 16. Given $3x+2y=54$, and $x : y :: 4 : 3$, to find the values of x and y. Ans. $x=12$, and $y=9$

Ex. 17. Given $\frac{x+8}{4}+6y=21$, and $\frac{y+6}{3}+5x=23$, to find the values of x and y. Ans. $x=4$, and $y=3$.

RULE III.

200. Find the value of the same unknown quantity in terms of the other and known quantities, in each of the equations ; then, let the two values, thus found, be put equal to each other ; an equation arises involving only one unknown quantity ; the value of which may be found, and therefore, that of the other unknown quantity, as in the preceding rules.

This rule depends upon the well-known axiom, (Art. 47) ; and the two preceding methods are founded on principles which are equally simple and obvious.

Ex. 1. Given $\left\{\begin{array}{l} x+3y=100, \\ 2x+\ \ y=100, \end{array}\right\}$ to find the values of x and y.

From the first equation, $x=100-3y$,

and from the second, $x=\frac{100-y}{2}$;

$$\therefore \frac{100-y}{2}=100-3y.$$

Multiplying by 2, $100-y=200-6y$,

by transposition, $6y-y=200-100$,

or, $5y=100$;

$\therefore$ by division, $y=20$,

whence, $x=100-3y=100-3\times 20$;

$\therefore x=40$.

Here, two values of y might have been found, which would have given an equation involving only x ; and from the solution of this new equation, a value of x, and therefore of y, might be found.

Ex. 2. Given $\frac{1}{2}x+\frac{1}{3}y=7$, and $\frac{1}{3}x+\frac{1}{2}y=8$, to find the values of x and y.

Multiplying both equations by 6, and we shall have

$3x+2y=42$, and $2x+3y=48$,

From the first of these equations, $x=\frac{42-2y}{3}$,

and from the second, $x=\frac{48-3y}{2}$;

$$\therefore \frac{42-2y}{3}=\frac{48-3y}{2};$$

Multiplying each member by 6, we shall have

$84-4y=114-9y$;

by transposition, $9y-4y=144-84$,

or $5y=60$; $\therefore y=12$

And, by substituting this value of y, in one of the values of x, the first, for instance, we shall have

$$x=\frac{42-24}{3}=\frac{18}{3}=6.$$

Ex. 3. Given $8x+18y=94$, and $8x-13y=1$, to find the values of x and y.

From the first equation, $x=\frac{47-9y}{4}$;

and from the second, $x=\frac{1+13y}{8}$:

$$\therefore \frac{47-9y}{4}=\frac{1+13y}{8};$$

And multiplying both sides of this equation by 8,

$$94-18y=1+13y;$$

$\therefore$ by transposition, $-18y-13y=-94+1$;

Changing the signs, or what amounts to the same thing, multiplying both sides by -1, and we shall have

$$18y+13y=94-1, \text{ or } 31y=93;$$

$$\therefore y=\frac{93}{31}=3;$$

whence $x=\frac{1+13y}{8}=\frac{1+39}{8}=\frac{40}{8}=5.$

Ex. 4. Given $\left.\begin{array}{r}x+y=a,\\ bx+cy=de,\end{array}\right\}$ to find the values of x and y.

From the first equation, $x=a-y$;

and from the second, $x=\frac{de-cy}{b}$;

$$\therefore a-y=\frac{de-cy}{b};$$

and multiplying by b, we shall have

$$ab-by=de-cy;$$

by transposition, $cy-by=de-ab$;

by collecting the coefficients, $(c-b)y=de-ab$;

$\therefore$ by division, $y=\frac{de-ab}{c-b}$;

whence $x=a-y=a-\frac{de-ab}{c-b}$;

that is, $x=\frac{ca-ab-de+ab}{c-b}=\frac{ca-de}{c-b}.$

Ex. 5. Given $3x+7y=79$, and $2y-\frac{1}{2}x=9$, to find the values of x and y.
Ans. $x=10$, and $y=7$.

Ex. 6. Given $\frac{x+y}{3}+1=6$, and $\frac{x-y}{7}+3=4$, to find the values of x and y. Ans. $x=11$, and $y=4$.

Ex. 7. Given $\frac{2x-3}{2}+y=7$, and $5x-13y=\frac{67}{2}$, to find the values of x and y. Ans. $x=8$, and $y=\frac{1}{2}$

Ex. 8. Given $\frac{3x-7y}{3}=\frac{2x+y+1}{5}$, and $8-\frac{x-y}{5}=6$, to find the values of x and y. Ans. $x=13$, and $y=3$.

Ex. 9. Given $x+y=10$, and $2x-3y=5$, to find the values of x and y. Ans. $x=7$, and $y=3$.

Ex. 10. Given $3x-5y=13$, and $2x+7y=81$, to find the values of x and y. Ans. $x=16$, and $y=7$.

Ex. 11. Given $\frac{x+2}{3}+8y=31$, and $\frac{y+5}{4}+10x=192$, to find the values of x and y. Ans. $x=19$, and $y=3$

Ex. 12. Given $\frac{2x-y}{2}+14=18$, and $\frac{2y+x}{3}=3$, to find the values of x and y. Ans. $x=5$, and $y=2$.

Ex. 13. Given $\left.\begin{aligned}\frac{2x+3y}{6}&=8-\frac{x}{3},\\ \text{and } \frac{7y-3x}{2}&=11+y,\end{aligned}\right\}$ to find the values of x and y.

Ans. $x=6$, and $y=8$.

201. Examples *in which the preceding Rules are applied, in the Solution of Simple Equations, Involving two unknown Quantities.*

Ex. 1. Given $\left.\begin{aligned}2y-\frac{x+3}{4}&=7+\frac{3x-2y}{5},\\ \text{and } 4x-\frac{8-y}{3}&=24\tfrac{1}{2}-\frac{2x+1}{2},\end{aligned}\right\}$ to find the values of x and y.

Multiplying the first equation by 20,

$$40y-5x-15=140+12x-8y;$$

$\therefore$ by transposition, $48y-17x=155$.

Multiplying the second equation by 6,

$$24x-16+2y=147-6x-3;$$

$\therefore$ by transposition, $2y+30x=160$ . . . (A).

Multiplying this by 24, we have

$$48y+720x=3840\,;$$
$$\text{but } 48y-\ 17x=\ 155\,;$$

$\therefore$ by subtraction, $737x=3685$,
and by division, $x=5$.
From equation (A), $2y=160-30x$;
$\therefore$ by substitution, $2y=160-150$.
by division, $y=\frac{10}{2}$; $\therefore y=5$.

The values of x and y might be found by any of the methods given in the preceding part of this Section; but in solving this example, it appears that Rule I, is the most expeditious method which we could apply.

Ex. 2. Given $\left.\begin{array}{l}\frac{2y}{18}-\frac{8x-2}{36}=1-\frac{4+y}{3}+\frac{x-y}{6},\\ \text{and } x:3y::4:7,\end{array}\right\}$
to find the values of x and y.

Reducing the first equation to lower terms,

$$\frac{y}{9}-\frac{4x-1}{18}=1-\frac{4+y}{3}+\frac{x-y}{6},$$

and therefore, multiplying by 18,

$$2y-4x+1=18-24-6y+3x-3y\,;$$

$\therefore$ by transposition, $7=7x-11y$

But from the second equation $7x=12y$.

Substituting therefore this value in the preceding equation, it becomes

$$12y-11y=7, \text{ or } y=7,$$

and $\therefore x=\frac{12y}{7}=\frac{84}{7}=12$.

Ex. 3. Given $\left.\begin{array}{l} x-\frac{3y-2+x}{11}=1+\frac{15x+\frac{4y}{3}}{33},\\ \text{and } \frac{3x+2y}{6}-\frac{y-5}{4}=\frac{11x+152}{12}-\frac{3y+1}{2},\end{array}\right\}$
to find the values of x and y.

Multiplying the first equation by 33,

$$33x-9y+6-3x=33+15x+\frac{4y}{3}\,;$$

multiplying again by 3, and transposing, we shall have $45x-31y=81$.

Multiplying the second equation by 12,

$$6x+4y-3y+15=11x+152-18y-6\,;$$

$\therefore$ by transposition, $19y-5x=131$.

Multiplying this by 9, $171y-45x=1179$;
but $45x-31y=\ \ 81$;

$\therefore$ by addition, $140y=1260$;
and by division, $y=9$.

Now, $5x=19y-131=171-131=40$;
$\therefore$ by division, $x=8$.

Ex. 4. Given $\left.\begin{array}{l}\frac{80+3x}{15}=18\frac{1}{3}-\frac{4x+3y-8}{7}, \\ \text{and } 10y+\frac{6x-35}{5}=55+10x,\end{array}\right\}$ to find the values of x and y.

Multiplying the first equation by 105, the least common multiple of 3, 7, and 15.

$$560+21x=1925-60x-45y+120;$$

$\therefore$ by transposition, $81x+45y=1485$,
and dividing by 9, $9x+5y=165$

From the second equation,

$$50y+6x-35=275+50x,$$

$\therefore$ by transposition, $50y-44x=310$;
and dividing by 2, $25y-22x=155$;
but multiplying the equation found above, by 5, $25y+45x=825$;

$\therefore$ by subtraction, $67x=670$,
and by division, $x=10$.

Now $5y=165-9x=165-90=75$, $\therefore y=15$.

Ex. 5. Given $\left.\begin{array}{l}\frac{4x}{x^2}+\frac{5y}{y^2}=\frac{9}{y}-1, \\ \text{and } \frac{5}{x}+\frac{4}{y}=\frac{7}{x}+\frac{3}{2},\end{array}\right\}$ to find the values of x and y.

Reducing the first equation to lower terms,

$$\frac{4}{x}+\frac{5}{y}=\frac{9}{y}-1;$$

$\therefore$ by transposition, $\frac{4}{x}-\frac{4}{y}=-1$;

from the 2nd equation, by transposition, $-\frac{2}{x}+\frac{4}{y}=\frac{3}{2}$;

$\therefore$ by addition, $\frac{2}{x}=\frac{1}{2}$.
and, consequently, $x=4$.

Now $\frac{4}{y}=\frac{4}{x}+1=2$; $\therefore 2y=4$, and $x=2$.

Ex. 6. Given $\left.\begin{array}{l}\frac{a}{x}+\frac{b}{y}=m, \\ \frac{c}{x}+\frac{d}{y}=n,\end{array}\right\}$ to find the values of x and y.

Multiplying the first equation by c, and the second by a, we shall have

$$\frac{ac}{x}+\frac{bc}{y}=mc,$$

$$\text{and } \frac{ac}{x}+\frac{ad}{y}=na,$$

$$\therefore \text{ by subtraction, } (bc-ad)\cdot\frac{1}{y}=mc-na;$$

$$\therefore y=\frac{bc-ad}{mc-na}.$$

$$\text{And } \frac{a}{x}=m-\frac{b}{y}=m-\frac{mbc-nab}{bc-ad}$$

$$=\frac{mbc-mad-mbc+nab}{bc-ad}=\frac{nab-mad}{bc-ad};$$

$$\therefore \frac{1}{x}=\frac{nb-md}{bc-ad}, \text{ and } x=\frac{bc-ad}{nb-md}.$$

Ex. 7. Given $\left.\begin{array}{l}3-\dfrac{7+\frac{2x}{y}}{5}=5-\dfrac{5x+9}{3y}, \\ \text{and } y-\dfrac{4+15y}{6x-2}=\dfrac{2xy-\frac{107}{8}}{2x+5},\end{array}\right\}$ to find the values of x and y.

Multiplying the first equation by $15y$,

$$\therefore 45y-21y-6x=75y-25x-45;$$

$$\text{and by transposition, } 51y-19x=45.$$

Multiplying the second equation by $2x+5$,

$$2xy+5y-\frac{8x+20+30xy+75y}{6x-2}=2xy-\frac{107}{8};$$

$$\therefore 5y+\frac{107}{8}=\frac{8x+20+30xy+75y}{6x-2};$$

and multiplying by $6x-2$, we shall have

$$30xy-10y+\frac{321x-107}{4}=8x+20+30xy+75y;$$

$$\therefore \frac{321x-107}{4}=8x+85y+20,$$

and $321x-107=32x+340y+80$;

$\therefore$ by transposition, $340y-289x=-187$.

The coefficients of y in this case, having aliquot parts; multiplying the first by 20, and the last by 3,

$$1020y-380x=\ \ 900,$$

and $1020y-867x=-561$;

$\therefore$ by subtraction, $487x=1461$,

and $x=3$;

consequently, $51y=45+19x=45+57=102$;

$\therefore y=2$.

Ex. 8. Given $8x-\frac{16+60x}{3y-1}=\frac{16xy-107}{5+2y}$, and $2+6y+9x=\frac{27x^2-12y^2+38}{3x-2y+1}$, to find the values of x and y.

Multiplying the first equation by $5+2y$,

$$40x+16xy-\frac{80+300x+32y+120xy}{3y-1}=16xy-107;$$

$$\therefore \text{by transposition } 40x+107=\frac{80+300x+32y+120xy}{3y-1}$$

and multiplying by $3y-1$, we shall have

$$120xy-40x+321y-107=80+300x+32y+120xy;$$

$\therefore$ by transposition, $289y-340x=187$.

And from the second equation,

$$27x^2-12y^2+15x+2y+2=27x^2-12y^2+38;$$

$\therefore$ by transposition, $15x+2y=36$;

whence, the coefficients of x having aliquot parts, multiplying the first equation by 3, and the second by 68,

$$867y-1020x=561,$$

and $136y+1020x=2448$;

$\therefore$ by addition, $1003y=3009$,

and $y=3$;

consequently, $15x=36-2y=36-6=30$;

and $\therefore$ by division, $x=2$.

Ex. 9. Given $x-\frac{2y-x}{23-x}=20-\frac{59-2x}{2}$, and $y+\frac{y-3}{x-18}=30-\frac{73-3y}{3}$, to find the values of x and y.

Ans. $x=21$, and $y=20$.

Ex. 10. Given $\frac{3x-1}{5}+3y-4=15$, and $\frac{3y-5}{6}+2x-8=7\frac{2}{3}$, to find the values of x and y.

Ans. $x=7$, and $y=5$.

Ex. 11. Given $9x+\frac{8y}{5}=70$, and $7y-\frac{13x}{3}=44$, to find the values of x and y.

Ans. $x=6$, and $y=10$.

Ex. 12. Given $\frac{7+x}{5}-\frac{2x-y}{4}=3y-5$,

and $\frac{5y-7}{2}+\frac{4x-3}{6}=18-5x$,

to find the values of x and y. Ans. $x=3$, and $y=2$.

Ex. 13. Given $x+1-\frac{3y+4x}{7}=7-\frac{9y+33}{14}$,

and $y-3-\frac{5x-4y}{2}=x-\frac{11y-19}{4}$,

to find the values of x and y. Ans. $x=6$, and $y=5$.

Ex. 14. Given $4x+\frac{15-x}{4}=2y+5+\frac{7x+11}{16}$,

and $3y-\frac{2x+y}{5}=2x+\frac{2y+4}{3}$,

to find the values of x and y. Ans. $x=3$, and $y=4$.

Ex. 15. Given $x-\frac{3x+5y}{17}+17=5y+\frac{4x+7}{3}$,

and $\frac{22-6y}{3}-\frac{5x-7}{11}=\frac{x+1}{6}-\frac{8y+5}{18}$,

to find the values of x and y. Ans. $x=8$, and $y=2$.

Ex. 16. Given $\frac{7x-21}{6}+\frac{3y-x}{3}=4+\frac{3x-19}{2}$,

and $\frac{2x+y}{2}-\frac{9x-7}{8}=\frac{3y+9}{4}-\frac{4x+5y}{16}$,

to find the values of x and y. Ans. $x=9$, and $y=4$.

Ex. 17. Given $\frac{7x+6}{11}+\frac{4y-9}{3}=3x-\frac{13-x}{2}-\frac{3y-x}{5}$, and $3x+4 : 2y-3 :: 5 : 3$, to find the values of x and y.

Ans. $x=7$, and $y=9$.

Ex. 18. Given $\frac{5x+13}{2}-\frac{8y-3x-5}{6}=9+\frac{7x-3y+1}{3}$, and $\frac{x+7}{3}:\frac{3y-8}{4}+4x::4:21$, to find the values of x and y.

Ans. $x=5$, and $y=4$.

Ex. 19. Given $\frac{3x+4y+3}{10}-\frac{2x+7-y}{15}=5+\frac{y-8}{5}$, and $\frac{9y+5x-8}{12}-\frac{x+y}{4}=\frac{7x+6}{11}$, to find the values of x and y.

Ans. $x=7$, and $y=9$.

Ex. 20. Given $3x-2y=15$, and $y+10:x-15::7:3$, to find the values of x and y.

Ans. $x=45$, and $y=60$.

Ex. 21. Given $x+150:y-50::3:2$, and $x-50:y+100::5:9$, to find the values of x and y.

Ans. $x=300$, and $y=350$.

Ex. 22. Given $(x+5).(y+7)=(x+1)(y-9)+112$, and $2x+10=3y+1$, to find the values of x and y.

Ans. $x=3$, and $y=5$.

Ex. 23. Given $3x+6y+1=\frac{6x^2+130-24y^2}{2x-4y+3}$,

and $3x-\frac{151-16x}{4y-1}=\frac{9xy-110}{3y-4}$,

to find the values of x and y. Ans. $x=9$, and $y=2$.

Ex. 24. Given $16x+6y-1=\frac{128x^2-18y^2+217}{8x-3y+2}$,

and $\frac{10x+10y-35}{2x+2y+3}=5-\frac{54}{3x+2y-1}$,

to find the values of x and y. Ans. $x=6$, and $y=5$.

§ III. RESOLUTION OF SIMPLE EQUATIONS,

Involving three or more unknown Quantities.

202. When there are three independent simple equations involving three unknown quantities.

RULE.

From *two* of the equations, find a third, which involves only two of the unknown quantities, by any of the rules in the preceding Section; and in like manner from the remaining equation, and one of the *others*, another equation which contains the same two unknown quantities may be deduced. Having therefore two equations, which involve only two unknown quantities, these may be determined; and, by substituting their values in any of the original equations, that of the third quantity will be obtained.

203. If there be four unknown quantities, their values may be found from four independent equations. For from the four given equations, by the rules in the last Section, three may be deduced which involve only three unknown quantities, the values of which may be found by the last Article; and hence the fourth may be found by substituting in any of the four given equations, the values of the three quantities determined.

If there be n unknown quantities, and n independent equations, the values of those quantities may be found in a similar manner. For from the n given equations, $n-1$ may be deduced, involving only $n-1$ unknown quantities; and from these $n-1$, $n-2$ may be obtained, involving only $n-2$ unknown quantities; and so on, till only one equation remains, involving one unknown quantity; which being found, the values of all the rest may be determined by substitution.

Ex. 1. Given $\left.\begin{aligned} x+y+z&=29, \\ x+2y+3z&=62, \\ \frac{x}{2}+\frac{y}{3}+\frac{z}{4}&=10, \end{aligned}\right\}$ to find the values of x, y, and z.

Subtracting the first equation from the second,

$$y+2z=33 \quad . \quad . \quad . \quad (A).$$

Multiplying the third equation by 12, the least common multiple of 2, 3, and 4,

$$6x+4y+3z=120$$

multiplying the 1st equation by 6, $\quad 6x+6y+6z=174$;

$\therefore$ by subtraction, $2y+3z=54$;

but, multiplying equation (A) by 2, $\quad 2y+4z=66$;

$\therefore$ by subtraction, $z=12$.

From equation (A), by transposition, $y=33-2z$;

$\therefore$ by substitution, $y=33-24$, or $y=9$.

From the first equation, by transposition,

$$x=29-y-z;$$

$\therefore$ by substitution, $x=29-9-12$,

and $x=29-21$, $\therefore x=8$.

In like manner, had the first equation been multiplied by 2, and subtracted from the second, an equation would have resulted, involving only x and z; and had it been multiplied by 4, and subtracted from the third when cleared of fractions, another equation would have been obtained, involving also x and z; whence by the preceding rules, the values of x and z could be found, and consequently the value of y also, by substitution.

Or if the first equation be multiplied by 3, and the second subtracted from it, an equation would arise, involving only x and y; and if the first when multiplied by 3, be subtracted from the third when cleared of fractions, another would arise involving only x and y; whence the values of x and y might be determined. And hence the third, that of z, might be found.

SECOND METHOD.

From the first equation, $x=29-y-z$;
substituting this value of x in the second equation,

$$29-y-z+2y+3z=62;$$

$\therefore$ by transposition, $y=33-2z$.

Also substituting, in the third equation, the value of x found from the first,

$$\frac{29-y-z}{2}+\frac{y}{3}+\frac{z}{4}=10;$$

multiplying this equation by 12, the least common multiple of 2, 3, and 4,

$$174-6y-6z+4y+3z=120,$$

and by transposition, $2y+3z=54$;

in which, substituting the value of y found above,

$$2(33-2z)+3z=54;$$

or $66-4z+3z=54$;

$\therefore$ by transposition, $z=12$;

whence $y=33-2z=33-24=9$,

and $x=29-y-z=29-9-12=8$.

It may be observed, that there will be the same variety of solution, as in the last case, according as x, y, or z, is exterminated.

THIRD METHOD.

The values of x, found in each of the equations, being compared, will furnish two equations each involving only y and z; from which the values of y and z may be deduced by any of the rules in the preceding Section, and hence, the value of x can be readily ascertained.

The same observation applies to this method of solution, as did to the last.

In some particular equations, two unknown quantities may be eliminated at once.

Ex. 2. Given $\left.\begin{array}{l} x+y+z=31 \\ x+y-z=25 \\ x-y-z=9 \end{array}\right\}$ to find the values of x, y, & z.

Adding the first and third equations, $2x=40$;

$$\therefore x=20.$$

Subtracting the second from the first, $2z=6$;

$$\therefore z=3;$$

and subtracting the third from the second,

$$2y=16; \quad \therefore y=8.$$

Ex. 3. Given $\left\{\begin{array}{l} x-y=2, \\ x-z=3, \\ y-z=1, \end{array}\right\}$ to find x, y, and z.

Here subtracting the first equation from the second, we have $y-z=1$; which is identically the third.

Therefore, the third equation furnishes no new condition; but what is already contained in the other two; and, consequently, the proposed equations are indeterminate; or, what is the same, we may obtain an infinite number of values which will satisfy the conditions proposed.

204. It is proper to remark, that in particular cases, Analysts make use of various other methods besides those pointed out in the practical rules; in the resolution of equations, which greatly facilitate the calculation, and by means of which, some equations of a degree superior to the first, may be easily resolved, after the same manner as simple equations.

We shall illustrate a few of those artifices by the following examples.

Ex. 4 Given $\left.\begin{array}{l} \frac{1}{x}+\frac{1}{y}=\frac{1}{8}, \\ \frac{1}{x}+\frac{1}{z}=\frac{1}{9}, \\ \text{and } \frac{1}{y}+\frac{1}{z}=\frac{1}{10}, \end{array}\right\}$ to find the values of x, y, and z.

By adding the three equations, we shall have

$$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{1}{8}+\frac{1}{9}+\frac{1}{10}=\frac{121}{360}.$$

Or, dividing by 2,

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{121}{720}.$$

From this subtracting each of the three first equations, and we shall have

$$\frac{1}{z}=\frac{31}{720}, \text{ or } z=\frac{720}{31}; \therefore z=23\frac{7}{31};$$

$$\frac{1}{y}=\frac{41}{720}, \text{ or } y=\frac{720}{41}; \therefore y=17\frac{23}{41};$$

$$\frac{1}{x}=\frac{49}{720}, \text{ or } x=\frac{720}{49}; \therefore x=14\frac{34}{49}.$$

Ex. 5. Given $2x=y+z+u$, $3y=x+z+u$, $4z=x+y+u$, and $u=x-14$, to find the values of x, y, z, and u.

By adding x to each member of the first equation, y to the second, and z to the third, we shall get

$$x+y+z+u=3x=4y=5z;$$

and from thence, $z=\frac{3x}{5}$, and $y=\frac{3x}{4}$;

which values being substituted in the first equation, we have

$$2x=\frac{3x}{4}+\frac{3x}{5}+u; \therefore u=\frac{13x}{20};$$

but, by the fourth equation, $u=x-14$;

$$\therefore x-14=\frac{13x}{20}, \text{ or } 20x-280=13x;$$

whence $x=40$: consequently $y=\frac{3x}{4}=30$, $z=24$, and $u=x-14=26$.

Ex. 6. Given $4x-4y-4z=24$, $6y-2x-2z=24$, and $7z-y-x=24$, to find the values of x, y, and z.

By putting $x+y+z=S$, the proposed equations become

$$8x-4S=24, 8y-2S=24, 8z-S=24;$$

$$\therefore x=3+\tfrac{1}{2}S, y=3+\tfrac{1}{4}S, z=3+\tfrac{1}{8}S.$$

By adding these three equations, we have

$$x+y+z=9+\tfrac{7}{8}S; \text{ whence } S=72.$$

Substituting this value for S, in x, y, and z, we shall find

$x=39$, $y=21$, and $z=12$.

Ex. 7. Given $\left.\begin{array}{r} x+y+z=90, \\ 2x+40=3y+20, \\ \text{and } 2x-4z+40=10, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=35$, $y=30$, and $z=25$.

Ex. 8. Given $\left.\begin{array}{r} x+a=y+z, \\ y+a=2x+2z, \\ \text{and } z+a=3x+3y, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=\frac{a}{11}$, $y=\frac{5a}{11}$, and $z=\frac{7a}{11}$.

Ex. 9. It is required to find the values of x, y, and z, in the following equations;

$x+y=13$, $x+z=14$, and $y+z=15$.

Ans. $x=6$, $y=7$, and $z=8$.

Ex. 10. In the following it is required to find the values of x, y, and z.

$$\left.\begin{array}{l} \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=124, \\ \frac{x}{3}+\frac{y}{4}+\frac{z}{5}=94, \\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}=76, \end{array}\right\} \qquad \left\{\begin{array}{l} x=48, \\ y=120, \\ z=240. \end{array}\right.$$

Ex. 11. Given $\left.\begin{array}{r} x+y+z=26, \\ x-y\quad=4, \\ \text{and } x-z\quad=6, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=12$, $y=8$, and $z=6$.

Ex. 12. Given $\left.\begin{array}{r} x+y+z=9, \\ x+2y+3z=16, \\ \text{and } x+y-2z=3, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=4$, $y=3$, and $z=2$.

Ex. 13. Given $\left.\begin{array}{r} x+y+z=12, \\ x+2y+3z=20, \\ \text{and } \frac{1}{3}x+\frac{1}{2}y+z=6, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=6$, $y=4$, and $z=2$.

Ex. 14. Given $x+y-z=8$, $x+z-y=9$, and $y+z-x=10$; to find the values of x, y, and z.

Ans. $x=8\frac{1}{2}$, $y=9$, and $z=9\frac{1}{2}$.

Ex. 15. Given $x+\frac{1}{2}y=100$, $y+\frac{1}{3}z=100$, and $z+\frac{1}{4}x=100$; to find the values of x, y, and z.

Ans. $x=64$, $y=72$, and $z=84$.

Ex. 16. Given $x+\frac{1}{2}y=357$, $y+\frac{1}{3}z=476$, $z+\frac{1}{4}u=595$, and $u+\frac{1}{5}x=714$; to find the values of x, y, z, and u.

Ans. $x=190$, $y=334$, $z=426$, and $u=676$.

§ IV. SOLUTION OF PROBLEMS PRODUCING SIMPLE EQUATIONS,

Involving more than one unknown Quantity.

205. The usual method of solving determinate problems of the first degree, is, to assume as many unknown letters, namely, x, y, z, &c., as there are unknown numbers to be found; then, having properly examined the meaning and conditions of the problem, translate the several conditions into as many distinct algebraic equations; and, finally, by the resolution of these equations according to the rules laid down in Chapter IV, the quantities sought will be determined. It is proper to observe that, in certain cases, other methods of proceeding may be used, which practice and observation alone can suggest.

PROBLEM I.

There are two numbers, such, that three times the greater added to one-third the lesser is equal 36; and if twice the greater be subtracted from 6 times the lesser, and the remainder divided by 8, the quotient will be 4. What are the numbers?

Let x designate the *greater* number, and y the *lesser* number.

Then $3x+\frac{y}{3}=36$, and $\frac{6y-2x}{8}=4$; $\therefore \begin{cases} 9x+y=108 & (A), \\ 6y-2x=32 & (B); \end{cases}$

Multiplying equation (A) by 6, $6y+54x=648$;
but $6y-2x=32$;

$\therefore$ by subtraction, $56x=616$,
and by division, $x=11$.

From equation (A), $y=108-9x$;
$\therefore$ by substitution, $y=108-99$, or $y=9$.

PROB. 2. After A had won four shillings of B, he had only half as many shillings as B had left. But had B won six shillings of A, then he would have three times as many as A would have had left. How many had each ?

Let $x=$ designate the number of shillings A had, and $y=$ the number B had;

$$\text{then } y-4=2x+8,$$
$$\text{and } y+6=3x-18;$$

$\therefore$ by subtraction, $10=x-26$,
and by transposition, $36=x$, or $x=36$;
by substitution, $y+6=3\times36-18$;
and by transposition, $y=84$;
$\therefore$ A had 36, and B 84.

PROB. 3. What fraction is that, to the numerator of which if 4 be added, the value is one-half, but if 7 be added to the denominator, its value is one-fifth ?

Let $x=$ its numerator, $y=$ denominator, then the fraction $\frac{x}{y}$.

Add 4 to the numerator, then $\frac{x+4}{y}=\frac{1}{2}$, $\therefore 2x+8=y$;

Add 7 to the denominator, then $\frac{x}{y+7}=\frac{1}{5}$, $\therefore 5x=y+7$;

by subtraction, $3x-8=7$;
by transposition, $3x=15$; $\therefore x=5$;
and $y=2x+8$; $\therefore$ by substitution, $y=10+8=18$,
and the fraction is $\frac{5}{18}$.

PROB. 4. A and B have certain sums of money, says A to B, give me $15l$ of your money, and I shall have 5 times as much as you have left: says B to A, give me $5l$ of your money, and I shall have exactly as much as you will have left. What sum of money had each ?

Let $x=$ A's money, $y=$ B's, then $x+15=$ what A would have, after receiving $15l$ from B.
$y-15=$ what B would have left.
Again, $y+5=$ what B would have after receiving $5l$ from A.
$x-5=$ what A would have left.
Hence, by the problem, $x+15=5\times(y-15)=5y-75$, and $y+5=x-5$.

by transposition, $5y-x=90$,
and $y-x=-10$;

$\therefore$ by subtraction, $4y=100$,
and by division, $y=25$ B's money.
From the second equation, $x=y+10$;
$\therefore$ by substitution, $x=25+10=35$ A's money.

PROB. 6. A person was desirous of relieving a certain number of beggars by giving them 2s. 6d. each, but found that he had not money enough in his pocket by 3 shillings; he then gave them 2 shillings each, and had four shillings to spare. What money had he in his pocket; and how many beggars did he relieve?

Let $x=$ money in his pocket (*in shillings*);
$y=$ the number of beggars.

Then $2\frac{1}{2}\times y$, or $\frac{5y}{2}=$ number of *shillings* which would have been given at 2s. 6d. each;
and $2\times y$, or $2y=$ . . . . at 2s. each.

Hence, by the problem, $\frac{5y}{2}=x+3$ (A),
and $2y=x-4$ (B).

$\therefore$ by subtraction, $\frac{y}{2}=7$,
or $y=14$, the number of beggars.
From equation (B), $x=2y+4=2\times 14+4$, by substitution,
$\therefore x=32$, the shillings in his pocket.

PROB. 6. There is a certain number, consisting of two digits. The *sum* of those digits is 5; and if 9 be added to the number itself, the digits will be inverted. What is the number?

Here it may be observed, that every number consisting of two digits is equal to 10 times the digit in the tens place, plus that in the units; thus, $24=2\times 10+4=20+4$.

Let $x=$ digit in the *units place*;
$y=$ that in the *tens*.
Then $10x+y=$ the number itself.
and $10y+x=$ the number with its digits *inverted*.
Hence, by the problem, $x+y=5$ (A),
and $10x+y+9=10y+x$, or by transposition, $9x-9y=-9$;
$\therefore$ by division, $x-y=-1$ (B).
Subtracting equation (B) from (A), $2y=6$;

$\therefore y=3$, and $x=5-y=5-3=2$;

$\therefore$ the number is $(10x+y)=23$.

Add 9 to this number, and it becomes 32, which is the number with the *digits inverted.*

PROB. 7. A sum of money was divided equally amongst a certain number of persons; had there been four more, each would have received one shilling less, and had there been four fewer, each would have received two shillings more than he did: required the number of persons, and what each received.

Let x designate the number of persons,

y the sum each received in shilllings;

then xy is the sum divided;

$\therefore (x+4)\times(y-1)=xy,$
and $(x-4)\times(y+2)=xy,$ } by the question;

$\therefore xy+4y-x-4=xy$, or $4y-x=4$,

and $xy-4y+2x-8=xy$, or $-4y+2x=8$;

$\therefore$ by addition, $x=12$;

and $4y=4+x=4+12$; $\therefore y=4$.

PROB. 8. A man, his wife, and son's years make 96, of which the father and son's equal the wife's and 15 years over, and the wife and son's equal the man's and two years over. What was the age of each?

Suppose x, y, and z = their respective ages.

1st condition $x+y+z=96$,
2nd . . . $x+z=y+15$,
3d . . . $y+z=x+2$, } by the problem.

Subtracting the 2nd from the 1st, $y=96-y-15$;

$\therefore 2y=81$, and $y=40\frac{1}{2}$ by division.

Subtracting the 3d from the 1st, $x=96-x-2$;

$\therefore$ by transposition and division, $x=47$,

And from the 1st, $z=96-y-x$; $\therefore z=8\frac{1}{2}$.

And their ages are 47, $40\frac{1}{2}$, and $8\frac{1}{2}$ respectively.

PROB. 9. A labourer working for a gentleman during 12 days, and having had with him, the first seven days, his wife and son, received 74 shillings; he wrought afterwards 8 other days, during 5 of which he had with him his wife and son, and he received 50 shillings. Required the gain of the labourer *per* day, and also that of his wife and son.

Let $x=$ the daily gain of the husband,

$y=$ that of the wife and son;

12 days work of the husband would produce $12x$,

7 of the wife and son would be $7y$;

$\therefore$ by the first condition, $12x+7y=74$;
and by the second, $8x+5y=50$;
Multiplying the 1st equation by 2, $24x+14y=148$;
2nd . . by 3, $24x+15y=150$;

$\therefore$ by subtraction, $y=2$.
And from the 2nd, $8x=50-5y=50-10$;
$\therefore$ by division $x=5$.

Consequently the husband would have gained alone 5s. *per* day, and the wife and son 2 shillings in the same time.

206. Let us now suppose that the first sum received by the workman was 46s, and the second 30s, the other circumstances remaining the same as before;

The equations of the question would be

$$12x+7y=46, \text{ and } 8x+5y=30.$$

From whence we find, by proceeding as above,

$$x=5, \text{ and } y=-2.$$

By putting in the place of x its value 5, in the above equations, they become

$$60+7y=46, \text{ and } 40+5y=30.$$

The inspection alone of these equations show an absurdity. In fact, it is impossible to form 46 by adding an absolute number to 60, which is already greater than it, and in like manner it is impossible to form 30 by adding an absolute number to 40.

Consequently what we attributed as a gain to the labour of the wife and son, must be an expense to the husband, which is also verified by the result $y=-2$.

207. The negative value of y makes known therefore a rectification in the enunciation of the problem; since that, instead of adding $7y$ to $12x$ in the first equation, and $5y$ to $8x$ in the second, y being considered a positive or an absolute number, we must subtract them in order to have the sum given for the common wages of these three persons; or, what is the same thing, if, in place of considering the money attributed to the wife and son as a gain, we would regard it as an expense made by them to the charge of the workman; then we must subtract this money from what the man would have gained alone, and there would be no contradiction in the equations, since they would become

$$60-7y=46, \text{ and } 40-5y=30;$$

from either of which we would derive $y=2$; and we should therefore conclude that if the workman gained 5s. *per* day, his wife and son's expense is 2s., which can be otherwise verified thus:

For 12 days work, he receives 5×12 or 60s.; the expense of his wife and son for 7 days, is 2×7 or 14s.; and there remains 46 shillings.

Again, he receives for 8 days work 5×8 or 40s.; the expense of his wife and son during 5 days, is 2×5 or 10s.; therefore his clear gain is 30 shillings.

208. It is very evident that, in place of the enunciation of (Prob. 9), we must substitute the following, in order that the problem proposed may be possible, with the above given quantities:

A labourer working for a gentleman during 12 *days, having had with him, the first* 7 *days, his wife and son, who occasion an expense to him, received* 46 *shillings; he has wrought, afterwards, for* 8 *other days, on* 5 *of which he had with him his wife and son, whose expenses he must still defray, and he received* 30 *shillings. Required the salary of the workman per day, and also the expense of his wife and son in the same time.*

Designating by x the daily wages of the workman, and by y the expense of his wife and son, for the same time; the equations of the problem shall be

$$12x-7y=46, \text{ and } 8x-5y=30;$$

which, being resolved, will give

$$x=5\text{s.}, \text{ and } y=2\text{s.}$$

209. Although negative values do not answer the enunciation of a concrete question, as has been observed (Art. 174), yet they satisfy the equations of the problem, as may be readily verified, by substituting 5 for x, and -2 for y, in the equations (Art. 206), since they would then become identically equal.

Prob. 10. Two pipes, the water flowing in each uniformly, filled a cistern containing 330 gallons, the one running during 5 hours, and the other during 4; the same two pipes, the first running during two hours, and the second three, filled another cistern containing 195 gallons. The discharge of each pipe is required.

Let x represent the discharge of the first in an hour; y that of the second in the same time.

And in order to have a general solution, put $a=5$, $b=4$, $c=330$, $a'=2$, $b'=3$, $c'=195$; then by the conditions of the problem we shall have these two equations,

$$ax+by=c, \text{ and } a'x+b'y=c';$$

which, being resolved (Art. 190), will give

$$x=\frac{b'c-bc'}{ab'-a'b}, \text{ and } y=\frac{ac'-a'c}{ab'-a'b}.$$

Now, by restoring the values of a, b, c, &c., we have

$$x=\frac{990-780}{15-8}=\frac{210}{7}=30;$$

$$\text{and } y=\frac{975-660}{15-8}=45.$$

Thus, the first pipe discharges 30 gallons *per* hour, and the second 45.

210. Let us now suppose that the first pipe running during 3 hours, and the second during 7, filled a cistern containing 190 gallons; that afterwards, the first running 4 hours, and the second 6, filled a cistern containing 120 gallons.

In this case, $a=3$, $b=7$, $c=190$, $a'=4$, $b'=6$, $c'=120$; and, consequently, $b'c-bc'=1140-840=300$, $ab'-a'b=18-28=-10$, $ac'-a'c=360-760=-400$, which will give $x=-30$, and $y=40$.

In order to understand the meaning of these results, we must return again to the conditions of the problem, or, what amounts to the same, we must try how these values of x and y satisfy the equations of the problem:

Thus, if we substitute -30 for x, and 40 for y, in the equations $3x+7y=190$ and $4x+6y=120$, resulting from the above problem, we find first, that $3x=-90$, and $7y=280$, consequently $3x+7y=-90+280$, which in effect is equal to 190. In like manner $4x+6y$ is found to be $-120+240$, which is equal to 120.

Having, therefore, discovered how the values -30 and $+40$ of x and y answer the equations $3x+7y=190$ and $4x+6y=120$, we perceive at the same time how they would answer the conditions of the problem; for since the use that has been made of the quantities $3x$ and $4x$, which express the quantities of water discharged by the first pipe in the first and second operation, was to subtract them from $7y$ and from $6y$, which express the quantities furnished in the same operations by the second pipe. The first pipe must be considered in this case as depriving the cisterns of water instead of furnishing any, as it did in the preceding problem, and as it was supposed in expressing the conditions of this problem.

211. Hence, in almost every question solved after a general manner, we may always conclude that when the value of the unknown quantity becomes negative, the quantity expressed by it should be considered as being of an opposite kind from what it was supposed in expressing the conditions of the problem.

What has been said with respect to unknown quantities, is equally applicable to known quantities, that is, when a general solution is applied to any particular case, if any of the given quantities a, b, c, &c. in the problem, are negative.

212. Let it be proposed, for example, to find what should be, in the foregoing problem, the discharges of two pipes, that the first furnishing water during 3 hours, and the second 4, may fill a cistern containing 320 gallons, and that the second pipe afterwards furnishing water during 6 hours, whilst the first discharges it during 3 hours, may fill a cistern containing 180 gallons.

We have only to put in the general solution (Art. 209), $a=3$, $b=4$, $c=320$, $a'=-3$, $b'=6$, $c'=180$, and there will result $x=40$, and $y=50$.

From whence it appears that the discharge of the first pipe is 40 gallons per hour, either to carry away the water as in the second operation, or to furnish it as in the first, and the discharge of the second, 50 gallons an hour, which it furnishes in both operations.

Prob. 11. A certain sum of money put out to interest, amounts in 8 months to 297*l.* 12*s.*; and in 15 months its amount is 306*l.* at simple interest. What is the sum and the rate per cent? Ans. 288*l.* at 5 per cent.

Prob. 12. There is a number consisting of two digits, the second of which is greater than the first, and if the number be divided by the sum of its digits, the quotient is 4; but if the digits be inverted, and that number divided by a number greater by 2 than the difference of the digits, the quotient becomes 14. Required the number. Ans. 48.

Prob. 13. What fraction is that, whose numerator being doubled, and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$? Ans. $\frac{4}{5}$.

Prob. 14. A farmer parting with his stock, sells to one person 9 horses and 7 cows for 300 dollars: and to another, at the same prices, 6 horses and 13 cows for the same sum. What was the price of each?

Ans. the price of a cow was 12 dollars, and of a horse 24 dollars.

Prob. 15. A Vintner has two casks of wine, from the greater of which he draws 15 gallons, and from the less 11; and finds the quantities remaining in the proportion of 8 to 3. After they became half empty, he puts 10 gallons of water into each, and finds that the quantities of liquor now in them are as 9 to 5. How many gallons will each hold?

Ans. the larger 79, and the smaller 35 gallons.

Prob. 16. A person having laid out a rectangular bowling-green, observed that if each side had been 4 yards longer, the adjacent sides would have been in the ratio of 5 to 4; but if each had been 4 yards shorter, the ratio would have been 4 to 3. What are the lengths of the sides?

Ans. 36, and 28 yards.

Prob. 17. A sets out express from C towards D, and three hours afterwards B sets out from D towards C, travelling 2 miles an hour more than A. When they meet it appears that the distances they have travelled are in the proportion of 13 to 15; but had A travelled five hours less, and B had gone 2 miles an hour more, they would have been in the proportion of 2 : 5. How many miles did each go *per* hour, and how many hours did they travel before they met?

Ans. A went 4, and B 6 miles an hour, and they travelled 10 hours after B set out.

Prob. 18. A Farmer hires a farm for 245*l*. *per annum*, the arable land being valued at 2*l*. an acre, and the pasture at 28 shillings: now the number of acres of arable is to half the excess of the arable above the pasture as 28 : 9. How many acres were there of each?

Ans. 98 acres of arable, and 35 of pasture.

Prob. 19. A and B playing at backgammon, A bets 3s. to 2s. on every game, and after a certain number of games found that he had lost 17 shillings. Now had A won 3 more from B, the number he would then have won, would be to the number B had won, as 5 to 4. How many games did they play?

Ans. 9.

Prob. 20. Two persons, A and B, can perform a piece of work in 16 days. They work together for 4 days, when A being called off, B is left to finish it, which he does in 36 days more. In what time would each do it separately?

Ans. A in 24 days, and B in 48 days.

Prob. 21. Some hours after a courier had been sent from A to B, which are 147 miles distant, a second was sent, who wished to overtake him just as he entered B; in order to which he found he must perform the journey in 28 hours less than the first did. Now the time in which the first travels 17 miles added to the time in which the second travels 56 miles, is 13 hours and 40 minutes. How many miles does each go *per* hour?

Ans. the first goes 3, and the second 7 miles an hour.

Prob. 22. Two loaded wagons were weighed, and their weights were found to be in the ratio of 4 to 5. Parts of their loads, which were in the proportion of 6 to 7, being taken out, their weights were then found to be in the ratio of 2 to 3; and the sum of their weights was then ten tons. What were the weights at first? Ans. 16, and 20 tons.

PROB. 23. A and B severally cut packs of cards; so as to cut off less than they left. Now the number of cards left by A added to the number cut off by B, make 50; also the number of cards left by both exceed the number cut off, by 64. How many did each cut off? Ans. A cut off 11, and B 9.

PROB. 24. A and B speculate with different sums; A gains 150*l*, B loses 50*l*, and now A's stock is to B's as 3 to 2. But had A lost 50*l*, and B gained 100*l*, then A's stock would have been to B's as 5 to 9. What was the stock of each?

Ans. A's was 300*l*, and B's 350*l*.

PROB. 25. A Vintner bought 6 dozen of port wine and 3 dozen of white, for 12*l*. 12*s*.; but the price of each afterwards falling a shilling *per* bottle, he had 20 bottles of port, and 3 dozen and 8 bottles of white more, for the same sum. What was the price of each at first?

Ans. the price of port was 2s. and of white 3s. *per* bottle.

PROB. 26. Find two numbers, in the proportion of 5 to 7, to which two other required numbers in the proportion of 3 to 5 being respectively added, the sums shall be in the proportion of 9 to 13: and the difference of those sums =16.

Ans. the two first numbers are 30 and 42; the two others, 6 and 10.

PROB. 27. A Merchant finds that if he mixes sherry and brandy in quantities which are in the proportion of 2 to 1, he can sell the mixture at 78s. *per* dozen; but if the proportion be as 7 to 2, he must sell it at 79 shillings a dozen. Required the price of each liquor.

Ans. the price of sherry was 81s., and of brandy 72s. *per* dozen.

PROB. 28. A number consisting of two digits when divided by 4, gives a certain quotient and a remainder of 3; when divided by 9 gives another quotient and a remainder of 8. Now the *value* of the digit on the left-hand is equal the quotient which was got when the number was divided by 9; and the other digit is equal $\frac{1}{17}$th of the quotient got when the number was divided by 4. Required the number. Ans. 71.

PROB. 29. To find three numbers, such, that the *first* with $\frac{1}{2}$ the sum of the *second* and *third* shall be 120; the second with $\frac{1}{5}$th the difference of the *third* and *first* shall be 70; and $\frac{1}{2}$ the sum of the three numbers shall be 95.

Ans. 50, 65, and 75.

PROB. 30. There are two numbers, such, that $\frac{1}{2}$ the greater added to $\frac{1}{3}$ the lesser is 13; and if $\frac{1}{2}$ the lesser be taken from $\frac{1}{3}$ the greater, the remainder is nothing. What are the numbers?

Ans. 18, and 12.

Prob. 31. There is a certain number, to the sum of whose digits if you add 7, the result will be three times the left-hand digit; and if from the number itself you subtract 18, the digits will be *inverted*. What is the number? Ans. 53.

Prob. 32. A person has two horses, and a saddle worth 10*l*; if the saddle be put on the *first* horse, his value becomes *double* that of the second; but if the saddle be put on the *second* horse, *his* value will not amount to that of the *first* horse by 13*l*. What is the value of each horse?

Ans. 56 and 33.

Prob. 33. A gentleman being asked the age of his two sons, answered, that if to the sum of their ages 18 be added, the result will be double the age of the elder; but if 6 be taken from the difference of their ages, the remainder will be equal to the age of the younger. What then were their ages?

Ans. 30 and 12.

Prob. 34. To find four numbers, such, that the sum of the 1st, 2d, and 3d, shall be 13; the sum of the 1st, 2d, and 4th, 15; the sum of the 1st, 3d, and 4th, 18; and lastly the sum of the 2d, 3d, and 4th, 20. Ans. 2, 4, 7, 9.

Prob. 35. A son asked his father how old he was. His father answered him thus. If you take away 5 from my years, and divide the remainder by 8, the quotient will be $\frac{1}{3}$ of your age; but if you add 2 to your age, and multiply the whole by 3, and then subtract 7 from the product, you will have the number of the years of my age. What was the age of the father and son? Ans. 53, and 18.

Prob. 36. Two persons, A and B, had a mind to purchase a house rated at 1200 dollars; says A to B, if you give me $\frac{2}{3}$ of your money, I can purchase the house alone; but says B to A, if you will give me $\frac{3}{4}$th of yours, I shall be able to purchase the house. How much money had each of them?

Ans. A had 800 and B 600 dollars.

Prob. 37. There is a cistern into which water is admitted by three cocks, two of which are exactly of the same dimensions. When they are all open, five-twelfths of the cistern is filled in 4 hours; and if one of the equal cocks be stopped, seven-ninths of the cistern is filled in 10 hours and 40 minutes. In how many hours would each cock fill the cistern?

Ans. Each of the equal ones in 32 hours, and the other in 24.

Prob. 38. Two shepherds, A and B, are intrusted with the charge of two flocks of sheep. A's consisting chiefly of ewes, many of which produced lambs, is at the end of the year increased by 80 ; but B finds his stock diminished by 20 : when their numbers are in the proportion of 8 : 3. Now had A lost 20 of his sheep, and B had an increase of 90, the numbers would have been in the proportion of 7 to 10. What were the numbers ? Ans. A's 160, and B's 110.

Prob. 39. At an election for two members of congress, three men offer themselves as candidates ; the number of voters for the two successful ones are in the ratio of 9 to 8 ; and if the first had had 7 more, his majority over the second would have been to the majority of the second over the third as 12 : 7. Now if the first and third had formed a coalition, and had one more voter, they would each have succeeded by a majority of 7. How many voted for each ?

Ans. 369, 328, and 300, respectively.

CHAPTER VI.

ON

INVOLUTION AND EVOLUTION

OF NUMBERS, AND OF ALGEBRAIC QUANTITIES.

213. *The powers of any quantity, are the successive products, arising from unity, continually multiplied by that quantity.* Or, the power of the order m of a quantity, m being a whole positive number, is the product of that quantity continually multiplied $m-1$ times into itself, or till the number of factors amounts to the number of units in that given power.

214. Involution is the method of raising any quantity to a given power ; Evolution, or the extraction of roots, being just the reverse of Involution, is the method of determining a quantity which, raised to a proposed power, will produce a given quantity.

Note.—The term root has been already defined, (Art. 12).

§ I. INVOLUTION OF ALGEBRAIC QUANTITIES.

215. It has been observed, (Art. 13), that the powers of algebraic quantities are expressed by placing the *index* or *exponent* of the power over the quantity.

Hence, *if a proposed root be a single letter, and without a coefficient, any required power of it will be expressed by the same letter with the index of the power written over it.* Thus, the nth power of a is $=a^n$, n being any positive number whatever.

216. *If the proposed root be itself a power, the required power will be obtained by multiplying the index of the given power into that of the required power.* Thus the mth power of a^p, or $(a^p)^m=a^{mp}$; for since, (Art. 213), $(a^p)^m=a^p\times a^p\times a^p$, &c. $= a^{p+p+p+\text{etc.}}=a^{pm}$. . . . . (1)
where the number of factors a^p is equal to m.

217. Also, *if a simple quantity be composed of several factors, it can be raised to any power by multiplying the index of every factor in the quantity by the exponent of the power.* Thus the mth power of $(a^rb^qc^r)$, or $(a^rb^qc^r)^m$ is $= a^{pm}b^{qm}c^{rm}$; for since (Art. 274), $(a^pb^qc^r)^m = (a^pb^qc^r) \times (a^pb^qc^r)$, &c. $= a^pa^p \ldots b^qb^q \ldots c^rc^r \ldots =(a^p)^m\times(b^q)^m\times(c^r)^m$; . . . . (2);
by observing that in each of these products, such as a^pa^r, &c., or b^qb^q, &c., there enter m equal factors.

Cor. Hence, *if the proposed quantity has a numerical coefficient, it must also be involved to the required power.* Thus the fourth power of $3a^2b^2$ is $=3^4a^{2\cdot 4}b^{2\cdot 4}=3\times 3\times 3\times 3\times a^8b^8= 81\,a^8b^8$. For the numerical coefficient is in this case the same as any other factor.

ROOTS AND POWERS OF NUMBERS.

1st.	2d	3d.	4th.	5th.	6th.	7th.	Square Root.	Cube Root.
1	1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	1.414213	1.26
3	9	27	81	243	729	2187	1.732	1.442
4	16	64	256	1024	4096	16384	2.	1.587
5	25	125	625	3125	15625	78125	2.236	1.71
6	36	216	1296	7776	46656	279936	2.449	1.817
7	49	343	2401	16807	117649	823543	2.646	1.913
8	64	512	4096	32768	262144	2097152	2.828	2.
9	81	729	6561	59049	531441	4782969	3.	2.08

218. *Any power of a fraction is equal to the same power of the numerator divided by the like power of the denominator.*

Thus the mth power of $\frac{a}{b}$, or $\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m}$; for $\left(\frac{a}{b}\right)^m=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}$, &c. $=$ (Art. 156), $\frac{a\times a\times a,\text{ etc.}}{b\times b\times b,\text{ etc.}}=\frac{a^m}{b^m}$; where the number of factors $\frac{a}{b}$ is equal to m.

And in like manner the mth power of $\frac{a^p b^q}{c^n c^r}$, or $\left(\frac{a^p b^q}{c^n c^r}\right)^{mc}=$

$$\frac{(a^p)^m(b^q)^m}{(c^n)^m(d^r)^m}=\frac{a^{pm}b^{qm}}{c^{nm}d^{rm}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

219. *Any even power of a positive or negative quantity, is necessarily positive.* In fact, $2m$ being the formula of even numbers, we have $(\pm a)^{2m}=[(\pm a)^2]^m=(+a^2)^m=+a^{2m}$ . . . (4).

220. *Any odd power of a quantity will have the same sign as the quantity itself.* For, the general formula of odd numbers, (Art. 111), being $2m+1$, we have $(\pm a)^{2m+1}=(\pm a)^{2m}\times(\pm a)$ $=a^{2m}\times\pm a=\pm a^{2m+1}$ . . . . (5).

The involution of algebraic quantities is generally divided into two cases.

CASE I.

To involve a simple algebraic Quantity.

RULE.

221. Raise the coefficient, if any, to the required power, then multiply the index of each factor, or letter, by the index of the required power, and write their several products over their respective factors; Let the quantities thus arising be annexed to each other and to the same power of the coefficient, prefixing the power sign, and it will be the power required. Or, multiply the quantity into itself as many times less one as is denoted by the index of the power, and the last product, with the proper sign prefixed, will be the answer.

Ex. 1. Required the square, or second power of $2ab$.

Here, $(2ab)^2=4\times a^2\times b^2=4a^2b^2$. *Ans.*

Ex. 2. What is the cube of $-3a^2b^2$?

Here, $(-3a^2b^2)^3=$ (Art. 220), $-(3a^2b^2)^3=-27\times a^{2\cdot3}\times b^{2\cdot3}$ $=-27a^6b^6$. *Ans.*

Ex. 3. What is the 4th power of $-2a^3x^2$?

Here, $(-2a^3x^2)^4=$ (Art. 219), $+(2a^3x^2)^4=16\times a^{3\cdot4}x^{2\cdot4}=$ $16a^{12}x^3$. *Ans.*

Ex. 4. What is the cube, or third power of abc?

Here, $abc\times abc\times abc=a\times a\times a\times b\times b\times b\times c\times c\times c=$ $a^3b^3c^3$. *Ans.*

222. *When the quantity to be involved is a fraction*, raise both the numerator and denominator to the power proposed.

Ex. 5. Required the 4th power of $-\frac{b}{2a}$.

Here, $\left(-\frac{b}{2a}\right)^4=+\left(\frac{b}{2a}\right)^4=\frac{b}{2a}\times\frac{b}{2a}\times\frac{b}{2a}\times\frac{b}{2a}=\frac{b^4}{16a^4}$. Or $\left(-\frac{b}{2a}\right)^4=+\frac{b^4}{(2a)^4}=\frac{b^4}{2^4\times a^4}=\frac{b^4}{16a^4}$.

Ex. 6. What is the 4th power of $-\frac{2a}{3x}$? Ans. $\frac{16a^4}{81x^4}$.

Ex. 7. What is the 8th power of $2a^2$? Ans. $256a^{16}$.

Ex. 8. What is the 7th power of $-x$? Ans. $-x^7$.

Ex. 9. What is the 6th power of $-\frac{a^2}{x^3}$? Ans. $\frac{a^{12}}{x^{18}}$.

Ex. 10. What is the 5th power of $\frac{c}{5}$? Ans. $\frac{c^5}{3125}$.

Ex. 11. What is the 4th power of $\frac{5x}{7}$? Ans. $\frac{625x^4}{2401}$.

Ex. 12. Required the cube of $-\frac{2ax^2}{3b}$? Ans. $-\frac{8a^3x^6}{27b^3}$.

Ex. 13. Required the square of $\pm a^2b^2$? Ans. a^4b^4.

Ex. 14. Required the 9th power of $-xy$? Ans. $-x^9y^9$.

Ex. 15. Required the 0th power of xy? Ans. 1.

Ex. 16. Required the 4th power of a^{-2}?

Ans, a^{-8}, or $\frac{1}{a^8}$.

CASE II.

To involve a compound algebraic Quantity.

RULE I.

223. Multiply the given quantity continually into itself as many times minus one as is denoted by the index of the power, as in the multiplication of compound algebraic quantities (Art. 79), and the last product will be the power required.

Ex. 1. What is the square of $a+2b$?

$$\begin{array}{l} a+2b \\ a+2b \\ \hline a^2+2ab \\ \quad\ +2ab+4b^2 \\ \hline \text{Square } =a^2+4ab+4b^2 \end{array}$$

Ex. 2. What is the cube of a^2-x^2?

$$\begin{array}{l} a^2-x^2 \\ a^2-x^2 \\ \hline a^4-a^2x^2 \\ \quad -a^2x^2+x^4 \\ \hline a^4-2a^2x^2+x^4 \\ a^2-x^2 \\ \hline a^6-2a^4x^2+\ a^2x^4 \\ \quad -\ a^4x^2+2a^2x^4-x^6 \\ \hline \end{array}$$

$$\text{Cube } =a^6-3a^4x^2+3a^2x^4-x^6$$

Ex. 3. Required the *fourth* power of $a+3b$.
Ans. $a^4+12a^3b+54a^2b^2+108ab^3+81b^4$.

Ex. 4. Required the *square* of $3x^2+2x+5$.
Ans. $9x^4+12x^3+34x^2+20x+25$.

Ex. 5. Required the *cube* of $3x-5$.
Ans. $27x^3-135x^2+225x-125$.

Ex. 6. Required the *cube* of x^2-2x+1.
Ans. $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

Ex. 7. Required the *fourth power* of $2+3x$.
Ans. $16+96x+216x^2+216x^3+81x^4$.

Ex. 8. Required the *fifth power* of $1-2x$.
Ans. $1-10x+40x^2-80x^3+80x^4-32x^5$.

Ex. 9. Required the *square* of $a+b+c+d$.
Ans. $a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd)$.

224. In the involution of a binomial or residual quantity of the form $a+b$, or $a-b$; the several terms in each successive power are found to bear a certain relation to each other, and observe a certain law, which the following Table is intended to explain.

TABLE OF THE POWERS OF $a+b$.

Powers.	Mode of expressing them.	Powers expanded.
Square.	$(a+b)^2$.	$a^2+2ab+b^2$.
Cube.	$(a+b)^3$.	$a^3+3a^2b+3ab^2+b^3$.
4th power.	$(a+b)^4$.	$a^4+4a^3b+6a^2b^2+4ab^3+b^4$.
5th power.	$(a+b)^5$.	$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4$ $+b^5$.
6th power.	$(a+b)^6$.	$a^6+6a^5b+15a^4b^2+20a^3b^3$ $+15a^2b^4+6ab^5+b^6$.

The successive powers of $a-b$ are precisely the same as those of $a+b$, except that the signs of the terms will be alternately $+$ and $-$. Thus, the *fifth power* of $a-b$ is $a^5-5a^4b+10a^3b^2-10a^2b^2+5ab^4-b^5$.

225. In reviewing that column of the above Table which contains the powers of $a+b$ *expanded*, we may observe,

I. That in each case, the *first* term is raised to the *given power*, and the last term is b raised to the *same* power; thus, in the *square*, the *first* term is a^2, and the *last* b^2; in the *cube*, the *first* term is a^3, and the *last* b^3; and so on of the rest.

II. That, with respect to the intermediate terms, the powers of a *decrease*, and the powers of b *increase*, by unity in each successive term. Thus, in the fifth power, we have

In the *second* term, ... a^4b;
third, ... a^3b^2;
fourth, ... a^2b^3;
fifth,. .... $a\,b^4$;

and so on in other powers.

III. That in each case, the *coefficient of the second term* is the same with the *index of the given power*. Thus, in the square, it is 2; in the cube, it is 3; in the fourth power, it is 4; and so on of the rest.

IV. That if the *coefficient* of a in any term be multiplied by its *index*, and the product divided by the *number of terms* to *that place*, this *quotient* will give the *coefficient* of the *next term*. Thus, in the fifth power, the coefficient of a in the second *term* multiplied by *its index*, and divided by the number of terms to that place $=\frac{4\times5}{2}=\frac{20}{2}=10=$ coefficient of the *third term.*

In the sixth power, $\frac{\text{Coeff. of } a \text{ in the 4th term . its index}}{\text{number of terms to that place.}}=\frac{20\times3}{4}=\frac{60}{4}=15=$ coefficient of the *fifth term.*

Hence, we are furnished with the following general rule for raising a binomial or residual quantity to any power, without the process of actual multiplication.

RULE II.

226. Find the terms without the coefficients, by observing that the index of the first, or leading quantity, begins with that of the given power, and decreases continually by 1, in every term to the last; and that, in the following quantity, its indices are 1, 2, 3, &c. Then, find the coefficients, by observing that those of the first and last terms are always 1; and that the coefficient of the second term is the index of the power of the first; and, for the rest, if the coefficient of any term be multiplied by the index of the leading quantity in it, and the product be divided by the number of terms to that place, it will give the coefficient of the term next following.

Ex. 1. Required the 8*th power* of $a+b$.

Here the terms, without the coefficients, are

$$a^8,\ a^7b,\ a^6b^2,\ a^5b^3,\ a^4b^4,\ a^3b^5,\ a^2b^6,\ ab^7,\ b^8.$$

And the coefficients, according to the rule, will be 1, 8, $\frac{8\times7}{2}=28$, $\frac{28\times6}{3}=56$, $\frac{56\times5}{4}=70$, $\frac{70\times4}{5}=56$, $\frac{56\times3}{6}=28$, $\frac{28\times2}{7}=8$, $\frac{8\times1}{8}=1$.

Then, the terms are thus:

The *first* term is . . . . a^8.

second, ... $8a^7b$.

third, ... $\frac{8\times7}{2}\times a^6b^2=28a^6b^2$.

fourth, ... $\frac{28\times6}{3}\times a^5b^3=56a^5b^3$.

fifth, ... $\frac{56\times5}{4}\times a^4b^4=70a^4b^4$.

sixth, ... $\frac{70\times4}{5}\times a^3b^5=56a^3b^5$.

seventh, ... $\frac{56\times3}{6}\times a^2b^6=28a^2b^6$

eighth, ... $\frac{28\times3}{7}\times ab^7 = 8ab^7$.

ninth, ... $\frac{8\times1}{8}\times b^8= b^8$.

And thus we have, $(a+b)^8=a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8$.

227. From this example and the foregoing Table the whole number of terms will evidently be one more than the index of the given power; after having calculated therefore as many terms as there are units in the index, of the given power, we may immediately proceed to the last term. And in like manner it may be observed, that when the number of terms in the resulting quantity is *even*, the coefficients of the two middle terms is the *same;* and that *in all cases* the coefficients *increase* as far as the *middle* term, and then *decrease* precisely in the same manner until we come to the last term. By attending to this *law of the coefficients*, it will be necessary to calculate them only as far as the *middle term*, and then set down the rest in an inverted order.

Thus in the above example, the middle term is $70a^4b^4$, and we have,

The *first* four coefficients, 1, 8, 28, 56.
The *last* four . . . . 56, 28, 8, 1.

228. But we are not yet arrived at the *most general* form in which this Rule may be exhibited. Suppose it was required to raise the binomial $a+b$ to any power denoted by the number (n). Proceeding with n as we have done with the several indices in the preceding examples, it appears that,

The *first* term would be a^n.

The *second*, . . . $na^{n-1}b$.

The *third*, . . . $\frac{n(n-1)}{2}a^{n-2}b$.

The *fourth*, . . . $\frac{n(n-1)\times(n-2)}{2\times3}a^{n-3}b^3$.

The *fifth*, . . . $\frac{n(n-1)\times(n-2)\times(n-3)}{2\times3\times4}a^{n-4}b^4$.

The *sixth*, $\frac{n(n-1)\times(n-2)\times(n-3)\times(n-4)}{2\times3\times4\times5}a^{n-5}b^5$.

. . . .

The *last*, ... b^n.

Or $(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2+\frac{n(n-1)\times(n-2)}{2.3}a^{n-3}b^3+\frac{n(n-1)\times(n-2)\times(n-3)}{2.3.4}a^{n-4}b^4$, &c. . . . . $+b^n$.

By the same process, $(a-b)^n=a^n-na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2-\frac{n(n-1)\times(n-2)}{2.3}a^{n-3}b^3+\frac{n(n-1)\times(n-2)\times(n-3)}{2.3.3}a^{n-4}b^4$—&c. ; the signs of the terms being alternately + and —1 ; and the sign of the last term is + or —1, according as n is even or odd ; we have the last term in the *former case*, $+b^n$, and in the latter $-b^n$.

This general and compendious method of raising a binomial quantity to any given power, is called from the name of its celebrated inventor, Sir Isaac Newton's " Binomial Theorem." The demonstration of this *Theorem*, with its application to the finding the powers and roots of compound quantities, forms the subject of another Chapter. Its present use will appear from the following Example.

Ex. 2. Required the fifth power of x^2+3y^2.

Substituting these quantities for a, b, n, in the foregoing general formula, it appears that

The *first* term, (a^n) . . . . $(x^2)^5$ . . . . . . $=x^{10}$.

2nd, . . $(na^{n-1}b)$. is $5\times(x^2)^4\times3y^2$ . . . $=15x^8y^2$.

3d, . $\left(\frac{n(n-1)}{2}a^{n-2}b^2\right)$. $5\times\frac{4}{2}\times(x^2)^3\times(3y^2)^2=90x^6y^4$.

4th, . $\left(\frac{n(n-1)\times(n-2)}{2.3}a^{n-3}b^3\right)$. is $5\times\frac{4}{2}\times\frac{3}{3}\times(x^2)^2\times(3y^2)^3$ . . . . $=270x^4y^6$.

5th, . $\left(\frac{n(n-1)(n-2)(n-3)}{2.3.4}a^{n-4}b^4\right)$ is $5\times\frac{4}{2}\times\frac{3}{3}\times\frac{2}{4}\times x^2\times(3y^2)^4$ . . . . $=405x^2y^8$.

Last, . (b^n) is $(3y^2)^5$. $=243y^{10}$.

So that $(x^2+3y^2)^5=x^{10}+15x^8y^2+80x^6y^4+270x^4y^6+405x^2y^8+243y^{10}$.

229. By means of this *Theorem*, we are enabled to raise a *trinomial*, or *quadrinomial* quantity to any power, without the process of actual multiplication.

Ex. 3. Required the square of $a+b+c$.

Here, including $a+b$ in a parentheses $(a+b)$, and considering it as *one* quantity, we should have $(a+b+c)^2=[(a+b)+c]^2$; and comparing them with the general formula ;

we have $\left.\begin{array}{ll}(a^n)=(a+b)^2=a^2+2ab+b^2 \\ (na^{n-1}b)c=2(a+b)\times c=2ac+2bc \\ (b^n)=c^2 \quad\quad =c^2\end{array}\right\}$

Hence, $(a+b+c)^2=(a+b)^2+2(a+b)\times c+c^2=a^2+2ab+b^2+2ac+2bc+c^2$.

Ex. 4. Required the seventh power of $a-b$.

Ans. $a^7-7a^6b+21a^5b^2-35a^4b^3+35a^3b^4-21a^2b^5+7ab^6-b^7$.

Ex. 5. Required the sixth power of $3x+2y$.

Ans. $729x^6+2916x^5y+4860x^4y^2+4320x^3y^3+2160x^2y^4+576xy^5+64y^6$.

Ex. 6. Required the square of $x+y+3z$.

Ans. $x^2+2xy+y^2+6xz+6yz+9z^2$.

Ex. 7. Required the fifth power of $1+2x$.

Ans. $1+10x+40x^2+80x^3+80x^4+32x^5$.

Ex. 8. Required the cube of $x^2-2xy+y^2$.

Ans. $x^6-6x^5y+15x^4y^2-20x^3y^3+15x^2y^4-6xy^5+y^6$.

§ II. EVOLUTION OF ALGEBRAIC QUANTITIES.

230. *The quantity which has been raised to any power is called the* root *of that power ;* thus the mth root of a power, is that quantity which we must continually multiply into itself, till the number of factors be equal to m, m being a positive whole number, in order to produce the power proposed. We may conclude from this definition, and from the Articles in the preceding section.

231. That *the* m*th root of a quantity such as a^{pm}, pm being a multiple of p, is obtained by dividing the exponent pm of this quantity, by the index of the required root.* Thus the mth root of $a^{pm}=a^{\frac{pm}{m}}=a^p$; the square root of $a^6=a^{\frac{6}{2}}=a^3$, and the cube root of $a^6=a^{\frac{6}{3}}=a^2$.

232. Also that *the* m*th root of a product such as $a^{2m}b^{3m}$, is equal to the* m*th root of each of its factors multiplied together.* Thus, the mth root of $a^{2m}b^{3m}$ is = the mth root of $a^{2m}\times$ the mth root of $b^{3m}=a^{\frac{2m}{m}}\times b^{\frac{3m}{m}}=a^2b^3$.

233. And that the mth *root of a fraction such as $\frac{a^m}{b^m}$, is equal to the* m*th root of the numerator divided by the* m*th root of its denominator.*

Thus the mth root of $\frac{a^m}{b^m}=\frac{a^{\frac{m}{m}}}{b^{\frac{m}{m}}}=\frac{a}{b}$.

234. *The square, the fourth root, or any even root of an affirmative quantity may be either + or — 1.* Thus the square root of $a^2 = a$ or $-a$; for $+a \times +a = +a^2$, and $-a \times -a = +a^2$. In fact, the $2m$th root of a^{2m} is equal to $+a$ or $-a$; for $(\pm a)^{2m} = (\pm a)^2 \times a^m = a^{2m}$.

235. *Any odd root of a quantity, will have the same sign as the quantity itself.* Thus the $(2m+1)$th root of $\pm a^{2m+1}$ is equal to $\pm a$; for $(\pm a)^{2m+1}$ is equal to $\pm a^{2m+1}$.

236. Evolution, or the rule for extracting the root of any algebraic Quantity whatever, is divided into the four following Cases.

CASE I.

To find any root of a simple algebraic Quantity.

RULE.

237. Extract the root of the coefficient for the numeral part, and the root of the quantity subjoined to it for the literal part, by the methods pointed out in the above propositions; then, these, joined together, will be the root required.

Ex. 1. It is required to find the square root of x^4.

Here, the square root of $x^4 = \pm\sqrt{x^4} = \pm x^{\frac{4}{2}} = \pm x^2$.

Ex. 2. Required the cube root of $-27x^3a^6$.

Here, the cube root of $-27x^3a^6 = -\sqrt[3]{27x^3a^6} = -\sqrt[3]{27} \times \sqrt[3]{x^3} \times \sqrt[3]{a^6} = -3 \times x \times a^2 = -3a^2x$.

Ex. 3. Required the square root of $\frac{a^2x^2}{b^2c^2}$.

Here, the square root of $a^2x^2 = \sqrt{a^2} \times \sqrt{x^2} = ax$, and the square root of $b^2c^2 = \sqrt{b^2} \times \sqrt{c^2} = bc$; $\therefore \pm\frac{ax}{bc}$ is the root required.

Ex. 5. It is required to find the square root of $64a^2x^4$.

Ans. $8ax^2$, or $-8ax^2$.

Ex. 6. It is required to find the cube root of $729a^6x^{12}$.

Ans. $9a^2x^4$.

Ex. 7. Required the fourth root of $256a^4b^8$.

Ans. $4ab^2$, or $-4ab^2$.

Ex. 8. Required the fifth root of $32a^5x^{10}$. Ans. $2ax^2$.

Ex. 9. Required the sixth root of $\frac{729a^6b^6}{4096x^{12}}$. Ans. $\pm\frac{3ab}{4x^2}$.

Ex. 10. Required the ninth root of $-\frac{x^9y^{18}}{a^9b^9}$. Ans. $-\frac{xy^2}{ab}$.

Ex. 11. Required the square root of $\frac{36a^6x^4}{4x^2y^2}$. Ans. $\pm\frac{6a^3x^2}{2xy}$.

Ex. 12. Required the cube root of $\frac{64x^3}{27a^6b^3}$. Ans. $\frac{4x}{3a^2b}$.

CASE II.

To extract the square root of a compound Quantity.

RULE

238. Observe in what manner the terms of the root may be derived from those of the power; and arrange the terms accordingly; then set the root of the first term in the quotient; subtract the square of the root, thus found, from the first term, and bring down the next two terms to the remainder for a dividend.

Divide the dividend, thus found, by double that part of the root already determined, and set down the result both in the quotient and divisor.

Multiply the divisor, so increased, by the term of the root last placed in the quotient, and subtract the product from the dividend, and to the remainder bring down as many terms as are necessary for a dividend, and continue the operation as before.

Ex. 1. Required the square root of $a^2+2ab+b^2$,

$$\begin{array}{r|l} & a^2+2ab+b^2 \quad (a+b \\ & a^2 \\ \hline 2a+b & 2ab+b^2 \\ & 2ab+b^2 \\ \hline \end{array}$$

On comparing $a+b$ with $a^2+2ab+b^2$, we observe that the first term of the power (a^2) is the square of the first term of the root (a). Put a therefore for the first term of the root, square it, and subtract that square from the first term of the power. Bring down the other two terms $2ab+b^2$, and double the first term (a) of the root; set down $2a$, and having divided the first term of the remainder ($2ab$) by it, we have b, the other term of the root; and since $2ab+b^2=(2a+b)\times b$, if to $2a$ the term b is added, and this sum multiplied by b, the result is $2ab+b^2$; which being subtracted from the terms brought down, nothing remains.

Ex. 2. Required the square root of $a^2+2ab+b^2+2ac+2bc+c^2$.

$$a^2+2ab+b^2+2ac+2bc+c^2(a+b+c$$
$$a^2$$
$$2a+b \;|\; 2ab+b^2$$
$$2ab+b^2$$
$$2a+2b+c \;|\; 2ac+2bc+c^2$$
$$2ac+2bc+c^2$$

On comparing the root $a+b+c$, thus found with its power, the reason of the rule for deriving the root from the power is evident. And the method of operation is the same as in the last example. Thus, having found the first two terms of the root as before, we bring down the remaining three terms $2ac+2bc+c^2$ of the power, and dividing $2ac$ by $2a$, it gives c, the third term of the root. Next, let the last term (b) of the preceding divisor be doubled, and add c to the divisor thus increased, and it becomes $2a+2b+c$; multiply this new divisor by c, and it gives $2ac+2bc+c^2$, which being subtracted from the terms last brought down, leaves no remainder. In like manner the following Examples are solved.

Ex. 3. Required the square root of $4x^4+6x^3+\frac{89}{4}x^2+15x+25$.

$$4x^4+6x^3+\frac{89}{4}x^2+15x+25\left(2x^2+\frac{3}{2}x+5\right.$$
$$4x^4$$
$$4x^2+\frac{3}{2}x\Big)\,6x^3+\frac{89}{4}x^2$$
$$6x^3+\frac{9}{4}x^2$$
$$4x^2+3x+5)\,20x^2+15x+25$$
$$20x^2+15x+25$$

Ex. 4. Required the square root of $x^6+4x^5+2x^4+9x^2-4x+4$. Ans. x^3+2x^2-x+2.

Ex. 5. Required the square root of $x^4+4ax^3+6a^2x^2+4a^3x+a^4$. Ans. $x^2+2ax+a^2$.

Ex. 6. Required the square root of $a^4-2a^3+\frac{3}{2}a^2-\frac{1}{2}a+\frac{1}{16}$. Ans. $a^2-a+\frac{1}{4}$.

Ex. 7. Required the square root of $4a^4+12a^3x+13a^2x^2+6ax^3+x^4$. Ans. $2a^2+3ax+x^2$.

Ex. 8. Required the square root of $9x^4+12x^3+34x^2+20x+25$. Ans. $3x^2+2x+5$.

Ex. 9. Required the square root of $a^2+2ab+b^2+2ac+2bc+c^2+2ad+2bd+2cd+d^2$. Ans. $a+b+c+d$.

Ex. 10. Required the square root of $a^4+12a^3b+54a^2b^2+108ab^3+81b^4$. Ans. $a^2+6ab+9b^2$.

Ex. 11. Required the square root of $a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6$. Ans. $a^3-3a^2x+3ax^2-x^3$.

Ex. 12. Required the square root of $a^4-2a^2x^2+x^4$.

Ans. a^2-x^2.

CASE III.

To extract the cube root of a compound Quantity.

RULE.

239. Arrange the terms as in the last case; and set the root of the first terms in the quotient; subtract the cube of the root, thus found, from the first term, and bring down three terms for a dividend.

Next, divide the first term of the dividend by 3 times the square of that part of the root already determined, and set the *result* in the quotient; then, to 3 times the square of that *part* of the *root*, annex 3 times the product of the *same part* and the last *result*, and also the square of the last *result*, with their proper signs; and it will give the divisor, multiply the divisor by the term of the root last placed in the quotient, and subtract the product from the dividend, bring down three terms or as many as may be necessary for a dividend, and proceed as before.

Ex. 1. Required the cube root of $a^3+3a^2b+3ab^2+b^3$.

$$
\begin{array}{rl}
 & a^3+3a^2b+3ab^2+b^3 \quad (a+b \\
 & a^3 \\
 & \overline{} \\
3a^2+3ab+b^2) & 3a^2b+3ab^2+b^3 \\
 & 3a^2b+3ab^2+b^3 \\
 & \overline{}
\end{array}
$$

The reason of the rule may be made evident from a comparison of the *roots* with *its* cube.

Or, thus, if the quantity whose root is to be extracted, has an exact root, the root of the leading term must be one term of its root; that is, the cube root of a^3, which is a, is one term of the root, and the remaining terms being brought down, the root of the last term b^3 is consequently another term of the root; but as the root may consist of more terms than two; the next term (b) of the root is always found by dividing $\left(\frac{3a^2b}{3a^2}=b\right)$ the first term of the dividend by three times the square of the divisor, and the two remaining terms of the dividend $3ab^2+b^3=(3ab+b^2)b$; hence $3ab+b^2$ must be added to $3a^2$ for a divisor; and so on.

Ex. 2. Required the cube root of $x^6+6x^5-40x^3+96x-64$.

$$
\begin{array}{l}
\qquad\qquad x^6+6x^5-40x^3+96x-64\ (x^2+2x-4 \\
\qquad\qquad x^2 \\
\hline
3x^4+6x^3+4x^2)6x^5-40x^3 \\
\qquad\qquad\qquad 6x^5+12x^4+8x^3 \\
\hline
3x^4+12x^3-24x+16)-12x^4-48x^3+96x-64 \\
\qquad\qquad\qquad\qquad -12x^4-48x^3+96x-64 \\
\hline
\end{array}
$$

Ex. 3. Required the cube root of $(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3$. Ans. $a+b+c$.

Ex. 4. Required the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$. Ans. x^2-2x+1.

Ex. 5. Required the cube root of $x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$. Ans. $x^2+2xy+y^2$.

Ex. 6. Required the cube root of $1-6x+12x^2-8x^3$.

Ans. $1-2x$.

CASE IV.

To find any root of a compound Quantity.

RULE.

240. Find the root of the first term, which place in the quotient; and having subtracted its corresponding power from that term, bring down the second term for a dividend. Divide this by twice the part of the root above determined, for the square root; by three times the square of it, for the cube root; by four times the cube of it, for the fourth root, &c. and the quotient will be the next term of the root.

Involve the whole of the root, thus found, to its proper power, which subtract from the given quantity, and divide the first term of the remainder by the same divisor as before.

Proceed in the same manner for the next following term of the root; and so on, till the whole is finished.

241. This rule may be demonstrated thus; $(a+b)^n=a^n+na^{n-1}b+$, &c. Here the nth root of a^n is a, and the next term $na^{n-1}b$ contains b, (the other term of the root) na^{n-1} times; hence, if we divide $na^{n-1}b$ by na^{n-1}, we have b, or $\frac{na^{n-1}b}{na^{n-1}}=b$; and so on, for any compound quantity, the root of which consists of more than two terms.

Now, if $n=2$; then, the divisor $na^{n-1}=2a$, for the square root;

if $n=3$; then, ... $na^{n-1}=3a^2$, for the cube root;

if $n=4$; then, ... $na^{n-1}=4a^3$, for the 4th root;

if $n=5$; then, ... $na^{n-1}=5a^4$, for the 5th root.

And so on for any other root, that is, involve the first term of the root, to the next lowest power, and multiply it by the index of the given power for a divisor.

Ex. 1. Required the square root of $a^4-2a^3x+3a^2x^2-2ax^3+x^4$.

$$a^4-2a^3x+3a^2x^2-2ax^3+x^4(a^2-ax+x^2.$$
$$a^4$$
$$2a^2)-2a^3x$$
$$(a^2-ax)^2=a^4-2a^3x+a^2x^2$$
$$2a^2)+2a^2x^2$$
$$(a^2-ax+x^2)^2=a^4-2a^3x+3a^2x^2-2ax^3+x^4.$$

Ex. 2. Required the 4th root of $16a^4-96a^3x+216a^2x^2-216ax^3+81x^4$.

$$16a^4-96a^3x+216a^2x^2-216ax^3+81x^4(2a-3x$$
$$16a^4$$
$$4\times(2a)^3=32a^3)-96a^3x$$
$$(2a-3x)^4=16a^4-96a^3x+216a^2x^2-216ax^3+81x^4.$$

242. As this rule, in high powers, is often found to be very laborious, it may be proper to observe, that the roots of certain compound quantities may sometimes be easily discovered:

thus, in the last example, the root is $2a-3x$, which is the difference of the roots of the first and last terms; and so on, for other compound quantities.

Hence, the following method in such cases; extract the roots of all the simple terms, and connect them together by the signs + or —, as may be judged most suitable for the purpose; then involve the compound root thus found, to its proper power, and if it be the same with the given quantity, it is the root required. But if it be found to differ only in some of the signs, change them from + to —, or from — to +, till its power agrees with the given one throughout. However, such artifices are not to be used by *learners*, because the regular mode of proceeding is more advantageous to *them*; besides, a knowledge of those artifices which are used by experienced Algebraists, can only be acquired from frequent practice.

Ex. 3. Required the square root of $a^2+2ab+b^2+2ac+2bc+c^2$.

Here, the square root of $a^2=a$; the square root of $b^2=b$; and the square root of $c^2=c$. Hence, $a+b+c$, is the root required, because $(a+b+c)^2=a^2+2ab+b^2+2ac+2bc+c^2$.

Ex. 4. Required the fifth root of $32x^5-80x^4+80x^3-40x^2+10x-1$. Ans. $2x-1$.

Ex. 5. Required the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-5x+1$. Ans. x^2-2x+1.

Ex. 6. Required the fourth root of $a^4-4a^3x+6a^2x^2-4ax^3+x^4$. Ans. $a-x$

Ex. 7. Required the square root of $x^8+2x^4y^4+y^3$. Ans. x^4+y^4

Ex. 8. Required the square root of $x^8-2x^4y^4+y^8$. Ans. x^4-y^4.

Ex. 9. Required the cube root of $a^3-6a^2x+12ax^2-8x^3$. Ans. $a-2x$.

Ex. 10. Required the sixth root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$. Ans. $x-1$.

Ex. 11. Required the fifth root of $x^{10}+15x^8y^2+90x^6y^4+270x^4y^6+405x^2y^8+243y^{10}$. Ans. x^2+3y^2.

Ex. 12. Required the square root of $x^2+2xy+y^2+6xz+6yz+9z^2$. Ans. $x+y+3z$.

§ III. INVESTIGATION OF THE RULES FOR THE EXTRACTION OF THE SQUARE AND CUBE ROOTS OF NUMBERS.

243. It has been observed, (Art. 104), that, a denoting the tens of a number, and b the units, the formula $a^2+2ab+b^2$ would represent the square of any number consisting of two

figures or digits; thus, for example, if we had to square 25 put $a=20$ and $b=5$, and we shall find

$$
\begin{aligned}
a^2&=400\\
2ab&=200\\
b^2&=\ \ 25\\
\hline
(a+b)^2&=(25)^2=625.
\end{aligned}
$$

244. Before we proceed to the investigation of these Rules, it will be necessary to explain the nature of the common arithmetical notation. It is very well known that the value of the figures in the common arithmetical scale increases in a tenfold proportion from the right to the left; a number, therefore, may be expressed by the addition of the *units*, *tens*, *hundreds*, &c. of which it consists; thus the number 4371 may be expressed in the following manner, viz. $4000+300+70+1$, or by $4\times1000+3\times100+7\times10+1$; also, in decimal arithmetic, each figure is supposed to be multiplied by that power of 10, positive or negative, which is expressed by its distance from the figure before the point: thus, $672.53=6\times10^2+7\times10^1+2\times10^0+5\times10^{-1}+3\times10^{-2}=6\times100+7\times10+2\times1+\frac{5}{10}+\frac{3}{100}=672+\frac{50}{100}+\frac{3}{100}=672\frac{53}{100}$. Hence, if the *digits* of a number be represented by $a, b, c, d, e,$ &c. beginning from the left-hand; then,

A number of 2 figures may be expressed by $10a+b$.
3 figures . . . by $100a+10b+c$.
4 figures . by $1000a+100b+10c+d$.
&c. &c. &c.

By the *digits* of a number are meant the figures which compose it, considered independently of the value which they possess in the arithmetical scale.

Thus the *digits* of the number 537 are simply the numbers 5, 3 and 7; whereas the 5, considered with respect to its place, in the numeration scale, means 500, and the 3 means 30.

245. Let a number of three figures, (viz. $100a+10b+c$) be squared, and its root extracted according to the *rule* in (Art. 288), and the operation stands thus;

$$
\begin{array}{ll}
\text{I.} & 10000a^2+2000ab+100b^2+200ac+20bc+c^2 \quad (100a+10b+c\\
& 10000a^2\\
\hline
& 200a+10b)2000ab+100b^2\\
& \qquad\qquad 2000ab+100b\\
\hline
& \qquad 200a+20b+c)200ac+20bc+c^2\\
& \qquad\qquad\qquad 200ac+20bc+c^2\\
\hline
\end{array}
$$

II. Let $\left.\begin{array}{l} a=2 \\ b=3 \\ c=1 \end{array}\right\}$ and the operation is transformed into the following one ;

$$\begin{array}{l}
40000+12000+900+400+60+1(200+30+1 \\
40000 \\
\hline
400+30)12000+900 \\
\qquad\qquad\qquad \overline{} \\
\qquad\qquad 400+60+1)400+60+1 \\
\qquad\qquad\qquad\qquad\quad 400+60+1 \\
\end{array}$$

III. But it is evident that this operation would not be affected by collecting the several numbers which stand in the same line into one sum, and leaving out the ciphers which are to be subtracted in the operation.

$$\begin{array}{r|l}
 & \dot{5}3\dot{3}6\dot{1}(231 \\
 & 4 \\
\hline
43 & 133 \\
 & 129 \\
\hline
461 & 461 \\
 & 461 \\
\hline
\end{array}$$

Let this be done ; and let two figures be brought down at a time, after the square of the first figure in the root has been subtracted ; then the operation may be exhibited in the manner annexed ; from which it appears, that the square root of 53361 is 231.

246. To explain the division of the given number into *periods* consisting of two figures each, by placing a dot over every second figure beginning with the units, as exhibited in the foregoing operation. It must be observed, that, since the square root of 100 is 10 ; of 10000 is 100 ; of 1000000 is 1000 ; &c. &c. it follows, that the square root of a number *less than* 100 must consist of *one* figure ; of a number *between* 100 *and* 10000, of *two* figures ; of a number *between* 10000 *and* 1000000, of *three* figures ; &c. &c., and consequently the number of these dots will show the number of figures contained in the square root of the given number. From hence it follows, that the *first* figure of the root will be the greatest square root contained in the first of those periods reckoning from the *left*.

Thus, in the case of $\dot{5}3\dot{3}6\dot{1}$ (whose square root is a number consisting of *three* figures); since the square of the figure standing in the *hundred's* place cannot be found either in the *last* period (61), or in the *last but one* (33), it must be found in the first period (5); consequently the first figure of the root will be the square root of the *greatest* square number contained in 5; and this number is 4, the first figure of the root will be 2. The remainder of the operation will be readily understood by comparing the steps of it with the several steps of the process for finding the square root of $(a+b+c)^2$ (Art. 238); for, having subtracted 4 from (5), there remains 1; bring down the next two figures (33), and the dividend is 133; double the first figure of the root (2), and place the result 4 in the divisor; 4 is contained in 13 three times; 3 is therefore the second figure of the root; place this both in the divisor and quotient, and the former is 43; multiply by 3, and subtract 129, the remainder is 4; to which bring down the next two figures (61), which gives 461 for a dividend. Lastly, double the last figure of the former divisor, and it becomes 46; place this in the next divisor, and since 4 is contained in 4 *once*, 1 is the third figure of the root; place 1 therefore both in the divisor and quotient; multiply and subtract as before, and nothing remains.

247. The method of extracting the *cube* root of numbers may be understood by comparing the process for extracting the cube root of $(a+b+c)^3$, (Art. 239), with the following operations, in which is deduced the cube root of the number 13997521.

$$
\begin{array}{rr}
 & \dot{1}39\dot{9}75\dot{2}1(200+40+1 \\
a^3=(200)^3= & 8000000 \\
\hline
\text{1st remainder} & 5997521 \\
\hline
\end{array}
$$

$$
\begin{array}{rr}
3a^2=3\times(200)^2=\text{divisor,} & \\
\because\ 3a^2b=3(200)^2\ \times 40= & 4800000 \\
3ab^2=3\times 200\times(40)^2= & 960000 \\
b^3=40\times 40\times 40= & 64000 \\
\hline
 & 5824000 \\
\hline
\text{2nd remainder} & 173521 \\
\hline
3(a+b)^2c=3(200+40)^2\times 1= & 172800 \\
3(a+b)c^2=3(200+40)\times 1= & 720 \\
c^3=1\times 1\times 1= & 1 \\
\hline
 & 173521 \\
\hline
\text{3d remainder} & 000000 \\
\hline
\end{array}
$$

Omitting the superfluous ciphers, and bringing down three figures at a time, the operation will stand thus;

$$
\begin{array}{rr}
 & \dot{1}39\dot{9}75\dot{2}1)241 \\
2^3= & 8 \\
\hline
 & 5997 \\
\hline
300\times 2^2\times 4= & 4800 \\
30\times 2\times 4^2= & 960 \\
4^3= & 64 \\
\hline
 & 5824 \\
\hline
 & 173521 \\
\hline
300\times(24)^2\times 1= & 172800 \\
30\times 24\times 1^2= & 720 \\
1^3= & 1 \\
\hline
 & 173521 \\
\hline
\end{array}
$$

248. These operations may be explained in the following manner ;

I. Since the cube root of 1000 is 10, of 1000000 is 100, &c. ; it follows, that the cube root of a number less than 1000 will consist of *one* figure ; of a number between 1000 and 1000000 of *two* figures, &c. &c. ; if, therefore, the given number be divided into *periods*, each consisting of *three figures*, by placing a dot over every third figure, beginning with the units, the number of those dots will show the number of figures of which the cube root consists ; and for the reason assigned in the preceding Article, (respecting the first figure of the square root), the *first figure* of the root will be the cube root of the greatest cube number contained in the first period.

II. Having *pointed* the number, we find that its cube root consists of *three* figures. The *first* figure is the cube root of the greatest cube number contained in 13 ; this being 2, the value of this figure is 200, or $a=200$, consequently $a^3=$ 8000000 ; subtract this number from 13997521, and the remainder is 5997521. Find the value of $3x^2$, and divide this latter number by it, and it gives 40 for the value of a, the second number of the root ; put this in the quotient, and then calculate the value of $3a^2b+3ab^2+b^3$, and subtract it, and there remains 173521. Find now the value of $3\times(a+b)^2$, and divide 173521 by it, and it gives 1 for the value of c, the *third* member of the root ; put this in the quotient, and then calculate the amount of $3(a+b)^2c+3(a+b)c^2+c^3$, which subtract, and nothing remains.

III. In reviewing the first of these two operations, it is evident that *six* ciphers might have been rejected in the value of a^3, and *three* in the value of $3a^2b+3ab^2+b^3$, without affecting the substance of the operation ; having therefore simplified the process as in the *second* operation, we are furnished with the following rule, for extracting the cube root of numbers.

RULE.

249. Point off every *third* figure, beginning with the units ; find the greatest cube number contained in the *first* period, and place the cube root of it in the quotient. Subtract its *cube* from the first period, and bring down the next three figures ; divide the number thus brought down by 300 times the square of the first figure of the root, and it will give the second figure ; add 300 times the square of the first figure,

30 times the product of the first and second figures, and the square of the second figure together, for a divisor; then multiply this divisor by the second figure, and subtract the result from the dividend, and then bring down the next period, and so proceed till all the periods are brought down.

The rules for extracting the higher powers of numbers, and of compound algebraic quantities, are very tedious, and of no great practical utility.

Examples for practice in the Square and Cube Roots of Numbers.

Ex. 1. Required the square root of 106929.

```
      106929(327
       9
      ----
  62 | 169
     | 124
     -----
  647 | 4529
      | 4529
      ------
```

Ex. 2. Required the cube root of 48228544.

```
   .  .  .
  48228544(364
  27
  ---
3276(21228                Divide by 300×3²=2700
     19656                         30×3×6= 540
     -----                            6×6=  36
393136) 1572544                     ----------
        1572544             1st Divisor =3276
        -------
     Divide by, (36)²×300=388800
               30×36×4=   4320
                   4×4=     16
                        ------
           2d Divisor 393135
```

Ex. 3. Required the square root of 152399025. Ans. 12345.

Ex. 4. Required the square root of 5499025. Ans. 2345.

Ex. 5. Required the cube root of 389017. Ans. 73.

Ex. 6. Required the cube root of 1092727. Ans. 103.

CHAPTER VII.

ON

IRRATIONAL AND IMAGINARY QUANTITIES.

§ I. THEORY OF IRRATIONAL QUANTITIES.

250. It has been demonstrated (Art. 231), that the mth root of a^p, the exponent p of the power being exactly divisible by the index m of the root, is $a^{\frac{p}{m}}$. Now in case that the exponent p of the power is not divisible by the index m of the root to be extracted, it appears very natural to employ still the same method of notation, since that it only indicates a division which cannot be performed: then the root cannot be obtained, but its approximate value may be determined to any degree of exactness. These *fractional* exponents will therefore denote imperfect powers with respect to the roots to be extracted; and quantities, having fractional exponents, are called *irrational quantities*, or *surds*.

It may be observed that the numerator of the exponent shows the power to which the quantity is to be raised, and the denominator its root. Thus, $a^{\frac{m}{n}}$ is the nth root of the mth power of a, and is usually read a in the power $\left(\frac{m}{n}\right)$.

251. In order to indicate any root to be extracted, the radical sign $\sqrt{}$ is used, which is nothing else but the initial of the word root, *deformed*, it is placed over the power, and in the opening of which the index m of the root to be extracted is written.

We have therefore $\sqrt[m]{a^p}=a^{\frac{p}{m}}$. For the square root, the sign $\sqrt{}$ is used without the index 2; thus, the square root of a^p is written $\sqrt{a^p}$, as has been already observed, (Art. 18).

Quantities having the radical sign $\sqrt{}$ prefixed to them, are called *radical quantities:* thus, $\sqrt[3]{a}$, $\sqrt{b}$, $\sqrt[4]{c^2}$, $\sqrt[n]{x^m}$, &c, are *radical quantities;* they are, also, commonly called *Surds.*

252. From the two preceding articles, and the rules given in the second section of the foregoing Chapter, we shall, in general, have,

$$\sqrt[m]{(a^p.b^q.c^r)}=\sqrt[m]{a^p}\times\sqrt[m]{b^q}\times\sqrt[m]{c^r}=a^{\frac{p}{m}}\times b^{\frac{q}{m}}\times c^{\frac{r}{m}};$$

$$\sqrt[m]{\frac{a^p.b^q}{c^r d^s}}=\frac{\sqrt[m]{(a^p.b^q)}}{\sqrt[m]{c\,d^s}}=\frac{\sqrt[m]{b^p}\times\sqrt[m]{b^q}}{\sqrt[m]{c^r}\times\sqrt[m]{d^s}}=\frac{a^{\frac{p}{m}}\times b^{\frac{q}{m}}}{c^{\frac{r}{m}}\times b^{\frac{s}{m}}}.$$

Therefore, $\sqrt[3]{a^3b}=\sqrt[3]{a^3}\times\sqrt[3]{b}=a\times\sqrt[3]{b}=a\sqrt[3]{b}$;

and $\sqrt[3]{\frac{a^6b^3c^2}{e^3x^4z}}=\frac{\sqrt[3]{a^6b^3c^2}}{\sqrt[3]{e^3x^4z}}=\frac{\sqrt[3]{a^6}\times\sqrt[3]{b^3}\times\sqrt[3]{c^2}}{\sqrt[3]{e^3}\times\sqrt[3]{x^3}\times\sqrt[3]{xz}}$

$=\frac{a^2b\sqrt[3]{c^2}}{ex\sqrt[3]{xz}}=\frac{a^2b}{ex}\sqrt[3]{\frac{c^2}{xz}}.$

253. Two or more radical quantities, having the same index, are said to be of the *same denomination*, or *kind ;* and they are of *different denominations*, when they have different indices.

In this last case, we can sometimes bring them to the same denòmination; this is what takes place with respect to the two following, $\sqrt{a^3b^2}$ and $\sqrt[4]{a^6b^4}=a^{\frac{6}{4}}\times b^{\frac{4}{4}}=a^{\frac{3}{2}}\,.\,b^{\frac{2}{2}}=\sqrt{a^3}\sqrt{b^2}$ $=\sqrt{a^3b^2}$. In like manner, the radical quantities $\sqrt[3]{2a^6b}$ and $\sqrt[3]{16a^3b}$, may be reduced to other equivalent ones, having the same radical quantity; thus, $\sqrt[3]{2a^6b}=\sqrt[3]{a^6}\times\sqrt[3]{2b}=a^2\sqrt[3]{2b}$, and $\sqrt[3]{16a^3b}=\sqrt[3]{8a^3\,.\,2b}=\sqrt[3]{8}\,.\sqrt[3]{a^3}\,.\sqrt[3]{2b}=2a\sqrt[3]{2b}$; where the radical factor $\sqrt[3]{2b}$ is common to both.

254. The addition and subtraction of radical quantities can in general be only indicated :

Thus, $\sqrt[3]{a^2}$ added to, or subtracted from $\sqrt{b}$, is written $\sqrt{b}$ $\pm\sqrt[3]{a^2}$, and no farther reduction can be made, unless we assign numeral values to a and b. But the sum of $\sqrt{a^2b}$, $\sqrt{a^2b}$, and $\sqrt{4a^2b}$ is $=a\sqrt{b}+a\sqrt{b}+2a\sqrt{b}=4a\sqrt{b}$; $3\sqrt[4]{ab}-\sqrt[4]{ab}$ $=2\sqrt[4]{ab}$; and $\sqrt{ab^2}+\sqrt[4]{a^6b^4}=b\sqrt{a}+ab\sqrt[4]{a^2}=b\sqrt{a}+ab\sqrt{a}$ $=(b+ab)\sqrt{a}$.

255. Hence we may conclude, that the addition and subtraction of radical quantities, having the same radical part, are performed like rational quantities.

Radical quantities are said to have the same *radical part*, when *like* quantities are placed under the same radical sign; in which case radical quantities are *similar* or *like.* It is sometimes necessary to simplify the radical quantities, (Art. 252), in order to discover this similitude, and it is independent of the coefficients.

Thus, for example, the radical quantities $3b\sqrt[3]{2a^5b^2}$, $8a\sqrt[3]{2a^2b^5}$, and $-7ab\sqrt[3]{2a^2b^2}$, become, by reduction, $3ab\sqrt[3]{2a^2b^2}$, $8ab\sqrt[3]{2a^2b^2}$, and $-7ab\sqrt[3]{2a^2b^2}$; which are similar quantities, and their sum is $=4ab\sqrt[3]{2a^2b^2}$.

256. We have demonstrated, (Art. 252), this formula, $\sqrt[m]{a^p b^q c^r}=\sqrt[m]{a^p}\times\sqrt[m]{b^b}\times\sqrt[m]{c^r}$; from which the rule for the multiplication of radical quantities, under the same radical sign, may be easily deduced.

257. Let us pass to radical quantities with different indices, and suppose that we had to find, for instance, the product of $\sqrt[m]{a^p}$ by $\sqrt[m']{b^q}$, or that of $a^{\frac{p}{m}}$ by $b^{\frac{q}{m'}}$: we can bring this case to the preceding, by reducing to the same denominator, (Art. 152), the fractions $\frac{p}{m}$ and $\frac{q}{m'}$; and we shall have $\sqrt[m]{a^p}\times\sqrt[m']{b^q}$

$$=a^{\frac{p}{m}}b^{\frac{q}{m'}}=a^{\frac{pm'}{mm'}}\times b^{\frac{qm}{mm'}}=\sqrt[mm']{a^{pm'}}\times\sqrt[mm']{b^{qm}}=\sqrt[mm']{a^{pm'}b^{qm}}.$$

258. The rule for dividing two radical quantities of the same kind, may be read in this formula (Art. 233).

$$\frac{\sqrt[m]{a^p}}{\sqrt[m]{b^q}}=\sqrt[m]{\frac{a^p}{b^q}},$$

and it only remains to extend it to two radical quantities of different denominations.

Let therefore $\sqrt[m]{a^p}$ be divided by $\sqrt[m']{b^q}$: by passing from radical signs to fractional exponents, we have

$$\frac{\sqrt[m]{a^p}}{\sqrt[m']{b^q}}=\frac{a^{\frac{p}{m}}}{b^{\frac{q}{m'}}}=\frac{a^{\frac{pm'}{mm'}}}{b^{\frac{qm}{mm'}}}=\frac{\sqrt[mm']{a^{pm'}}}{\sqrt[mm']{b^{qm}}}=\sqrt[mm']{\frac{a^{pm'}}{b^{qm}}}.$$

We may likewise suppose, under the radical signs, any number of factors whatever, and it shall be easy to assign the quotient, (Art. 252).

Let now $a=b$ in the formula

$$\sqrt[m]{a^p}\times\sqrt[m]{b^q}=\sqrt[m]{a^p\,.\,b^q};$$

it becomes, by passing from radical signs to fractional exponents,

$$a^{\frac{p}{m}}\times a^{\frac{q}{m}}=\sqrt[m]{a^{p+q}}=a^{\frac{p+q}{m}}=a^{\frac{p}{m}+\frac{q}{m}}.$$

Therefore the rule demonstrated (Art. 71), with regard to whole positive exponents, extends to fractional exponents.

259. In the same hypotheses $b=a$, the quotient $\frac{\sqrt[m]{a^p}}{\sqrt[m]{b^q}}$ becomes $\frac{a^{\frac{p}{m}}}{a^{\frac{q}{m}}}=\sqrt[m]{\frac{a^p}{a^q}}=\sqrt[m]{a^{p-q}}=a^{\frac{p-q}{m}}=a^{\frac{p}{m}-\frac{q}{m}}$;

another extension of the rule given (Art. 86), to fractional positive exponents.

260. We may, in the preceding formula, suppose $p=o$; and it becomes, (since $a^{\frac{p}{m}}=a^{\frac{o}{m}}=a^{o}=1$) $\frac{1}{a^{\frac{q}{m}}}=a^{\frac{-q}{m}}$, a transformation demonstrated, (Art. 86) in the case of whole exponents, and which still takes place when the exponents are fractional.

261. If we now admit the two equalities,

$$\frac{1}{a^{\frac{p}{m}}}=a^{\frac{-p}{m}};\quad \frac{1}{a^{\frac{q}{m}}}=a^{\frac{-q}{m}};$$

and if we multiply them member by member, we shall have the equal products,

$$\frac{1}{a^{\frac{p}{m}}}\times\frac{1}{a^{\frac{q}{m}}}=\frac{1}{a^{\frac{p}{m}+\frac{p}{m}}};\ \text{or}\ a^{\frac{p}{m}}\times a^{\frac{q}{m}}=a^{\frac{p}{m}-\frac{q}{m}}.$$

It appears therefore evident, that exponentials with fractional negative exponents, follow the same rule in their multiplication, as those with whole positive exponents.

262. The division of $a^{-\frac{p}{m}}$ by $a^{-\frac{p}{m}}$, gives for the quotient,

$$\frac{a^{-\frac{p}{m}}}{a^{-\frac{q}{m}}}=\frac{a^{\frac{q}{m}}}{a^{\frac{p}{m}}}=a^{\frac{q}{m}-\frac{p}{m}}=a^{-\frac{p}{m}+\frac{q}{m}}.$$

Now the exponent of the qnotient, namely $-\frac{p}{m}+\frac{q}{m}$, is the exponent of the dividend, minus that of the divisor, which is still a generality of the rule (Art. 86), relative to the division of exponentials.

263. The rules that have been demonstrated in the preceding articles may be extended to radical quantities having *irrational* exponents: For instance, $\frac{1}{a\sqrt{2}}, \frac{1}{b\sqrt{3}}$, &c. since that the roots of $\sqrt{2}$ and $\sqrt{3}$ might be obtained with a sufficient degree of approximation, and such that the error may be neglected; so that these exponents shall be terminated decimal fractions, which can be always replaced by ordinary fractions.

264. The formation of the powers of radical quantities, is nothing else but the multiplication of a number of radical quantities of the same denomination, marked by the degree of the power; so that it is sufficient to raise the quantity under the radical sign to the proposed power, and afterwards

to affect this power with the common radical sign. If the index of the radical sign is divisible by the exponent of the power in question, the operation then is performed by dividing that index by the exponent of the power. Let us give two examples for these two cases, $(\sqrt[m]{a^pb^q})^s=\sqrt[m]{a^{ps}b^{bs}}$; $(\sqrt[ms]{a^pb^q})^s=\sqrt[m]{a^pb^q}$.

265. If the exponent of the power is equal to the index of the radical sign, the power is the quantity under the radical sign. In fact, the indication $\sqrt[m]{a^p}$, shows that a^p is the mth power of a certain number $\sqrt[m]{a^p}$, which we can always assign, either rigorously, or by an approximation, so that the mth power of $\sqrt[m]{a^p}$ is a^p. In like manner, the square of $\sqrt{a}$ is a; the cube of $\sqrt[3]{a}$ is a; the 5th power of $\sqrt[5]{(-a^2)}$ is $-a^2$; and so on.

266. A *rational quantity may be reduced to the form of a given surd, by raising it to the power whose root the surd expresses, and prefixing the radical sign.* Thus $a^2=\sqrt[5]{a^{10}}=\sqrt[4]{a^8}=\sqrt[3]{a^6}$, &c. and $a+x=(a+x)^{\frac{m}{m}}$. In the same manner, *the form of any radical quantity may be altered*; thus, $\sqrt{(a+x)}=\sqrt[4]{(a+x)^2}=\sqrt[6]{(a+x)^3}$, &c. or $(a+x)^{\frac{1}{2}}=(a+x)^{\frac{2}{4}}=(a+x)^{\frac{3}{6}}$, &c. Since the quantities are here raised to certain powers, and the roots of those powers are again taken; therefore the values of the quantities are not altered. Also, *the coefficient of a surd may be introduced under the radical sign, by first reducing it to the form of the surd, and then multiplying as in* (Art. 257). Thus, $a\sqrt{x}=\sqrt{a^2}\times\sqrt{x}=\sqrt{a^2x}$; $6\sqrt{2}=\sqrt{36}\times\sqrt{2}=\sqrt{72}$; and $x(2a-x)^{\frac{1}{2}}=(x^2)^{\frac{1}{2}}\times(2a-x)^{\frac{1}{2}}=\sqrt{(2ax^2-x^3)}$.

267. *Conversely, any quantity may be made the coefficient of a surd, if every part under the sign be divided by this quantity, raised to the power whose root the sign expresses.* Thus, $\sqrt{(a^3-a^2x)}=\sqrt{a^2}\times\sqrt{(a-x)}=a\sqrt{(a-x)}$; $\sqrt{60}=\sqrt{(4\times 15)}=\sqrt{4}\times\sqrt{15}=2\sqrt{15}$; and $\sqrt[m]{(a^{mn}-a^mx^n)}=\sqrt[m]{[a^m\times(a^n-x^n)]}=\sqrt[m]{a^m}\times\sqrt[m]{(a^n-x^n)}=a\sqrt[m]{(a^n-x^n)}$.

268. Let us pass to the extraction of roots of radical quantities, and let the mth root of $\sqrt[n]{a^t}$ be required, which we indicate thus, $\sqrt[m]{\sqrt[n]{a^t}}$. We shall put $\sqrt[m]{\sqrt{a^t}}=x$, or $\sqrt[m]{a'}=x$, by making $\sqrt[n]{a^t}=a'$. Involving both sides to the power m, we find a' or $\sqrt[n]{a^t}=x^m$, raising again to the power n, we obtain $a^t=x^{mn}$. If the mnth root of both sides be extracted, we have another enunciation of x; namely,

$\sqrt[mn]{a^t}=x=\sqrt[m]{\sqrt[n]{a^t}}$.

We shall find, by a like calculation,

$$\sqrt[m]{\sqrt[n]{\sqrt[p]{\sqrt[q]{a^t}}}}=\sqrt[mnpq]{a^t}.$$

And, in fact, we make 1st, $\sqrt[n]{\sqrt[p]{\sqrt[q]{a^t}}}=a'$, whence $\sqrt[m]{a'}=x$, and $a'=\sqrt[n]{\sqrt[p]{\sqrt[q]{a^t}}}=x^m$; 2d, by putting $\sqrt[p]{\sqrt[q]{a^t}}=a''$, whence $\sqrt[n]{a''}=x^m$, and $a''=x^{mn}$; 3d, making $\sqrt[q]{a^t}=a'''$, whence $\sqrt[p]{a'''}=x^{mn}$, and $a'''=\sqrt[q]{a^t}=x^{mnp}$; and finally $a^t=x^{mnpq}$, $\therefore x=\sqrt[mnpq]{a^t}$. Thus, for example, the 12th root of the number a can be transformed into $\sqrt[3]{\sqrt[2]{\sqrt[2]{a}}}$.

169. It is to be observed, that radical quantities or surds, when properly reduced, are subject to all the ordinary rules of arithmetic. This is what appears evident from the preceding considerations. It may be likewise remarked, that, in the calculations of surds, fractional exponents are frequently more convenient than radical signs.

§ II. REDUCTION OF RADICAL QUANTITIES OR SURDS.

CASE I.

To reduce a rational quantity to the form of a given Surd.

RULE.

270. Involve the given quantity to the power whose root the surd expresses; and over this power place the radical sign, or proper exponent, and it will be of the form required.

Ex. 1. Reduce a to the form of the *cube root.*

Here, the given quantity a raised to the third power is a^3, and prefixing the sign $\sqrt[3]{}$, or placing the fractional exponent $(\frac{1}{3})$ over it, we have $a=\sqrt[3]{a^3}=(a^3)^{\frac{1}{3}}$ (Art. 251).

271. A rational coefficient may, in like manner, be reduced to the form of the surd to which it is joined; by raising it to the power denoted by the index of the radical sign.

Ex. 2. Let $5\sqrt{a}=\sqrt{25}\times\sqrt{a}=\sqrt{25a}$.

Ex. 3. Reduce $-3a^2b$ to the form of the *cube root.*

Here, $(-3a^2b)^3=-27a^6b^3$; $\therefore -\sqrt[3]{27a^6b^3}$ is the surd required.

Ex. 4. Reduce $-4xy$ to the form of the *square root.*

Here, $(-4xy)^2=16x^2y^2$; $\therefore -4xy=-\sqrt{16x^2y^2}$.

Ex. 5. Reduce $\frac{1}{2}x$ to the form of the *cube root.*

Ans. $(\frac{1}{8}x^3)^{\frac{1}{3}}$.

Ex. 6. Reduce $a+z$ to the form of the *square root*.

Ans. $(a^2+2az+z^2)^{\frac{1}{2}}$

Ex. 7. Reduce $4x^{\frac{1}{4}}$ to the form of the *cube root*.

Ans. $(\sqrt[3]{64x^{\frac{3}{4}}})$ or $(64x^{\frac{3}{4}})^{\frac{1}{3}}$.

Ex. 8. Reduce $-x^{\frac{1}{2}}y^{\frac{1}{2}}$ to the form of the *square root*.

Ans. $-\sqrt{xy}$.

Ex. 9. Reduce $-ab$ to the form of the *square root*.

Ans. $-\sqrt{a^2b^2}$.

CASE II.

To reduce Surds of different indices to other equivalent ones, having a common index.

RULE.

272. Reduce the indices of the given quantities to fractions having a common denominator, and involve each of them to the power denoted by its numerator; then 1 set over the common denominator will form the common index.

Or, if the common index be given, divide the indices of the quantities by the given index, and the quotients will be the new indices for those quantities. Then over the said quantities, with their new indices, set the given index, and they will make the equivalent quantities sought.

Ex. 1. Reduce $\sqrt{a}$ and $\sqrt[3]{b}$ to surds of the same radical sign.

Here, $\sqrt{a}=a^{\frac{1}{2}}$, and $\sqrt[3]{b}=b^{\frac{1}{3}}$. Now, the fractions $\frac{1}{2}$ and $\frac{1}{3}$ reduced to the least common denominator, are $\frac{3}{6}$ and $\frac{2}{6}$;

$\therefore a^{\frac{1}{2}}=a^{\frac{3}{6}}=(a^3)^{\frac{1}{6}}=\sqrt[6]{a^3}$, and $b^{\frac{1}{3}}=b^{\frac{2}{6}}=(b^2)^{\frac{1}{6}}=\sqrt[6]{b^2}$.

Consequently $\sqrt[6]{a^3}$ and $\sqrt[6]{b^2}$ are the surds required

Ex. 2. Reduce $\sqrt{a}$ and $\sqrt[4]{x}$ to surds of the same radical sign $\sqrt[6]{\ }$, or to the common index $\frac{1}{6}$.

(Art. 251), $\sqrt{a}=a^{\frac{1}{2}}$, and $\sqrt[4]{x}=x^{\frac{1}{4}}$; then $\frac{1}{2}\div\frac{1}{6}=\frac{1}{2}\times6=3$; and $\frac{1}{4}\div\frac{1}{6}=\frac{1}{4}\times6=\frac{3}{2}$; $\therefore \sqrt[6]{a^3}$ and $\sqrt[6]{x^{\frac{3}{2}}}$, or $(a^3)^{\frac{1}{6}}$ and $(x^{\frac{3}{2}})^{\frac{1}{6}}$, are the quantities required.

Ex. 3. Reduce a^2 and $b^{\frac{1}{2}}$ to the same radical sign $\sqrt[3]{\ }$.

Ans. $\sqrt[3]{a^6}$, and $\sqrt[3]{b^{\frac{3}{2}}}$.

Ex. 4. Reduce $a^{\frac{1}{4}}$ and $x^{\frac{1}{3}}$ to surds of the same radical sign
Ans. $\sqrt[12]{x^3}$ and $\sqrt[12]{x^4}$.

Ex. 5. Reduce $\sqrt[n]{a}$ and $\sqrt[m]{y}$ to surds of the same radical sign.
Ans. $\sqrt[mn]{a^m}$ and $\sqrt[mn]{y^n}$.

Ex. 6. Reduce $a^{\frac{1}{3}}$ and $b^{\frac{1}{5}}$ to surds of the same radical sign.
Ans. $\sqrt[15]{a^5}$ and $\sqrt[15]{b^3}$.

Ex. 7. Reduce $3\sqrt[3]{2}$ and $2\sqrt{5}$ to the same radical sign.
Ans. $3\sqrt[6]{4}$ and $2\sqrt[6]{125}$.

Ex. 8. Reduce $\sqrt[3]{xy}$ and $\sqrt[4]{ax}$ to the same radical sign.
Ans $\sqrt[12]{x^4y^4}$ and $\sqrt[12]{a^3x^3}$.

CASE III.

To reduce radical Quantities or Surds, to their most simple forms.

RULE.

273. Resolve the given number, or quantity, under the radical sign, if possible, into two factors, so that one of them may be a perfect power; then extract the root of that power, and prefix it, as a coefficient to the irrational part.

Ex 1. Reduce $\sqrt{a^2b}$ to its most simple form.

Here $\sqrt{a^2b}=\sqrt{a^2}\times\sqrt{b}=a\times\sqrt{b}=a\sqrt{b}$.

Ex. 2. Reduce $\sqrt[m]{a^mx}$ to its most simple form.

Here $\sqrt[m]{a^mx}=\sqrt[m]{a^m}\times\sqrt[m]{x}=a^{\frac{m}{m}}\times\sqrt[m]{x}=a\times\sqrt[m]{x}$.

Ex. 3. Reduce $\sqrt{72}$ to its most simple form.

Here $\sqrt{72}=\sqrt{(36\times2)}=\sqrt{36}\times\sqrt{2}=6\sqrt{2}$.

274. When the radical quantity has a rational coefficient prefixed to it; that coefficient must be multiplied by the root of the factor above mentioned; and then proceed as before.

Ex. 4. Reduce $5\sqrt[3]{24}$ to its simplest form.

Here $5\sqrt[3]{24}=5\sqrt[3]{(8\times3)}=5\sqrt[3]{8}\times\sqrt[3]{3}=5\times2\times\sqrt[3]{3}=10\sqrt[3]{3}$.

Ex. 5. Reduce $\sqrt{a^4bc}$ and $\sqrt{98a^2x}$ to their most simple form.
Ans. $a^2\sqrt{bc}$ and $7a\sqrt{2x}$.

Ex. 6. Reduce $\sqrt[4]{243}$ and $\sqrt[5]{96}$ to their most simple form.
Ans. $3\sqrt[4]{3}$ and $2\sqrt[5]{3}$.

Ex. 7. Reduce $\sqrt[3]{(a^3+a^3b^2)}$ to its most simple form.
Ans. $a\sqrt[3]{(1+b^2)}$.

Ex. 8. Reduce $\sqrt{\left(\frac{a^3b-4a^2b^2+4ab^3}{c^2}\right)}$ to its most simple form.
Ans. $\frac{a-2b}{c}\sqrt{ab}$.

Ex. 9. Reduce $(a+b)\sqrt[3]{[(a-b)^3\times x^2]}$ to its most simple form. Ans. $(a^2-b^2)\sqrt[3]{x^2}$.

275. *If the quantity under the radical sign be a fraction, it may be reduced to a whole quantity*, thus :

Multiply both the numerator and denominator by such a quantity as will make the denominator a complete power corresponding to the root ; then extract the root of the fraction whose numerator and denominator are complete powers, and take it from under the radical sign.

Ex. 1. Reduce $\frac{c}{d}\times\sqrt{\frac{a^2}{b}}$ to an *integral* surd in its most simple form.

Here, $\frac{c}{d}\sqrt{\frac{a^2}{b}}=\frac{c}{d}\sqrt{\frac{a^2b}{b^2}}=\frac{c}{d}\sqrt{\frac{a^2}{b^2}\times b}=\frac{c}{d}\times\frac{a}{b}\sqrt{b}=\frac{ca}{bd}\sqrt{b}$.

Ex. 2. Reduce $\frac{1}{3}\sqrt[3]{\frac{16}{81}}$ to an *integral* surd in its simplest form.

Here, $\frac{1}{3}\sqrt[3]{\frac{16}{81}}=\frac{1}{3}\sqrt[3]{\left(\frac{8\times2}{27\times3}\right)}=\frac{1}{3}\sqrt[3]{\frac{8}{27}}\times\sqrt[3]{\frac{2}{3}}=\frac{1}{3}\times\frac{2}{3}\sqrt[3]{\frac{2\times3^2}{3^3}}$

$=\frac{2}{9}\sqrt[3]{(\frac{1}{27}\times18)}=\frac{2}{9}\times\frac{1}{3}\sqrt[3]{18}=\frac{2}{27}\sqrt[3]{18}$.

Ex. 3. Reduce $\frac{3}{4}\sqrt{\frac{2}{7}}$ to an *integral* surd in its most simple form. Ans. $\frac{3}{28}\sqrt{14}$.

Ex. 4. Reduce $x\sqrt{\frac{b}{y}}$ and $a\sqrt[3]{\frac{c^2}{a}}$ to *integral* surds in their most simple form. Ans. $\frac{x}{y}\sqrt{by}$ and $\sqrt[3]{c^2a^2}$.

Ex. 5. Reduce $\sqrt[4]{\frac{1}{3}}$ and $\frac{2}{3}\sqrt{\frac{1}{2}}$ to *integral* surds in their most simple form. Ans. $\frac{1}{3}\sqrt[4]{27}$ and $\frac{1}{3}\sqrt{2}$.

Ex. 6. Reduce $\sqrt[3]{\frac{54}{125}}$ and $\sqrt{\frac{a^3}{8x^4}}$ to their most simple form.

Ans. $\frac{3}{5}\sqrt[3]{2}$ and $\frac{a}{4x^2}\sqrt{2a}$.

276. The utility of reducing surds to their most simple forms, especially when the surd part is fractional, will be readily perceived from the 3d example above given, where it is found that $\frac{3}{4}\sqrt{\frac{2}{7}}=\frac{3}{28}\sqrt{14}$, in which case it is only necessary to extract the square root of the whole number 14, (or to find it in some of the tables that have been calculated for that purpose), and then multiply it by $\frac{3}{28}$; whereas we must, otherwise, have first divided the numerator by the denominator, and then have found the root of the quotient, for the surd part ; or else have determined the root of both the numerator and denominator, and then divide the one by the other ; which are each of them troublesome pro-

cesses; and the labour would be much greater for the cube and other higher roots.

277. There are other cases of reducing algebraic Surds to simpler forms, that are practised on several occasions; for instance, to reduce a fraction whose denominator is irrational, to another that shall have a rational denominator. But, as this kind of reduction requires some farther elucidation, it shall be treated of in one of the following sections.

§ III. APPLICATION OF THE FUNDAMENTAL RULES OF ARITHMETIC TO SURD QUANTITIES.

CASE I.

To add or subtract Surd Quantities.

RULE.

278. Reduce the radical parts to their simplest terms, as in the last case of the preceding section; then, if they are *similar*, annex the common surd part to the sum, or difference of the rational parts, and it will give the sum, or difference required.

Ex. 1. Add $4\sqrt{x}$, $\sqrt{x}$, and $5\sqrt{x}$ together.

Here the radical parts are already in their simplest terms, and the surd part the same in each of them; $\therefore 4\sqrt{x}+\sqrt{x}+5\sqrt{x}=(4+1+5)\times\sqrt{x}=10\sqrt{x}$ the sum required.

Ex. 2. Find the *sum* and *difference* of $\sqrt{16a^2x}$ and $\sqrt{4a^2x}$.

$$\sqrt{16a^2x}=\sqrt{16a^2}\times\sqrt{x}=4a\sqrt{x},$$

and $\sqrt{4a^2x}=\sqrt{4a^2}\times\sqrt{x}=2a\sqrt{x}$;

$\therefore$ the *sum* $=(4a+2a)\times\sqrt{x}=6a\sqrt{x}$;

and the *difference* $=(4a-2a)\times\sqrt{x}=2a\sqrt{x}$.

Ex. 3. Find the *sum* and *difference* of $\sqrt[3]{108}$ and $9\sqrt[3]{32}$.

Here $\sqrt[3]{108}=\sqrt[3]{27}\times\sqrt[3]{4}=3\times\sqrt[3]{4}=3\sqrt[3]{4}$,

and $9\sqrt[3]{32}=9\sqrt[3]{8}\times\sqrt[3]{4}=18\times\sqrt[3]{4}=18\sqrt[3]{4}$,

the *sum* $=(18+3)\times\sqrt[3]{4}=21\sqrt[3]{4}$;

and the *difference* $=(18-3)\times\sqrt[3]{4}=15\sqrt[3]{4}$.

279. If the surd part be not the *same* in each of the quantities, after having reduced the radical parts to their simplest terms, it is evident that the addition or subtraction of such quantities can only be indicated by placing the signs + or $\sim$ between them.

Ex. 4. Find the *sum* and *difference* of $3\sqrt[3]{a^3b}$ and $b\sqrt{c^2d}$.

Here $3\sqrt[3]{a^3b}=3\sqrt[3]{a^3}\times\sqrt[3]{b}=3a\times\sqrt[3]{b}=3a\sqrt[3]{b}$,

and $b\sqrt{c^2d}=b\sqrt{c^2}\times\sqrt{d}=bc\times\sqrt{d}=bc\sqrt{d}$;

the *sum* $=3a\sqrt[3]{b}+bc\sqrt{d}$·

and the *difference* $=3a\sqrt[3]{b}\sim bc\sqrt{d}$.

Ex. 5. Find the *sum* and *difference* of $\sqrt{\frac{8}{27}}$ and $\sqrt{\frac{1}{6}}$.

Ans. The *sum* $=\frac{7}{18}\sqrt{6}$, and *difference* $=\frac{1}{18}\sqrt{6}$.

Ex. 6. Find the *sum* and *difference* of $\sqrt{27a^4x}$ and $\sqrt{3a^4x}$.

Ans. The *sum* $=4a^2\sqrt{3x}$, and *difference* $=2a^2\sqrt{3x}$.

Ex. 7. Find the *sum* and *difference* of $\frac{1}{2}\sqrt{a^2b}$ and $\frac{1}{3}\sqrt{bx^4}$.

Ans. The *sum* $=\left(\frac{2x^2+3a}{6}\right)\sqrt{b}$, and *difference* $\left(\frac{2x^2\sim 3a}{6}\right)$ $\sqrt{b}$.

Ex. 8. Required the *sum* and *difference* of $3\sqrt[3]{625}$ and $2\sqrt[3]{135}$.

Ans. The *sum* $=21\sqrt[3]{5}$, and *difference* $=9\sqrt[3]{5}$.

Ex. 9. Required the *sum* and *difference* of $\sqrt[4]{a^6b^2}$ and $\sqrt[3]{x^5y^2}$.

Ans. The *sum* $=a\sqrt{ab}+x\sqrt[3]{x^2y^2}$, and *difference* $=a\sqrt{ab}\sim x\sqrt[3]{x^2y^2}$.

CASE II.

To multiply or divide Surd Quantities.

RULE.

280. Reduce them to equivalent ones of the same denomination, and then multiply or divide both the rational and the irrational parts by each other respectively.

The product or quotient of the irrational parts may be reduced to the most simple form, by the last case in the preceding section.

Ex. 1. Multiply $\sqrt{a}$ by $\sqrt[3]{b}$, or $a^{\frac{1}{2}}$ by $b^{\frac{1}{3}}$.

The fractions $\frac{1}{2}$ and $\frac{1}{3}$, reduced to *a common denominator*, are $\frac{3}{6}$ and $\frac{2}{6}$.

$$\therefore a^{\frac{1}{2}}=a^{\frac{3}{6}}=\sqrt[6]{a^3}\text{; and } b^{\frac{1}{3}}=b^{\frac{2}{6}}=\sqrt[6]{b^2}.$$

Hence $\sqrt{a}\times\sqrt[3]{b}=\sqrt[6]{a^3}\times\sqrt[6]{b^2}=\sqrt[6]{a^3b^2}$.

Ex. 2. Multiply $2\sqrt{3}$ by $3\sqrt[3]{4}$.

By *reduction*, $2\sqrt{3}=2\times 3^{\frac{3}{6}}=2\times\sqrt[6]{3^3}=2\sqrt[6]{27}$;

and $3\sqrt[3]{4}=3\times 4^{\frac{2}{6}}=3\sqrt[6]{4^2}=3\sqrt[6]{16}$.

$\therefore 2\sqrt{3}\times 3\sqrt[3]{4}=2\sqrt[6]{27}\times 3\sqrt[6]{16}=6\sqrt[6]{432}$.

Ex. 3. Divide $8\sqrt[3]{512}$ by $4\sqrt[3]{2}$.

Here $8\div 4=2$, and $\sqrt[3]{512}\div\sqrt[3]{2}=\sqrt[3]{256}=4\sqrt[3]{4}$.

$\therefore 8\sqrt[3]{512}\div 4\sqrt[3]{2}=2\times 4\sqrt[3]{4}=8\sqrt[3]{4}$.

Ex. 4. Divide $2\sqrt[3]{bc}$ by $3\sqrt{ac}$.

Now $2\sqrt[3]{bc}=2\times(bc)^{\frac{1}{3}}=2\times(bc)^{\frac{2}{6}}=2\sqrt[6]{b^2c^2}$, and $3\sqrt{ac}=3\times(ac)^{\frac{1}{2}}=3\times(ac)^{\frac{3}{6}}=3\sqrt[6]{a^3c^3}$;

$$\therefore \frac{2\sqrt[3]{bc}}{3\sqrt{ac}}=\frac{2}{3}\times\sqrt[6]{\frac{b^2c^2}{a^3c^3}}=\frac{2}{3}\sqrt[6]{\frac{b^2}{a^3c}}=\frac{2}{3}\sqrt[6]{\frac{b^2a^3c^5}{a^6c^6}}=\frac{2}{3ac}\sqrt[6]{b^2a^3c^5}.$$

281. *If two surds have the same rational quantity under the radical signs, their product, or quotient, is obtained by making the sum, or difference, of the indices, the index of that quantity.*

Ex. 5. Multiply $\sqrt[3]{a^4}$ by $\sqrt[3]{a^2}$ or $a^{\frac{4}{3}}$ by $a^{\frac{2}{3}}$.

Here, $a^{\frac{4}{3}}\times a^{\frac{2}{3}}=a^{\frac{4}{3}+\frac{2}{3}}=a^{\frac{6}{3}}=a^2$. Or $\sqrt[3]{a^4}\times\sqrt[3]{a^2}=\sqrt[3]{(a^4\times a^2)}$ $=\sqrt[3]{a^6}=a^2$, as before.

Ex. 6. Divide $\sqrt[4]{a^3}$ by $\sqrt[3]{a^4}$, or $a^{\frac{3}{4}}$ by $a^{\frac{4}{3}}$.

Here, $a^{\frac{3}{4}}\div a^{\frac{4}{3}}=a^{\frac{3}{4}-\frac{4}{3}}=a^{\frac{9}{12}-\frac{16}{12}}=a^{-\frac{7}{12}}=\frac{1}{a^{\frac{7}{12}}}=\sqrt[12]{\frac{1}{a^7}}$.

282. *If compound surds are to be multiplied, or divided, by each other, the operation is usually performed as in the multiplication, or division of compound algebraic quantities.* It frequently happens that the *division* of compound surds can only be indicated.

Ex. 7. Multiply $\sqrt{3}-\sqrt[3]{a^2}$ by $\sqrt[3]{3}+\sqrt[3]{a}$.

$$\begin{array}{l} \sqrt{3}-\sqrt[3]{a^2} \\ \sqrt[3]{3}+\sqrt[3]{a} \\ \hline \sqrt[6]{243}-\sqrt[3]{(3a^2)} \\ \quad +\sqrt[6]{(27a^2)}-a \\ \hline \end{array}$$

Since $\sqrt{3}\times\sqrt[3]{3}=3^{\frac{3}{6}}\times3^{\frac{2}{6}}=\sqrt[6]{(3^3\times3^2)}=\sqrt[6]{(27\times9)}=\sqrt[6]{243}$

Product $=\sqrt[6]{243}-\sqrt[3]{(3a^2)}+\sqrt[6]{27a^2}-a$.

Ex. 8. Divide $\sqrt{b^2ca}+\sqrt{a^2b}-bc-\sqrt{abc}$ by $\sqrt{bc}+\sqrt{a}$.

$$\begin{array}{r|l} \sqrt{b^2ca}+\sqrt{a^2b}-bc-\sqrt{abc} & \sqrt{bc}+\sqrt{a} \\ \sqrt{b^2ca}+\sqrt{a^2b} \quad\quad\quad\quad & \text{Quot.}=\sqrt{ba}-\sqrt{bc}. \\ \hline -bc-\sqrt{abc} & \\ -bc-\sqrt{abc} & \\ \hline \end{array}$$

Ex. 9. Multiply $\sqrt[3]{15}$ by $\sqrt{10}$. Ans. $\sqrt[6]{225000}$.

Ex. 10. Multiply $\frac{1}{2}\sqrt[3]{6}$ by $\frac{2}{3}\sqrt[3]{18}$. Ans. $\sqrt[3]{4}$.

Ex. 11. Multiply $\sqrt[3]{18}$ by $\sqrt[3]{4}$. Ans. $2\sqrt[3]{9}$.

Ex. 12. Multiply $\frac{1}{4}\sqrt[3]{6}$ by $\frac{2}{15}\sqrt[3]{9}$. Ans. $\frac{1}{10}\sqrt[3]{2}$.

Ex. 13. Divide $4\sqrt{50}$ by $2\sqrt{5}$. Ans. $2\sqrt{10}$.

Ex. 14. Divide $\frac{3}{4}\sqrt{\frac{5}{6}}$ by $\frac{1}{2}\sqrt{\frac{1}{3}}$. Ans $\frac{3}{4}\sqrt{10}$.

Ex. 15. Divide $\sqrt[6]{a^2d^3b^2}$ by $\sqrt{d}$. Ans. $\sqrt[3]{ab}$.

Ex. 16. Multiply $a^{\frac{2}{3}}x^{\frac{1}{2}}$ by $a^{\frac{1}{4}}x^{\frac{1}{6}}$. Ans. $a^{\frac{11}{12}}x^{\frac{2}{3}}$.

Ex. 17. Multiply $\sqrt[5]{a^2b^3c^4}$ by $\sqrt[\frac{5}{4}]{a^2b^3c^4}$. Ans. $a^2b^3c^4$.

Ex. 18. Divide $(a^4+b^3)^{\frac{1}{2}}$ by $(a^4+b^3)^{\frac{1}{3}}$

Ans. $\sqrt[6]{(a^4+b^3)}$.

Ex 19. Multiply $4+2\sqrt{2}$ by $2-\sqrt{2}$. Ans. 4.

Ex. 20. Multiply $\sqrt{(a-\sqrt{(b-\sqrt{3}))}}$ by $\sqrt{(a+\sqrt{(b-\sqrt{3}))}}$.

Ans. $\sqrt{(a^2-b+\sqrt{3})}$.

Ex. 21. Divide a^3b-ab^2c by $a^2+a\sqrt{bc}$.

Ans. $ab-b\sqrt{bc}$.

Ex. 22. Divide a^4+x^4 by $a^2+ax\sqrt{2}+x^2$

Ans. $a^2-ax\sqrt{2}+x^2$.

283. It is proper to observe, since the powers and roots of quantities may be expressed by negative exponents, *that any quantity may be removed from the denominator of a fraction into the numerator; and the contrary, by changing the sign of its index or exponent*; which transformation is of frequent occurrence in several *analytical* calculations.

Ex. 1. Thus, (since $\frac{1}{b^3}=b^{-3}$), $\frac{a^2}{b^3}$ may be expressed by a^2b^{-3}; and (since $a^2=\frac{1}{a^{-2}}$), we have $\frac{a^2}{b^3}=\frac{1}{b^3a^{-2}}$.

Ex. 2. The quantity $\frac{a^2b^3}{c^4e^5}$ may be expressed by $a^2b^3c^{-4}e^{-5}$.

Ex. 3. Let the denominator of $\frac{a^{\frac{1}{2}}x^{\frac{2}{3}}}{c\,b^2}$ be removed into the numerator. Ans. $a^{\frac{1}{2}}x^{\frac{2}{3}}c^{-1}b^{-2}$.

Ex. 4. Let the numerator of $\frac{a^2x^3}{b}$ be removed into the denominator. Ans. $\frac{1}{a^{-2}x^{-3}b}$.

Ex. 5. Let $x^2y^2a^{\frac{1}{3}}$ be expressed with a negative exponent.

Ans. $\frac{1}{x^{-2}y^{-2}a^{-\frac{1}{3}}}$.

CASE III.

To involve or raise Surd Quantities to any power.

RULE.

284. Involve the rational part into the proposed power, then multiply the fractional exponents of the surd part by the index of that power, and annex it to the power of the rational part, and the result will be the power required.

Compound surds are involved as integers, observing the rule of multiplication of simple radical quantities.

Ex. 1. What is the square of $2\sqrt{a}$?

The *square* of $2\sqrt{a}=(2a^{\frac{1}{2}})^2=2^2\times a^{\frac{1\cdot 2}{2}}=4a$.

Ex. 2. What is the cube of $\sqrt[3]{(a^2-b^2+\sqrt{3})}$?

The cube of $\sqrt[3]{(a^2-b^2+\sqrt{3})}=(a^2-b^2+\sqrt{3})^{\frac{1\cdot 3}{3}}=a^2-b^2+\sqrt{3}$.

285. *Cor.* Hence, if the quantities are to be involved to a power denoted by the index of the surd root, the power required is formed by taking away the radical sign, as has been already observed.

Ex. 3. What is the *cube* of $\frac{1}{2}\sqrt{2ax}$?

Here $(\frac{1}{2})^3=\frac{1}{8}$, and $(\sqrt{2ax})^3=(2ax)^{\frac{1\cdot 3}{2}}=(2ax)^{\frac{3}{2}}$
$=(2ax)\times(2ax)^{\frac{1}{2}}$; $\therefore \frac{1}{8}\times 2ax\times(2ax)^{\frac{1}{2}}=$
$\frac{1}{4}ax\sqrt{2ax}$ is the power required.

Ex. 4. It is required to find the square of $\sqrt{a}-\sqrt{b}$.

$$\begin{array}{l} \sqrt{a}-\sqrt{b} \\ \sqrt{a}-\sqrt{b} \\ \hline a-\sqrt{ab} \\ \quad -\sqrt{ab}+b \\ \hline \end{array}$$

The square $a-2\sqrt{ab}+b$.

Ex. 5. It is required to find the square of $3\sqrt[3]{3}$.
Ans. $9\sqrt[3]{9}$.

Ex. 6. Find the cube of $\sqrt{a}$. Ans. $a\sqrt{a}$

Ex. 7. Find the 4th power of $-\sqrt[3]{a^2}$. Ans. $a^2\sqrt[3]{a^2}$.

Ex. 8. Find the 5th power of $-\sqrt[5]{ab}$. Ans. $-ab$.

Ex. 9. Required the cube of $a-\sqrt{b}$.
Ans. $a^3-3a^2\sqrt{b}+3ab-b\sqrt{b}$

Ex. 10. Required the square of $3+\sqrt{5}$.

Ans. $14+6\sqrt{5}$.

Ex. 11. Required the cube of $-\sqrt[3]{(\sqrt{a}-\sqrt{bc})}$.

Ans. $\sqrt{bc}-\sqrt{a}$.

CASE IV.

To evolve or extract the Roots of Surd Quantities.

RULE.

286. Divide the index of the irrational part by the index of the root to be extracted; then annex the result to the proper root of the rational part, and they will give the root required.

If it be a compound surd quantity, its root, if it admits of any, may be found, as in Evolution. And if no such root can be found, prefix the radical sign, which indicates the root to be extracted.

Ex. 1. What is the *square root* of $81\sqrt{a}$?

Here $\sqrt{81}=9$, and the square root of $\sqrt{a}$ or $a^{\frac{1}{2}}=a^{\frac{1}{2}}\div 2=a^{\frac{1}{2}}\times{}^{\frac{1}{2}}=a^{\frac{1}{4}}=\sqrt[4]{a}$; $\therefore \sqrt{(81\sqrt{a})}=9\sqrt[4]{a}$, or $9a^{\frac{1}{4}}$.

Ex. 2. What is the *square root* of $a^2-6a\sqrt{b}+9b$.

$$\begin{array}{l} a^2-6a\sqrt{b}+9b(a-3\sqrt{b} \\ a^2 \\ \hline 2a-3\sqrt{b})-6a\sqrt{b}+9b \\ \quad\quad -6a\sqrt{b}+9b \\ \hline \end{array}$$

Ex. 3. Find the *square root* of $9\sqrt[3]{3}$. Ans. $3\sqrt[6]{3}$.

Ex. 4. Find the 4th *root* of $\frac{16}{81}\sqrt[3]{a^2}$. Ans. $\frac{2}{3}\sqrt[6]{a}$.

Ex. 5. Find the *cube root* of $(5a^2-3x^2)^{\frac{3}{2}}$.

Ans. $\sqrt{(5a^2-3x^2)}$.

Ex. 6. Required the *cube root* of $\frac{1}{8}a^3b$. Ans. $\frac{1}{2}a\sqrt[3]{b}$.

Ex. 7. What is the *fifth root* of $32\sqrt[3]{x^5}$? Ans. $2\sqrt[3]{x}$.

Ex. 8. What is the 4th root of $16a\sqrt[2]{x}$? Ans. $2\sqrt[8]{a^4x}$.

Ex. 9. What is the *n*th root of $\sqrt[m]{a^nx^2}$?

Ans. $a^{\frac{1}{m}}x^{\frac{2}{mn}}$.

Ex. 10. It is required to find the cube root of $a^3-3a^2\sqrt{x}+3ax-x\sqrt{x}$. Ans. $a-\sqrt{x}$.

§ IV. METHOD OF REDUCING A FRACTION, WHOSE DENOMINATOR IS A SIMPLE OR A BINOMIAL SURD, TO ANOTHER THAT SHALL HAVE A RATIONAL DENOMINATOR.

287. A fraction, whose denominator is a simple surd, is of the form $\frac{a}{\sqrt[n]{x}}$; where x may represent any rational quantities whatever, either simple or compound; thus,

$$\frac{bc}{\sqrt{ab}},\ \frac{a}{\sqrt[n]{(a^2-b)}},\ \frac{c-d}{\sqrt[3]{(a+y)}},\ \&c.$$

are fractions, whose denominators are simple surd quantities.

288. It is evident that, if a surd of the form $\sqrt[n]{x}$ be multiplied by $\sqrt[n]{x^{n-1}}$, the product shall be rational; since $\sqrt[n]{x}\times\sqrt[n]{x^{n-1}}=\sqrt[n]{(x\times x^{n-1})}=\sqrt[n]{x^n}=x$; in like manner, if $\sqrt[3]{(a+x)}$ be multiplied by $\sqrt[3]{(a+x)^2}$, the product will be $a+x$.

289. Hence, *if the numerator and denominator of a fraction of the form* $\frac{a}{\sqrt[n]{x}}$ *be multiplied by* $\sqrt[n]{x^{n-1}}$, *the result will be a fraction, whose denominator shall be rational.*

Thus, let both the numerator and denominator of the fraction $\frac{a}{\sqrt{x}}$ be multiplied by $\sqrt{x}$, and it becomes $\frac{a\sqrt{x}}{x}$; and by multiplying the numerator and denominator of the fraction $\frac{\sqrt[3]{b}}{\sqrt[3]{(a+x)}}$, by $\sqrt[3]{(a+x)^2}$, it becomes $\frac{\sqrt[3]{[b(a+x)^2]}}{\sqrt[3]{(a+x)^3}}=\frac{b^{\frac{1}{3}}(a+x)^{\frac{2}{3}}}{a+x}$. Or, in general, if both the numerator and denominator of a fraction of the form $\frac{a}{\sqrt[n]{x}}$ be multiplied by $\sqrt[n]{x^{n-1}}$, it becomes $\frac{a\sqrt[n]{x^{n-1}}}{x}$, a fraction whose denominator is a rational quantity.

290. Compound surd quantities are such as consist of two or more terms, some or all of which are irrational; and if a quantity of this kind consist only of *two* terms, it is called a binomial surd; and a fraction whose denominator is a *binomial* surd, is, in general, of the form $\frac{x}{\sqrt[n]{a}\pm\sqrt[n]{b}}$.

291. If a multiplier be required, that shall render any binomial surd, whether it consist of *even* or *odd* roots, rational, it may be found by substituting the given numbers, or letters, of

which it is composed, in the places of their equals, in the following general formula:

Binomial, $\sqrt[n]{a} \pm \sqrt[n]{b}$.

Multiplier, $\sqrt[n]{a^{n-1}} \mp \sqrt[n]{a^{n-2}b} + \sqrt[n]{a^{n-3}b^2} \mp \sqrt[n]{a^{n-4}b^3} +$, &c., where the upper sign of the multiplier must be taken with the upper sign of the binomial, and the lower with the lower; and the series continued to n terms. This multiplier is derived from observing the quotient which arises from the actual division of the numerator by the denominator of the following fractions: thus,

I. $\frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 +$, &c. . $+y^{n-1}$ to n terms, whether n be *even* or *odd*, (Art. 108).

II. $\frac{x^n - y^n}{x + y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 -$, &c. . . $-y^{n-1}$ to n terms, when n is an *even* number, (Art. 109).

III. $\frac{x^n + y^n}{x + y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 -$, &c. . . $+y^{n-1}$ to n terms, when n is an *odd* number, (Art. 110).

292. Now let $x^n = a$, $y^n = b$; then, (Art. 116), $x = \sqrt[n]{a}$, $y = \sqrt[n]{b}$, and these fractions severally become $\frac{a-b}{\sqrt[n]{a} - \sqrt[n]{b}}$, $\frac{a-b}{\sqrt[n]{a} + \sqrt[n]{b}}$, and $\frac{a+b}{\sqrt[n]{a} + \sqrt[n]{b}}$; and by the application of the rules in the preceding section we have $x^{n-1} = \sqrt[n]{a^{n-1}}$; $x^{n-2} = \sqrt[n]{a^{n-2}}$, $x^{n-3} = \sqrt[n]{a^{n-3}}$, &c. also, $y^2 = \sqrt[n]{b^2}$; $y^3 = \sqrt[n]{b^3}$; &c.; hence, $x^{n-2}y = \sqrt[n]{a^{n-2}} \times \sqrt[n]{b} = \sqrt[n]{a^{n-2}b}$; $x^{n-3}y^2 = \sqrt[n]{a^{n-3}} \times \sqrt[n]{b^2} = \sqrt[n]{a^{n-3}b^3}$; &c. By substituting these values of x^{n-1}, $x^{n-2}y$, $x^{n-3}y^2$, &c., in the several quotients, we have $\frac{a-b}{\sqrt[n]{a} - \sqrt[n]{b}} = \sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}b} + \sqrt[n]{a^{n-3}b^2} +$, &c. . . . . $+\sqrt[n]{b^{n-1}}$ to n terms; where n may be any whole number whatever. And $\frac{a \pm b}{\sqrt[n]{a} + \sqrt[n]{b}} = \sqrt[n]{a^{n-1}} - \sqrt[n]{a^{n-2}b} + \sqrt[n]{a^{n-3}b^2} -$, &c. . . . $\pm \sqrt[n]{b^{n-1}}$ to n terms; where the terms b and $\sqrt[n]{b^{n-1}}$ have the sign $+$, when n is an *odd* number: and the sign $-$, when n is an *even* number.

293. Since the *divisor* multiplied by the *quotient* gives the *dividend*, it appears from the foregoing operations that, *if a binomial surd of the form* $\sqrt[n]{a} - \sqrt[n]{b}$ *be multiplied by* $\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}b} +$, &c. . $+\sqrt[n]{b^{n-1}}$ (*n being any* whole number whatever), *the product will* be $a-b$, a rational quantity; and if a binomial surd of the form $\sqrt[n]{a} + \sqrt[n]{b}$ be multiplied by $\sqrt[n]{a^{n-1}} - \sqrt[n]{a^{n-2}b} + \sqrt[n]{{}^{n-3}b^2} -$, &c. . . . $\pm \sqrt[n]{b^{n-1}}$, the product will be

$a+b$ or $a-b$, according as the index n is an *odd* or an *even* number.

294. Hence it follows, that, *if the numerator and denominator of the fraction* (Art. 290), *be multiplied by the multiplier*, (Art. 291), *it becomes another equivalent fraction, whose denominator shall be rational.*

There are some instances, in which the reduction may be performed without the formal application of the rule, which will be illustrated in the following examples.

Ex. 1. Reduce $\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}$ to a fraction with a rational denominator.

To find the multiplier which shall make $\sqrt{5}-\sqrt{3}$ *rational*, we have $n=2$, $a=5$, $b=3$; $\therefore$ (Art. 291), $\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}b}$ $=$(since $a^{n-2}=a^{2-2}=a^0=1$) $\sqrt{5}+\sqrt{3}$; $\therefore \frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}\times\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}=\frac{16+4\sqrt{15}}{2}=8+2\sqrt{15}$.

295. This multiplier, $\sqrt{5}+\sqrt{3}$, could be readily ascertained, without the application of the formula, by inspection only; since the *sum* into the *difference* of two quantities gives the *difference of their squares*; also the multiplier that shall render $\sqrt{a}+\sqrt{b}$ *rational*, is evidently $\sqrt{a}-\sqrt{b}$. In like manner, a trinomial surd may also be rendered rational, by changing the sign of one of its terms for a multiplier; and a quadrinomial surd by changing the signs of two of its terms, &c.

Ex. 2. Reduce $\frac{2}{\sqrt{5}+\sqrt{3}-\sqrt{2}}$ to a fraction with a rational denominator.

In the first place, $\frac{2}{\sqrt{5}+\sqrt{3}-\sqrt{2}}\times\frac{\sqrt{5}+\sqrt{3}+\sqrt{2}}{\sqrt{5}+\sqrt{3}+\sqrt{2}}=$ $\frac{2(\sqrt{5}+\sqrt{3}+\sqrt{2})}{6+2\sqrt{15}}$; $\therefore \frac{\sqrt{5}+\sqrt{3}+\sqrt{2}}{3+\sqrt{15}}\times\frac{-3+\sqrt{15}}{-3+\sqrt{15}}=$ $\frac{(\sqrt{5}+\sqrt{3}+\sqrt{2})\times(-3+\sqrt{15})}{6}$ is the fraction required.

Ex. 3. Reduce $\frac{1}{\sqrt[3]{3}-\sqrt[3]{2}}$ to a fraction with a rational denominator.

To find the multiplier which shall make $\sqrt[3]{3}-\sqrt[3]{2}$ *rational*, we have $n=3$, $a=3$, $b=2$; $\therefore \sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}b}+\sqrt[n]{b^{n-1}}=$ $\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4}$.

Now $(\sqrt[3]{3}-\sqrt[3]{2})(\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4})=a-b=3-2=1$;

∴ the *denominator* is 1, and the fraction is reduced to $\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4}$.

296. Hence for the *sum*, or *difference*, of two cube roots, which is one of the most useful cases, the multiplier will be a trinomial surd consisting of the squares of the two given terms, and their product, with its sign changed.

Ex. 4. Reduce $\frac{3\sqrt{15}-4\sqrt{5}}{\sqrt{15}+\sqrt{5}}$ to a fraction with a rational denominator. Ans. $\frac{13-7\sqrt{3}}{2}$.

Ex. 5. Reduce $\frac{3}{\sqrt{5}-\sqrt{x}}$ to a fraction with a rational denominator. Ans. $\frac{3\sqrt{5}+3\sqrt{x}}{5-x}$.

Ex. 6. Reduce $\frac{8}{\sqrt{3}+\sqrt{2}+1}$ to a fraction whose denominator shall be rational. Ans. $4+2\sqrt{2}-2\sqrt{6}$.

Ex. 7. Reduce $\frac{a}{\sqrt[3]{x}+\sqrt[3]{y}}$ to a fraction whose denominator shall be rational.

Ans. $\frac{a}{x+y}(\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2})$.

Ex. 8. Reduce $\frac{2}{\sqrt[4]{5}+\sqrt[4]{3}}$ to a fraction whose denominator shall be rational. Ans. $\sqrt[4]{125}-\sqrt[4]{75}+\sqrt[4]{45}-\sqrt[4]{27}$.

297. It may not be improper to take notice here of another transformation which binomial surd quantities may undergo by equal involution, and evolution.

Ex. 1. To transform $\sqrt{2}+\sqrt{3}$ to a universal surd.

Its square $=5+2\sqrt{6}$; ∴ the root $=\sqrt{(5+2\sqrt{6})}$.

Ex. 2. To reduce $\sqrt{27}+\sqrt{48}$ to a universal surd.

Here $(\sqrt{27}+\sqrt{48})^2=27+2\sqrt{1296}+48=147$; ∴ $\sqrt{27}+\sqrt{48}=\sqrt{147}=\sqrt{49\times 3}=7\sqrt{3}$.

Ex. 3. To transform $\sqrt[3]{320}-\sqrt[3]{40}$ to a general surd.

Here $(\sqrt[3]{320}-\sqrt[3]{40})^3=320-3\sqrt[3]{4096000}+3\sqrt[3]{512000}-40=40$; ∴ $\sqrt[3]{320}-\sqrt[3]{40}=2\sqrt[3]{5}$.

298. This transformation is very useful, since, by means of it, we can always reduce the sum or difference of any two surd quantities, if they admit of the same irrational part, to a single surd. This may be proved, in general, thus; if $\sqrt[n]{a}$ and $\sqrt[n]{b}$ admit of the same irrational part, they must be of the form $\sqrt[n]{a'^n m}$ and $\sqrt[n]{b'^n m}$; and $(\sqrt[n]{a'^n m}+\sqrt[n]{b'^n m})^n=a'^n m+n$

$\sqrt[n]{}(a'^{n(n-1)}m^{n-1}b'^{n}m)+\frac{n(n-1)}{2}\sqrt[n]{}(a'^{n(n-2)}m^{n-2}b'^{2n}m^3)+\&c.$. $b'^n m=a'^n m+na'^{n-1}\times mb'^n+\&c.\ldots\ldots b'^n m$ $\therefore\sqrt[n]{a}+\sqrt[n]{b}=\sqrt[n]{}(a'^n m+nma'^{n-1}b'^n+\&c.\ldots\ldots b'^n m)=$ the nth root of a rational quantity. Hence the product of $\sqrt{a}$ by $\sqrt{b}$ is rational if $\sqrt{a}$ and $\sqrt{b}$ admit of the same irrational part; also, $\sqrt[3]{a^2}\times\sqrt[3]{b}$, or $\sqrt[3]{a}\times\sqrt[3]{b^2}$, is rational, if $\sqrt[3]{a}$ and $\sqrt[3]{b}$ admit of the same irrational part; and, in general, $\sqrt[n]{a^{-1}}\times\sqrt[n]{b}$, or $\sqrt[n]{a}\times\sqrt[n]{b^{n-1}}$, is rational, if $\sqrt[n]{a}$ and $\sqrt[n]{b}$ admit of the same irrational part.

299. It is proper to observe, that, for the addition or subtraction of two quadratic surds, the following method is given in the BIJA GANITA, or the *Algebra* of the HINDOOS, translated by STRACHEY. Thus, *to find the sum or difference of two surds, $\sqrt{a}$ and $\sqrt{b}$, for instance.*

RULE.

Call $a+b$ the greater surd; and, if $a\times b$ is rational, (that is, *a square*), call $2\sqrt{ab}$ the less surd, the sum will be $\sqrt{(a+b+2\sqrt{ab})}$, $(=(\sqrt{a}\pm\sqrt{b})^2)$, and the difference $\sqrt{(a+b-2\sqrt{ab})}$. If $a\times b$ is irrational, the addition and subtraction are impossible; that is, they can only be indicated.

Example. Required the sum and difference of $\sqrt{2}$ and $\sqrt{8}$. Here $2+8=10=>$ surd; $2\times8=16$, $\therefore\sqrt{16}=4$, and $2\sqrt{16}=2\times4=8=<$surd. Then $10+8=18$, and $10-8=2$; $\therefore\sqrt{18}=$ sum, and $\sqrt{2}=$ difference.

ANOTHER RULE.

Divide a by b, and write $\sqrt{\frac{a}{b}}$ in two places. In the first place add 1, and in the second subtract 1; then we shall have $\sqrt{[\left(\sqrt{\frac{a}{b}}+1\right)^2\times b]}=\sqrt{a}+\sqrt{b}$, and $\sqrt{[\left(\sqrt{\frac{a}{b}}-1\right)^2\times b]}=\sqrt{a}-\sqrt{b}$.

If $\sqrt{\frac{a}{b}}$ is irrational, (that is, not a square), the addition or subtraction can be only made by connecting the surds by the signs + or − 1, as they are.

STURMIUS, in his *Mathesis Enucleata*, has also given a method similar to the above.

Ex. 4. To transform $\sqrt{2+\sqrt{3}}$ to a general surd.

Ans. $\sqrt{(5+2\sqrt{6})}$.

Ex. 5. To transform $\sqrt{a}-2\sqrt{x}$ to a universal surd.

Ans. $\sqrt{(a+4x-4\sqrt{ax})}$.

Ex. 6. To transform $3\sqrt[3]{\frac{1}{3}}+\sqrt[3]{72}$ to a universal surd.

Ans. $3\sqrt[3]{9}$.

§ V. METHOD OF EXTRACTING THE SQUARE ROOT OF BINOMIAL SURDS.

300. *The square root of a quantity cannot be partly rational and partly a quadratic surd.* If possible, let $\sqrt{n}=a+\sqrt{m}$; then by squaring both sides, $n=a^2+2a\sqrt{m}+m$, and $2a\sqrt{m}=n-a^2-m$; therefore, $\sqrt{m}=\frac{n-a^2-m}{2a}$, a rational quantity, which is contrary to the supposition.

A quantity of the form $\sqrt{a}$ is called a *quadratic surd.*

301. *If any equation $x+\sqrt{y}=a+\sqrt{b}$, consisting of rational quantities and quadratic surds, the rational parts on each side are equal, and also the irrational parts.*

If x be not equal to a, let $x=a+m$; then $a+m+\sqrt{y}=a+\sqrt{b}$, or $m+\sqrt{y}=\sqrt{b}$; that is, $\sqrt{b}$ is partly rational, and partly a quadratic surd, which is impossible, (Art. 300); $\therefore x=a$, and $\sqrt{y}=\sqrt{b}$.

302. *If two quadratic surds $\sqrt{x}$ and $\sqrt{y}$, cannot be reduced to others which have the same irrational part, their product is irrational.*

If possible, let $\sqrt{xy}=rx$, where r is a whole number or a fraction. Then $xy=r^2x^2$, and $y=r^2x$; $\therefore \sqrt{y}=r\sqrt{x}$; that is, $\sqrt{y}$ and $\sqrt{x}$ may be so reduced as to have the same irrational part, which is contrary to the supposition.

303. *One quadratic surd, $\sqrt{x}$, cannot be made up of two others, $\sqrt{m}$ and $\sqrt{n}$, which have not the same irrational part.*

If possible, let $\sqrt{x}=\sqrt{m}+\sqrt{n}$; then by squaring both sides, $x=m+2\sqrt{mn}+n$, and $x-m-n=2\sqrt{mn}$, a rational quantity equal to an irrational, which is absurd.

304. *Let $(a+b)^{\frac{1}{c}}=x+y$, where* c *is an even number,* a *a rational quantity,* b *a quadratic surd, x and y, one or both of them, quadratic surds, then $(a-b)^{\frac{1}{c}}=x-y$.*

By involution, $a+b=x^c+cx^{c-1}y+c.\frac{c-1}{2}x^{c-2}y^2+$ &c., and since c is even, the odd terms of the series are rational, and the even terms irrational; $\therefore a=x^c+c.\frac{c-1}{2}x^{c-2}y^2+$ &c., and $b=cx^{c-1}y+c.\frac{c-1}{2}.\frac{c-2}{3}x^{c-3}y^3+$ &c., (Art. 301); hence, $a-b$

$=x^c-cx^{c-1}y+c\cdot\frac{c-1}{2}x^{c-2}y^2-$, &c.; and consequently, by evolution, $(a-b)^{\frac{1}{c}}=x-y$.

305. *If c be an odd number, a and b, one or both quadratic surds, and x and y involve the same surds that a and b do respectively, and also $(a+b)^{\frac{1}{c}}=x+y$, then $(a-b)^{\frac{1}{c}}=x-y$.*

By involution, $a+b=x^c+cx^{c-1}y+c\cdot\frac{c-1}{2}x^{c-2}y^2+$, &c., where the odd terms involve the same surd that x does, because c is an odd number, and the even terms, the same surd that y does; and since no part of a can consist of y and its parts, (Art. 301), $a=x^c+c\cdot\frac{c-1}{2}x^{c-2}y^2+$, &c., and $b=cx^{c-1}y$ $+c\cdot\frac{c-1}{2}\cdot\frac{c-2}{3}\cdot x^{c-3}y^3+$, &c.; hence, $a-b=x^c-cx^{c-1}y$ $+c\cdot\frac{c-1}{2}x^{c-2}y^2-$, &c.; $\therefore$ by evolution, $(a-b)^{\frac{1}{c}}=x-y$.

306. *The square root of a binomial, one of whose terms is a quadratic surd, and the other rational, may sometimes be expressed by a binomial, one or both of whose terms are quadratic surds.*

Let $a+\sqrt{b}$ be the given binomial, and suppose $\sqrt{(a+\sqrt{b})}$ $=x+y$; where x and y are one or both quadratic surds; then $\sqrt{(a-\sqrt{b})}=x-y$; $\therefore$ by multiplication, $\sqrt{(a^2-b)}=x^2-y^2$, also, by squaring both sides of the first equation,
$a+\sqrt{b}=x^2+2xy+y^2$, and $a=x^2+y^2$;
$\therefore$ by addition, $a+\sqrt{(a^2-b)}=2x^2$, and by subtraction, $a-\sqrt{(a^2-b)}=2y^2$; and the root $x+y=\sqrt{[\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-b)}]}+\sqrt{[\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-b)}]}$.

From this conclusion it appears, that the square root of $a+\sqrt{b}$ can only be expressed by a binomial of the form $x+y$, one or both of which are quadratic surds, when a^2-b is a perfect square.

By a similar process it might by shown that the square root of $a-\sqrt{b}$ is $\sqrt{[\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-b)}]}-\sqrt{[\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-b)}]}$, subject to the same limitation.

Ex. 1. Required the square root of $3+2\sqrt{2}$.

Let $\sqrt{(3+2\sqrt{2})}=x+y$; then $\sqrt{(3-2\sqrt{2})}=x-y$; by multiplication, $\sqrt{(9-8)}=x^2-y^2$; that is, $x^2-y^2=1$.

Also, by squaring both sides of the first equation, $3+2\sqrt{2}$ $=x^2+2xy-y^2$, and $x^2+y^2=3$; $\therefore$ by addition, $2x^2=4$, and $x=\sqrt{2}$.

Again, by subtraction, $2y^2=2$; $\therefore y=1$, and $x+y=\sqrt{2}+1$ = the root required.

Or, the root may be found by substituting 3 for a, $2\sqrt{2}=\sqrt{8}$ for $\sqrt{b}$, or 8 for b, in the above formula; thus,
$x+y=\sqrt{[\frac{3}{2}+\frac{1}{2}\sqrt{(9-8)}]}+\sqrt{[\frac{3}{2}-\frac{1}{2}\sqrt{(9-8)}]}=\sqrt{(\frac{3}{2}+\frac{1}{2})}+\sqrt{(\frac{3}{2}-\frac{1}{2})}=\sqrt{2}+1$.

Ex. 2. Required the square root of $19+8\sqrt{3}$.

Ans. $4+\sqrt{3}$.

Ex. 3. What is the square root of $12-\sqrt{140}$?

Ans. $\sqrt{7}-\sqrt{5}$.

Ex. 4. Find the square root of $7+4\sqrt{3}$.

Ans. $2+\sqrt{3}$.

Ex. 5. Find the square root of $7-2\sqrt{10}$.

Ans. $\sqrt{5}-\sqrt{2}$.

Ex. 6. Find the square root of $31+12\sqrt{-5}$.

Ans. $6+\sqrt{-5}$.

Ex. 7. Find the square root of $18-10\sqrt{-7}$.

Ans. $5-\sqrt{-7}$.

Ex. 8. Find the square root of $-1+4\sqrt{-5}$.

Ans. $2+\sqrt{-5}$.

307. *The* c*th root of a binomial, one or both of whose terms are possible quadratic surds, may sometimes be expressed by a binomial of that description.*

Let A+B be the given binomial surd, in which both terms are possible; the quantities under the radical signs whole numbers; and A is greater than B.

Let $\sqrt[c]{[(A+B)\times\sqrt{Q}]}=x+y$;
then $\sqrt[c]{[(A-B)\times\sqrt{Q}]}=x-y$;

$\therefore$ by multiplication, $\sqrt[c]{[(A^2-B^2)\times Q]}=x^2-y^2$; now let Q be so assumed, that $(A^2-B^2)\times Q$ may be a perfect cth power $=n^c$, then $x^2-y^2=n$.

Again, by squaring both sides of the first two equations, we have

$\sqrt[c]{[(A+B)^2\times Q]}=x^2+2xy+y^2$
$\sqrt[c]{[(A-B)^2\times Q]}=x^2-2xy+y^2$;

$\therefore \sqrt[c]{[(A+B)^2\times Q]}+\sqrt[c]{[(A-B)^2\times Q]}=2x^2+2y^2$; which is always a whole number when the root is a binomial surd; take therefore s and t, the nearest integer values of $\sqrt[c]{[(A+B)^2\times Q]}$ and $\sqrt[c]{[(A-B)^2\times Q]}$, one of which is greater and the other less than the true value of the corresponding quantity; then since the sum of these surds is an integer, the fractional parts must destroy each other, and $2x^2+2y^2=s+t$, exactly, when the root of the proposed quantity can be obtained. We have therefore these two equations, $x^2-y^2=n$, and $x^2+y^2=\frac{1}{2}s$

$+\frac{1}{2}t$; $\therefore$ by addition, $2x^2=n+\frac{1}{2}s+\frac{1}{2}t$, and $x=\frac{1}{2}\sqrt{(2n+s+t)}$; and by subtraction, $2y^2=\frac{1}{2}s+\frac{1}{2}t-n$, and $y=\frac{1}{2}\sqrt{(s+t-2n)}$.

Consequently, if the root of the binomial $\sqrt[c]{[(A+B)\times\sqrt{Q}]}$ be of the form $x+y$, it is $\frac{1}{2}\sqrt{(2n+s+t)}+\frac{1}{2}\sqrt{(s+t-2n)}$; and the cth root of $A+B$ is $\dfrac{\sqrt{(2n+s+t)}+\sqrt{(s+t-2n)}}{2\sqrt[2c]{Q}}$.

Ex. 1. Required the cube root of $10+\sqrt{108}$.

In this case, $\sqrt{108}$ is >10; $\therefore A=\sqrt{108}$, $B=10$, $A^2-B^2=108-100=8$, and $8Q=n^3$. Now, since 8 is a cube number, Q may be taken equal to 1; then $8Q=8=n^3$; $\therefore n=2$. Also, $\sqrt[3]{[(A+B)^2]}=7+f$; $\sqrt[3]{[(A-B)^2]}=1-f$, where f is some fraction less than unity; $\therefore s=7$, $t=1$; and $x+y=\dfrac{\sqrt{12}+2}{2}$ $=\sqrt{3}+1$.

If therefore the cube of $10+\sqrt{108}$ can be expressed in the proposed form, it is $\sqrt{3}+1$; which on trial is found to succeed.

Ex. 2. Find the cube root of $26+15\sqrt{3}$.

Ans. $2+\sqrt{3}$.

Ex. 3. Find the cube root of $9\sqrt{3}-11\sqrt{2}$.

Ans. $\sqrt{3}-\sqrt{2}$.

Ex. 4. Find the cube root of $4\sqrt{5}+8$.

Ans. $\dfrac{\sqrt{5}+1}{\sqrt[3]{2}}$.

308. In the operation, it is required to find a number Q, such that $(A^2-B)^2\times Q$ may be a perfect cth power; this will be the case, if Q be taken equal to $(A^2-B^2)^{c-1}$; but to find a less number which will answer this condition, let A^2-B^2 be divisible by $a, a, \ldots (m)$; $b, b, \ldots (n)$; $d, d, \ldots (r)$; &c. in succession, that is, let $A^2-B^2=a^m b^n d^r$, &c; also, let $Q=a^x b^y d^z$, &c. Then $(A^2-B^2).Q=a^{m+x}\times b^{n+y}\times d^{r+z}$, &c. which is a perfect cth power, if x, y, z, &c., be so assumed that $m+x$, $n+y$, $r+z$, &c. are respectively equal to c, or some multiple of c. Thus, to find a number which multiplied by 2250 will produce a perfect cube, divide 2250 as often as possible by the prime numbers 2, 3, 5, &c. and it appears that $2\times3\times3\times5\times5\times5=2\times3^2\times5^3=2250$; if, therefore, it be multiplied by $2^2\times3$, it becomes $2^3\times3^3\times5^3$, or $(2.3.5)^3$; a perfect cube. See WOOD'S ALGEBRA

§ VI. CALCULATION OF IMAGINARY QUANTITIES.

309. In the Involution of negative quantities, it was observed, that the even powers were all affected with the sign +, and the odd powers, with —; there is consequently no quan-

tity which, multiplied into itself in such a manner that the number of factors shall be *even*, can generate a negative quantity. Hence quantities of the form $\sqrt{-a^2}$, $\sqrt[4]{-16}$, $\sqrt[6]{-a^3}$, $\sqrt[4]{-a^4}$, and in general $\sqrt[m]{-a}$, have no *real* roots; and are therefore usually called *impossible* or *imaginary*.

It is to be observed that all quantities, either *positive* or *negative*, or even *irrational*, are considered to be *real*.

310. Although the values of imaginary quantities are unassignable in numbers, they are yet of great use in some of the higher branches of analysis, as well as in showing when a result of this kind occurs, that the question, under the proposed conditions, is impossible.

Thus, if it should be required *to find a number whose square subtracted from* 3, *gives* 7 *for a remainder*. We have for a translation

$$3-x^2=7\ ;\ \therefore\ x^2=3-7=-4.$$

The unknown quantity x is therefore the square root of the number -4, a root which is imaginary; and in fact, the enunciation comprehends an impossibility. If we had thus proposed the question, *to find a number whose square added to* 3, *gives* 7 *for a sum*, we should have had for the translation $x^2+3=7$; $\therefore$ $x^2=4$ and $x=2$, which is a *real* root.

Thus negative isolated results arise from the subtraction of a greater number from a lesser, and imaginary quantities are given by a new operation to be performed upon these kind of remainders.

311. This being premised, it is only necessary farther to observe, that the method of adding and subtracting imaginary radicals, is the same as for real quantities.

Thus, $\sqrt[4]{-a}+2\sqrt[4]{-a}=3\sqrt[4]{-a}$; $6+\sqrt{-4}+6-\sqrt{-4}=12$; and $3\sqrt{-ax}+\sqrt[4]{-y}-(\sqrt{-ax}-\sqrt[4]{-y})=2\sqrt{-ax}+2\sqrt[4]{-y}$.

312. *Every imaginary radical quantity of the form* $\sqrt{-a}$, *can be reduced to the form* $\sqrt{a}\times\sqrt{-1}$, *or* $a^{\frac{1}{2}}\sqrt{-1}$.

In order to demonstrate this, let the identical equality be, $(c-b)a=(c-b)a$; by extracting the root of both sides, we shall have $\sqrt{(c-b)}\times\sqrt{a}=\sqrt{[(c-b)a]}$; which under the relation $b>c$, or in the hypotheses, for instance, $b=c+1$, becomes $\sqrt{-1}\times\sqrt{a}=\sqrt{-a}$; and, in general, $\sqrt[2n]{-a}=\sqrt[2n]{a}\times\sqrt{-1}$.

It may be demonstrated, in a similar manner, that $\sqrt{-\dfrac{c}{m}}$

$=\sqrt{\frac{c}{m}}\times\sqrt{-1}$; and in general, that $\sqrt[2n]{-\frac{c}{m}}=\sqrt[2n]{\frac{c}{m}}\times\sqrt[2n]{-1}$.

313. Hence, in the calculation of imaginary radicals, it is sufficient to demonstrate the rules for multiplying and involving the imaginary radical $\sqrt{-1}$; since imaginary quantities can be always resolved into factors; so that -1 only shall remain under the radical sign.

314. In the first place, then it may be observed, when a^2 is considered abstractedly, or without any regard to its generation, then $\sqrt{a^2}$ may be either $+a$ or $-a$ there being nothing in the nature of the quantity so taken, to denote from which of these two expressions it was derived.

315. But this ambiguity, which, in the above mentioned case, arises from our being unacquainted with the origin of the quantity whose root is to be extracted, will not take place when the sign of the quantity from which it was produced is known; as there can, then, be only one root, which must evidently be taken in *plus* or *minus*, according to the state it existed in before it was involved.

316. Thus, $\sqrt{[(+a)\times(a)]}$, or $\sqrt{[(+a^2)]}$ cannot be of the ambiguous form $\pm a$, as it would have been if a^2 had been unconditionally assumed, but it is simply a; and, for a like reason, $\sqrt{[(-a)\times(-a)]}$, or $\sqrt{(-a)^2}$ is $=-a$, and not $\pm a$; since the value of the equivalent expression $+\sqrt{a^2}$, or $-\sqrt{a^2}$ in these cases, is determined, from the circumstance of its being known how a^2 is derived.

317. Hence *the product of* $\sqrt{-1}$ *by* $\sqrt{-1}$, or which is the same, $(\sqrt{-1})^2$ is $=-\sqrt{1}=-1$. This is what appears evident from, since that in squaring a quantity with the radical sign $\sqrt{}$, we have only to take it away, that is, to pass the quantity from under the radical sign.

318. Also, *if the factors, in this case, be both negative, the result will be the same as before;* since $-(\sqrt{-1})\times-(\sqrt{-1})$ $=+(\sqrt{-1})^2=-1$; but *if one of the factors be positive and the other negative*, we shall have $+(\sqrt{-1})\times-(\sqrt{-1})=-(\sqrt{-1})^2=+1$.

319. *All whole positive numbers are comprised in one of these four formulæ;*

$$4n,\ 4n+1,\ 4n+2,\ 4n+3,$$

n being a whole positive number; since that, if any whole number be divided by 4, the remainder must be 0, 1, 2, or 3.

If we designate $\sqrt{-1}$ by x, the several powers of $\sqrt{-1}$ shall be therefore represented by one of these four formulæ:
$(\sqrt{-1})^{4n}=x^{4n}=(x^4)^n=(+1)^n=+1$;
$(\sqrt{-1})^{4n+1}=x^{4n+1}=x^{4n}.x=x=+\sqrt{-1}$;
$(\sqrt{-1})^{4n+2}=x^{4n+2}=x^{4n}.x^2=x^2=-1$.
$(\sqrt{-1})^{4n+3}=x^{4n+3}=x^{4n}.x^3=-1.x=-\sqrt{-1}$.

Thus, in order to know any given power of $\sqrt{-1}$, *it is sufficient to divide the exponent of the power proposed by* 4, *and the power of* $\sqrt{-1}$ *indicated by the remainder shall be that which is required.*

320. *When one imaginary quantity is to be multiplied by another, the result whether they be both positive or both negative, is equal to* minus *the square root of the product, taking them as real quantities.*

Thus, $(+\sqrt{-a})\times(+\sqrt{-b})=-\sqrt{ab}$; since, $(+\sqrt{-a})\times(+\sqrt{-b})=\sqrt{a}\times\sqrt{-1}\times\sqrt{b}\times\sqrt{-1}=\sqrt{a}\times\sqrt{b}\times(\sqrt{-1})^2=-1\times\sqrt{ab}=-\sqrt{ab}$. And, in a similar manner, it may be proved that $(-\sqrt{-a})\times(-\sqrt{-b})=-\sqrt{ab}$.

321. *And if one of the imaginary radicals be positive, and the other negative, the* result arising from their multiplication *will be* plus *the square root of their product, taking them as before.*

Thus, $(+\sqrt{-a})\times(-\sqrt{-b}=+\sqrt{ab}$; since $+\sqrt{-a}=+\sqrt{a}\times\sqrt{-1}$, and $-\sqrt{-b}=(-\sqrt{-1})\times\sqrt{b}$; $\therefore(\sqrt{a}\times\sqrt{-1})\times(-\sqrt{-1})\times\sqrt{b})=[(+\sqrt{-1}).(-\sqrt{-1})]\sqrt{ab}=+1\times\sqrt{ab}=+\sqrt{ab}$.

322. *When one imaginary radical is to be divided by another, the result, whether they be both positive or both negative, will be equal to* plus *the square root of their quotient, taking them as real quantities.*

Thus, $\frac{+\sqrt{-a}}{+\sqrt{-b}}$ or $\frac{-\sqrt{-a}}{-\sqrt{-b}}=+\sqrt{\frac{a}{b}}$; and $\frac{+\sqrt{-a}}{+\sqrt{-a}}$ or $\frac{-\sqrt{-a}}{-\sqrt{-a}}=1$.

323. *And if one of the imaginary radicals be positive and the other negative, the result arising from division, will be* minus *the square root of their quotient, taking them as before.*

Thus, $\frac{+\sqrt{-a}}{-\sqrt{-b}}$ or $\frac{-\sqrt{-a}}{+\sqrt{-b}}=-\sqrt{\frac{a}{b}}$; and $\frac{+\sqrt{-a}}{-\sqrt{-a}}$ or $\frac{-\sqrt{-a}}{+\sqrt{-a}}=-1$.

324. *If an imaginary radical is to be divided by a real radical, or a real radical by an imaginary one, the result will be equal to plus or minus the square root of their quotient, according as the radical is affirmative or negative.*

Thus, $\frac{\sqrt{-a}}{\sqrt{b}}$ or $\frac{\sqrt{a}}{\sqrt{-b}}=+\sqrt{-\frac{a}{b}}$; and $\frac{\sqrt{-a}}{-\sqrt{a}}$ or $\frac{-\sqrt{a}}{\sqrt{-a}}=$ $-\sqrt{-1}$.

The several powers of imaginary radicals can be readily derived from the formulæ (Art. 319); it only now remains to illustrate the preceding rules by a few practical examples.

Ex. 1. It is required to multiply $a-\sqrt{-b}$ by $a-\sqrt{-b}$, or to find the square of $a-\sqrt{-b}$.

$$\begin{array}{l} a-\sqrt{-b} \\ a-\sqrt{-b} \\ \hline a^2-a\sqrt{-b} \\ \quad -a\sqrt{-b}-b \\ \hline a^2-2a(\sqrt{-b})-b \text{ Ans} \end{array}$$

Ex. 2. It is required to find the quotient of $1+\sqrt{-1}$ divided by $1-\sqrt{-1}$.

Here $\frac{1+\sqrt{-1}}{1-\sqrt{-1}}=\frac{1+\sqrt{-1}}{1-\sqrt{-1}}\times\frac{1+\sqrt{-1}}{1+\sqrt{-1}}=\frac{2\sqrt{-1}}{2}=\sqrt{-1}$. Ans.

Ex. 3. It is required to multiply $1+\sqrt{-1}$ by $1+\sqrt{-1}$; or to find the square of $1+\sqrt{-1}$. Ans. $2\sqrt{-1}$.

Ex. 4. It is required to find the product arising from multiplying $1+\sqrt{-1}$ by $1-\sqrt{-1}$ Ans. 2.

Ex. 5. It is required to find the square, or second power of $a+\sqrt{-b^2}$. Ans. $a^2-b^2+2ab\sqrt{-1}$.

Ex. 6. It is required to multiply $5+2\sqrt{-3}$ by $2-\sqrt{-3}$. Ans. $16-\sqrt{-3}$.

Ex. 7. It is required to find the cube, or third power, of $a-\sqrt{-b^2}$. Ans. $a^3-3ab^2+(b^3-3a^2b)\sqrt{-1}$.

Ex. 8. It is required to find the quotient of $3+\sqrt{-4}$ divided by $3-2\sqrt{-1}$. Ans. $\frac{1}{13}(5+12\sqrt{-1})$

Ex. 9. It is required to find the square of $\sqrt{(a+b\sqrt{-1})}+\sqrt{(a-b\sqrt{-1})}$. Ans. $2a+2\sqrt{(a^2+b^2)}$

CHAPTER VIII.

ON

PURE EQUATIONS.

325. Equations are considered as of two kinds, called simple or *pure*, and *adfected;* each of which are differently denominated according to the dimensions of the unknown quantity.

326. If the equation, when cleared of fractions and radical signs or fractional exponents, contain only the *first power* of the unknown quantity, it is called a *simple equation.*

327. If the unknown quantity rises to the *second power* or *square*, it is called a *quadratic equation.*

328. If the unknown quantity rises to the *third power* or *cube*, it is called a *cubic equation*, &c.

329. *Pure equations*, in general, *are those wherein only one complete power of the unknown quantity is concerned.* These are called *pure equations* of the *first degree*, *pure quadratics*, *pure cubics*, *pure biquadratics*, &c., according to the dimension of the unknown quantity.

Thus, $x=a+b$ is a *pure* equation of the *first degree;*

$x^2=a^2+ab$ is a *pure quadratic;*

$x^3=a^3+a^2b+c$ is a *pure cubic;*

$x^4=a^4+a^3b+ac^2+d$ is a *pure biquadratic;* &c.

330. *Adfected equations are those wherein different powers of the unknown quantity are concerned, or are found in the same equation.* These are called *adfected quadratics*, *adfected cubics*, *adfected biquadratics*, &c., according to the highest dimension or power of the unknown quantity.

Thus, $x^2+ax=b$, is an adfected quadratic

$x^3+ax^2+bx=c$, an adfected cubic;

$x^4+ax^3+bx^2+cx=d$, an adfected biquadratic.

In like manner other adfected equations are denominated according to the highest power of the unknown quantities.

§ I. SOLUTION OF PURE EQUATIONS OF THE FIRST DEGREE BY INVOLUTION.

331. We have already delivered, under the denomination of Simple Equations, the methods of resolving *pure equations* of the

first degree, in all cases, except when the quantity is affected with radical signs or fractional exponents, in which case the following rule is to be observed.

RULE.

332. If the equation contains a single radical quantity, transpose all the other terms to the contrary side; then involve each side into the power denominated by the index of the surd; from whence an equation will arise free from radical quantities, which may be resolved by the rules pointed out in Chap. III.

If there are more than one radical sign over the quantity, the operation must be repeated; and if there are more than one surd quantity in the equation, let the most complex of those surds be brought by itself on one side, and then proceed as before.

Ex. 1. Given $\sqrt{(4x+16)}=12$, to find the value of x.

Squaring both sides of the equation, $4x+16=144$;

by transposition, $4x=144-16$; $\therefore x=32$.

Ex. 2. Given $\sqrt[3]{(2x+3)}+4=7$, to find the value of x.

By transposition, $\sqrt[3]{(2x+3)}=7-4=3$;

cubing both sides, $2x+3=27$;

by transposition, $2x=27-3$; $\therefore x=12$.

Ex. 3. Given $\sqrt{(12+x)}=2+\sqrt{x}$, to find the value of x.

By squaring, $12+x=4+4\sqrt{x}+x$;

by transposition, $8=4\sqrt{x}$, or $\sqrt{x}=2$;

$\therefore$ by squaring, $x=4$.

Ex. 4. Given $\sqrt{(x+40)}=10-\sqrt{x}$, to find the value of x.

By squaring, $x+40=100-20\sqrt{x}+x$;

by transposition, $20\sqrt{x}=60$, or $\sqrt{x}=3$;

$\therefore$ by squaring, $x=9$.

Ex. 5. Given $\sqrt{(x-16)}=8-\sqrt{x}$, to find the value of x.

By squaring both sides of the equation,

$x-16=64-16\sqrt{x}+x$; $\therefore 16\sqrt{x}=64+16=80$;

by division, $\sqrt{x}=5$; $\therefore x=25$.

Ex. 6. Given $\sqrt{(x-a)}=\sqrt{x}-\frac{1}{2}\sqrt{a}$, to find the value of x.

Squaring both sides of the equation,

$x-a=x-\sqrt{(ax)}+\frac{1}{4}a$;

$\therefore$ by transposition, $\sqrt{(ax)}=\frac{5}{4}a$;

by squaring, $ax=\dfrac{25a^2}{16}$; $\therefore x=\dfrac{25a}{16}$.

Ex. 7. Given $\sqrt{5}\times\sqrt{(x+2)}=\sqrt{5x}+2$, to find the value of x.

By squaring, $5x+10=5x+4\sqrt{5x}+4$;
by transposition, $6=4\sqrt{5x}$; $\therefore \sqrt{5x}=\frac{3}{2}$;
by squaring again, $5x=\frac{9}{4}$; $\therefore x=\frac{9}{20}$.

Ex. 8. Given $\frac{x-ax}{\sqrt{x}}=\frac{\sqrt{x}}{x}$, to find the value of x.

Multiplying both sides of the equation by $\sqrt{x}$,

$$x-ax=\frac{x}{x}=1, \text{ or } (1-a)x=1; \therefore x=\frac{1}{1-a}.$$

Ex. 9. Given $\frac{\sqrt{x}+28}{\sqrt{x}+4}=\frac{\sqrt{x}+38}{\sqrt{x}+6}$, to find the value of x.

Multiplying both sides by $(\sqrt{x}+4)\times(\sqrt{x}+6)$,
we have $x+34\sqrt{x}+168=x+42\sqrt{x}+152$;
by transposition, $16=8\sqrt{x}$, or $2=\sqrt{x}$;
$\therefore$ by squaring, $x=4$.

Ex. 10. Given $\frac{\sqrt{ax}-b}{\sqrt{ax}+b}=\frac{3\sqrt{ax}-2b}{3\sqrt{ax}+5b}$, to find the value of x.

Multiplying both sides by $(\sqrt{ax}+b)\times(3\sqrt{ax}+5b)$,
$3ax+2b\sqrt{ax}-5b^2=3ax+b\sqrt{ax}-2b^2$,
$\therefore$ by transposition, $b\sqrt{ax}=3b^2$;
by division, $\sqrt{ax}=3b$;
$\because$ by squaring, $ax=9b^2$, and $x=\frac{9b^2}{a}$.

Ex. 11. Given $\sqrt{(x+\sqrt{x})}-\sqrt{(x-\sqrt{x})}=\frac{3}{2}\sqrt{\left(\frac{x}{x+\sqrt{(x)}}\right)}$, to find the value of x.

Multiply both sides of the equation by $\sqrt{(x+\sqrt{x})}$, $x+\sqrt{(x)}-\sqrt{(x^2-x)}=\frac{3\sqrt{x}}{2}$,

$\therefore$ by transposition, $x-\frac{\sqrt{(x)}}{2}=\sqrt{(x^2-x)}$;
and dividing by $\sqrt{x}$, $\sqrt{x}-\frac{1}{2}=\sqrt{(x-1)}$;
$\therefore$ by squaring, $x-\sqrt{x}+\frac{1}{4}=x-1$; $\therefore \sqrt{x}=\frac{5}{4}$,
and by squaring, $x=\frac{25}{16}$.

Ex. 12. Given $\sqrt{(x-24)}=\sqrt{x}-2$, to find the value of x.
Ans. $x=49$.

Ex. 13. Given $\sqrt{(4a+x)}=2\sqrt{(b+x)}-\sqrt{x}$, to find the value of x.
Ans. $x=\frac{(b-a)^2}{2a-b}$.

Ex. 14. Given $x+a+\sqrt{(2ax+x^2)}=b$, to find the value of x.
Ans. $\frac{(b-a)^2}{2b}$.

Ex. 15. Given $\frac{\sqrt{x+2a}}{\sqrt{x+b}}=\frac{\sqrt{x+4a}}{\sqrt{x+3b}}$, to find the value of x.

Ans. $x=\left(\frac{ab}{a-b}\right)^2$.

Ex. 16. Given $\frac{3x-1}{\sqrt{3x+1}}=1+\frac{\sqrt{3x-1}}{2}$, to find the value of x.

Ans. $x=3$.

Ex. 17. Given $x=\sqrt{[a^2+x\sqrt{(b^2+x^2)}]}-a$, to find the value of x.

Ans. $x=\frac{b^2-4a^2}{4a}$.

Ex. 18. Given $\sqrt{(2+x)}+\sqrt{x}=\frac{4}{\sqrt{(2+x)}}$, to find the value of x.

Ans. $x=\frac{2}{3}$.

Ex. 19. Given $\sqrt[3]{(10x+35)}-1=4$, to find the value of x.

Ans. $x=9$.

Ex. 20. Given $\sqrt[5]{(9x-4)}+6=8$, to find the value of x.

Ans. $x=4$.

Ex. 21. Given $\sqrt{(x+16)}=2+\sqrt{x}$, to find the value of x.

Ans. $x=9$.

Ex. 22. Given $\sqrt{(x-32)}=16-\sqrt{x}$, to find the value of x.

Ans. $x=81$.

Ex. 23. Given $\sqrt{(4x+21)}=2\sqrt{x}+1$, to find the value of x.

Ans. $x=25$.

Ex. 24. Given $\sqrt{[1+x\sqrt{(x^2+12)}]}=1+x$, to find the value of x.

Ans. $x=2$.

Ex. 25. Given $\sqrt{x}+\sqrt{(x-9)}=\frac{36}{\sqrt{(x-9)}}$, to find the value of x.

Ans. $x=25$.

Ex. 26. Given $\sqrt[m]{(a+x)}=\sqrt[2m]{(x^2+5ax+b^2)}$, to find the value of x.

Ans. $x=\frac{a^2-b^2}{3a}$.

Ex. 27. Given $\frac{\sqrt{9x}-4}{\sqrt{x}+2}=\frac{15+\sqrt{9x}}{\sqrt{x}+40}$, to find the value of x.

Ans. $x=4$.

Ex. 28. Given $\frac{\sqrt{6x}-2}{\sqrt{6x}+2}=\frac{4\sqrt{6x}-9}{4\sqrt{6x}+6}$, to find the value of x.

Ans. $x=6$.

Ex. 29. Given $\frac{5x-9}{\sqrt{5x+3}}-1=\frac{\sqrt{5x-3}}{2}$, to find the value of x.

Ans. $x=5$.

Ex. 30. Given $\frac{ax-b^2}{\sqrt{ax}+b}=c+\frac{\sqrt{ax}-b}{c}$, to find the value of x.

Ans. $x=\frac{1}{a}\cdot\left(b+\frac{c^2}{c-1}\right)^2$.

§ II. SOLUTION OF PURE EQUATIONS OF THE SECOND, AND OTHER HIGHER DEGREES, BY EVOLUTION.

RULE.

333. Transpose the terms of the equation in such a manner, that the given power of the unknown quantity may be on one side of the equation, and the known quantities on the other; then extract the root, denoted by the exponent of the power, on each side of the equation, and the value of the unknown quantity will be determined. In the same way any adfected equation, having that side which contains the unknown quantity, a complete power, may be reduced to a simple equation, from which the value of the unknown quantity will be ascertained, by the rules in Chap. III.

Ex. 1. Given $x^2-17=130-2x^2$, to find the values of x.

By transposition, $3x^2=147$;

$\therefore$ by division, $x^2=49$,

and by evolution, $x=\pm 7$.

334. It has been already observed, that $\sqrt[2n]{a}$ may be either $+$ or $-$, where n is any whole number whatever; and, consequently, all pure equations of the second degree admit of two solutions. Thus, $+7\times+7$, and -7×-7, are both equal to 49; and both, when substituted for x in the original equation, answer the condition required.

Ex. 2. Given $x^2+ab=5x^2$, to find the values of x.

By transposition, $4x^2=ab$;

$\therefore 2x=\pm\sqrt{ab}$, and $x=\pm\frac{1}{2}\sqrt{ab}$.

Ex. 3. Given $x^2-6x+9=a^2$, to find the values of x.

By evolution, $x-3=\pm a$; $\therefore x=3\pm a$.

Ex. 4. Given $4x^2-4ax+a^2=x^2+12x+36$, to find the value of x.

By extracting the square root on both sides, we have $2x-a=x+6$;

$\therefore$ by transposition, $x=a+6$.

Ex. 5. Given $\left.\begin{array}{r}x^2+y^2=13,\\ \text{and } x^2-y^2=5,\end{array}\right\}$ to find the values of x and y.

By addition, $2x^2=18$; $\therefore x=\pm\sqrt{9}=\pm 3$.

By subtraction, $2y^2=8$; $\therefore y=\pm\sqrt{4}=\pm 2$.

Ex. 6. Given $81x^4=256$, to find the values of x.

By extracting the square root, $9x^2=\pm 16$;
By extracting again, $3x=\pm\sqrt{\pm 16}=\pm 4$, or $\pm 4\sqrt{-1}$;
$\therefore x=\pm\frac{4}{3}$, or $x=\pm\frac{4}{3}\sqrt{-1}$.

Ex. 7. Given $x^6-3x^4+3x^2-1=27$, to find the values of x.
By evolution, $x^2-1=3$; $\therefore x^2=4$, and $x=\pm 2$.

Ex. 8. Given $36x^2=a^2$, to find the values of x.
Ans. $x=\pm\frac{1}{6}a$.

Ex. 9. Given $x^3=27$, to find the value of x. Ans. $x=3$.

Ex. 10. Given $x^2+6x+9=25$, to find the values of x.
Ans. $x=2$, or -8.

Ex. 11. Given $3x^2-9=21+3$, to find the values of x.
Ans. $x=\pm\sqrt{11}$.

Ex. 12. Given $x^3-x^2+\frac{1}{3}x-\frac{1}{27}=a^3$, to find the values of x.
Ans. $x=a+\frac{1}{3}$.

Ex. 13. Given $x^2+\frac{2}{3}x+\frac{1}{9}=a^2b^2$, to find the values of x.
Ans. $x=\pm ab-\frac{1}{3}$.

Ex. 14. Given $x^2+bx+\frac{1}{4}b^2=a^2$, to find the values of x.
Ans. $x=\pm a-\frac{1}{2}b$.

Ex. 15. Given $x^4-2x^2+1=9$, to find the values of x.
Ans. $x=\pm 2$, or $\pm\sqrt{-2}$.

Ex. 16. Given $x^4-4x^2+4=4$, to find the values of x.
Ans. $x=\pm 2$, or $\pm\sqrt{0}$.

Ex. 17. Given $5x^2-27=3x^2+215$, to find the values of x.
Ans. $x=\pm 11$.

Ex. 18. Given $5x^2-1=244$, to find the values of x.
Ans. $x=\pm 7$.

Ex. 19. Given $9x^2+9=3x^2+63$, to find the values of x.
Ans. $x=\pm 3$.

Ex. 20. Given $2ax^2+b-4=cx^2-5+d-ax^2$, to find the values of x. Ans. $x=\pm\sqrt{\dfrac{d-b-1}{3a-c}}$.

Ex. 21. Given $x^4+y^4=a$ and $x^4-y^4=b$, to find the values of x and y.
Ans. $x=\pm\sqrt{(\pm\frac{1}{2}\sqrt{(2a+2b)})}$ and $y=\pm\sqrt{(\pm\frac{1}{2}\sqrt{(2a-2b)})}$.

§ III. EXAMPLES IN WHICH THE PRECEDING RULES ARE APPLIED IN THE SOLUTION OF PURE EQUATIONS.

335. *When the terms of an equation involve powers of the unknown quantity placed under radical signs.*

Let the equation be cleared of radical signs, as in Sect. I; then, the value of the unknown quantity will be determined by extracting the root, as in Sect. II.

And by a similar process, any equation containing the pow ers of a function of the unknown quantity, or containing the powers of two unknown quantities, may frequently be reduced to lower dimensions.

Ex. 1. Given $\sqrt[3]{x^2}=\sqrt[3]{(a+b)}$, to find the values of x.

Cubing both sides, $x^2=a+b$;

$$\therefore x=\pm\sqrt{(a+b)}.$$

Ex. 2. Given $\sqrt[4]{(x^2-9)}=\sqrt{(x-3)}$; to find the values of x.

Here, the given quantity may be exhibited under the form $(x^2-9)^{\frac{1}{4}}=(x-3)^{\frac{1}{2}}$; then, by squaring both sides, $(x^2-9)^{\frac{1}{4}\times 2}=(x-3)^{\frac{1}{2}\times 2}$, or $(x^2-9)^{\frac{1}{2}}=x-3$;

by squaring again, $x^2-9=x^2-6x+9$;

$\therefore$ by transposition, $6x=18$; and $x=3$.

Ex. 3. Given $x^2-y^2=9$, and $x-y=1$; to find the values of x and y.

Dividing the corresponding members of the first equation by those of the second, we have $x+y=9$;

adding this equation to the second, $2x=10$;

$\therefore x=5$, and $y=9-x$; $\therefore y=4$.

Ex. 4. Given $\left.\begin{array}{l}\sqrt{x}+\sqrt{y}=5,\\ \text{and } \sqrt{x}-\sqrt{y}=1,\end{array}\right\}$ to find the values of x and y.

Adding the two equations, $2\sqrt{x}=6$, $\therefore \sqrt{x}=3$,

and by involution, $x=9$.

Subtracting the two equations, $2\sqrt{y}=4$, and $\sqrt{y}=2$;

$\therefore$ by involution, $y=4$.

Ex. 5. Given $\left.\begin{array}{l}x^2+xy=12,\\ \text{and } y^2+xy=24,\end{array}\right\}$ to find the values of x and y.

By addition, $x^2+2xy+y^2=36$;

$\therefore$ extracting the square root, $x+y=\pm 6$.

Now $x^2+xy=x\,.\,(x+y)=\pm 6x$;

$\therefore \pm 6x=12$, and $x=\pm 2$;

$\therefore y=\pm 6\mp 2=\pm 4$.

Ex. 6. Given $x+\sqrt{(a^2+x^2)}=\dfrac{2a^2}{\sqrt{(a^2+x^2)}}$, to find the values of x.

Multiplying by $\sqrt{(a^2+x^2)}$, we have $x\sqrt{(a^2+x^2)}+a^2+x^2=2a^2$;

by transposition, $x\sqrt{(a^2+x^2)}=a^2-x^2$,

and squaring both sides, $a^2x^2+x^4=a^4-2a^2x^2+x^4$;

$$\therefore 3a^2x^2=a^4, \text{ and } x=\pm\frac{a}{\sqrt{3}}.$$

Ex. 7. Given $\left.\begin{array}{l}x^2+y^2=\dfrac{13}{x-y}\\ \text{and } xy=\dfrac{6}{x-y}\end{array}\right\}$ to find the values of x and y.

From the 1st equation subtracting twice the 2d.

$x^2-2xy+y^2=(x-y)^2=\frac{1}{x-y}$, $\therefore (x-y)^3=1$,

and $x-y=1$; $\therefore x^2+y^2=13$;

and $2xy=12$;

$\therefore$ by addition, $x^2+2xy+y^2=25$,

$\therefore$ by extracting the square root, $x+y=\pm 5$;

but $x-y=1$;

$\therefore$ by addition, $2x=6$, or -4; and $x=3$, or -2;

by subtraction, $2y=4$, or -6; and $y=2$, or -3.

Ex. 8. Given $x^{\frac{2}{3}}+y^{\frac{2}{3}}=13$, and $x^{\frac{1}{3}}+y^{\frac{1}{3}}=5$; to find the values of x and y.

Squaring the second equation, $x^{\frac{2}{3}}+2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}=25$

but $x^{\frac{2}{3}} \qquad +y^{\frac{2}{3}}=13$

$\therefore$ by subtraction, $2x^{\frac{1}{3}}y^{\frac{1}{3}}=12$.

Subtracting this from the 1st equation,

$x^{\frac{2}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}=1$;

$\therefore$ extracting the square root, $x^{\frac{1}{3}}-y^{\frac{1}{3}}=\pm 1$;

but $x^{\frac{1}{3}}+y^{\frac{1}{3}}=5$;

$\therefore$ by addition, $2x^{\frac{1}{3}}=6$, or 4;

and $x^{\frac{1}{3}}=3$, or 2; $\therefore x=27$, or 8;

$\therefore$ by subtraction, $2y^{\frac{1}{3}}=4$, or 6,

and $y^{\frac{1}{3}}=2$, or 3; $\therefore y=8$, or 27.

Ex. 9. Given $x^8+x^4y^4+y^8=273$, and $x^4+x^2y^2+y^4=21$, to find the values of x and y.

Dividing the first equation by the second, $x^4-x^2y^2+y^4=13$;

subtracting this from the second equation, $2x^2y^2=8$;

$\therefore x^2y^2=4$;

by adding this equation to the second, $x^4+2x^2y^2+y^4=25$;

$\therefore x^2+y^2=\pm 5$.

Subtracting the equation $x^2y^2=4$, from $x^4-x^2y^2+y^4=13$,

$x^4-2x^2y^2+y^4=9$; $\therefore x^2-y^2=\pm 3$,

$\therefore$ by addition, $2x^2=\pm 8$, and $x=\pm 2$, or $\pm 2\sqrt{+1}$; and by subtraction, $2y^2=\pm 2$, and $y=\pm 1$, or $\pm\sqrt{-1}$.

Ex. 10. Given $\dfrac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}=b$, to find the value of x.

Multiply the numerator and denominator by $\sqrt{(a+x)}+\sqrt{(a-x)}$, $\dfrac{[\sqrt{(a+x)}+\sqrt{(a-x)}]^2}{2x}=b$, or $2a+2\sqrt{(a^2-x^2)}=2bx$;

$\therefore \sqrt{(a^2-x^2)}=bx-a$, and squaring both sides, $a^2-x^2=b^2x^2-2abx+a^2$,

$$\therefore b^2x^2+x^2=2abx, \text{ and } x=\frac{2ab}{b^2+1}.$$

Ex. 11. Given $(x^2-y^2)\times(x-y)=3xy$, and $(x^4-y^4)\times(x^2-y^2)=45x^2y^2$, to find the values of x and y.

Dividing the second equation by the first,

$$(x^2+y^2)\,.\,(x+y)=15xy;\ \therefore x^3+x^2y+xy^2+y^3=15xy;$$

but from the first, $x^3-x^2y-xy^2+y^3=3xy$;

$\therefore$ by addition, $2x^3+2y^3=18xy$, and $x^3+y^3=9xy$.

But by subtraction, $2x^2y+2xy^2=12xy$, and $x+y=6$;

$\therefore$ by cubing, $x^3+3x^2y+3xy^2+y^3=216$;

$x^3+y^3=9xy$;

$\therefore$ by subtraction, $3x^2y+3xy^2=216-9xy$, or $3\,.\,(x+y)\,.\,xy=3\times 6\,.\,xy=216-9xy$; $\therefore 27xy=216$, and $xy=8$.

Now $x^2+2xy+y^2=36$,

and $4xy=32$;

$\therefore$ by subtraction, $x^2-2xy+y^2=4$,

and by extracting the square root, $x-y=\pm 2$,

by $x+y=6$,

$\therefore$ by addition, $2x=8$, or 4; and $x=4$, or 2;

and by subtraction, $2y=4$, or 8; $\therefore y=2$, or 4.

Ex. 12. Given $\dfrac{a}{x}+\dfrac{\sqrt{(a^2-x^2)}}{x}=\dfrac{x}{b}$, to find the values of x.

Ans. $x=\pm\sqrt{(2ab-b^2)}$.

Ex. 13. Given $x^2+3x-7=x+2+\dfrac{18}{x}$, to find the values of x.

Ans. $x=3$, or -3.

Ex. 14. Given $\sqrt{\left(\dfrac{x+a}{x}\right)}+2\sqrt{\left(\dfrac{a}{x+a}\right)}=b^2\times$

$\sqrt{\left(\frac{x}{x+a}\right)}$, to find the values of x. Ans. $x=\frac{a}{(b\mp 1)^2}$.

Ex. 15. Given $x+y : x :: 5 : 3$, and $xy=6$, to find the values of x and y. Ans. $x=\pm 3$, and $y=\pm 2$.

Ex. 16. Given $x-y : x :: 5 : 6$, and $xy^2=384$, to find the values of x and y. Ans. $x=24$, and $y=4$.

Ex. 17. Given $x+y : x :: 7 : 5$, and $xy+y^2=126$, to find the values of x and y. Ans. $x=\pm 15$, and $y=\pm 6$.

Ex. 18. Given $xy^2+y=21$, and $x^2y^4+y^2=333$, to find the values of x and y. Ans. $x=2$, or $\frac{1}{108}$; and $y=3$, or 18.

Ex. 19. Given $x^2y+xy^2=180$, and $x^3+y^3=189$, to find the values of x and y. Ans. $x=5$, or 4; and $y=4$, or 5.

Ex. 20. Given $x+\sqrt{xy}+y=19$, and $x^2+xy+y^2=133$, to find the values of x and y. Ans. $x=9$, or 4; and $y=4$, or 9.

Ex. 21. Given $x^2y+xy^2=6$, and $x^3y^2+x^2y^3=12$, to find the values of x and y. Ans. $x=2$, or 1; and $y=1$, or 2.

Ex. 22. Given $(x^2+y^2)\times(x+y)=2336$, and $(x^2-y^2)(x-y)=576$, to find the values of x and y.

Ans. $x=11$, or 5; and $y=5$, or 11.

Ex. 23. Given $x^3+y^3=(x+y)\,.\,xy$, and $x^2y+xy^2=4xy$, to find the values of x and y. Ans. $x=2$, and $y=2$.

Ex. 24. Given $2\,.\,(x^2+y^2)\,.\,(x+y)=15xy$, and $4\,(x^4-y^4)(x^2+y^2)=75x^2y^2$, to find the values of x and y.

Ans. $x=2$, and $y=1$

Ex. 25. Given $x-y : y :: 4 : 5$, and $x^2+4y^2=181$, to find the values of x and y. Ans. $x=\pm 9$, and $y=\pm 5$.

Ex. 26. Given $x^2+y^2 : x^2-y^2 :: 17 : 8$, and $xy^2=45$, to find the values of x and y. Ans. $x=5$, and $y=3$.

Ex. 27. Given $\sqrt[4]{x}-\sqrt[4]{y}=3$, and $\sqrt[4]{x}+\sqrt[4]{y}=7$; to find the values of x and y. Ans. $x=625$, and $y=16$.

Ex. 28. Given $\sqrt{x}+\sqrt{y} : \sqrt{x}-\sqrt{y} :: 4 : 1$, and $x-y=16$, to find the values of x and y. Ans. $x=25$, and $y=9$.

Ex. 29. Given $x^3+y^3 : x^3-y^3 :: 559 : 127$, and $x^2y=294$; to find the values of x and y. Ans. $x=7$, and $y=6$.

Ex. 30. Given $x^{\frac{4}{3}}+y^{\frac{2}{5}}=20$, and $x^{\frac{2}{3}}+y^{\frac{1}{5}}=6$; to find the values of x and y.

Ans. $x=\pm 8$, or $\pm\sqrt{8}$, and $y=32$, or 1024

Ex. 31. Given $x^4+2x^2y^2+y^4=1296-4xy(x^2+xy+y^2)$, and $x-y=4$; to find the values of x and y.

Ans. 5, or -1, and $y=1$, or -5.

Ex. 32. Given $\frac{\sqrt{(4x+1)}+\sqrt{4x}}{\sqrt{(4x+1)}-\sqrt{4x}}=9$, to find the value of x.

Ans. $x=\frac{4}{9}$.

Ex. 33. Given $xy=a^2$, and $x^2+y^2=s^2$; to find the values of x and y. Ans. $x=\pm\frac{1}{2}[\sqrt{(s^2+2a^2)}+\sqrt{(s^2-2a^2)}]$,
and $y=\pm\frac{1}{2}[\sqrt{(s^2+2a^2)}-\sqrt{(s^2-2a^2)}]$.

Ex. 34. Given $x^2+x\sqrt[3]{xy^2}=208$, and $y^2+y\sqrt[3]{x^2y}=1053$, to find the values of x and y. Ans. $x=\pm 8$, and $y=\pm 27$.

Ex. 35. Given $x^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}}=$ 1009,
and $x^3+x^{\frac{3}{2}}y^{\frac{3}{2}}+y^3=582193$,
to find the values of x and y.

Ans. $x=81$, or 16; and $y=16$, or 81.

CHAPTER IX.

ON

THE SOLUTION OF PROBLEMS,

PRODUCING PURE EQUATIONS.

336. In addition to what has been already said, with respect to the translation of problems into algebraic equations, it is very proper to observe, that, when two quantities are required which are in the given proportion of m to n, the unknown quantities are represented by mx and nx; then the values of x, found from the equation of the problem by the methods in the preceding chapter, being multiplied by m and n respectively, will give the numbers required.

If three quantities are required, which have given ratios to one another, assume mx, nx, and px, m to n being the ratio of the first to the second, and n to p being that of the second to the third; then proceed as before.

Problem 1. There are two numbers in the proportion of 4 to 5, the difference of whose square is 81. What are those numbers?

Let $4x$ and $5x=$ the numbers;
then $(25x^2-16x^2=)\ 9x^2=81$; $\therefore x^2=9$, and $x=\pm 3$. Consequently the numbers are ± 12 and ± 15.

Prob. 2. It is required to divide 18 into two such parts, that the squares of those parts may be in the proportion of 25 to 16.

Let $x=$ the greater part; then $18-x=$ the less;
$\therefore x^2 : (18-x)^2 :: 25 : 16$, and $16x^2=25(18-x)^2$;

$\therefore$ extracting the square root, $4x=5(18-x)$, and $9x=90$; $\therefore x=10$, and the parts are 10 and 8.

Prob. 3. What two numbers are those whose difference, multiplied by the greater, produces 40, and by the less 15 ?

Let $x=$ the greater, and $y=$ the less ;

$\therefore x^2-xy=40$, and $xy-y^2=15$;

$\therefore$ by subtraction, $x^2-2xy+y^2=25$,

and $x-y=\pm 5$.

$\therefore$ from the first equation, $x(x-y)=\pm 5x=40$,

and $x=\pm 8$.

From the 2d, $y(x-y)=\pm 5y=\pm 15$; $\therefore y=\pm 3$.

Prob. 4. What two numbers are those whose difference, multiplied by the less, produces 42, and by their sum 133 ?

Let $x=$ the greater, and $y=$ the less ;

$\therefore (x-y)\cdot y=42$, and $(x-y)\cdot(x+y)=133$;

$\therefore$ by subtracting twice the first from the second,

$x^2-2xy+y^2=49$; $\therefore x-y=\pm 7$;

whence $\pm 7y=42$, and $y=\pm 6$;

but $x=y\pm 7$; $\therefore x=\pm 6\pm 7=\pm 13$.

Prob. 5. What two numbers are those, which being both multiplied by 27, the first product is a square, and the second the root of that square ; but being both multiplied by 3, the first product is a cube, and the second the root of the cube ?

Let x and y be the numbers ;

then $\sqrt{27x}=27y$, and $\therefore x=27y^2$,

also $\sqrt[3]{3x}=3y$; and $\therefore x=9y^3$;

whence $9y^3=27y^2$, and $y=3$; $\therefore x=9\times 27=243$;

$\therefore$ the numbers are 243, and 3.

Prob. 6. Two travellers, A and B, set out to meet each other ; A leaving the town C at the same time that B left D. They travelled the direct road, CD : and, on meeting, it appeared that A had travelled 18 miles more than B : and that A could have gone B's journey in $15\frac{3}{4}$ days, but B would have been 28 days in performing A's journey. What was the distance between C and D ?

Let $x=$ the number of miles A has travelled ;

$\therefore x-18=$ the number B has travelled ;

and $x-18 : x :: 15\frac{3}{4}$: the number of days A travelled, $= \frac{63x}{4.(x-18)}$; also $x : x-18 :: 28$: to the number of days B travelled $=\frac{28.(x-18)}{x}$; $\therefore \frac{28.(x-18)}{x}=\frac{63x}{4(x-18)}$; or $16.(x-18)^2=9x^2$; $\therefore 4.(x-18)=\pm 3x$, and $x=72$, or $10\frac{2}{7}$; whence

A travelled 72, and B 54 miles; and, the whole distance, CD 126 miles.

Prob. 7. Two partners, A and B, dividing their gain (60*l*), B took 20*l*. A's money continued in trade 4 months; and if the number 50 be divided by A's money, the quotient will give the number of months that B's money, which was 100*l*., continued in trade. What was A's money, and how long did B's money continue in trade?

Suppose A's money was x pounds; $\therefore \frac{50}{x}=$ the number of months B's money was in trade; and since B gained 20*l*., A gained 40*l*.

$\therefore\ 4x : \frac{50\times 100}{x} :: 2 : 1$, and $4x=\frac{10000}{x}$;

$\therefore\ 4x^2=10000$, and $x^2=2500$; $\therefore\ x=\pm 50$.

$\therefore$ A's money was 50*l*., and B's money was one month in trade.

Prob. 8. A detachment from an army was marching in regular column, with 5 men more in depth than in front; but upon the enemy coming in sight, the front was increased by 845 men; and by this movement the detachment was drawn up in five lines. Required the number of men.

Let $x=$ the number in front;

$\therefore\ x+5=$ the number in depth,

and $x(x+5)=$ the whole number of men;

also, $(x+845)\times 5=$ the whole number of men;

$\therefore\ x^2+5x=5x+4225$, and $x^2=4225$; $\therefore\ x=\pm 65$.

And, consequently, $5x+4225 = 325+4225 = 4550$, the number of men. Here, although the negative value of x will not answer the conditions of the problem, yet it will satisfy the above equation; for, if we substitute -65 for x, we shall have $(-65)^2+5(-65)=5(-65)+4225$; that is, or $4225-325=-325+4225$; $\therefore\ 4225=4225$, or $4225-4225=o$, that is, $o=o$.

Prob. 9. It is required to divide the number a into two such parts, that the squares of those parts may be in the proportion of m to n.

Let $x=$ one of these parts; then $a-x=$ the other; and according to the enunciation of the problem, we shall have the equation,

$\frac{x^2}{(a-x)^2}=\frac{m}{n}$; $\therefore\ \frac{x}{a-x}=\pm\sqrt{\frac{m}{n}}$, or (putting $\frac{m}{n}=m'$), $x=\pm(a-x)\sqrt{m'}$.

By resolving separately the two equations of the first degree comprised in the above formula, namely,

$$x=+(a-x)\sqrt{m'}, \text{ and } x=-(a-x)\sqrt{m'},$$

we shall have, from the first,

$x=\dfrac{a\sqrt{m'}}{1+\sqrt{m'}}$, and from the second $x=\dfrac{-a\sqrt{m'}}{1-\sqrt{m'}}$.

By the first solution, the second part of the proposed number is $a-\dfrac{a\sqrt{m'}}{1+\sqrt{m'}}=\dfrac{a}{1+\sqrt{m'}}$; and the two parts, $\dfrac{a\sqrt{m'}}{1+\sqrt{m'}}$ and $\dfrac{a}{1+\sqrt{m'}}$, are, as was required in the enunciation of the question, both less than the number proposed.

By the second solution, we have

$a-\left(\dfrac{-a\sqrt{m'}}{1-\sqrt{m'}}\right)=a+\dfrac{a\sqrt{m'}}{1-\sqrt{m'}}=\dfrac{x}{1-\sqrt{m'}}$; and the two parts are $-\dfrac{a\sqrt{m'}}{1-\sqrt{m'}}$ and $\dfrac{a}{1-\sqrt{m'}}$.

Their signs being contrary, *the number* a *is not*, properly speaking, *their sum*, but *their difference*.

Now, if $a=18$, $m=25$, and $n=16$; then substituting these values in the formula $\dfrac{a\sqrt{m'}}{1+\sqrt{m'}}$ and $\dfrac{a}{1+\sqrt{m'}}$, we shall find 10 and 8 equal to the two parts required, the same as in Ex. 2., which is a particular case of this general problem.

Prob. 10. What two numbers are those, whose sum is to the greater as 10 to 7; and whose sum, multiplied by the less, produces 270? Ans. ± 21 and ± 9.

Prob. 11. What two numbers are those, whose difference is to the greater as 2 to 9, and the difference of whose squares is 128? Ans. ± 18 and ± 14.

Prob. 12. A mercer bought a piece of silk for 16*l.* 4*s.*; and the number of shillings which he paid for a yard was to the number of yards as 4 : 9. How many yards did he buy, and what was the price of a yard?

Ans. 27 yards, at 12*s.* per yard.

Prob. 13. Find three numbers in the proportion of $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$: the sum of whose squares is 724.

Ans. ± 12, ± 16, and ± 18.

Prob. 14. It is required to divide the number 14 into two such parts, that the quotient of the greater part, divided by the less, may be to the quotient of the less divided by the greater as 16 : 9. Ans. The parts are 8 and 6.

PROB. 15. What two numbers, are those whose difference is to the less, as 4 to 3; and their product, multiplied by the less, is equal to 504? Ans. 14 and 6.

PROB. 16. Find two numbers, which are in the proportion of 8 to 5, and whose product is equal to 360.

Ans. ± 24, and ± 15.

PROB. 17. A person bought two pieces of linen, which, together, measured 36 yards. Each of them cost as many shillings per yard, as there were yards in the piece; and their whole prices were in the proportion of 4 to 1. What were the lengths of the pieces? Ans. 24 and 12 yards.

PROB. 18. There is a number consisting of two digits, which being multiplied by the digit on the left hand, the product is 46; but if the sum of the digits be multiplied by the same digit, the product is only 10. Required the number.

Ans. 23.

PROB. 19. From two towns, C and D, which were at the distance of 396 miles, two persons, A and B, set out at the same time, and met each other, after travelling as many days as are equal to the difference of the number of miles they travelled *per* day; when it appears that A has travelled 216 miles. How many miles did each travel *per* day?

Ans. A went 36, and B 30.

PROB. 20. There are two numbers, whose sum is to the greater as 40 is to the less, and whose sum is to the less as 90 is to the greater. What are the numbers?

Ans. 36, and 24.

PROB. 21. There are two numbers, whose sum is to the less as 5 to 2; and whose difference, multiplied by the difference of their squares, is 135. Required the numbers.

Ans. 9, and 6.

PROB. 22. There are two numbers, which are in the proportion of 3 to 2; the difference of whose fourth powers is to the sum of their cubes as 26 to 7. Required the numbers.

Ans. 6, and 4.

PROB. 23. A number of boys set out to rob an orchard, each carrying as many bags as there were boys in all, and each bag capable of containing 4 times as many apples as

there were boys. They filled their bags, and found the number of apples was 2916. How many boys were there?
Ans. 9 boys.

Prob. 24. It is required to find two numbers, such that the product of the greater, and square of the less, may be equal to 36; and the product of the less, and square of the greater, may be 48. Ans. 4, and 3.

Prob. 25. There are two numbers, which are in the proportion of 3 to 2; the difference of whose fourth powers is to the difference of their squares as 52 to 1. Required the numbers. Ans. 6, and 4.

Prob. 26. Some gentlemen made an excursion, and every one took the same sum. Each gentleman had as many servants attending him as there were gentlemen; and the number of dollars which each had was double the number of all the servants; and the whole sum of money taken out was $3456. How many gentlemen were there? Ans. 12.

Prob. 27. A detachment of soldiers from a regiment, being ordered to march on a particular service, each company furnished four times as many men as there were companies in the regiment; but those becoming insufficient, each company furnished 3 more men; when their number was found to be increased in the ratio of 17 to 16. How many companies were there in the regiment? Ans. 12.

Prob. 28. A charitable person distributed a certain sum among some poor men and women, the numbers of whom were in the proportion of 4 to 5. Each man received one-third of as many shillings as there were persons relieved; and each woman received twice as many shillings as there were women more than men. Now the men received all together 18*s*. more than the women. How many were there of each? Ans. 12 men, and 15 women.

Prob. 29. Bought two square carpets for 62*l*. 1*s*.; for each of which I paid as many shillings *per* yard as there were yards in its side. Now had each of them cost as many shillings *per* yard as there were yards in a side of the other, I should have paid 17*s*. less. What was the size of each?
Ans. One contained 81, and the other 64 square yards.

Prob. 30. A and B carried 100 eggs between them to market, and each received the same sum. If A had carried as many as B, he would have received 18 pence for them; and if B had only taken as many as A, he would have received 8 pence. How many had each? Ans. A 40, and B 60

PROB. 31. The sum of two numbers is 5 (s), and their product 6 (p): What is the sum of their 5th powers?

Ans. $275=(s^5-5ps^3+5p^2s)$.

CHAPTER X.

ON

QUADRATIC EQUATIONS.

337. Quadratic equations, as has been already observed, are divided into pure and adfected. All pure equations of the second degree are comprehended in the formula $x^2=n$, where n may be any number whatever, *positive* or *negative*, *integral* or *fractional*. And the value of x is obtained by extracting the square root of the number n; this value is double, for we have, $x=\pm\sqrt{n}$, and in fact, $(\pm\sqrt{n})^2=n$. This may be otherwise explained, by observing, (Art. 106), that $x^2-n=(x+\sqrt{n}).(x-\sqrt{n})=o$, and that any product consisting of two factors becomes nought, when there is no restriction in the equality to zero of that product, by making each of its factors equal to zero.

We have, therefore, $x=-\sqrt{n}$, $x=+\sqrt{n}$, or $x=\pm\sqrt{n}$.

338. Now, since the square root is taken on both sides of the equation, $x^2=n$, in order to arrive at $x=\pm\sqrt{n}$; it is very natural to suppose that, x being the square root of x^2, we should also affect x with the double sign $\pm$; and, therefore, in resolving the equation $x^2=n$, we should write $\pm x=\pm\sqrt{n}$; but by arranging these signs in every possible manner, namely;

$$+x=+\sqrt{n},\quad +x=-\sqrt{n},$$
$$-x=-\sqrt{n},\quad -x=+\sqrt{n},$$

we would still have no more than the two first equations, that is, $+x=\pm\sqrt{n}$; for if we change the signs of the equations $-x=-\sqrt{n}$ and $-x=+\sqrt{n}$, they become $+x=+\sqrt{n}$ and $+x=-\sqrt{n}$, or $x=\pm\sqrt{n}$.

339. If, in the formula $x^2=n$, n be negative, or, which is the same thing, if we have $x^2=-n$, where n is positive; then, $x=\pm\sqrt{-n}=\pm\sqrt{n}\times\sqrt{-1}$, and in fact $(\pm\sqrt{n})^2\times(\sqrt{-1})^2$

$=n\times-1=-n$; therefore, the two roots of a pure equation are either both real or both imaginary.

340. All adfected quadratic equations, after being properly reduced according to the rules pointed out in the reduction of simple equations, may be exhibited under the following general forms; namely, $x^2+nx=o$, and $x^2+nx=n'$; where n and n' may be any numbers whatever, *positive*, or *negative*, *integral* or *fractional*.

341. The solution of adfected quadratic equations of the form $x^2+nx=o$, is attended with no difficulty; for the equation $x^2+nx=o$, being divided by x, becomes $x+n=o$, from which we find $x=-n$, though we find only one value of x, according to this mode of solution, still there may be two values of x, which will satisfy the proposed equation.

In the equation, $x^2=3x$, for example, in which it is required to assign such a value of x, that x^2 may become equal to $3x$, this is done by supposing $x=3$, a value which is found by dividing the equation by x; but besides this value, there is also another which is equally satisfactory; namely, $x=o$; for then $x^2=o$, and $3x=o$.

342. An adfected quadratic equation is said to be complete, when it is of the form $x^2+nx=n'$; that is, when three terms are found in it; namely, that which contains the square of the unknown quantity, as x; that in which the unknown quantity is found only in the first power, as nx; and lastly, the term which is composed only of known quantities; and, as there is no difficulty attending the reduction of adfected quadratic equations to the above form by the known rules: the whole is at present reduced to determining the true value of x from the equation $x^2+nx=n'$.

We shall begin with remarking, that if x^2+nx were a real square, the resolution would be attended with no difficulty, because that it would be only required to extract the square root on both sides, in order to reduce it to a simple equation.

343. But it is evident that x^2+nx cannot be a square; since we have already seen, that if a root consists of two terms, for example, $x+a$, its square always contains three terms, namely, twice the product of the two parts, besides the square of each part, that is to say, the square of $x+a$ is $x^2+2ax+a^2$.

344. Now, we have already on one side x^2+nx; we may, therefore consider x^2 as the square of the first part of the root, and in this case nx must represent twice the product of x, the first part of the root, by the second part: consequently, this second part must be $\frac{1}{2}n$, and in fact the square of $x+\frac{1}{2}n$ is found to be $x^2+nx+\frac{1}{4}n^2$.

345. Now $x^2+nx+\frac{1}{4}n^2$, being a real square, which has for its root $x+\frac{1}{2}n$, if we resume our equation $x^2+nx=n'$, we have only to add $\frac{1}{4}n^2$, to both sides, which gives us $x^2+nx+\frac{1}{4}n^2=n'+\frac{1}{4}n^2$, the first side being actually a square, and the other containing only known quantities. If, therefore, we take the square root of both sides, we find $x+\frac{1}{2}n=\sqrt{(\frac{1}{4}n^2+n')}$; and as every square root may be taken either affirmatively or negatively, we shall have for x two values expressed thus;

$$x=-\tfrac{1}{2}n\pm\sqrt{(\tfrac{1}{4}n^2+n')}.$$

346. This formula contains the rule by which all quadratic equations may be resolved, and it will be proper, as EULER justly observes, to commit it to memory, that it may not be necessary to repeat, every time, the whole operation which we have gone through. We may always arrange the equation in such a manner, that the pure square x^2 may be found on one side, and the above equation have the form $x^2=-nx+n'$, where we see immediately that $x=-\frac{1}{2}n\pm\sqrt{(\frac{1}{4}n^2+n')}$.

347 The general rule, therefore, which may be deduced from that, in order to resolve $x^2=-nx+n'$, is founded on this consideration. That the unknown x is equal to half the coefficient or multiplier of x on the other side of the equation, *plus* or *minus* the root of the square of this number, and the known quantity, which forms the third term of the equation.

Thus, if we had the equation $x^2=6x+7$, we should immediately say, that $x=3\pm\sqrt{(9+7)}=3\pm4$; whence we have these two values of x; namely, $x=7$, and $x=-1$.

348. The method of resolving adfected quadratic equations will be still better understood by the four following forms; in which n and n' may be any *positive* numbers whatever, *integral* or *fractional.*

I. In the case $x^2+nx=n'$, where $x=-\frac{1}{2}n+\sqrt{(\frac{1}{4}n^2+n')}$, or $-\frac{1}{2}n-\sqrt{(\frac{1}{4}n^2+n')}$, the first value of x must be positive, because $\sqrt{(\frac{1}{4}n^2+n')}$ is $>\sqrt{\frac{1}{4}n^2}$, or its equal $\frac{1}{2}n$; and its second value will evidently be negative, because each of the terms of which it is composed is negative.

II. In the case $x^2-nx=n'$, from which we find $x=\frac{1}{2}n+\sqrt{(\frac{1}{4}n^2+n')}$, or $\frac{1}{2}n-\sqrt{(\frac{1}{4}n^2+n')}$, the first value of x, is manifestly positive, being the sum of two positive terms; and the second value will be negative, because $\sqrt{(\frac{1}{4}n^2+n')}$ is $>\sqrt{(\frac{1}{4}n^2)}$, or its equal $\frac{1}{2}n$.

III. In the case $x^2-nx=-n'$, we have $x=\frac{1}{2}n+\sqrt{(\frac{1}{4}n^2-n')}$, or $\frac{1}{2}n-\sqrt{(\frac{1}{4}n^2-n')}$; both the values of x will be positive, when $\frac{1}{4}n^2$ is $>n'$; for its first value is then evidently positive, being composed of two positive terms; and its second value will also be positive, because $\sqrt{(\frac{1}{4}n^2-n')}$ is less than $\sqrt{(\frac{1}{4}n^2)}$, or its equal $\frac{1}{2}n$. But if $\frac{1}{4}n^2$, in this case, be less than n', both the values of x will be imaginary; because the quantity, $\frac{1}{4}n^2 - n'$ under the radical sign, is then negative; and consequently $\sqrt{(\frac{1}{4}n^2 - n')}$ will be imaginary, or of no assignable value.

IV. Also, in the fourth case, $x^2 + nx = -n'$, where $x = -\frac{1}{2}n + \sqrt{(\frac{1}{4}n^2 - n')}$, or $-\frac{1}{2}n - \sqrt{(\frac{1}{4}n^2 - n')}$, the two values of x will be both negative, or both imaginary, according as $\frac{1}{4}n^2$ is greater or less than n'.

349. Hence we may conclude, from the constant occurrence of the double sign before the radical part of the preceding expressions, that every quadratic equation must have two roots; which are both real, or both imaginary; and though the latter of these cannot be considered as real quantities, but merely as pure algebraic symbols, of no determinate value, yet when they are submitted to the operations indicated by the equation, the two members of that equation will be always identical, or which is the same, it shall be always reduced to the form $0 = 0$

350. It may here also be further observed, that, in some equations involving, radical quantities of the form $\sqrt{(ax+b)}$ both values of x, found by the ordinary process, will not answer the proposed equation, except that we take the radical quantity with the double sign $\pm$. Let, for example, the values of x be found in the equation $x + \sqrt{(5x+10)} = 8$.

Here, by transposition, $\sqrt{(5x+10)} = 8 - x$,

therefore by squaring, $5x + 10 = 64 - 16x + x^2$,

or $x^2 - 21x = -54$; and $\therefore x = 18$, or 3.

Now, since these two values of x are found from the resolution of the equation $x^2 - 21x = -54$; it necessarily follows, that each of them, when substituted for x, must satisfy that equation; which may be verified thus; in the first place, by substituting 18 for x, in the equation $x^2 - 21x = -54$, we have $(18)^2 - 21 \times 18 = -54$, or $324 - 378 = -54$; that is, $-54 = -54$, or $0 = 0$.

Again, substituting, 3 for x, we have $(3)^2 - 21 \times 3 = -54$, or $9 - 63 = -54$; $-54 = -54$, or $54 - 54 = 0$; $\therefore 0 = 0$.

351. And as the equation $x^2 - 21x = -54$, may be deduced from the equation $+\sqrt{(5x+10)} = 8 - x$, or $-\sqrt{(5x+10)} = 8 - x$; it is evident that the radical quantity $\sqrt{(5x+10)}$, must be taken, with the double sign $\pm$, in the primitive equation, in order that it would be satisfied by the values, 18 and 3, of x, above found; that is, 18 answers to the sign $-$, and 3 to the sign $+$. But if one of these signs be excluded by the nature of the question; then only one of the values will sa-

tisfy the original equation; for instance, if in the equation $x+\sqrt{(5x+10)}=8$, the sign $-$ be excluded from the radical quantity, then the square root of $5x+10$ must be considered as a positive quantity; and because it is equal to $8-x$; the value of x, since both are positive, which will answer the proposed equation, must be less than 8; therefore, 3 is the value of x, which will satisfy the equation $x+\sqrt{(5x+10)}=8$, which can be readily verified thus; substituting 3 for x, we have $3+\sqrt{(15+10)}=8$, or $3+5=8$. And for a similar reason, 18 is the value of x, which will answer the equation $x-\sqrt{(5x+10)}=8$; for $18-\sqrt{(90+10)}=18-10=8$; $\therefore 8=8$, or $0=0$.

352. It is proper to take notice here of the following method of resolving quadratic equations, the principle of which is given in the *Bija Ganita*, before mentioned: thus, if a quadratic equation be of the form $4a^2x^2\pm 4abx=\pm 4ac$, it is evident that, by adding b^2 to both sides, the left-hand member will be a complete square, since it is the square of $2ax\pm b$; and, therefore, by extracting the square root of both sides, there will arise a simple equation, from which the values of x may be determined.

353. Now, any quadratic equation of the form $ax^2\pm bx=\pm c$, (to which every quadratic may be reduced by the known rules), by multiplying both sides by $4a$, will become $4a^2x^2\pm 4abx=\pm 4ac$. From which we infer, that *if each side of the equation be multiplied by four times the coefficient of* x^2, *and to each side there be added the square of the coefficient of* x, *the quantity on the left-hand side of the resulting equation will always be a complete square; from which, by extracting the square root, the values of* x *will be determined.* If the coefficient $a=1$, then *both sides of the equation is multiplied by* 4, *and the square of the coefficient of* x *is added*, as before.

§ I. SOLUTION OF ADFECTED QUADRATIC EQUATIONS, INVOLVING ONLY ONE UNKNOWN QUANTITY.

354. RULE I. Let the terms be arranged on one side of the equation, according to the dimensions of the unknown quantity, beginning with the highest; and the known quantities be transposed to the other; then, if the square of the unknown quantity has any coefficient, either positive or negative, let all the terms be divided by this coefficient. If the square of half the coefficient of the second term be now added to both sides of the equation, that side which involves the unknown quantity will become a complete square; and extracting the

square root on both sides of the equation, a simple equation will be obtained, from which the values of the unknown quantity may be determined.

355. Rule II. The terms of the equation being arranged as above, let each side be multiplied by four times the coefficient of x^2, and to each side add the square of the coefficient of x; then the left-hand member, being a complete power, extract the square root on each side of the equation, and there arises a simple equation, from which the values of x may be determined.

356. It may be observed, that all equations may be solved as quadratics, by completing the square, in which there are two terms involving the unknown quantity, or any function of it, and the exponent of one is double that of the other. Thus, $x^6+px^3=q$, $x^{2n}-px^n=q$, $x^{\frac{n}{2}}+x^{\frac{n}{4}}=a$, $a^2x^2+ax=b$, $x^{3n}+ax^{\frac{3n}{2}}=b$, $p^2x^{4n}-px^{2n}=d$, $(x^2+px+q)^2+(x^2+px+q)=r$, $x.(x+ax)^2+bx.(x^2+ax)=d$, are of the same form as quadratics, and the values of the unknown quantity may be determined in the same manner.

357. Many equations also, in which more than one unknown quantity are involved, may, in a similar manner, be reduced to lower dimensions by completing the square, as $x^2y^2+pxy=q$, $(x^3+y^3)^2+p.(x^3+y^3)=r$. Instances of this kind will occur in the next section.

358. And many adfected equations of the third, and other higher degrees, may be exhibited under the form of a quadratic, from which, by completing the square, the value of the unknown quantity will be determined. The biquadratic equation $x^4-8ax^3+8a^2x^2+32a^3x=d$, for instance, may be reduced to the form $(x^2-4ax)^2-8a^2(x^2-4ax)=d$. Thus the two first terms (x^2-4ax) of the square root of the left-hand member being found according to the rule (Art. 299), and the remainder $-8a^2x^2+32a^3x$, being evidently equal to $-8a^2(x^2-4ax)$; therefore $x^4-8ax^3+8a^2x^2+32a^3x=(x^2-4ax)^2-8a^2(x^2-4ax)=d$. Hence it follows, that if the remainder, after having found the first two terms of the square root (Art. 238), can be resolved into two factors, so that the factor containing the unknown quantity, shall be equal to the terms of the root thus found; the proposed biquadratic may always be reduced to a quadratic form.

359. In a similar manner, the cubic equation $x^3+2ax^2+5a^2x+4a^3=o$, may be reduced to the form $(x^2+ax)^2\times 4a^2(x^2+ax)=o$; thus, multiplying every term of the proposed

equation by x, it becomes $x^4+2ax^3+5a^2x^2+4a^3x=0$, which can be reduced to the above form, as in the preceding article. There are a variety of other artifices for reducing equations to lower dimensions, which will be illustrated in the following examples.

Ex. 1. Given $x^2+8x=20$, to find the values of x.

Completing the square, $x^2+8x+16=36$;

and extracting the root, $x+4=\pm 6$;

Whence, by transposition, $x=2$, or -10.

Ex. 2. Given $x^2-8x+5=14$, to find the values of x.

By transposition, $x^2-8x=9$;

and completing the square, $x^2-8x+16=25$;

$\therefore$ extracting the root, $x-4=\pm 5$,

and $x=9$, or -1.

Ex. 3. Given $\frac{x+\sqrt{(x^2-9)}}{x-\sqrt{(x^2-9)}}=(x-2)^2$, to find the values of x.

Multiplying the numerator and the denominator of the fraction by $x+\sqrt{(x^2-9)}$, $\frac{(x+\sqrt{(x^2-9)})^2}{9}=(x-2)^2$; $\therefore \frac{x+\sqrt{(x^2-9)}}{3}$ $=\pm(x-2)$: Taking the positive sign, $x+\sqrt{(x^2-9)}=3x-6$, or $\sqrt{(x^2-9)}=2x-6$; $\therefore x^2-9=4x^2-24x+36$; by transposition and division, $x^2-8x=-15$; $\therefore$ completing the square, &c. $x=5$, or 3.

But, by taking the negative sign, $x+\sqrt{(x^2-9)}=-3x+6$; $\therefore$ by transposing and squaring, $x^2-9=16x^2-48x+36$, and by transposition and division, $x^2-\frac{16}{5}x=-3$; completing the squaring, $x^2-\frac{16}{5}x+\frac{64}{25}=-\frac{11}{25}$; $\therefore$ taking the root and transposing, $x=\frac{8\pm\sqrt{-11}}{5}$.

Ex. 4. Given $x^4+\frac{17}{2}x^3-34x=16$, to find the values of x.

By transposition, $x^4+\frac{17}{2}x^3=34x+16$; completing the square, $x^4+\frac{17}{2}x^3+\left(\frac{17}{4}x\right)^2=\left(\frac{17}{4}x\right)^2+34x+16$; $\therefore$ extracting the root, $x^2+\frac{17}{4}x=\pm\left(\frac{17}{4}x+4\right)$.

Let the positive root be taken; then, by transposition, $x^2=4$; $\therefore x=2$, or -2.

But if the negative value be taken, $x^2+\frac{17}{4}x=-\frac{17}{4}x-4$;

$\therefore x^2+\frac{17}{2}x=-4$; and $x^2+\frac{17}{2}x+\frac{289}{16}=\frac{289}{16}-4=\frac{225}{16}$; $\therefore$ extracting the root, $x+\frac{17}{4}=\pm\frac{15}{4}$, and by transposition, $x=-8$, or $-\frac{1}{2}$.

Ex. 5. Given $4x^2-3x=85$, to find the values of x.

Multiplying by 16, $64x^2-48x=1360$, and, adding the square of 3, $64x^2-48x+9=1369$;
$\therefore$ extracting the square root, $8x-3=\pm 37$;
by transposition, $8x=40$, or -34, $\therefore x=5$, or $-4\frac{1}{4}$.

Ex. 6. Given $6x+\frac{35-3x}{x}=44$, to find the values of x.

Multiplying by x, $6x^2+35-3x=44x$;
$\therefore$ by transposition, $6x^2-47x=-35$;
and by division, $x^2-\frac{47}{6}x=-\frac{35}{6}$; therefore completing the square, $x^2-\frac{47}{6}x+\left(\frac{47}{12}\right)^2=\frac{2209}{144}-\frac{35}{6}=\frac{1369}{144}$; $\therefore$ extracting the root, $x-\frac{47}{12}=\pm\frac{37}{12}$, and $x=7$, or $\frac{5}{6}$.

Ex. 7. Given $5x-\frac{3x-3}{x-3}=2x+\frac{3x-6}{2}$, to find the values of x.

Multiplying by $2x-6$, we have $10x^2-36x+6=4x^2-12x+3x^2-15x+18$;
$\therefore$ by transposition, $3x^2-9x=12$;
and by division, $x^2-3x=4$:
$\therefore$ completing the square, $x^2-3x+\frac{9}{4}=4+\frac{9}{4}=\frac{25}{4}$,
and extracting the root, $x-\frac{3}{2}=\pm\frac{5}{2}$;
$\therefore x=4$, or -1.

Ex. 8. Given $3x-\frac{3x-10}{9-2x}=2+\frac{6x^2-40}{2x-1}$, to find the values of x.

Multiplying by $2x-1$,

$$6x^2-3x-\frac{6x^2-23x+10}{9-2x}=4x-2+6x^2-40,$$

or $7x+\frac{6x^2-23x+10}{9-2x}=42$;

$\therefore 63x-14x^2+6x^2-23x+10=378-84x$;
by transposition, $124x-8x^2=368$,

and $x-\frac{31}{2}x=-46$; $\therefore$ by completing the square,

$$x^2-\frac{31}{2}x+\frac{961}{16}=\frac{961}{16}-46=\frac{225}{16};$$

$\therefore$ extracting the root, $x-\frac{31}{4}=\pm\frac{15}{4}$;

and therefore $x=\frac{23}{2}$, or 4.

Ex. 9. Given $\sqrt{x^5}+\sqrt{x^3}=6\sqrt{x}$, to find the values of x.

Dividing by $\sqrt{x}$, $x^2+x=6$:

$\therefore$ completing the square, $x^2+x+\frac{1}{4}=6+\frac{1}{4}=\frac{25}{4}$;

and extracting the root, $x+\frac{1}{2}=\pm\frac{5}{2}$;

$\therefore x=2$, or -3.

Ex. 10. Given $x^n-2ax^{\frac{n}{2}}=b$, to find the values of x.

Completing the square, $x^n-2ax^{\frac{n}{2}}+a^2=a^2+b$;

$\therefore$ extracting the root, $x^{\frac{n}{2}}-a=\pm\sqrt{(a^2+b)}$,

and $x^{\frac{n}{2}}=a\pm\sqrt{(a^2+b)}$; $\therefore x=(a\pm\sqrt{(a^2+b)})^{\frac{2}{n}}$.

Ex. 11. Given $x^2-2x+6\sqrt{(x^2-2x+5)}=11$, to find the values of x.

Adding 5 to each side of the equation,

$$(x^2-2x+5)+6\sqrt{(x^2-2x+5)}=16;$$

$\therefore$ by completing the square,

$$(x^2-2x+5)+6\sqrt{(x^2-2x+5)}+9=25;$$

and extracting the root, $\sqrt{(x^2-2x+5)}+3=\pm 5$;

$\therefore \sqrt{(x^2-2x+5)}=2$, or -8;

$\therefore$ squaring both sides, $x^2-2x+5=4$, or 64;

whence $x^2-2x+1=0$, or 60;

and extracting the root, $x-1=0$, or $\pm\sqrt{60}$;

$\therefore x=1$, or $1\pm\sqrt{60}$.

Schol. It is proper to observe, that the equation, x^2-2x+1, has two equal roots, although x appears to have only one value; but it is because x is twice found $=1$, as the common method of resolution shows; for we have $x=1\pm\sqrt{0}$, that is to say, x is in two ways $=1$.

Ex. 12. Given $x^4+4x^3+12x^2+16x=a$, to find the values of x.

Here the two first terms of the square root of the left-hand member (Art. 238), is found to be x^2+2x, and the remainder is $8x^2+16x$, which can be readily resolved into the factors 8 and x^2+2x, since $(8x^2+16x)\div(x^2+2x)$ gives 8 for the quotient. Consequently the proposed equation may be exhibited under the quadratic form $(x^2+2x)^2+8(x^2+2x)=a$;

$\therefore$ by completing the square, $(x^2+2x)^2+8(x^2+2x)+16=a+16$; and extracting the root, $x^2+2x+4=\pm\sqrt{(a+16)}$.

Now by taking the positive sign,

$$x^2+2x+4=+\sqrt{(a+16)};$$

by transposition, $x^2+2x=-4+\sqrt{(a+16)}$;

$\therefore$ completing the square, $x^2+2x+1=-3+\sqrt{(a+16)}$;

and extracting the root, $x+1=\pm\sqrt{(-3+\sqrt{a+16})}$;

$$\therefore x=-1\pm\sqrt{(-3+\sqrt{(a+16)})}.$$

Again, by taking the negative sign,

$$x^2+2x+4=-\sqrt{(a+16)};$$

$\therefore x^2+2x=-4-\sqrt{(a+16)}$; and

completing the square, $x+2x+1=-3-\sqrt{(a+16)}$;

$\therefore$ extracting the root, $x+1=\pm\sqrt{(-3-\sqrt{(a+16)})}$;

and $x=-1\pm\sqrt{(-3-\sqrt{(a+16)})}$.

Ex. 13. Given $3x^2-12x+12=16-4$, to find the values of x.

By transposition, $3x^2-12x=16-4-12=0$;

and by division, $x^2-4x=0$;

$\therefore$ by completing the sqnare, $x^2-4x+4=4$;

and extracting the root, $x-2=\pm 2$;

$\therefore x=4$, or 0.

Ex. 14. Given $x^3-4x^2+6x=4$, to find the values of x.

Multiplying both sides by x, $x^4-4x^3+6x^2-4x=0$,

$$\therefore (x^2-2x)^2+2(x^2-2x)=0.$$

$$\therefore x^2-2x+1=\pm 1, \text{ and } x=1\pm\sqrt{\pm 1};$$

$\therefore$ the three roots of the proposed equation, are 1, $1+\sqrt{-1}$, and $1-\sqrt{-1}$. The other value of x, which is equal to $1-1$, or 0, belongs to the equation $(x^2-2x)^2+2(x^2-2x)=0$; hence there are four roots, or four values of x, which will satisfy this last equation.

Ex. 15. Given $27x^2-\frac{841}{3x^2}+\frac{17}{3}=\frac{232}{3x}-\frac{1}{3x^2}+5$, to find the values of x.

Multiplying every term by 3,

$$81x^2-\frac{841}{x^2}+17=\frac{232}{x}-\frac{1}{x^2}+15;$$

$$\therefore \text{by transposition, } 81x^2+17+\frac{1}{x^2}=\frac{841}{x^2}+\frac{232}{x^2}+15.$$

Adding unity to each side, in order to complete the square;

$$81x^2+18+\frac{1}{x^2}=\frac{841}{x^2}+\frac{232}{x}+16;$$

$$\text{and extracting the root, } 9x+\frac{1}{x}=\pm(\frac{29}{x}+4).$$

Let the positive value be taken; then by transposition, $9x-4=\frac{28}{x}$, and $\therefore 9x^2-4x=28$; by completing the square, &c., we shall have $x=2$, or $-\frac{14}{9}$. But if the negative value be taken, $9x^2+4x=-30$ and completing the square, &c., $x=\frac{-2\pm\sqrt{(-266)}}{9}$.

Ex. 16. Given $3x^2+2x-9=76$, to find the walues of x.

Ans. $x=5$, or $-\frac{17}{3}$.

Ex. 17. Given $\frac{8-x}{2}-\frac{2x-11}{x-3}=\frac{x-2}{6}$ to find the values of x.

Ans. $x=6$, or, $\frac{1}{2}$.

Ex. 18. Given $\frac{3x+4}{5}-\frac{30-2x}{x-6}=\frac{7x-14}{10}$, to find the values of x.

Ans. $x=36$, or 12.

Ex. 19. Given $\frac{x^3-10x^2+1}{x^2-6x+8}=x-3$, to find the values of x.

Ans. $x=1$, or -28.

Ex. 20. Given $\sqrt{(x+5)}\times\sqrt{(x+12)}=12$, to find the values of x

Ans. $x=4$, or -21.

Ex. 21. Given $2x^2+3x-5\sqrt{(2x^2+3x+9)}+3=0$, to find the values of x.

Ans. $x=3$, or $-\frac{9}{2}$, or $\frac{-3\pm\sqrt{-55}}{4}$

Ex. 22. Given $9x+\sqrt{(16x^2+36x^3)}=15x^2-4$, to find the values of x.

Ans. $x=\frac{4}{3}$, or $-\frac{1}{3}$; or $\frac{9\pm\sqrt{481}}{50}$

Ex. 23. Given $\frac{49x^2}{4}+\frac{48}{x^2}-49=9+\frac{6}{x}$, to find the values of x.

Ans. $x=2$, or $-\frac{8}{7}$, or $\frac{-3\pm\sqrt{93}}{7}$.

Ex. 24. Given $x^4-2x^3+x=132$, to find the values of x.

Ans. $x=4$, or -3, or $\frac{1\pm\sqrt{(-43)}}{2}$.

Ex. 25. Given $x^{\frac{6}{5}}+x^{\frac{3}{5}}=756$, to find the values of x.

Ans. $x=243$, or $(-28^{\frac{5}{3}}$.

Ex. 26. Given $x^3-x^{\frac{3}{2}}=56$, to find the values of x.

Ans. $x=4$, or $(-7)^{\frac{2}{3}}$.

Ex. 27. Given $x+5=\sqrt{(x+5)}+6$, to find the values of x.
Ans. $x=4$, or -1.

Ex. 28. Given $x+16-7\sqrt{(x+16)}=10-4\sqrt{(x+16)}$, to find the values of x.
Ans. $x=9$, or -12.

Ex. 29. Given $x+4+\frac{7x-8}{x}=13$, to find the values of x.
Ans. $x=4$, or -2.

Ex. 30. Given $14+4x-\frac{x+7}{x-7}=3x+\frac{9+4x}{3}$, to find the values of x.
Ans. $x=28$, or 9.

Ex. 31. Given $\frac{x+4}{3}-\frac{7-x}{x-3}=\frac{4x+7}{9}-1$, to find the values of x.
Ans. $x=21$, or 5.

Ex. 32. Given $2x+18-\frac{8x^2+16}{4x+7}=27-\frac{12x-11}{2x-3}$, to find the values of x.
Ans. $x=8$, or 5.

Ex. 33. Given $\frac{x^4+2x^3+8}{x^2+x-6}=x^2+x+8$, to find the values of x.
Ans. $x=4$, or $-4\frac{2}{3}$.

Ex. 34. Given $\sqrt{(4x+5)}\times\sqrt{(7x+1)}=30$, to find the values of x.
Ans. $x=5$, or $-6\frac{11}{28}$.

Ex. 35. Given $\frac{x+12}{x}+\frac{x}{x+12}=\frac{78}{15}$, to find the values of x.
Ans. $x=3$, or -15.

Ex. 36. Given $x^{\frac{4}{3}}+7x^{\frac{2}{3}}=44$, to find the values of x.
Ans. $x=\pm 8$, or $\pm(-11)^{\frac{3}{2}}$.

Ex. 37. Given $4x^{\frac{1}{3}}+x^{\frac{1}{6}}=39$, to find the values of x.
Ans. $x=729$, or $\left(\frac{13}{4}\right)^6$.

Ex. 38. Given $3x^6+42x^3=3321$, to find the values of x.
Ans. $x=3$, or $-\sqrt[3]{41}$.

Ex. 39. Given $\frac{8}{x^3}+2=\frac{17}{x^{\frac{3}{2}}}$, to find the values of x.
Ans. $x=4$, or $\frac{1}{2}\sqrt[3]{2}$.

Ex. 40. Given $x^2+11+\sqrt{(x^2+11)}=42$, to find the values of x.
Ans. $x=\pm 5$, or $\pm\sqrt{38}$.

Ex. 41. Given $x^2-12x+50=0$, to find the values of x.
Ans. $x=6\pm\sqrt{(-14)}$.

Ex. 42. Given $3x-\frac{1}{4}x^2=10$, to find the values of x.
Ans. $x=6\pm\sqrt{-4}$.

Ex. 43. Given $x^6-2x^3=48$, to find the values of x.

Ans. $x=2$, or $\sqrt[3]{-6}$.

Ex. 44. Given $x^4+2x^2-7x^2-8x=-12$, to find the values of x.

Ans. 2, or -3, or 1, or -2.

Ex. 45. Given $x^4-10x^3+35x^2-50x+24=0$, to find the values of x.

Ans. $x=1$, 2, 3, or 4.

Ex. 46. Given $x^3-8x^2+19x-12=0$, to find the values of x.

Ans. $x=1$, 3, or 4.

Ex. 47. Given $\frac{x+\sqrt{x}}{x-\sqrt{x}}=\frac{x^2-x}{4}$, to find the values of x.

Ans. $x=4$, or 1, or $\frac{3}{2}\pm\frac{1}{2}\sqrt{-7}$.

Ex. 48. Given $4x^4+\frac{x}{2}=4x^3+33$, to find the values of x.

Ans. $x=2$, or $-\frac{3}{2}$; or $\frac{1\pm\sqrt{(-43)}}{4}$.

§ II. SOLUTION OF ADFECTED QUADRATIC EQUATIONS, INVOLVING TWO UNKNOWN QUANTITIES.

360. When there are two equations containing two unknown quantities, a single equation, involving only one of the unknown quantities, may sometimes be obtained, by the rules laid down for the solution of simple equations; from which equation the values of the unknown quantity may be found, as in the preceding Section. Whence, by substitution, the values of the other may also be determined. In many cases, however, it may be more convenient to solve one or both of the equations first; that is, to find the values of one of the unknown quantities, in terms of the other and known quantities, as before; when the rules for eliminating unknown quantities, (§ I. Chap. IV), may be more easily applied.

The solution will sometimes be rendered more simple by particular artifices; the proper application of which shall be illustrated in the following examples.

Ex. 1. Given $x+2y=7$, and $x^2+3xy-y^2=23$, to find the values of x and y.

From the 1st equation $x=7-2y$;

$\therefore x^2=49-28y+4y^2$;

Substituting these values for x and x^2 in the 2d equation, then $49-28y+4y^2+21y-6y^2-y^2=23$,

or $3y^2+7y=49-23=26$.

$36y^2+84y+49=312+49=361$;

$\therefore$ extracting the square root, $6y+7=19$,

and $6y=19-7=12$; $y=2$,

and $x=7-2y=7-4=3$

Ex. 2. Given $4xy=96-x^2y^2$, and $x+y=6$, to find the values of x and y.

From the first equation $x^2y^2+4xy+4=100$,
and extracting the root, $xy+2=\pm 10$;
$\therefore xy=8$, or -12.

Now squaring the second equation,

$$x^2+2xy+y^2=36;$$
$$\text{but}\quad 4xy\quad =32, \text{ or } -48.$$

$\therefore$ by subtraction, $x^2-2xy+y^2=4$, or 84;
and extracting the root, $x-y=\pm 2$, or $\pm\sqrt{84}$;
but $x+y=\ 6$;

$\therefore$ by addition, $2x=8$, or 4, or $6\pm\sqrt{84}$;
whence, $x=4$, or 2, or $3\pm\sqrt{21}$;
and by subtraction, $2y=4$, or 8, or $6\mp\sqrt{84}$;
$\therefore y=2$, or 4, or $3\mp\sqrt{21}$.

Ex. 3. Given $x^2+x+y=18-y^2$, and $xy=6$, to find the values of x and y.

By transposition, $x^2+y^2+x+y=18$;
and from the second equation, $2xy\quad =12$;

$\therefore$ by addition, $x^2+2xy+y^2+x+y=30$;
and completing the square,

$$(x+y)^2+(x+y)+\tfrac{1}{4}=30+\tfrac{1}{4}=\frac{121}{4};$$

$\therefore$ extracting the root, $x+y+\frac{1}{2}=\pm\frac{11}{2}$,
and $x+y=5$, or -6;
whence, from the first equation, $x^2+y^2=13$, or 24;
but $2xy=12$;

$\therefore$ by subtraction, $x^2-2xy+y^2=1$, or 12;
$\therefore$ extracting the root, $x-y=\pm 1$, or $\pm 2\sqrt{3}$.
Now $x+y=5$, or -6;
$\therefore$ by addition, $2x=6$, or 4, or $-6\pm 2\sqrt{3}$;
$\therefore x=3$, or 2, or $-3\pm\sqrt{3}$;
and by subtraction, $2y=4$, or 6, or $-6\mp 2\sqrt{3}$;
$\therefore y=2$, or 3, or $-3\mp\sqrt{2}$.

Ex. 4. Given $x-2\sqrt{xy}+y-\sqrt{x}+\sqrt{y}=0$, and $\sqrt{x}+\sqrt{y}=5$, to find the values of x and y.

Completing the square in the first equation,

$$(\sqrt{x}-\sqrt{y})^2-(\sqrt{x}-\sqrt{y})+\tfrac{1}{4}=\tfrac{1}{4};$$

and extracting the root, $\sqrt{x}-\sqrt{y}-\frac{1}{2}=\pm\frac{1}{2}$;

$\therefore \sqrt{x} - \sqrt{y}, = 1$ or 0,

but from the second equation, $\sqrt{x} + \sqrt{y} = 5$;

$\therefore$ by addition, $2\sqrt{x} = 6$, or 5,

and $\sqrt{x} = 3$, or $\frac{5}{2}$, $\therefore x = 9$, or $\frac{25}{4}$.

By subtraction, $2\sqrt{y} = 4$, or 5; $\therefore y = 4$, or $\frac{25}{4}$.

Ex. 5. Given $x^{\frac{2}{3}} y^{\frac{3}{2}} = 2y^2$, and $8x^{\frac{1}{3}} - y^{\frac{1}{2}} = 14$, to find the values of x and y.

From the 1st equation, $x^{\frac{2}{3}} = 2y^{\frac{1}{2}}$; and $\therefore \frac{1}{2}x^{\frac{2}{3}} = y^{\frac{1}{2}}$; substituting this value in the second equation,

$8x^{\frac{1}{3}} - \frac{1}{2}x^{\frac{2}{3}} = 14$; and $\therefore 16x^{\frac{1}{3}} - x^{\frac{2}{3}} = 28$;

or, by changing the signs, $x^{\frac{2}{3}} - 16x^{\frac{1}{3}} = -28$;

completing the square, $x^{\frac{2}{3}} - 16x^{\frac{1}{3}} + 64 = 36$;

and extracting the root, $x^{\frac{1}{3}} - 8 = \pm 6$;

$\therefore x^{\frac{1}{3}} = 14$, or 2, and $x = 2744$, or 8.

Ex. 6. Given $x^{\frac{3}{2}} + y^{\frac{2}{3}} = 3x$, and $x^{\frac{1}{2}} + y^{\frac{1}{3}} = x$, to find the values of x and y.

By squaring the second equation, $x + 2x^{\frac{1}{2}}y^{\frac{1}{3}} + y^{\frac{2}{3}} = x^2$;

but $x^{\frac{3}{2}} \qquad + y^{\frac{2}{3}} = 3x$;

$\therefore$ by subtraction, $x - x^{\frac{3}{2}} + 2x^{\frac{1}{2}}y^{\frac{1}{3}} = x^2 - 3x$;

but from the second equation, $y^{\frac{1}{3}} = x - x^{\frac{1}{2}}$;

Let this value be substituted in the preceding equation; then $x - x^{\frac{3}{2}} + 2x^{\frac{3}{2}} - 2x = x^2 - 3x$;

$\therefore$ by transposition, $2x = x^2 - x^{\frac{3}{2}}$;

and dividing by x, $2 = x - x^{\frac{1}{2}}$;

completing the square, $x - x^{\frac{1}{2}} + \frac{1}{4} = 2 + \frac{1}{4} = \frac{9}{4}$;

and extracting the root $x^{\frac{1}{2}} - \frac{1}{2} = \pm \frac{3}{2}$;

$\therefore x^{\frac{1}{2}} = 2$, or -1; and $x = 4$, or 1.

By taking the former value of x, we have $y^{\frac{1}{3}} = x - x^{\frac{1}{2}} = 4 - 2 = 2$; $\therefore y = 8$.

and by taking the latter value, $y^{\frac{1}{3}} = x - x^{\frac{1}{2}} = 1 + 1 = 2$,

(since $x^{\frac{1}{2}} = -1$, $-x^{\frac{1}{2}} = +1$); $\therefore y = 8$

Ex. 7. Given $y^2-64=8x^{\frac{1}{2}}y$, and $y-4=2y^{\frac{1}{2}}x^{\frac{1}{2}}$, to find the values of x and y.

From the first equation, $y^2-8x^{\frac{1}{2}}y=64$;

completing the square, $y^2-8x^{\frac{1}{2}}y+16x=16x+64$;

extracting the root, $y-4x^{\frac{1}{2}}=\pm 4\sqrt{(x+4)}$;

and $\therefore y=4x^{\frac{1}{2}}\pm 4\sqrt{(x+4)}$.

Also, from the second equation, $y-2y^{\frac{1}{2}}x^{\frac{1}{2}}=4$;

$\therefore$ completing the square, &c., $y^{\frac{1}{2}}=x^{\frac{1}{2}}\pm\sqrt{(x+4)}$;

multiplying by 4, $4y^{\frac{1}{2}}=x^{\frac{1}{2}}\pm 4\sqrt{(x+4)}$;

$\therefore y=4y^{\frac{1}{2}}$, and $y=16$.

But, from the second equation, $x^{\frac{1}{2}}=\dfrac{y-4}{2y^{\frac{1}{2}}}=\dfrac{12}{8}=\dfrac{3}{2}$;

$\therefore$ by involution, $x=\dfrac{9}{4}$.

361. When the equations are homogeneous, that is, when x^2, y^2, or xy, is found in every term of the two equations, they assume the form of

$ax^2+bxy+cy^2=d$,

$a'x^2+b'xy+c'y^2=d'$; and their solution may be effected in the following manner:

Assume $x=vy$, then $x^2=v^2y^2$; by substituting these values for x^2 and x in both equations, we have

$$av^2y^2+bvy^2+cy^2=d\,;\ \therefore y^2=\frac{d}{av^2+bv+c}\quad . \quad . \quad . \quad (1),$$

$$a'v^2y^2+b'vy^2+c'y^2=d'\,;\ \therefore y^2=\frac{d'}{a'v^2+b'v+c'}\quad . \quad . \quad . \quad (2).$$

$$\text{Hence}\ \frac{d}{av^2+bv+c}=\frac{d'}{a'v^2+b'v+c'}\,;$$

$\therefore (a'd-ad')v^2+(b'd-bd')v=cd'-c'd$; which is a quadratic equation, from whence the value of v may be determined. Having the value of v, the value of y may be found from either of the equations (1) or (2); and then the value of x, from the equation $x=vy$.

Ex. 8. Given $2x^2+3xy+y^2=20$, and $5x^2+4y^2=41$, to find the values of x and y.

Let $x=vy$, then $2v^2y^2+3vy^2+y^2=20$;

$\therefore y^2=\frac{20}{2v^2+3v+1}$, and $5v^2y^2+4y^2=41$;

$\therefore y^2=\frac{41}{5v^2+4}$; Hence $\frac{20}{2v^2+3v+4}=\frac{41}{5v^2+4}$, or $6v^2-41v=-13$;

$\therefore$ by division, completing the square, &c. $v=\frac{13}{2}$ or $\frac{1}{3}$.

Let $v=\frac{1}{3}$, then $y^2=\frac{41}{5v^2+4}=\frac{369}{41}=9$; $\therefore y=3$, or -3,

and $x=vy=1$, or -1.

Again, let $v=\frac{13}{2}$; then $y=\pm\sqrt{\frac{164}{861}}$, and $x=\pm\frac{13}{2}\sqrt{\frac{164}{861}}$.

Consequently there are four values, both of x and y, which satisfy the proposed equations.

362. When the unknown quantities in each equation are similarly involved, the operation may sometimes be shortened, by substituting for the unknown quantities the sum and difference of two others.

Ex. 9. Given $\left.\begin{aligned}\frac{x^2}{y}+\frac{y^2}{x}=18,\\ \text{and } x+y=12,\end{aligned}\right\}$ to find the values of x and y.

Assume $x=z+v$, and $y=z-v$; $\therefore x+y=2z=12$;
or $z=6$; $\therefore x=6+v$, and $y=6-v$.

Also, since $\frac{x^2}{y}+\frac{y^2}{x}=18$, $x^3+y^3=18xy$;

$\therefore (6+v)^3+(6-v)^3=18(6+v)\times(6-v)$;
or $432+36v^2=648-18v^2$;
and by transposition, $54v^2=216$;
$\therefore v^2=4$; and $v=\pm2$; $\therefore x=6\pm2=8$, or 4;
and $y=6\pm2=4$, or 8.

363. In all quadratics of this kind, in which x may be changed for y, and y for x, in the original equations, without altering their form, the two values of one of the quantities may be taken for the values of the two quantities sought.

Ex. 10. Given $x+y=2a$, and $x^5+y^5=b$, to find the values of x and y.

Let $x-y=2z$; then $x=a+z$, and $y=a-z$;
$\therefore$ by substitution, $(a+z)^5+(a-z)^5=b$, or, by involution and addition, $2a^5+20a^2z^2+10az^4=b$;

$\therefore z^4+2a^2z^2=\frac{b-2a^5}{10a}$, and $z=\pm\sqrt{[-a^2\pm\sqrt{(\frac{b+8a^5}{10a})}]}$.

$\therefore x=a\pm\sqrt{[-a^2\pm\sqrt{(\frac{b+8a^5}{10a})}]}$, and $y=a\mp\sqrt{[-a^2\pm\sqrt{(\frac{b+8a^5}{10a})}]}$.

Now, let $x+y=6$, and $x^5+y^5=1056$; then by substituting 3 for a, and 1056 for b, in the formula of roots, the values of x and y will be found; that is, $x=3\pm1$, or $3\pm\sqrt{-19}$; and $y=3\mp1$, or $3\mp\sqrt{-19}$. Or, by substituting the above values of a and b in the equation $10az^4-20a^3z^2+2a^5=b$, it becomes $30z^4+540z+486=1056$; from which the values of z may be found; whence, by substitution, the values of x and y will be determined, as before.

Ex. 11. Given $x+4y=14$, and $y^2+4x=2y+11$, to find the values of x and y.

Ans. $x=-46$, or 2; and $y=15$, or 3.

Ex. 12. Given $2x+3y=118$, and $5x^2-7y^2=4333$, to find the values of x and y.

Ans. $x=35$, or $-\frac{3899}{17}$; and $y=16$, or $\frac{3268}{17}$.

Ex. 13. Given $x^2+4y^2=256-4xy$, and $3y^2-x^2=39$, to find the values of x and y.

Ans. $x=\pm6$, or ±102; and $y=\pm5$, or ±59.

Ex. 14. Given $x^n+y^n=2a^n$, and $xy=c^2$, to find the values of x and y.

$$\text{Ans.}\ \begin{cases} x=[a^n\pm\sqrt{(a^{2n}-c^{2n})}]^{\frac{1}{n}}; \\ y=\dfrac{c^2}{[a^n\pm\sqrt{(a^{2n}-c^{2n})}]^{\frac{1}{n}}} \end{cases}$$

Ex. 15. Given $x^2+2xy+y^2+2x=120-2y$, and $xy-y^2=8$, to find the values of x and y.

Ans. $x=6$, or 9, or $-9\mp\sqrt{5}$; and $y=4$, or 1, or $-3\pm\sqrt{5}$.

Ex. 16. Given $x^2+y^2-x-y=78$, and $xy+x+y=39$, to find the values of x and y.

Ans. $x=9$, or 3; or $-6\frac{1}{2}\pm\frac{1}{2}\sqrt{-39}$; and $y=3$, or 9, or $-6\frac{1}{2}\mp\frac{1}{2}\sqrt{-39}$.

Ex. 17. Given $\left.\begin{array}{l}\dfrac{x^2}{y^2}+\dfrac{4x}{y}=\dfrac{85}{9}, \\ \text{and } x-y=2,\end{array}\right\}$ to find the values of x and y.

Ans. $x=5$, or $\frac{17}{10}$; and $y=3$, or $-\frac{3}{10}$.

Ex. 18. Given $\left.\begin{array}{l}x^4-2x^2y+y^2=49, \\ \text{and } x^4-2x^2y^2+y^4-x^2+y^2=20,\end{array}\right\}$ to find the values of x and y.

Ans. $x=\pm3$, or $\pm\sqrt{6}$, or $\pm\frac{1}{2}\sqrt{(30\pm6\sqrt{5})}$; and $y=2$, or -1, or $\frac{1}{2}(1\pm3\sqrt{5})$.*

* There are four other values, both of x and y, which are all imaginary.

Ex. 19. Given $4-x^{\frac{1}{2}}=3-y$, and $4-x=y-y^{\frac{1}{2}}$, to find the values of x and y. Ans. $x=4$, or $\frac{1}{4}$; and $y=1$, or $2\frac{1}{4}$.

Ex. 20. Given $x^{\frac{3}{2}}+x-4x^{\frac{1}{2}}=y^2+y+2$, and $xy=y^2+3y$, to find the values of x and y.

Ans. $x=4$, or 1; and $y=1$, or -2.

Ex. 21. Given $x^2+xy=56$, and $xy+2y^2=60$, to find the values of x and y. Ans. $x=\pm 4\sqrt{2}$, or ∓ 14; and $y=\pm 3\sqrt{2}$, or ± 10.

Ex. 22. Given $x-y=15$, and $xy=2y^3$, to find the values of x and y. Ans. $x=18$, or $12\frac{1}{2}$; and $y=3$, or $-2\frac{1}{2}$.

Ex. 23. Given $10x+y=3xy$, and $9y-9x=18$, to find the values of x and y. Ans. $x=2$, or $-\frac{1}{3}$; and $y=4$, or $\frac{5}{3}$.

Ex. 24. Given $x+y : x-y :: 13 : 5$, and $y^2+x=25$, to find the values of x and y.

Ans. $x=9$, or $-14\frac{1}{16}$; and $y=4$, or $-6\frac{1}{4}$.

Ex. 25. Given $x^2y^4-7xy^2=1710$, and $xy-y=12$, to find the values of x and y.

Ans. $x=5$, or $\frac{1}{5}$, or $\dfrac{-19}{17\mp 6\sqrt{-2}}$; and $y=3$, or -15, or $-6\pm\sqrt{-2}$.

Ex. 26. Given $xy+xy^2=12$, and $x+xy^3=18$, to find the values of x and y. Ans. $x=2$, or 16; and $y=2$, or $\frac{1}{2}$.

Ex. 27. Given $x+y+\sqrt{(x+y)}=6$, and $x^2+y^2=10$, to find the values of x and y.

Ans. $x=3$, or 1; or $4\frac{1}{2}\pm\frac{1}{2}\sqrt{-61}$; and $y=1$, or, 3, or $4\frac{1}{2}\mp\frac{1}{2}\sqrt{-61}$.

Ex. 28. Given $x^2+4\sqrt{(x^2+3y+5)}=55-3y$, and $6x-7y=16$, to find the values of x and y.

$$\text{Ans.}\begin{cases} x=5, \text{ or } \dfrac{-53}{7}; \text{ or } \dfrac{-9\pm\sqrt{5072}}{7}. \\ y=2, \text{ or } -\dfrac{430}{7}; \text{ or } \dfrac{-166\pm 6\sqrt{5072}}{49}. \end{cases}$$

Ex. 29. Given $x^2+2x^3y=441-x^4y^2$, and $xy=3+x$, to find the values of x and y.

$$\text{Ans.}\begin{cases} x=3, \text{ or } -7; \text{ or } -2\pm\sqrt{-17}, \\ y=2, \text{ or } \frac{4}{7}; \text{ or } \frac{5}{7}\mp\frac{1}{7}\sqrt{-17}. \end{cases}$$

Ex. 30. Given $(x+y)^2-3y=28+3x$, and $2xy+3x=35$, to find the values of x and y.

$$\text{Ans.}\begin{cases} x=5, \text{ or } \frac{7}{2}, \text{ or } -\frac{5}{4}\pm\frac{1}{4}\sqrt{(-255)}, \\ y=2, \text{ or } \frac{7}{2}, \text{ or } -\frac{11}{4}\mp\sqrt{\tfrac{1}{4}(-255.)} \end{cases}$$

Ex. 31. Given $x^2+3x+y=73-2xy$, and $y^2+3y+x=44$, to find the values of x and y.

$$\text{Ans.}\begin{cases} x=4, \text{ or } 16; \text{ or } -12\mp\sqrt{58}, \\ y=5, \text{ or } -7; \text{ or } -1\pm\sqrt{58}. \end{cases}$$

Ex. 32. Given $\frac{x^4}{y^2}+\frac{y^4}{x^2}=136\frac{1}{9}-2xy$, and $x+y=10$, to find the values of x and y.

Ans. $\begin{cases} x=6, \text{ or } 4; \text{ or } 5\pm5\sqrt{(-\frac{13}{11})} \\ y=4, \text{ or } 6; \text{ or } 5\mp5\sqrt{-(\frac{13}{11})}. \end{cases}$

Ex. 33. Given $y^4-432=12xy^2$, and $y^2=12+2xy$, to find the values of x and y.

Ans. $x=2$, or 3; and $y=6$, or $\sqrt{(21)}+3$.

CHAPTER XI.

ON

THE SOLUTION OF PROBLEMS,

PRODUCING QUADRATIC EQUATIONS.

§ I. SOLUTION OF PROBLEMS PRODUCING QUADRATIC EQUATIONS, INVOLVING ONLY ONE UNKNOWN QUANTITY.

364. It may be observed, that, in the solution of problems which involve quadratic equations, we sometimes deduce, from the algebraical process, answers which do not correspond with the conditions. The reason seems to be, that the algebraical expression is more general than the common language, and the equation, which is a proper representation of the conditions, will express other conditions, and answer other suppositions.

Prob. 1. A person bought a certain number of oxen for 80 guineas, and if he had bought four more for the same sum, they would have cost a guinea a piece less. Required the number of oxen and price of each.

Let $x=$ the number; then $\frac{80}{x}=$ the price of each;

$\therefore \frac{80}{x+4}=\frac{80}{x}-1$, by the problem,

and by reduction, $x^2+4x=320$;

$\therefore x^2+4x+4=324$, and $x+2=\pm18$;

$\therefore x=16$, or -20.

And $\frac{80}{x}=\frac{80}{16}=5$ guineas, the price of each.

The negative value (-20) of x, will not answer the condition of the problem.

PROB. 2. There are two numbers whose difference is 9, and their sum multiplied by the greater produces 266. What are those numbers?

Let $x=$ the greater; $\therefore x-y=$ the less.

and $x.(2x-9)=266$; $\therefore x^2-\frac{9}{2}.x=\frac{266}{2}$.

completing the square, &c. $x-\frac{9}{4}=\pm\frac{47}{4}$;

$\therefore x=14$, or $-9\frac{1}{2}$; and $x-9=5$, or $-18\frac{1}{2}$.

Here both values answer the conditions of the problem.

PROB. 3. A set out from C towards D, and travelled 7 miles a day. After he had gone 32 miles, B set out from D towards C, and went every day one-nineteenth of the whole journey; and after he had travelled as many days as he went miles in one day, he met A. Required the distance of the places C and D.

Suppose the distance was x miles.

$\therefore \frac{x}{19}=$ the number of miles B travelled *per* day; and also $=$ the number of days he travelled before he met A.

$$\therefore \frac{x^2}{361}+32+\frac{7x}{19}=x;$$

by transposition and completing the square,

$$3\frac{x^2}{161}-\frac{12x}{19}+36=36-32=4;$$

extracting the root, $\frac{x}{19}-6=\pm2$;

$\therefore \frac{x}{19}=8$, or 4; and $x=152$, or 76, both which values answer the conditions of the problem. The distance therefore of C from D was 152, or 76 miles.

PROB. 4. To divide the number 30 into two such parts, that their product may be equal to *eight* times their difference.

Let $x=$ the *lesser* part; $\therefore 30-x=$ the greater part, and $30-x-x$, or $30-2x=$ their difference.

Hence, by the problem, $x(30-x)=8(30-2x)$, or $30x-x^2=240-16x$; $\therefore x^2-46x=-240$.

$\therefore$ completing the square, $x^2-46x+529=289$;
$\therefore x=23\pm17=40$, or $6=$ *lesser* part;
and $30-x=30-6=24=$ *greater* part.

In this case, the solution of the equation gives 40 and 6 for the *lesser* part. Now as 40 cannot possibly be a *part* of 30, we take 6 for the *lesser* part, which gives 24 for the *greater* part; and the two numbers, 24 and 6, answer the conditions required.

Prob. 5. Some bees had alighted upon a tree; at one flight the square root of half of them went away; at another eight-ninths of them; two bees then remained. How many then alighted on the tree?

Let $2x^2=$ the number of bees; $x+\frac{16x^2}{9}+2=2x^2$,

or $9x+16x^2+18=18x^2$; $\therefore 2x^2-9x=18$;
Multiplying by 8, $16x^2-72x=144$;
adding 81 to both sides, $16x^2-72x+81=225$;
$\therefore 4x=9\pm15=24$, or -6; and $x=6$, or $-1\frac{1}{2}$.
$\therefore 2x^2=72$, or $4\frac{1}{2}$.

But the negative value $-1\frac{1}{2}$ of x, is excluded by the nature of the problem; therefore, $72=$ number of bees.

365. If, in a problem proposed to be solved, there are two quantities sought, whose sum, or difference, is equal to a given quantity, for instance, $2a$; let half their difference, or half their sum, be denoted by x; then $x+a$ will represent the greater, and $x-a$ the lesser, (Art. 102). According to this method of notation, the calculation will be greatly abridged, and the solution of the problem will often be rendered very simple.

Prob. 6. The sum of two numbers is 6, and the sum of their 4th power is 272. What are the numbers?

Let $x=$ half the difference of the two numbers; then $3+x=$ the greater number, and $3-x=$ the lesser.

$\therefore$ by the problem, $(3+x)^4+(3-x)^4=272$,
or $162+108x^2+2x^4=272$; from which, by transposition and division, $x^4+54x^2=55$:

$\therefore$ completing the square, $x^4+54x^2+729=784$,
and extracting the root, $x^2+27=\pm28$;
$\therefore x^2=-27\pm28$, and $x=\pm1$, or $\pm\sqrt{-55}$.

Now, by taking the positive value, $+1$, for x, (since in this case, it is the only value of x which will answer the problem); we shall have $3+1=4=$ the greater, and $3-1=2=$ the lesser.

Prob. 7. To divide the number 56 into two such parts, that their product shall be 640. Ans. 40, and 16.

Prob. 8. There are two numbers whose difference is 7, and half their product *plus* 30, is equal to the square of the lesser number. What are the numbers?

Ans. 12, and 19.

Prob. 9. A and B set out at the *same time* to a place at the distance of 150 miles. A travelled 3 miles an hour faster than B, and arrives at his journey's end 8 hours and 20 minutes before him. At what rate did each person travel per hour?

Ans. A 9, and B 6 miles an hour.

Prob. 10. The difference of two numbers is 6; and if 47 be added to *twice the square of the lesser*, it will be equal to the *square of the greater*. What are the numbers?

Ans. 17, and 11.

Prob. 11. There are two numbers whose product is 120, if 2 be added to the lesser, and 3 subtracted from the greater, the product of the sum and remainder will also be 120. What are the numbers? Ans. 15, and 8.

Prob. 12. A person bought a certain number of sheep for 120*l*. If there had have been 8 more, each would have cost him ten shillings less. How many sheep were there?

Ans. 40.

Prob. 13. A Merchant sold a quantity of brandy for 39*l*. and gained as much per cent as the brandy cost him. What was the price of the brandy? Ans. 30*l*.

Prob. 14. Two partners, A and B, gained 18*l*. by trade. A's money was in trade 12 months, and he received for his principal and gain 26*l*. Also, B's money, which was 30*l*. was in trade 16 months. What money did A put into trade?

Ans. 20*l*.

Prob. 15. A and B set out from two towns which were at the distance of 247 miles, and travelled the direct road till they met. A went 9 miles a day; and the number of days, at the end of which they met, was greater by 3 than the number of miles which B went in a day. How many miles did each go?

Ans. A 117, and B 130 miles.

Prob. 16. A man playing at hazard won at the first throw, as much money as he had in his pocket; at the second throw, he won 5 shillings more than the square root of what he then had; at the third throw, he won the square of all he then had; and then he had 112*l*. 16*s*. What had he at first?

Ans. 18 shillings.

Prob. 17. If the square of a certain number be taken from 40, and the square root of this difference be increased by 10, and the sum multiplied by 2, and the product divided by the number itself, the quotient will be 4. Required the number.

Ans. 6.

Prob. 18. There is a field in the form of a rectangular parallelogram, whose length exceeds the breadth by 16 yards; and it contains 960 square yards. Required the length and breadth. Ans. 40 and 24 yards.

Prob. 19. A person being asked his age, answered, if you add the square root of it to half of it, and subtract 12, there will remain nothing. Required his age. Ans. 16.

Prob. 20. To find a number from the cube of which, if 19 be subtracted, and the remainder multiplied by that cube, the product shall be 216. Ans. 3, or -2.

Prob. 21. To find a number, from the double of which if you subtract 12, the square of the remainder, *minus* 1, will be 9 times the number sought. Ans. 11, or $3\frac{1}{4}$.

Prob. 22. It is required to divide 20 into two such parts, that the product of the whole and one of the parts, shall be equal to the square of the other.

Ans. $10\sqrt{5}-10$, and $30-10\sqrt{5}$.

Prob. 23. A labourer dug two trenches, one of which was 6 yards longer than the other, for 17*l.* 16*s.*, and the digging of each of them cost as many shillings *per* yard as there were yards in its length. What was the length of each?

Ans. 10, and 16 yards.

Prob. 24. A company at a tavern had 8*l.* 15*s.* to pay, but before the bill was paid, two of them sneaked off, when those who remained had each 10*s.* more to pay. How many were there in the company at first? Ans. 7.

Prob. 25. There are two square buildings, that are paved with stones, a foot square each. The side of one building exceeds that of the other by 12 feet, and both their pavements taken together contain 2120 stones. What are the lengths of them separately? Ans. 26, and 38 feet.

Prob. 26. In a parcel which contains 24 coins of silver and copper, each silver coin is worth as many pence as there are copper coins, and each copper coin is worth as many pence as there are silver coins, and the whole is worth 18 shillings. How many are there of each?

Ans. 6 of one, and 18 of the other.

Prob. 27. Two messengers, A and B, were despatched at the same time to a place 90 miles distant; the former of whom riding one mile an hour more than the other, arrived at the end of his journey an hour before him. At what rate did each travel per hour?

Ans. A went 10, and B 9 miles *per* hour.

Prob. 28. A man travelled 105 miles, and then found that if he had not travelled so fast by 2 miles an hour, he should have been 6 hours longer in performing the journey. How many miles did he go *per* hour? Ans. 7 miles.

Prob. 29. Bought two flocks of sheep for 65*l.* 13*s*., one containing 5 more than the other. Each sheep cost as many shillings as there were sheep in the flock. Required the number in each flock. Ans. 23, and 28.

Prob. 30. A regiment of soldiers, consisting of 1066 men, is formed into two squares, one of which has 4 men more in a side than the other. What number of men are in a side of each of the squares? Ans. 21, and 25.

Prob. 31. What number is that, to which if 24 be added, and the square root of the sum extracted, this root shall be less than the original quantity by 18? Ans. 25.

Prob. 32. A Poulterer going to market to buy turkeys, met with four flocks. In the second were 6 more than three times the square root of double the number in the first. The third contained three times as many as the first and second; and the fourth contained 6 more than the square of one-third the number in the third; and the whole number was 1938. How many were there in each flock?

Ans. The numbers were 18, 24, 126, and 1770, respectively.

Prob. 33. The sum of two numbers is 6, and the sum of their 5th powers is 1056. What are the numbers?

Ans. 4, and 2.

§ II. SOLUTION OF PROBLEMS PRODUCING QUADRATIC EQUATIONS, INVOLVING MORE THAN ONE UNKNOWN QUANTITY.

366. It is very proper to observe, that the solution of a problem, producing quadratic equations, involving two unknown quantities, will sometimes be very much facilitated by assuming x equal to their half sum, and y equal to their half difference; then, (Art. 102), $x+y$ will denote the greater, and $x-y$ the lesser. The solution, according to this method of notation, will, in general, be more simple than that which would have been found, if the two unknown quantities were represented by x and y respectively.

PROBLEM 1. Required two numbers, such, that their sum, their product, and the difference of their squares, may be all equal.

Let $x+y=$ the greater; and $x-y=$ the lesser;

$\therefore 2x=(x+y).(x-y)=x^2-y^2,$ and $2x=(x+y)^2-(x-y)^2=4xy,$ } by the problem.

From the 2d equation, $y=\frac{1}{2}$; $\therefore y^2=\frac{1}{4}$:

Now, by substituting this value of y^2, in the first we have $2x=x^2-\frac{1}{4}$; $\therefore x^2-2x=\frac{1}{4}$, and $x=1\pm\frac{1}{2}\sqrt{5}$.

367. The preceding problem leads also to the solution of the following.

PROB. 2. To find two numbers, such, that their sum, their product, and the sum of their squares, may be all equal.

Let, as in the last problem, $x+y=$ the greater, and $x-y=$ the lesser; then, by the problem,

$2x=x^2-y^2$, and $2x=(x+y)^2+(x-y)^2=2x^2+2y^2$;

$$\therefore x=x^2+y^2;$$
$$\text{but } 2x=x^2-y^2;$$

$$\therefore \text{by addition, } 3x=2x^2, \text{ and } x=\tfrac{3}{2};$$
$$\therefore \text{by substitution, } \tfrac{3}{2}=\tfrac{9}{4}+y^2; \text{ and } y=\pm\tfrac{1}{2}\sqrt{-3};$$
$$\therefore x+y=\tfrac{3}{2}\pm\tfrac{1}{2}\sqrt{-3}, \text{ and } x-y=\tfrac{3}{2}\mp\tfrac{1}{2}\sqrt{-3}.$$

Hence it follows, that no two numbers can be found to answer the conditions; and therefore the problem is impossible: Although the above values of x and y are imaginary, still they will satisfy the equations, $2x=x^2-y^2$, and $2x=2x^2+2y^2$, which may be readily verified by substitution.

368. It is sometimes more expedient to represent one of the unknown quantities by x, and the other by xy. The utility of this method of notation for eliminating one of the unknown quantities, will appear evident, from the solution of the following problem.

PROB. 3. What two numbers are those, whose sum multiplied by the greater is 77; and whose difference, multiplied by the lesser, is equal to 12?

Let $xy=$ the greater, and $x=$ the lesser; then by the problem, $x^2y^2+xy=77$, and $x^2y-x^2=12$;

$$\therefore x^2=\frac{77}{y^2+y}, \text{ and } x^2=\frac{12}{y-1}; \therefore \frac{12}{y-1}=\frac{77}{y^2+y},$$

and clearing of fractions, $12y^2+12y=77y-77$;

by transposition and division, $y^2-\frac{65}{12}y=-\frac{77}{12}$;

$\therefore$ completing the square, and extracting the root, $y=\frac{11}{3}$, or $\frac{7}{4}$. Either value of y will answer the conditions of the problem; Let $y=\frac{7}{4}$; then $x=\frac{12}{y-1}=16$; $\therefore x=\pm 4$, and $xy=\pm 7$. Hence the numbers, by taking the positive values, are 4 and 7. Let also $y=\frac{11}{3}$; then $x^2=\frac{9}{2}$; $\therefore x=\pm\frac{3}{2}\sqrt{2}$, and $xy=\frac{11}{3}\times\pm\frac{3}{2}\sqrt{2}=\pm\frac{11}{2}\sqrt{2}$. Hence the irrational numbers, $\frac{3}{2}\sqrt{2}$ and $\frac{11}{2}\sqrt{2}$, will also answer the conditions of the problem.

369. When a problem expresses more than two distinct conditions, which require to be translated into as many equations; the solution cannot be obtained by means of quadratics, unless that some of the equations are of the first degree; for the final equation resulting from the elimination of the unknown quantities will, in general, be of a higher degree than the second. There are, however, some *particular cases* in which the unknown quantities may be eliminated by certain artifices, (which are best learned by experience), so as to leave the final equation of a quadratic form.

Prob. 4. It is required to find three numbers, such, that the product of the first and second, added to the sum of their squares, shall be equal to 37; the product of the first and third added to the sum of their squares, shall be equal to 49; and the product of the second and third added to the sum of their squares, shall be equal to 61.

Let $x=$ the first number, $y=$ the second, and $z=$ the third.

Then, $\left.\begin{aligned} x^2+y^2+xy&=37;\\ x^2+z^2+xz&=49;\\ \text{and } y^2+z^2+yz&=61;\end{aligned}\right\}$ by the problem.

By subtracting the first equation from the second, $x^2-y^2+(z-y)x=12$; $\therefore z+y+x=\frac{12}{x-y}$ . . . . . (a).

By subtracting the second equation from the third, $y^2-x^2+(y-x)z=12$; $\therefore y+x+z=\frac{12}{y-x}$ . . . . . (b);

$\therefore \frac{12}{z-y}=\frac{12}{y-x}$, and $y-x=z-y$; $\therefore 2y=x+z$.

By substituting $2y$ for $x+z$, in equations (a) and (b), we find $3y=\frac{12}{z-y}$, and $3y=\frac{12}{y-x}$;

$\therefore zy-y^2=4$, and $y^2-yx=4$;

$\therefore z=\frac{y^2+4}{y}$, and $x=\frac{y^2-4}{y}$; $\therefore x^2=\left(\frac{y^2-4}{y}\right)^2$.

Now, by substituting these values of x and x^2 in the first of the original equations, it becomes

$\left(\frac{y^2-4}{y}\right)^2+y^2+y.\frac{y^2-4}{y}=37$; $\therefore$ by reduction,

$y^4-\frac{49}{3}y^2=-16$; and, by completing the square,

$y^4-\frac{49}{3}y^2+\left(\frac{49}{3}\right)^2=\frac{2401-192}{36}$; $\therefore y^2=\frac{49}{6}\pm\frac{47}{6}$;

and, by taking the positive sign, $y=\pm 4$;

$\therefore$ by taking $y=4$, $x=\frac{y^2-4}{y}=\frac{16-4}{4}=3$, and

$$z=\frac{y^2+4}{y}=\frac{16+4}{4}=\frac{20}{4}=5.$$

Hence the three numbers sought are 3, 4, and 5, which are in arithmetical progression. This relation appears also evident from the result $2y=x+z$, found in the beginning of the solution.

Prob. 5. There are three numbers, the difference of whose differences is 8; their sum is 41; and the sum of their squares 669. What are the numbers?

Let $x=$ the second number,
and $y=$ the difference of the second and least;
$\therefore x-y$, x, and $x+y+8$ are the numbers,
and their sum $=3x+8=41$; $\therefore 3x=33$, and $x=11$;
$\therefore (11-y)^2+121+(19+y)^2=669$, or $y^2+8y=48$;
$\therefore$ completing the square, and extracting the root,
$y+4=\pm 8$, and $y=4$, or -12, both which values answer the conditions; and the numbers are 7, 11, and 23.

Prob. 6. What number is that, which being divided by the product of its two digits, the quotient is 2; and if 27 be added to it, the digits will be inverted? Ans. 36.

Prob. 7. There are three numbers, the difference of whose differences is 5; their sum is 44; and continual product is 1950. What are the numbers? Ans. 6, 13, and 25.

Prob. 8. A farmer received 7*l.* 4*s.* for a certain quantity of wheat, and an equal sum at a price less by 1*s.* 6*d.* per bushel, for a quantity of barley, which exceeded the quantity of wheat by 16 bushels. How many bushels were there of each? Ans. 32 bushels of wheat, and 48 of barley.

Prob. 9. A poulterer bought 15 ducks and 12 turkeys for five guineas. He had two ducks more for 18 shillings, than he had of turkeys for 20 shillings. What was the price of each? Ans. the price of a duck was 3*s*. and of a turkey 5*s*.

Prob. 10. There are three numbers, the difference of whose differences is 3; their sum is 21; and the sum of the squares of the greatest and least is 137. Required the numbers.

Ans. 4, 6, and 11.

Prob. 11. There is a number consisting of 2 digits, which, when divided by the sum of its digits, gives a quotient greater by 2 than the first digit. But if the digits be inverted, and then divided by a number greater by unity than the sum of the digits, the quotient is greater by 2 than the preceding quotient. Required the number. Ans. 24.

Prob. 12. What two numbers are those, whose product is 24, and whose sum added to the sum of their squares is 62?

Ans. 4, and 6.

Prob. 13. A grocer sold 80 pounds of mace, and 100 pounds of cloves, for 65*l*.; but he sold 60 pounds more of cloves for 20*l*. than he did of mace for 10*l*. What was the price of a pound of each?

Ans. the mace cost 10*s*. and the cloves 5*s*. per pound.

Prob. 14. To divide the number 134 into three such parts, that once the first, twice the second, and three times the third, added together, may be equal to 278; and that the sum of the squares of the three parts may be equal to 6036.

Ans. 40, 44, and 50, respectively.

Prob. 15. Find two numbers, such, that the square of the greater *minus* the square of the lesser, may be 56; and the square of the lesser *plus* one third, their product may be 40.

Ans. 9, and 5.

Prob. 16. There are two numbers, such, that three times the square of the greater *plus* twice the square of the less is 110; and half their product, *plus* the square of the lesser, is 4. What are the numbers? Ans. 6, and 1.

Prob. 17. What number is that, the sum of whose digits is 15; and if 31 be added to their *product*, the digits will be inverted? Ans. 78.

Prob. 18. There are two numbers, such, that if the lesser be taken from three times the greater, the remainder will be 35; and if four times the greater be divided by three times the lesser *plus* one, the quotient will be equal to the lesser number. What are the numbers? Ans. 13, and 4.

Prob. 19. To find two numbers, the first of which, *plus* 2, multiplied into the second, *minus* 3, may produce 110; and the first *minus* 3, multiplied by the second *plus* 2, may produce 80. Ans. 8, and 14.

Prob. 20. Two persons, A and B, comparing their wages, observe, that if A had received *per* day, in addition to what he does receive, a sum equal to one-fourth of what B received *per* week, and had worked as many days as B received shillings *per* day, he would have received 48*s.*; and had B received 2 shillings a day more than A did, and worked for a number of days equal to half the number of shillings he received *per* week, he would have received 4*l.* 18*s.* What were their daily wages? Ans. A's 5 shillings, and B's 4.

Prob. 21. Bacchus caught Silenus asleep by the side of a full cask, and seized the opportunity of drinking, which he continued for two-thirds of the time that Silenus would have taken to empty the whole cask. After that Silenus awoke, and drank what Bacchus had left. Had they drunk both together, it would have been emptied two hours sooner, and Bacchus would have drunk only half what he left Silenus. Required the time in which they could have emptied the cask separately. Ans. Silenus in 3 hours, and Bacchus in 6.

Prob. 22. Two persons, A and B, talking of their money, A says to B, if I had as many dollars at 5*s.* 6*d.* each, as I have shillings, I should have as much money as you; but, if the number of my shillings were squared, I should have twice as much as you, and 12 shillings more. What had each?
Ans. A had 12, and B 66 shillings.

Prob. 23. It is required to find two numbers, such, that if their product be added to their sum it shall make 62; and if their sum be taken from the sum of their squares, it shall leave 86. Ans. 8, and 6.

Prob. 24. It is required to find two numbers, such, that their difference shall be 98, and the difference of their cube roots 2. Ans. 125, and 27.

Prob. 25. There is a number consisting of two digits. The left-hand digit is equal to 3 times the right-hand digit; and if 12 be subtracted from the number itself, the remainder will be equal to the square of the left-hand digit. What is the number? Ans. 93.

Prob. 26. A person bought a quantity of cloth of two sorts for 7*l.* 18 shillings. For every yard of the better sort he gave as many shillings as he had yards in all; and for every yard of the worse as many shillings as there were yards of the better sort more than of the worse. And the whole price of the better sort was to the whole price of the worse as 72 to 7. How many yards had he of each?

Ans. 9 yards of the better, and 7 of the worse.

PROB. 27. There are four towns in the order of the letters, A, B, C, D. The difference between the distances, from A to B, and from B to C, is greater by four miles than the distance from B to D. Also the number of miles between B and D is equal to two-thirds of the number between A to C. And the number between A and B is to the number between C and D as seven times the number between A and C : 26. Required the respective distances.

Ans. AB$=$42, BC$=$6, and CD$=$26 miles.

CHAPTER XII.

ON

THE EXPANSION OF INFINITE SERIES.

§ I. RESOLUTIONS OF ALGEBRAIC FRACTIONS.

370. An *infinite series* is a continued rank, or progression of quantities, connected together by the signs $+$ or $-$; and usually proceeds according to some regular, or determined *law.*

Thus, $\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+$, &c.

Or, $\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+$, &c.

In the first of which, the several terms are the reciprocals of the odd numbers, 1, 3, 5, 7, &c.; and in the latter the reciprocals of the even numbers, 2, 4, 6, 8, &c., with alternate signs.

371. We have already observed (Art. 96), that if the first or leading term of the remainder, in the division of algebraic quantities, be not divisible by the divisor, the operation might be considered as terminated; or, which is the same, that the integral part of the quotient has been obtained. And it has also been remarked, (Art. 89), that the division of the remainder by the divisor can be only indicated, or expressed, by a fraction: thus, for example, if we have to divide a^o by $a+1$,

we write for the quotient $\frac{1}{a+1}$: This, however, does not prevent us from attempting the division according to the rules that have been given, nor from continuing it as far as we please, and we shall thus not fail to find the true quotient, though under different forms.

372. To prove this, let us actually divide a^0 or 1, by $1-a$, thus;

$$\begin{array}{r|l} 1 & 1-a \\ 1-a & \overline{\phantom{Quot. 1+\frac{a}{1+a}}} \\ \hline \text{remainder} \quad a & \textit{Quot.}\ 1+\frac{a}{1+a} \end{array}$$

Therefore $\frac{1}{1-a}=1+\frac{a}{1-a}$; but $\frac{a}{1-a}=a+\frac{a^2}{1-a}$; $\frac{a^2}{1-a}=a^2+\frac{a^3}{1-a}$; $\frac{a^3}{1-a}=a^3+\frac{a^4}{1-a}$; $\frac{a^4}{1-a}=a^4+\frac{a^5}{1-a}$, &c.

This shows that the fraction $\frac{1}{1-a}$ may be exhibited under all the following forms:

$$\frac{1}{1-a}=1+\frac{a}{1-a};\ =1+a+\frac{a^2}{1-a};$$
$$=1+a+a^2+\frac{a^3}{1-a};\ =1+a+a^2+a^3+\frac{a^4}{1-a};$$
$$=1+a+a^2+a^3+a^4+\frac{a^5}{1-a}\text{\&c.}$$

Now, by considering the first of these formulæ, which is $1+\frac{a}{1-a}$, and observing that $1=\frac{1-a}{1-a}$, we have $1+\frac{a}{1-a}=\frac{1-a}{1-a}+\frac{a}{1-a}=\frac{1-a+a}{1-a}=\frac{1}{1-a}$.

If we follow the same process with regard to the second expression, that is to say, if we reduce the integral part $1+a$ to the same denominator, $1-a$, we shall have the fraction $\frac{1-a^2}{1-a}$, to which if we add $\frac{a^2}{1-a}$, we shall have $\frac{1-a^2+a^2}{1-a}=\frac{1}{1-a}$.

In the third formula of the quotient, the integers $1+a+a^2$ reduced to the denominator $1-a$ make $\frac{1-a^3}{1-a}$, and if we add to it the fraction $\frac{a^3}{1-a}$ the sum will be $\frac{1}{1-a}$.

Therefore each of these formula is in fact the value of the proposed fraction $\frac{1}{1-a}$.

273. This being the case, we may continue the series as far as we please, without being under the necessity of performing any more calculations; by observing, in the first place, that each of these formulæ is composed of an integral part which is the sum of the successive powers of a, beginning with $a^0=1$ inclusively;

Secondly, of a fraction which has always for the denominator $1-a$, and for the numerator the letter a, with an exponent greater, by unity, than that of the same letter in the last term of the integral part.

This constant formation of the successive formulæ, is what *Analysts* call a *law*. And the manner of deducing general laws by the consideration of certain particular cases, is usually called *induction;* which, though not a strict method of proof, says LAPLACE, has been the source of almost all the discoveries that have hitherto been made, both in analysis and physics, of which all the phenomena are the mathematical results of a small number of invariable laws. It is thus that NEWTON, by following the law of the numeral coefficients, in the square, the cube, the fourth power, &c. of a binomial, arrived soon at the general law, that is to say, at the general formula that bears his name, and which will be demonstrated in one of the following Sections: This *Geometer* has carefully added, that in following this mode of investigation, we must not generalize too hastily; as it often happens, that a law, which appears to take place in the first part of a process, is not found to hold good throughout. Thus, in the simple instance of reducing $\frac{531251}{3093750}$ to a decimal, its equivalent value is 17174949, &c., of which the real, repeating period is 49, and not 17, as might, at first, be imagined.

374. From what has been observed with regard to the successive quotients, we can, in general, put

$$\frac{1}{1-a}=1+a+a^2+a^3+a^4 \quad . \; . \; . \; . \; . \; . \; . \; . \quad a^n+\frac{a^n+1}{1+a},$$

n being a whole positive number, which augmented by unity, gives the place of the term. In fact, making $n=3$, a^n becomes a^3, which is the fourth term of the quotient, for $n=4$, a^n becomes a^4, which is the fifth term. But as nothing hinders us from removing indefinitely the fractional term which terminates the series, that is, of adding always a term to the integral part;

so that we might still go on without end; for which reason it may be said that the proposed fraction has been resolved into an infinite series; which is, $1+a+a^2+a^3+^4+a^5+a^6+a^7+a^8+a^9+a^{10}+a^{11}+a^{12}+$, &c. to infinity: and there are sufficient grounds to maintain that the value of this infinite series is the same as that of the fraction $\frac{1}{1-a}$.

Or that, $\frac{1}{1-a}=1+a+a^2+a^3+a^4+$; &c.

375. What has been just observed may at first appear strange; but the consideration of some particular cases will make it easily understood.

Let us suppose, in the first place, $a=1$; the general quotient above will become a particular quotient corresponding to the fraction $\frac{1}{1-1}$. The series taken indefinitely, shall be

$$\frac{1}{0}=1+1+1+1+1+1+, \text{ \&c.}$$

In order to see clearly the meaning of this result, let us suppose that we have to divide unity or 1 successively by the numbers 1, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, $\frac{1}{10000}$, &c., we will have the quotient, 1, 10, 100, 1000, 10000, &c., continually and indefinitely increasing; because the divisors are continually and indefinitely decreasing; but these divisors tend towards zero, which they cannot attain, although they approach to it continually, or that the difference becomes less and less; and at the same time the value of the fraction increases continually, and tends to that which corresponds to the divisor zero or 0; and it is as much impossible that the fraction in its successive augmentations, attains $\frac{1}{0}$, as it is that the denominator in its successive diminutions arrives at zero. Thus $\frac{1}{0}$ *is the last term or limit* of the *increasing values of the fraction*: at this period, it has received all its augmentations: $\frac{1}{0}$ is not therefore a number, it is the superior limit of numbers; such is the notion that we must have of this result $\frac{1}{0}$, which the analysts call, for abbreviation, *infinity*, and which is denoted by the character ∞, (Art. 35). It is frequently given as an answer to an impossible question, (which will be noticed in a subsequent part of the Work); and in fact, it is very proper to announce this circumstance, since that we cannot assign the number denoted by this sign.

It may still be remarked, that if we would take but the first six terms of the series, we must close the development by the corresponding remainder divided by this divisor, which gives,

$$\frac{1}{1-1}=\frac{1}{0}=1+1+1+1+1+1+\frac{1}{0};$$

this equality, absurd in appearance, proves that six terms at least do not hinder the series from being indefinitely continued. And in fact, if after having taken away six terms from this series, it would cease to be infinite, or become terminated, in restoring to it these six terms, it should be composed of a definite or assignable number of terms, which it is not. Therefore the surplus of the series must have the same sum as the total. We can yet say that $\frac{1}{0}$, inasmuch as it is not a magnitude, can receive no augmentation, so that $1+1+1+$, &c. $+\frac{1}{0}$ must remain equal to $\frac{1}{0}$.

Hence, we might conclude that a finite quantity added to, or subtracted from infinity, makes no alteration.

Thus, $\infty \pm a = \infty$.

However, it may be necessary in this place to observe, that, although an infinity cannot be increased, or decreased, by the addition, or subtraction, of finite quantities; still, it may be increased or decreased, by multiplication or division, in the same manner as any other quantity; Thus, if $\frac{1}{0}$ be equal to infinity, $\frac{2}{0}$ will be the double of it, $\frac{3}{0}$ thrice, and so on. See EULER'S *Algebra*, Vol. I.

NOTE.—$\frac{1}{1}$, $\frac{1}{\frac{1}{10}}$, $\frac{1}{\frac{1}{100}}$, $\frac{1}{\frac{1}{1000}}$, &c. are considered to be fractions, in which the denominators are 1, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c.

Now, as 1 divided by any assignable quantity, however great it may be, can never arrive completely at 0, consequently the fractions in their successive augmentations can never arrive at infinity, except that unity or 1, be divided by a quantity infinitely great; that is to say, it must be divided by *infinity*; hence we may conclude that $\frac{1}{\infty}$ is in reality equal to nothing, or $\frac{1}{\infty}=0$.

360. It may not be improper to take notice in this place of other properties of *nought* and *infinity*.

I. That nought added to or subtracted from any quantity, makes it neither greater nor less; that is,

$$a+0=a, \text{ and } a-0=a.$$

II. Also, if nought be multiplied or divided by any quantity, both the product and quotient will be nought; because any number of times 0, or any part of 0, is 0: that is,

$$0\times a, \text{ or } a\times 0=0, \text{ and } \frac{0}{a}=0.$$

III. From the last property, it likewise follows, that nought divided by nought, is a finite quantity, of some kind or other. For since $0\times a=0$, or $0=0\times a$, it is evident from the ordinary rules of division, that

$$\frac{0}{0}=a.$$

IV. Farther, if nought be multiplied by infinity, the product will be some infinite quantity. For since $\frac{1}{0}$ or $\frac{a}{0}=\infty$; therefore, $0\times\infty=a$.

361. It may be also remarked, that nought multiplied by 0 produces 0; that is,

$$0\times 0=0.$$

For, since $0\times a=0$, whatever quantity a may be, then, supposing $a=0$, $0\times 0=0$.

From this we might infer, according to the rules of division, that the value of $\frac{0}{0}=0$, or that nought divided by nought is nought, in this particular case.

Also, that 0, raised to any power, is 0; that is, $0^m=0$; it follows that $\frac{0^m}{0^m}=\frac{0}{0}$; but if in $a^{m-m}=\frac{a^m}{a^m}$ (Art. 86), we suppose $a=0$, which may be allowed, since a designates any number, we have $0^0=\frac{0}{0}$.

If we really effect the division of 0 by 0, we could put for the quotient any number whatever, since any number, multiplied by zero, gives for the product zero, which is here the dividend.

This expression, 0^0, appears therefore to admit of an infinity of numerical values; and yet such a result as $\frac{0}{0}$ can, in many cases, admit of a finite and determined value. It is thus, for example, that the fraction $\frac{Ka^m}{a^n}$, in the hypothesis of $a=0$, becomes $\frac{K\times 0}{0}=\frac{0}{0}$.

But, if at first we write this fraction under the form Ka^{m-n}, and that we put $a=0$, we find that it becomes $K\times 0^{m-n}$, which is 0 for $m>n$; in case of $m<n$, or $m=n-d$, we shall have (Art. 86), $\frac{K}{0^d}=\frac{K}{0}$; which is equal to infinity, as has been already observed; finally, for $m=n$, we can divide above and below by a^m, and the fraction is reduced to K, which is a finite quantity.

362. If we suppose, in the fraction (Art. 358), $a=2$, we find

$$\frac{1}{1-2}=1+2+4+8+16+32+64+, \&c.,$$

which at first sight it will appear absurd. But it must be remarked, that if we wish to stop at any term of the above series, we cannot do so without joining the fraction which remains. Suppose, for example, we were to stop at 64; after having written $1+2+4+8+16+32+64$; we must join the fraction $\frac{128}{1-2}$, or $\frac{128}{-1}$, or -128; we shall therefore have for the complete quotient $127-128$, than is in fact -1.

Here, however far the fractional term may be extended, its numerical value, which is negative, will always surpass, by a unit, that of the integral part, so that this is totally destroyed; and as in the hypotheses of $a>1$, we shall always subtract more than what we will add, we shall never meet with the result $\frac{1}{0}$.

363. These are the considerations which are necessary when we assume for a numbers greater than unity; but if we now suppose a less than 1, the whole becomes more intelligible; for example, let $a=\frac{1}{2}$, and we shall have $\frac{1}{1-a}=\frac{1}{1-\frac{1}{2}}=\frac{1}{\frac{1}{2}}=2$, which will also be equal to the following se-

ries, $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}$, &c., to infinity (Art. 358). Now, if we take only two terms of the series, we shall have $1+\frac{1}{2}$, and it wants $\frac{1}{2}$ of being equal to 2; if we take three terms, it wants $\frac{1}{4}$, for the sum is $1\frac{3}{4}$; if we take four terms, we have $1\frac{7}{8}$, and the deficiency is only $\frac{1}{8}$: Therefore, we see very clearly that the more terms of the quotient we take, the less the difference becomes; and that, consequently, if we continue to take successive portions of this series, the differences between those consecutive sums and the fraction $\frac{1}{1-\frac{1}{2}}=2$, decrease, and end by becoming less than any given number, however small it may be. The number 2 is therefore still a *limit*, according to the acceptation of this word.

Now, it may be observed, that if the preceding series be continued to infinity, there will be no difference at all between its sum and the value of the fraction $\frac{1}{1-\frac{1}{2}}$, or 2.

364. *A limit, according to the notion of the ancients, is some fixed quantity, to which another of variable magnitude can never become equal, though, in the course of its variation, it may approach nearer to it than any difference that can be assigned; always supposing that the change, which the variable quantity undergoes, is one of continued increase, or continued diminution.* Such, for example, is the area of a circle, with regard to the areas of the circumscribed and inscribed polygons, for, by increasing the number of sides of these figures, their difference may be made less than any assigned area, however small; and since the circle is necessarily less than the first, and greater than the second, it must differ from either of them by a quantity less than that by which they differ from each other. The circle will thus answer all the conditions of a limit, which is included in the above definition.

365. The preceding considerations are very proper to define the nature of the word limit; but as algebra, which is the subject we are treating of here, needs no foreign aid to demonstrate its principles, it is necessary, therefore, to explain the nature of the word limit, by the consideration of algebraic expressions. For this purpose, let, in the first place, the very simple fraction be $\frac{ax}{x+a}$, in which we suppose that x may be positive, and augmented indefinitely; in dividing both terms of this fraction by x, the result $\frac{x}{1+\frac{a}{x}}$, evidently shows that

the function remains always less than a, but that it approaches continually to a, since that the part $\frac{a}{x}$, of its denominator, diminishes more and more, and can be reduced to such a degree of smallness as we would wish.

366. The difference between a and the proposed fraction being in general expressed by $a-\frac{ax}{x+a}=\frac{a^2}{x+a}$, becomes so much smaller, according as x is larger, and can be rendered less than any given magnitude, however small it may be; so that the proposed fraction can approach to a as near as we would wish: a is therefore the *limit* of the fraction $\frac{ax}{x+a}$, relatively to the indefinite augmentation which x can receive. It is in the characters which we have just expressed, that the true acceptation, which we must give to the word limit, consists, in order to comprehend every thing which can relate to it.

367. If we had remarked in the preceding example, that by carrying on, as far as we would wish, the augmentation of x, we could never regard, as nothing, the fraction $\frac{a^2}{x+a}$; therefore we would reasonably conclude, that the fraction $\frac{ax}{x+a}$, though it would approach indefinitely to the limit a, could never attain a, and, consequently, cannot surpass it; but it would be wrong to insert this circumstance as a condition in the general definition of the word *limit*; we would thereby exclude the ratios of vanishing quantities, ratios whose existence is incontestable, and from which we derive much in analysis.

368. In fact, when we compare the functions ax and $ax+x^2$, we find that their ratio, reduced to its most simple expression, is $\frac{a}{a+x}$, and that it approaches nearer and nearer to unity, according as x diminishes. It becomes exactly 1, when $x=0$; but the quantities ax and $ax+x^2$, which are then rigorously nothing, can they have a determinate ratio? This is what appears difficult to conceive; and we cannot give a clear idea of it but by presenting the quantity 1 as a limit to which the ratio of the functions ax and $ax+x^2$ can approach as near as we would wish, since the difference, $1-\frac{a}{a+x}=\frac{x}{a+x}$, can be rendered less than any assignable magnitude, however small this magnitude may be.

On the other hand, the ratio $\frac{a}{a+x}$, of the quantities ax and $ax+x^2$ can not only attain unity when we make $x=0$, but surpass it when we suppose x negative, since it becomes then $\frac{a}{a-x}$, a quantity which is greater than 1, when $x<a$. This circumstance appears not at all contrary to the idea of limit; for we can regard the value 1, which answers to $x=0$, as a term towards which the ratio of the functions ax and $ax+x^2$ tends, by the diminutions of the values of x, whether positive or negative. For further illustrations of the world limit, and what is meant by infinity, and infinitely small quantities or infinitesimals, the intelligent reader is referred to LACROIX's *Introduction to the Traité du Calcul Differentiel et du Calcul Integral*, 4to. where these subjects are clearly elucidated.

369. Now, let $a=\frac{1}{3}$, in the fraction $\frac{1}{1-a}$, and we shall have $\frac{1}{1-\frac{1}{3}}=\frac{3}{2}=1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+$, &c. If we take two terms, we find $1+\frac{1}{3}$, and the difference $=\frac{1}{6}$; three terms give $1+\frac{4}{9}$, the error $=\frac{1}{18}$; for four terms the error is no more than $\frac{1}{54}$. Since, therefore, the error always becomes three times less, it tends towards zero, which it cannot attain, and the sum tends toward $\frac{3}{2}$, which is the *limit*.

370. Again, let us take $a=\frac{2}{3}$, and we shall have $\frac{1}{1-\frac{2}{3}}=3$ $=1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}+\frac{32}{343}+$&c.; here, in the first place, the sum of two terms, which is $1+\frac{2}{3}$, is less than 3 by $1+\frac{1}{3}$; taking three terms, which make $2\frac{1}{9}$, the error is $\frac{8}{9}$; for four terms, whose sum is $2\frac{11}{27}$ the error is $\frac{16}{27}$.

371. Finally, for $a=\frac{1}{4}$, we find $\frac{1}{1-\frac{1}{4}}=1+\frac{1}{3}=1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}$ $+\frac{1}{256}+$, &c.; the first two terms are equal to $1\frac{1}{4}$, which gives $\frac{1}{12}$ for the error; and taking one term more, we shall have only an error of $\frac{1}{48}$.

372. From the preceding considerations we may readily conclude, that any fraction having a compound denominator may be converted into an infinite series by the following rule: and if the denominator be a simple quantity, it may be divided into two or more parts.

RULE.

Divide the numerator by the denominator, as in the division of integral quantities, and the operation continued as far as may be thought necessary, will give the series required.

Ex. 1. It is required to reduce $\frac{ax}{a-x}$ into an infinite series.

$$\begin{array}{l|l}
ax & a-x \\
\underline{ax-x^2} & \\
x^2 & \text{Quotient.} \\
\underline{x^2-\frac{x^3}{a}} & x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}, \text{ \&c.} \\
\text{1st rem.} \ldots \frac{x^3}{a} & \\
\underline{\frac{x^3}{a}-\frac{x^4}{a^2}} & \\
\text{2d rem.} \ldots \frac{x^4}{a^2} & \\
\underline{\frac{x^4}{a^2}-\frac{x^5}{a^3}} & \\
\frac{x^5}{a^3}, \text{ \&c.} &
\end{array}$$

The terms in the quotient are found thus; dividing the first remainder x^2, by a, the first term of the divisor $a-x$, we shall have $\frac{x^2}{a}$ for the second term of the quotient, because the division can be only indicated; multiplying the divisor by $\frac{x^2}{a}$, and subtracting the product from x^2, the remainder is $\frac{x^3}{a}$, again, dividing this remainder by a, the result will be $\frac{x^3}{a^2}$, which is the third term in the quotient; and, in like manner, we might continue the operation as far as we please: But the *law of continuation* is evident, because the powers of x increase by unity in each successive term of the quotient, and the powers of a increase by unity in the denominator of each of the terms after the first.

And the sum of the terms infinitely continued is said to be equal to the original fraction $\frac{ax}{a-x}$. Thus we say that the numerical fraction $\frac{2}{3}$, when reduced to a decimal, is equal to .6666, &c., continued to infinity.

Ex. 2. It is required to convert $\frac{a}{a-x}$ into an infinite series.

$$\begin{array}{r|l} a & a-x \\ a-x & \overline{\qquad\qquad} \\ \hline x & \textit{Quotient.} \\ x-\frac{x^2}{a} & 1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+, \&c. \\ \hline \frac{x^2}{a} & \\ \frac{x^2}{a}-\frac{x^3}{a} & \\ \hline \frac{x^3}{a^2}, \&c. & \end{array}$$

In this example, if x be less than a, the series is *convergent*, or the value of the terms continually diminishes; but, when x is greater than a, it is said to *diverge*: Thus, let $a=3$ and $x=2$, then $1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+$, &c. $=1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+$, &c.; where the fractions or terms of the series grow less and less, and the farther they are extended the more they *converge* or *approximate* to 0, which is supposed to be the last term or limit.

But if $a=2$, and $x=3$, then $1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+$, &c. $=1+\frac{3}{2}+\frac{9}{4}+\frac{27}{8}+$, &c., in which the terms become larger and larger. This is called a *diverging* series.

Ex. 3. It is required to convert $\frac{1}{1+a}$ into an infinite series.

$$\begin{array}{l|l} 1 & 1+a \\ 1+a & \overline{\quad\quad} \\ \hline -a & \textit{Quotient.} \\ -a-a^2 & 1-a+a^2-a^3+a^4-a^5+a^6-, \&c. \\ \hline \quad a^2 & \\ \quad a^2+a^3 & \\ \hline \quad\quad -a^3 & \\ \quad\quad -a^3-a^4 & \\ \hline \quad\quad\quad a^4 & \\ \quad\quad\quad a^4+a^5 & \\ \hline \quad\quad\quad\quad -a^5, \&c. & \end{array}$$

Whence it follows, that the fraction $\frac{1}{1+a}$ is equal to the series, $1-a+a^2-a^3+a^4-a^5+a^6-a^7+$, &c.

372. If we make $a=1$, we have this remarkable comparison: $\frac{1}{1+a}=1-1+1-1+1-1+1-$, &c. to infinity; which appears rather contradictory; for if we stop at -1, the series gives 0; and if we finish at $+1$, it gives $+1$. The real question, however, results from the fractional parts, which (by division) is always $+\frac{1}{2}$ when the sum of the terms is 0, and $-\frac{1}{2}$ when the sum is $+1$: because the complete quotient is found by placing the remainder over the divisor, in the form of a fraction, and annexing it to the terms in the quotient with its proper sign; but the remainder in the present case is $+1$, or -1; hence the fraction to be added is $+\frac{1}{2}$, or $-\frac{1}{2}$; and, consequently, $\frac{1}{2}$ is the true quotient in the former case, and $1-\frac{1}{2}$, or $\frac{1}{2}$ in the other. This will appear evident by taking successive portions of the series; thus, for six terms, we shall have $1-1+1-1+1-1+\frac{1}{2}=\frac{1}{2}$, and for seven terms, $1-1+1-1+1-1+1-\frac{1}{2}=\frac{1}{2}$.

Scholium. Here we might infer, by conversion, that the sum of an infinite series is found, when we know the fraction which would produce such a series by actual division; but, although it is a fact that the fraction is a value of the series, still it may not be the only one which would produce the same series: Thus, the above series, $1-1+1-1+1-1+1-1+$, &c., to infinity, can be produced by several other fractions besides the fraction $\frac{1}{2}$.

Let, for example, $\frac{1}{3}$ be converted into an infinite series by actual division: Now, it is plain that $\frac{1}{3}=\frac{1}{1+1+1}$, and the operation will stand thus:

$$\begin{array}{l|l}
1 & 1+1+1 \\
1+1+1 & \overline{} \\
\hline
\quad -1-1 & \textit{Quotient.} \\
\quad -1-1-1 & 1-1+1-1+1-1+, \&c. \\
\hline
\qquad\quad +1 & \\
\qquad\quad +1+1+1 & \\
\qquad\quad \overline{\qquad\quad -1-1} & \\
\qquad\qquad\quad -1-1-1 & \\
\qquad\qquad\quad \overline{\qquad\quad +1, \&c.} &
\end{array}$$

In like manner, $\frac{1}{4}$ will produce the above series, and so on.

374. Let us now make $a=\frac{1}{2}$; and the preceding development shall be

$$\frac{1}{1+\frac{1}{2}}=\frac{2}{3}=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+, \&c.:$$

The sum of two terms is $\frac{1}{2}$, which is too small by $\frac{1}{6}$; three terms give $\frac{3}{4}$, which is too much by $\frac{1}{12}$; for the sum of four terms, we have $\frac{5}{8}$, which is too small by $\frac{1}{24}$, &c.

We see here that the successive portions of the series are alternately greater and less than the fraction $\frac{2}{3}$, which represent it; but that the difference, whether it be in excess or deficiency, becomes less and less.

375. Suppose again $a=\frac{1}{3}$, and we shall have

$$\frac{1}{1+a}=\frac{1}{1+\frac{1}{3}}=\frac{3}{4}=1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\frac{1}{81}-\frac{1}{243}+, \&c.$$

Now, by considering only two terms, we have $\frac{2}{3}$, which is too small by $\frac{1}{12}$; three terms make $\frac{7}{9}$, which is too much by $\frac{1}{36}$; four terms give $\frac{20}{27}$, which is too small by $\frac{1}{108}$, and so on.

376. The fraction $\frac{1}{1+a}$ may also be resolved into an infinite series another way; namely, by dividing 1 by $a+1$, as follows:

$$
\begin{array}{l|l}
1 & a+1 \\
1+\frac{1}{a} & \overline{\textit{Quot.}} \\
\overline{\phantom{1+\frac{1}{a}}} & \frac{1}{a}-\frac{1}{a^2}+\frac{1}{a^3}-\frac{1}{a^4}+\frac{1}{a^5}-, \ \&c. \\
-\frac{1}{a} & \\
-\frac{1}{a}-\frac{1}{a^2} & \\
\overline{\phantom{-\frac{1}{a}-\frac{1}{a^2}}} & \\
\qquad \frac{1}{a^2} & \\
\qquad \frac{1}{a^2}+\frac{1}{a^3} & \\
\qquad \overline{\phantom{\frac{1}{a^2}+\frac{1}{a^3}}} & \\
\qquad \quad -\frac{1}{a^3}, \ \&c. &
\end{array}
$$

It is however unnecessary to carry the actual division any farther, as we are enabled already to continue the series to any length, from the law which may be observed in those terms we have obtained ; the signs are alternately *plus* and *minus*, and each term is equal to the preceding one multiplied by $\frac{1}{a}$.

It is thus by changing the order of the terms of the denominator, we obtain the quotient under different forms, and that we pass from a diverging series, for certain values of a, to a converging series for the same values.

It may also be here observed, that in the division of the two polynomials, if we deviate from the established rule (Art. 93), we arrive at quotients which do not terminate :

Thus, for example, a^2-b^2, divided by $a+b$, according to the rule above quoted, gives for the quotient $a-b$; but if we divide a^2-b^2 by $b+a$, we shall arrive at a quotient which does not terminate : thus,

$$
\begin{array}{l|l}
a^2-b^2 & b+a \\
a^2+\frac{a^3}{b} & \overline{} \\
\overline{\phantom{a^2+\frac{a^3}{b}}} & \textit{Quot.} = \frac{a^2}{b}-\frac{a^3}{b^2}+\frac{a^4}{b^3}-\frac{a^5}{b^4}+, \ \&c. \\
-\frac{a^3}{b}-b^2 & \\
-\frac{a^3}{b}-\frac{a^4}{b^2} & \\
\overline{\phantom{-\frac{a^3}{b}-\frac{a^4}{b^2}}} &
\end{array}
$$

$$\frac{a^4}{b^2}-b^2$$
$$\frac{a^4}{b^2}+\frac{a^5}{b^3}$$
$$\overline{\qquad\qquad\qquad}$$
$$-\frac{a^5}{b^3}-b^2$$
&c.

Here, we can clearly see that the quotient will not terminate, however far we may continue the operation, because we have always a remainder.

In this case, by taking $b+a$ for a divisor, we must, in order to find the quotient $a-b$, divide the whole dividend by all the divisor, that is to say, a^2-b^2 or $(a+b)\times(a-b)$ by $a+b$.

377. When there are more than two terms in the divisor, we may also continue the division to infinity in the same manner.

Ex. 4. It is required to convert $\dfrac{1}{1-a+a^2}$ into an infinite series.

$$\begin{array}{l|l}
1 & 1-a+a^2 \\
\underline{1-a+a^2} & \overline{\text{Quot.}} \\
\quad a-a^2 & 1+a-a^3-a^4+a^6+a^7, \text{ \&c.} \\
\quad \underline{a-a^2+a^3} & \\
\qquad -a^3 & \\
\qquad \underline{-a^3+a^4-a^5} & \\
\qquad\quad -a^4+a^5 & \\
\qquad\quad \underline{-a^4+a^5-a^6} & \\
\qquad\qquad a^6 & \\
\qquad\qquad \underline{a^6-a^7+a^8} & \\
\qquad\qquad\quad a^7-a^8 & \\
\qquad\qquad\quad \underline{a^7-a^8+a^9} & \\
\qquad\qquad\qquad -a^9 & \\
\qquad\qquad\qquad \text{\&c.} &
\end{array}$$

We have therefore $\dfrac{1}{1-a+a^2}=1+a-a^3-a^4+a^6+a^7$, &c.

to infinity: where, if we make $a=1$, we have $\frac{1}{1-1+1}=1=1+1-1-1+1+1$, &c., which series contains twice the series found, (Art. 372), $1-1+1-1+1$, &c. Now, as we have found this to be equal to $\frac{1}{2}$, it is not extraordinary that we should find $\frac{2}{2}$, or 1, for the value of that which we have just determined.

By making $a=\frac{1}{2}$, we shall have $\frac{1}{\frac{3}{4}}=\frac{4}{3}=1+\frac{1}{2}-\frac{1}{8}-\frac{1}{16}+\frac{1}{64}+\frac{1}{128}-\frac{1}{512}$, &c.

If $a=\frac{1}{3}$, we shall have

$$\frac{1}{\frac{7}{9}}=\frac{9}{7}=1+\frac{1}{3}-\frac{1}{27}-\frac{1}{81}+\frac{1}{729}, \text{ \&c.}$$

And if we take the four leading terms of this series, we have $\frac{104}{81}$, which is only $\frac{1}{567}$ less than $\frac{9}{7}$.

Let us suppose again $a=\frac{2}{3}$, and we shall have $\frac{1}{\frac{7}{9}}=\frac{9}{7}=1+\frac{2}{3}-\frac{8}{27}-\frac{16}{81}+\frac{64}{729}+$, &c. this series is therefore equal to the preceding one, and by subtracting one from the other, we obtain $\frac{1}{3}-\frac{7}{27}-\frac{15}{81}+\frac{64}{729}$, &c., which is necessarily $=0$.

378. The method which has been here explained, serves to resolve, generally, all fractions into infinite series; which is often found, as has been observed by EULER in his *Algebra*, to be of the greatest utility; it is also remarkable, that an infinite series, though it never ceases, may have a determiate value. It should likewise be observed, that from this branch of Mathematics, inventions of the utmost importance have been derived, on which account the subject deserves to be studied with the greatest attention.

Ex. 5. It is required to convert $\frac{a}{a+x}$ into an infinite series.

Ans. $1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+$, &c.

Ex. 6. It is required to convert $\frac{c}{a+b}$ into an infinite series.

Ans. $\frac{c}{a}-\frac{bc}{a^2}+\frac{b^2c}{a^3}-\frac{b^3c}{a^4}+$, &c.

Ex. 7. It is required to convert $\frac{b}{a+x}$ into an infinite series.

Ans. $\frac{b}{a}(1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+$, &c

Ex. 8. It is required to convert $\frac{b}{a-x}$ into an infinite series.

Ans. $\frac{b}{a}(1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+$, &c.)

Ex 9. It is required to convert $\frac{1+x}{1-x}$ into an infinite series.

Ans. $1+2x+2x^3+2x^4+2x^5+$, &c.

Ex. 10. It is required to convert $\frac{a^2}{(a+x)^2}$ into an infinite series.

Ans. $1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}+$, &c.

Ex. 11. It is required to convert $\frac{a}{c-x}$ into an infinite series.

Ans. $\frac{a}{c}+\frac{ax}{c^2}+\frac{ax^2}{c^3}+\frac{ax^3}{c^4}+$, &c.

Ex. 12. It is required to convert $\frac{a^2+x^2}{a^4+x^4}$ into an infinite series.

Ans. $\frac{1}{a^2}-\frac{x^4}{a^6}+\frac{x^8}{a^{10}}-\frac{x^{12}}{a^{14}}+\frac{x^{16}}{a^{18}}-$, &c.

Ex. 13. It is required to convert $\frac{6}{9}$, or $\frac{6}{10-1}$, into an infinite series.

Ans. $\frac{6}{10}+\frac{6}{10^2}+\frac{6}{10^3}+\frac{6}{10^4}+$, &c.

Ex. 14. It is required to convert $\frac{1}{4}$ or $\frac{1}{5-1}$ into an infinite series.

Ans. $\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\frac{1}{625}+$, &c. $=\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\frac{1}{5^4}+$, &c.

§ II. INVESTIGATION OF THE BINOMIAL THEOREM.

379. Previous to the investigation of the *Binomial Theorem*, it is necessary to observe, that *any two algebraic expressions are said to be identical, when they are of the same value, for all values of the letters of which they are composed.* Thus, $x-1=x-1$, is an *identical equation:* and shows that x is indeterminate; or that the equation will be satisfied by substituting, for x, any quantity whatever.

Also, $(x+a)\times(x-a)$ and x^2-a^2, are identical expressions; that is, $(x+a)\times(x-a)=x^2-a^2$; whatever numeral values may be given to the quantities represented by x and a.

380. *When the two members of any identity consist of the same successive powers of some indefinite quantity x, the coefficient of all the like powers of x, in that identity, will be equal to each other.*

For, let the proposed identity consist of an indefinite number of terms; as,

$$a+bx+cx^2+dx^3+\&c. =a'+b'x+c'x^2+d'x^3+\&c.$$

Then since it will hold good, whatever may be the value of x, let $x=0$, and we shall have, from the vanishing of the rest of the terms, $a=a'$.

Whence, suppressing these two terms, as being equal to each other, there will arise the new identity $bx+cx^2+dx^3+\&c. =b'x+c'x^2+d'x^3+\&c.$ which, by dividing each of its terms by x, becomes

$$b+cx+dx^2+\&c. =b'+c'x+d'x^2+\&c.$$

And, consequently, if this be treated in the same manner as the former, by taking $x=0$, we shall have $b=b'$, and so on; the same mode of reasoning giving $c=c'$, $d=d'$, &c., as was to be shown.

381. Newton, as is well known, left no demonstration of this celebrated theorem, but appears, as has already been observed, to have deduced it merely from an induction of particular cases, and though no doubt can be entertained of its truth from its having been found to succeed in all the instances in which it has been applied, yet, agreeably to the rigour that ought to be observed in the establishment of every mathematical theory, and especially in a fundamental proposition of such general use and application, it is necessary that as regular and strict a proof should be given of it as the nature of the subject and the state of analysis will admit.

382. In order to avoid entering into a too prolix investigation of the simple and well-known elements, upon which the general formulæ depends, it will be sufficient to observe, that it can be easily shown, from some of the first and most common rules of Algebra, that whatever may be the operations which the index (m) directs to be performed upon the expression $(a+x)^m$, whether of elevation, division, or extraction of roots, the terms of the resulting series will necessarily arise, by the regular integral powers of x; and that the first two terms of this series will always be $a^m+ma^{m-1}x$; so that the entire expansion of it may be represented under the form

$$a^m+ma^{m-1}x+Ba^{m-2}x^2+Ca^{m-3}+Da^{m-4}x^3+\&c.$$

Where B, C, D, &c. are certain numerical coefficients, that are independent of the values of a and x; which two latter may be considered as denoting any quantities whatever.

383. For supposing the index m to be an integer, and taking $a=1$, which will render the following part of the investigation more simple, and equally answer the purpose intended; it is plain that we shall have, according to what has been shown,

$$(1+x)^m=1+mx+bx^2+cx^3+dx^4+, \text{ \&c.} \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

384. And if the index m, of the given binomial, be negative, it will be found by division, that $(1+x)^{-m}$, or the equivalent expression

$$\frac{1}{(1+x)^m}=\frac{1}{1+mx+bx^2+cx^3, \text{ \&c.}}=1-mx-b'x^2-c'x^2-, \text{ \&c.}$$

where the law of the terms, in each of these cases is similar, to that above mentioned.

385. Again, let there be taken the binomial $(1+x)^{\frac{m}{n}}$, having the fractional index $\frac{m}{n}$; where m and n are whole positive numbers.

Then, since $(1+x)^m$ is the nth power of $(1+x)^{\frac{m}{n}}$; and, as above shown, $(1+x)^m=1+ax+b^2+cx^3+dx^4+$, &c., such a series must be assumed for $(1+x)^{\frac{m}{n}}$, that, when raised to the nth power, will give a series of the form $1+ax+bx^2+cx^3+dx^4+$, &c.

But the nth or any other integral power of the series $1+px+qx^2+rx^3+sx^4+$, &c. will be found, by actual multiplication, to give a series of the form here mentioned; whence, in this case, also, it necessarily follows, that

$$(1+x)^{\frac{m}{n}}=1+px+qx^2+rx^3+sx^4+, \text{ \&c.}$$

And if each side of this last expression be raised to the nth power, we shall have $(1+x)^m=[1+(px+qx^2+rx^3+sx^4+$, &c.$)]^n$; or, by actual involution,

$$1+mx+bx^2+cx^3+, \text{ \&c.} =1+n(px+qx^2+, \text{ \&c.})+, \text{ \&c.}$$

Whence, by comparing the coefficients of x, on each side of this last equation, we shall have, according to (Art. 380),

$np=m$, or $p=\frac{m}{n}$; so that, in this case,

$$(1+x)^{\frac{m}{n}}=1+\frac{m}{n}x+qx^2+rx^3+sx^4+, \text{ \&c.} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2);$$

where the coefficient of the second term, and the several powers of x, follow the same law as in the case of integral powers

386. Lastly, if the index $\frac{m}{n}$ be negative, it will be found by division as above, that $(1+x)^{-\frac{m}{n}}$ or the equivalent expressions,

$$\frac{1}{(1+x)^{\frac{m}{n}}}=\frac{1}{1+\frac{m}{n}x+qx^2,\ \&c.}=1-\frac{m}{n}x-q'x^2-,\ \&c.\ (3),$$

where the series still follows the same law as before.

387. And as the several cases, (1, 2, 3), here given, are of the same kind with those that are designed to be expressed in universal terms, by the general formula; it is in vain, as far as regards the first two terms, and the general form of the series, to look for any other origin of them than what may be derived from these, or other similar operations.

388. Hence, because $(a+x)^m=a^m\left(1+\frac{x}{a}\right)^m$, if there be assumed $(a+x)^m=a^m+ma^{m-1}x+Bx^2+Cx^3+Dx^4$, &c.; or which will be more commodious, and equally answer the design proposed,

$$\left(1+\frac{x}{a}\right)^m=1+A_1\left(\frac{x}{a}\right)+A_2\left(\frac{x}{a}\right)^2+A_3\left(\frac{x}{a}\right)^3+,\ \&c.\ \ldots\ (4),$$

it will only remain to determine the values of the coefficients A_1, A_2, A_3, &c. and to show the law of their dependence on the index (m) of the operation by which they are produced.

389. For this purpose, let m denote any number whatever, whole or fractional, positive or negative; and for $\frac{x}{a}$, in the above formula, put $y+z$; then, there will arise $\left(1+\frac{x}{a}\right)^m=[1+(y+z)]^m=[(1+y)+z]^m$, which being all identical expressions, when taken according to the above form, will evidently be equal to each other.

390. Whence, as the numeral coefficients, A_1, A_2, A_3, &c. of the developed formulæ, will not change for any value that can be given to a and x, provided the index (m), remains the same, the two latter may be exhibited under the forms

$$[1+(y+z)]^m=1+A_1\,(y+z)+A_2\,(y+z)^2+,\ \&c.$$
$$[(1+y)+z]^m=(1+y)^m+A_1z(1+y)^{m-1}+A_2z^2(1+y)^{m-2}+\ \&c.$$

And, consequently, by raising the several terms of the first of these series to their proper powers, and putting $1+y=p$ in the latter, we shall have

$$1+A_1(y+z)+A_2(y^2+2yz+z^2)+A_3(y^3+3y^2z+3yz^2+z^3)+,\ \&c.\ =p^m+A_1p^{m-1}z+A_2p^{m-2}z^2+A_3p^{m-3}z^3+,\ \&c.$$

391. Or, by ordering the terms, so that those which are affected with the same power of z may be all brought together, and arranged under the same head, this last expression will stand thus:

$$\begin{array}{c|c|c|l} 1+A_1 & z+A^2 & z^2+A_3 & z^3+\&c.\ (5). \\ A_1y+2A_2y & +3A_3y & +4A_4y & \\ A_2y^2+3A_3y^2 & +6A_4y^2 & +10A_5y^2 & \\ A_3y^3+4A_4y^3 & +10A_5y^3 & +30A_6y^3 & \\ \&c. & \&c. & \&c. & \end{array}$$

$$=p+A_1p^{m-1}z+A_2p^{m-2}z^2+A_3p^{m-3}z^3+\ \&c.$$

In which equation it is evident, that both y and z are indeterminate, and independent of the values A_1, A_2, A_3, &c.; since the result here obtained arises solely from the substitution of the sum of these quantities for $\frac{x}{a}$ in equation (4).

392. Hence, as the first terms and the coefficients, or multipliers of the like powers of z, in these two expressions, are, in this case, identical, we shall have, by comparing the first column of the left-hand member with the first term of that on the right,

$$1+A_1y+A_2y^2+A_3y^3+A_4y^4+\ \&c.\ =p^m,$$

which is an identity that verifies itself; since, by hypothesis, $(1+y)^m=p^m$, and, according to the general formula, $(1+y)^m =1+A_1y+A_2y^2+A_3y^3+\ \&c.$

393. Also, if the second of these columns be compared in like manner, with the second on the right, there will arise the new identity,

$A_1+2A_2y+3A_3y^2+4A_4y^3=A_1p^{m-1}$; which will be sufficient, independently of the rest of the terms for determining the values of the coefficients A_1, A_2, A_3, &c.

For since $A_1p^{m-1}=A_1\frac{p^m}{p}=\frac{A_1}{1+y}$ $(1 + A_1y + A_2y^2+A_3y^3+$ &c.), the equating this series with the last, and multiplying the left-hand side by $1+y$, will give

$[A_1+2A_2y+3A_3y^2+\&c.]\,(1 + y)=A_1+A_1A_1y+A_1A_2y^2+A_1A_3y^3+\&c.$

And, therefore, by actually performing the operation, and arranging the terms accordingly, we shall have

$$\begin{array}{c|c|c|l} A_1+2A_2 & y+3A_3 & y^2+4A_4 & y^3+\ \&c. \\ +A_1 & +2A_2 & +3A_3 & \end{array}$$

$$=A_1+A_1A_1y+A_1A_2y^2+A_1A_3y^3+\ \&c.$$

394. From which last identity, there will obviously arise, by equating the homologous terms of its two members, the following relations of the coefficients:

$$\begin{array}{l|c|l} A_1=A_1 & & A_1=A_1 \\ 2A_2=A_1A_1-A_1 & & A_2=\dfrac{A_1(A_1-1)}{2} \\ 3A_3=A_1A_2-2A_2 & \text{or} & A_3=\dfrac{A_2(A_1-2)}{3} \\ 4A_4=A_1A_3-3A_3 & & A_4=\dfrac{A_3(A_1-3)}{4} \\ \cdots & & \cdots \\ nA_n=A_1A_{n-1}-(n-1)A_{n-1} & & A_n=\dfrac{A_{n-1}[A_1-(n-1)]}{n}. \end{array}$$

And, consequently, as the coefficient A_1 of the second term of the expanded binomial, has been shown to be equal, in all cases, to the index (m) of the proposed binomial, the last of these expressions will become of the form

$$A_1=m$$

$$A_2=\frac{m(m-1)}{2}$$

$$A_3=\frac{m(m-1).(m-2)}{2.3}$$

$$A_4=\frac{m(m-1).(m-2).(m-3)}{2.3.4}$$

$$A_n=\frac{m(m-1).(m-2).(m-3)\ \ldots\ [m-(n-1)]}{2.3.4.5\ \ldots\ n};$$

where the law of the continuation of the terms, from A_4 to the general term A_n, is sufficiently evident.

395. Whence it follows, that, whether the index m be *integral* or *fractional*, *positive* or *negative*, the proposed binomial $(a+x)^m$, when expanded, may always be exhibited under the form

$$a^m\left(1+\frac{x}{a}\right)^m=$$

$$a^m+[1+m\left(\frac{x}{a}\right)+\frac{m(m-1)}{2}\left(\frac{x}{a}\right)^2+\frac{m(m-1).(m-2)}{2.3}\left(\frac{x}{a}\right)^3+$$

&c.];

$$\text{or } (a+x)^m=$$

$$a^m+ma^{m-1}x+\frac{m(m-1)}{2}a^{m-2}x+\frac{m(m-1)\ (m-2)}{2.3.}a^{m-3}x^3 \ \&c.$$

And if $-\frac{x}{a}$ be substituted in the place of $+\frac{x}{a}$, the same formula will, in that case, be expressed as follows :

$$a\left(1-\frac{x}{a}\right)^m = a^m\left[1-m\left(\frac{x}{a}\right)+\frac{m(m-1)}{2}\left(\frac{x}{a}\right)^2-\right.$$
$$\left.\frac{m(m-1)\,.\,(m-2)}{2.3}\left(\frac{x}{a}\right)^3+,\ \&c.\right];$$

$$\text{or } (a-x)^m = a^m - ma^{m-1}x + \frac{m(m-1)}{2}a^{m-2}x^2 -$$
$$\frac{m(m-1)\,.\,(m-2)}{2.4}a^{m-3}x^3,\ \&c.$$

Where it is to be observed, that the series, in each of these cases, will terminate at the $(m+1)$th term, when m is a whole positive number; but if m be fractional or negative, it will proceed *ad infinitum;* as neither the factors $m-1$, $m-2$, $m-3$, &c. can then become $=0$.

396. To this we may add, that in the two last instances here mentioned, the second term $\left(\frac{a}{x}\right)$ of the binomial must be less than 1, or otherwise the series, after a certain number of terms, will diverge, instead of converging.

397. It may also be farther remarked, that when a and x in these formulæ, are each equal to 1, we shall have, agreeably to such a substitution, $(a+n)^m=(1+1)^m=2^m=1+m+\frac{m(m-1)}{2}+\frac{m(m-1)\,.\,(m-2)}{2.3}+\frac{m(m-1)\,.\,(m-2)\,.\,(m-3)}{2.3.4}+$, &c., and

$$(a-x)^m=(1-1)^m=0^m=0=1-m+$$
$$\frac{m(m-1)}{2}-\frac{m(m-1)\,.\,(m-2)}{2.3}+\frac{m(m-1)\,.\,(m-2)\,.\,(m-3)}{2.3.4}-,\ \&c.$$

From which it appears, that the sum of the coefficients arising out of the development of the mth power, or root of any binomial, is equal to 2^m; and that the sum of the coefficients of the odd terms of the mth power, or root of a residual quantity, is equal to the sum of the coefficients of the even terms.

398. Finally, let $m=0$; then $(a+x)^m = a^0 + o \times a^{0-1}x + \frac{0(0-1)}{2}a^{0-2}x^2+$, &c., $=a^0+0\,.\,\frac{x}{a}+0\,.\,\frac{x^2}{a^2}+$, &c.

where it is evident that the series terminates at the first term (a^0); since the coefficient of every successive term involves 0 for one of its factors; therefore $(a+x)^0=a^0=1$, (Art. 86). And, if $a=x$; then $(a-x)^0=a^0=1$, that is, $0^0=1$. Hence, it follows, that any quantity, either simple or compound, raised to the power 0, is equal to unity or 1; and also that 0^0 is in all cases equal to unity or 1.

399. Although it has been observed, that 0^0 appears to admit of an infinity of numerical values; because it is equal to $\frac{0}{0}$, which is the mark of indetermination; yet it is plain, from what is above shown, that 0^0 is only one of the values of $\frac{0}{0}$, which, in that particular case, where $\frac{0^m}{0^m}=0^0=\frac{0}{0}$, is equal to unity. The intelligent reader is referred to BONNYCASTLE's *Algebra*, 8vo. vol. ii. Also, LAGRANGE's *Theorie des Fonctions Analytiques*, and *Lecons sur le Calcul des Fonctions.*

§ III. APPLICATION OF THE BINOMIAL THEOREM TO THE EXPANSION OF SERIES.

400. The method of expanding any binomial of the form $(a\pm x)^m$, when m is any whole number whatever, has been already pointed out, and it has also been observed, that the series will always terminate, when m is a whole number: But when m is a negative number, or a *fraction*, then the series expressing the value of $(a+x)^m$ does not terminate.

Let $m=\frac{n}{r}$, and substitute $\frac{n}{r}$ for m in the series then

$$(a+x)^{\frac{n}{r}}=a^{\frac{n}{r}}+\frac{n}{r}a^{\frac{n}{r}-1}x+\frac{\frac{n}{r}\left(\frac{n}{r}-1\right)}{2}a^{\frac{n}{r}-2}x^2+, \text{\&c.}$$

$$=a^{\frac{n}{r}}+\frac{na^{\frac{n}{r}}}{r}\left(\frac{x}{a}\right)+\frac{n(n-r)a^{\frac{n}{r}}}{2r^2}\left(\frac{x^2}{a^2}\right)+, \text{\&c.} =a^{\frac{n}{r}}\left(1+\frac{n}{r}\left(\frac{x}{a}\right)\right.$$

$$\left.+\frac{n(n-r)}{2r^2}\left(\frac{x^2}{a^2}\right)+, \text{\&c.}\right)$$ which is a general expression for finding the *approximate* value of any binomial surd quantity, $\frac{n}{r}$ being either *positive* or *negative*, n and r any whole numbers whatever.

Ex. 1. Find the approximate value of $\sqrt[3]{(b^3+c^3)}$ or $(b^3+c^3)^{\frac{1}{3}}$.

$$\left.\begin{array}{l} \text{Here } a=b^3 \\ x=c^3 \\ n=1 \\ r=3 \end{array}\right\} \begin{array}{l} \therefore a^{\frac{n}{r}}=\sqrt[3]{b^3}=b\,; \\ \frac{n}{r}\left(\frac{x}{a}\right)=\frac{1}{3}\left(\frac{c^3}{b^3}\right)=\frac{c^3}{3b^3}\,; \\ \frac{n(n-1)}{2r^2}\left(\frac{x^2}{a^2}\right)=\frac{1(-3)}{2.3^2}\left(\frac{c^6}{b^6}\right)=-\frac{c}{3^2b^6}\,; \end{array}$$

$$\frac{n(n-r).(n-2r)}{2.3r^3}\left(\frac{x^2}{a^3}\right)=\frac{1(1-3).(1-6)}{2\,.\,3\,.\,3^3}\left(\frac{x^9}{b^9}\right)=\frac{5c^9}{3^4b^9};$$

&c. = &c.

Hence $\sqrt[3]{(b^3+c)^3}=b(1+\frac{c^3}{3b^3}-\frac{c^6}{3^2.b^6}+\frac{5c^9}{3^4.b^9}$, &c.).

Ex. 2. Find the value of $\frac{1}{(b+c)^2}$ or $(b+c)^{-2}$ in a series.

Here $a=b$, $c=x$, $n=-2$, $r=1$ } $\therefore a^{\frac{n}{r}}=b^{-2}=\frac{1}{b^2}$;

$$\frac{n}{r}\left(\frac{x}{a}\right)=-\frac{2c}{b};$$

$$\frac{n(m-r)}{2r}\left(\frac{x^2}{a}\right)=\frac{-2(-2-1)}{2}\left(\frac{c^2}{b^2}\right)=\frac{3c^2}{b^2};$$

&c. = &c.

Hence $\frac{1}{(b+c)^2}=\frac{1}{b^2}\left(1-\frac{2c}{b}+\frac{3c^2}{b^2}-\frac{4c^3}{b^3}+\&c.\right)$

401. Now let $n=1$, $(a+x)^{\frac{n}{r}}=(a+x)^{\frac{1}{r}}=\sqrt[r]{(a+x)}$; and $a^{\frac{n}{r}}=\sqrt[r]{a}$; hence the series (Art. 398) is transformed into $\sqrt[r]{(a+x)}=\sqrt[r]{a}(1+\frac{1}{r}\left(\frac{x}{a}\right)+\frac{1-r}{2r^2}\left(\frac{x^2}{a^2}\right)+\frac{(1-r).(1-2r)}{2\,.\,3\,.\,r^3}\left(\frac{x^3}{a^3}\right)$ $+\&c.)$ . . . . (A).

Let $a=1$, $x=1$; then $\sqrt[r]{2}=1+\frac{1}{r}+\frac{1-r}{2r^2}+\frac{(1-r).(1-2r)}{2\,.\,3\,.\,r^3}$ $+\&c.$ . . . . (B).

Thus, if $r=2$, then $\sqrt{2}=1+\frac{1}{2}-\frac{1}{2^3}+\frac{1}{2^4}-\frac{5}{2^7}+\frac{7}{2^8}-\frac{3\,.\,7}{2^{10}}$ $+\&c.$ And if $r=3$,

then $\sqrt[3]{3}=1+\frac{1}{3}-\frac{1}{3^2}+\frac{5}{3^4}-\frac{2.5}{3^5}+\frac{2.11}{3^6}+\frac{2.7.11}{3^8}+$ &c

By means of the series marked A, the rth root of many other numbers may be found; if a and x be so assumed, that x is a small number with respect to a, and $\sqrt[r]{a}$, a whole number.

Ex. 3. It is required to convert $\sqrt{5}$, or its equal $\sqrt{(4+1)}$, into an infinite series.

Here $a=4$, $x=1$, $r=2$; then $\sqrt[r]{a}=\sqrt{4}=2$, and we have

$$\sqrt{(4+1)}=2(1+\frac{1}{2^3}-\frac{1}{2^7}+\frac{1}{2^{10}}-\frac{5}{2^{15}}+\&c.)$$

Ex. 4. It is required to convert $\sqrt[3]{9}$, or its equal $\sqrt[3]{(8+1)}$ into an infinite series.

Here $a=8$, $x=1$, $r=3$; then $\sqrt[r]{a}=\sqrt[3]{8}=2$, and we obtain

$$\sqrt[3]{(8+1)}=\sqrt[3]{9}=2\left\{1+\frac{1}{3.8}-\frac{1}{3^2.8^2}+\frac{1}{3^4.8^3}-\frac{2.5}{3^5.8^4}+\&c.\right.$$

402. The several terms of these series are found by substituting for a, x, and r, their values in the general series marked (A) or (B), and then rejecting the factors common to both the numerators and denominators of the fractions.

Thus, for instance, to find the 5th term of the series expressing the approximate value of $\sqrt[3]{9}$, we take the 5th term of the general series marked (A), which is

$\frac{(1-r).(1-2r).(1-3r)}{3\,.\,3\,.\,4r^4}\left(\frac{x^4}{a^4}\right)$, where $a=8$, $x=1$, and $r=3$;

$\therefore$ the value of the fraction is $-\frac{2\,.\,5\,.\,8}{2.3.4.3^4}\left(\frac{1}{8^4}\right)=-\frac{2\,.\,5}{2.3.4.3^4.8^3}$

$=-\frac{2\,.\,5}{3.3^4.8.8^3}=-\frac{2\,.\,5}{3^5\,.\,8^4}$. In this manner each term of the series is calculated; and the law which they observe is, that the *numerators* of the fractions, consist of certain combinations of prime numbers, and the *denominators* of combinations of certain powers of a and r.

Ex. 5. Find the value of $(c^2-x^2)^{\frac{3}{4}}$ in a series.

Ans. $\sqrt{c^3}\left(1-\frac{3x^2}{2^2.c^2}-\frac{3x^4}{2^5.c^4}-\frac{5x^6}{2^7.c^6}-\&c.\right)$

Ex. 6. It is required to convert $\sqrt[3]{6}$, or its equal $\sqrt[3]{(8-2)}$, into an infinite series.

Ans. $2\left(1-\frac{1}{3.4}-\frac{1}{3^2.4^2}-\frac{5}{3^4.4^3}-\&c.\right)$

Ex. 7. It is required to extract the square root of 10, in an infinite series. Ans. $3+\frac{1}{2.3}-\frac{2}{2.4.3^3}+\frac{1.3}{2.4.6.3^5}-\&c.$

Ex. 8. To expand $a^2(a^2-x)^{-\frac{1}{2}}$ in a series.

Ans. $a+\frac{1}{2}\left(\frac{x}{a}\right)+\frac{3}{2.4}\left(\frac{x^2}{a^3}\right)+\frac{3.5}{2.4.6}\left(\frac{x^3}{a^5}\right)+\&c.$

Ex. 9. To find the value of $\sqrt[5]{(a^5+x^5)}$ in a series.

Ans. $a+\frac{x^5}{5a^4}-\frac{2x^{10}}{25a^9}+\frac{6x^{15}}{125a^{14}}-\&c.$

Ex. 10. Fnd the cube root of $1-x^3$, in a series.

Ans. $1-\frac{x^3}{3}-\frac{x^6}{9}-\frac{5x^9}{81}-\frac{10x^{12}}{243}-\&c.$

CHAPTER XIII.

ON

PROPORTION AND PROGRESSION.

§ I. ARITHMETICAL PROPORTION AND PROGRESSION.

403. Arithmetical Proportion is the relation which two numbers, or quantities, of the same kind, have to two others, when the difference of the first pair is equal to that of the second.

404. Hence, three quantities are in arithmetical proportion, when the difference of the first and second is equal to the difference of the second and third. Thus, 2, 4, 6; and a, $a+b$, $a+2b$, are quantities in arithmetical proportion.

405. And four quantities are in arithmetical proportion, when the difference of the first and second is equal to the difference of the third and fourth. Thus, 3, 7, 12, 16; and a, $a+b$, c, $c+b$, are quantities in arithmetical proportion.

406. Arithmetical Progression is, when a series of numbers or quantities increase or decrease by the same common difference. Thus 1, 3, 5, 7, 9, &c. and a, $a+d$, $a+2d$, $a+3d$, &c. are an increasing series in arithmetical progression, the common differences of which are 2 and d. And 15, 12, 9, 6, &c. and a, $a-d$, $a-2d$, $a-3d$, &c. are decreasing series in arithmetical progression, the common differences of which are 3 and d.

407. It may be observed, that Garnier, and other European writers on Algebra, at present, treat of arithmetical proportion and progression under the denomination of equi-differences, which they consider, as Bonnycastle justly observes, not without reason, as a more appropriate appellation than the former, as the term *arithmetical* conveys no idea of the nature of the subject to which it is applied.

408. They also represent the relations of these quantities under the form of an equation, instead of by points, as is usually done; so that if a, b, c, d, taken in the order in which they stand, be four quantities in arithmetical proportion, this

relation will be expressed by $a-b=c-d$; where it is evident that all the properties of this kind of proportion can be obtained by the mere transposition of the terms of the equation.

409. Thus, by transposition, $a+d=b+c$. From which it appears, that *the sum of the two extremes is equal to the sum of the two means:* And if the third term in this case be the same as the second, or $c=b$, the equi-difference is said to be continued, and we have

$$a+d=2b\ ;\ \text{or}\ b=\tfrac{1}{2}(a+d)\ ;$$

where it is evident, that *the sum of the extremes is double the mean;* or *the mean equal to half the sum of the extremes.*

410. In like manner, by transposing all the terms of the original equation, $a-b=c-d$, we shall have $b-a=d-c$; which shows that the consequents b, d, can be put in the places of the antecedents a, c; or, conversely, a and c in the places of b and d.

411. Also, from the same equality $a-b=c-d$, there will arise, by adding $m-n$ to each of its sides,

$$(a+m)-(b+n)=(c+m)-(d+n)\ ;$$

where it appears that the proportion is not altered, by augmenting the antecedents a and c by the same quantity m, and the consequents b and d by another quantity n. In short, every operation by way of addition, subtraction, multiplication, and division, made upon each member of the equation, $a-b=c-d$, gives a new property of this kind of proportion, without changing its nature.

412. The same principles are also equally applicable to any continued set of equi-differences of the form $a-b=b-c=c-d=d-e$, &c. which denote the relations of a series of terms in what has been usually called arithmetical progression.

413. But these relations will be more commodiously shown, by taking a, b, c, d, &c. so that each of them shall be greater or less than that which precedes it by some quantity d'; in which case the terms of the series will become

$$a,\ a\pm d',\ a\pm 2d',\ a\pm 3d',\ a\pm 4d',\ \&c.$$

Where, if l be put for that term in the progression of which the rank is n, its value, according to the law here pointed out, will evidently be

$$l=a\pm(n-1)d'\ ;$$

which expression is usually called the general term of the series; because, if 1, 2, 3, 4, &c. be successively substituted for n, the results will give the rest of the terms.

Hence *the last term of any arithmetical series is equal to the first term plus or minus, the product of the common difference, by the number of terms less one.*

414. Also, if s be put equal to the sum of any number of terms of this progression, we shall have

$$s=a+(a\pm d')+(a\pm 2d)+ \ldots\ldots +[a\pm(n-1)d'].$$

And by reversing the order of the terms of the series,

$$s=[a\pm(n-1)d']+[a\pm(n-2)d']+ \ldots (a\pm d')+a.$$

Whence, by adding the corresponding terms of these two equations together, there will arise

$$2s=[2a\pm(n-1)d']+[2a\pm(n-1)d'], \text{ \&c. to } n \text{ terms.}$$

And, consequently, as all the n terms of this series are equal to each other, we shall have

$$2s=n[2a\pm(n-1)d'], \text{ or } s=\tfrac{n}{2}[2a\pm(n-1)d'] \ldots (1).$$

415. Or, by substituting l for the last term $a\pm(n-1)d'$, as found above, this expression (1) will become

$$s=\tfrac{n}{2}(a+l) \quad \ldots\ldots \quad (2).$$

Hence, *the sum of any series of quantities in arithmetical progression is equal to the sum of the two extremes multiplied by half the number of terms.*

It may be observed, that from equations (1) and (2), if any three of the five quantities, a, d', n, l, s, be given, the rest may be found.

416. Let l, as before, be the *last* term of an arithmetic series, whose *first* term is (a), *common difference* (d'), and *number of terms* (n); then $l=a+(n-1)d'$; $\therefore d'=\dfrac{l-a}{n-1}$. Now the *intermediate* terms between the first and the last is $n-2$; let $n-2=m$, then $n-1=m+1$. Hence, $d'=\dfrac{l-a}{m+1}$, which gives the following rule for finding *any number* of arithmetic means between two numbers. *Divide the difference of the two numbers by the given number of means increased by unity, and the quotient will be the common difference.* Having the common difference, the *means* themselves will be known.

Example 1. Find the sum of the series 1, 3, 5, 7, 9, 11, &c. continued to 120 terms.

$$\left.\begin{array}{r}\text{Here } a=1,\\ d'=2,\\ n=120\end{array}\right\} \therefore s=[2a+(n-1)d']\tfrac{n}{2}=\tfrac{120}{2}[2\times1+(120-1)2]=14400.$$

Ex. 2. The *sum* of an arithmetic series is 567, the *first term* 7, and the *common difference* 2. What are the *number of terms* ?

Here $s=567$, $a=7$, $d'=2$; $\therefore 2s=n[2a+(n-1)d]=n[14+(n-1)2]=14n+2n^2-2n=1134$; $\therefore n^2+6n+9=576$, and $n=21$.

Ex. 3. The *sum* of an arithmetic series is 1455, the *first term* 5, and the *number of terms* 30. What is the *common difference?* Ans. 3.

Ex. 4. The *sum* of an arithmetic series is 1240, *common difference* 4, and *number of terms* 20. What is the *first term?* Ans. 100.

Ex. 5. Find the *sum* of 36 terms of the series, 40, 38, 36, 34, &c. Ans. 108.

Ex. 6. The *sum* of an arithmetic series is 440, *first term* 3, and *common difference* 2. What are the *number* of terms ? Ans. 20.

Ex. 7. A person bought 47 sheep, and gave 1 shilling for the *first* sheep, 3 for the *second*, 5 for the *third*, and so on. What did *all* the sheep cost him? Ans. 110*l.* 9*s*.

Ex. 8. Find *six* arithmetic means between 1 and 43. Ans. 7, 13, 19, 25, 31, 37.

§ II. GEOMETRICAL PROPORTION AND PROGRESSION.

417. Geometrical Proportion, is the relation which two numbers, or quantities, of the same kind, have to two others, when the antecedents or leading terms of each pair, are the same parts of their consequents, or the consequents of their antecedents.

418. And if two quantities only are to be compared together, the part, or parts, which the antecedent is of the consequent, or the consequent of the antecedent, is called the *ratio ;* observing, in both cases, to follow the same method.

419. *Direct proportion*, is when the same relation subsists between the first of four quantities, and the second, as between the third and fourth.

Thus, a, ar, b, br, as in direct proportion.

420. *Inverse*, or *reciprocal proportion*, is when the first and second of four quantities are directly proportional to the reciprocals of the third and fourth.

Thus, a, ar, br, b, are inversely proportional ; because a, ar $\frac{1}{br}$, $\frac{1}{b}$, are directly proportional.

421. The same reason that induced the writers mentioned

in (Art. 407), to give the name of equi-differences to arithmetical proportionals, also led them to apply that of equi-quotients to geometrical proportionals, and to express their relations in a similar way by means of equations.

Thus, if there be taken any four proportionals, a, b, c, d, which it has been usual to express by means of points, as below,

$$a : b :: c : d.$$

This relation, according to the method above-mentioned, will be denoted by the equation $\frac{a}{b}=\frac{c}{d}$, (Art. 24); where the equal ratios are represented by fractions, the numerators of which are the antecedents, and the denominators the consequents. Hence, $ad=bc$.

422. And if the third term c, in this case, be the same as the second, or $c=b$, the proportion is said to be continued, and we have $ad=b^2$, $b=\sqrt{ad}$; where it is evident, *that the product of the extremes of three proportionals, is equal to the square of the mean:* or, *that the mean is equal to the square root of the product of the two extremes.*

423. Also, from the equality, $\frac{a}{b}=\frac{c}{d}$, there will result $\frac{a\pm b}{b}=\frac{c\pm d}{d}$: for, by *adding* or *subtracting* 1 from each side of the equation; then $\frac{a}{b}\pm 1=\frac{c}{d}\pm 1$; $\therefore \frac{a\pm b}{b}=\frac{c\pm d}{d}$, and $a\pm b : b :: c\pm d : d$.

Hence, *when four quantities are proportionals, the sum or difference of the first and second is to the second as the sum or difference of the third and fourth, is to the fourth.*

424. In like manner, if $a : b :: c : d$; then, $ma : mb :: \frac{1}{n}c : \frac{1}{n}d$. For $\frac{a}{b}=\frac{c}{d}$; $\therefore$ (Art. 118), $\frac{ma}{mb}=\frac{\frac{1}{n}c}{\frac{1}{n}d}$; and, $ma : mb :: \frac{1}{n}c : \frac{1}{n}d$.

Hence, *when four quantities are proportionals, if the first and second be multiplied, or divided by any quantity, and also the second and fourth, the resulting quantities will still be proportionals.*

425. Also, if $a : b :: c : d$; then $\frac{a}{b}=\frac{c}{d}$; $\therefore \frac{a^n}{b^n}=\frac{c^n}{d^n}$, and $a^n : b^n :: c^n : d^n$; where n may be any number either *integral or fractional.*

Hence, *if four quantities be proportionals, any power or root of those quantities will be proportionals.*

And, by proceeding in a similar manner, all the properties and transformations of *ratios* and *proportion*, can be easily obtained from the equality $\frac{a}{b}=\frac{c}{d}$, or $ad=bc$.

426. In addition to what is here said, it may be observed, that the ratio of two squares is frequently called *duplicate ratio*; of two square roots, *subduplicate ratio*; of two cubes, *triplicate ratio*; and of two cube roots, *subtriplicate ratio*. See the APPENDIX at the end of this Treatise, where the doctrine of *ratios* and *proportion* is fully explained and clearly illustrated.

427. GEOMETRICAL PROGRESSION, is when a series of numbers, or quantities, have the same constant ratio, or which increase, or decrease, by a common multiplier, or divisor. Thus, the numbers 1, 2, 4, 8, 16, &c. (which *increase* by the continual *multiplication* of 2), and the numbers 1, $\frac{1}{3}$, $\frac{1}{7}$, $\frac{1}{27}$, &c. (which *decrease* by the continued *division* of 3, or multiplication of $\frac{1}{3}$), are in Geometrical Progression.

428. In general, if a represents the *first term* of such a series, and r the *common multiplier* or *ratio*; then may the series itself be represented by a, ar, ar^2, ar^3, ar^4, &c., which will evidently be an *increasing* or *decreasing* series, according as r is a *whole number*, or a *proper fraction*. In the foregoing series, the *index of r* in any term is *less by unity* than the number which denotes *the place of that term in the series*. Hence, if the number of terms in the series be denoted by (n), the last term will be ar^{n-1}.

429. Let l be the last term of a *geometric* series, then $l=ar^{n-1}$ and $r^{n-1}=\frac{l}{a}$; $\therefore r=\sqrt[n-1]{\frac{l}{a}}$ The number of intermediate terms between the first and last is $n-2$; let $n-2=m$, then $n-1=m+1$, and $r=\sqrt[m+1]{\frac{l}{a}}$, which gives the following rule for finding *any number* of geometric means between two numbers; viz. *Divide one number by the other, and take that root of the quotient which is denoted* by $m+1$; *the result will be the common ratio.* Having the *common ratio*, the means are found by multiplication.

430. Let S be made to denote the sum of n terms of the series, including the first, then

$$a+ar+ar^2+ar^3+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad +ar^{n-2}+ar^{n-1}=S$$

Multiply the equation by r, and it becomes
$ar+ar^2+ar^3+ \quad . \; . \; . \; . \; . \; . \; . \; . \quad +ar^{n-2}=ar^{n-1}+ar^n=rS.$

Whence, subtracting the first of these equations from the second, observing that all the terms except a and ar^n destroy each other, we shall have

$$ar^n-a=rS-S=(r-1)S\;;\text{ and }\therefore S=\frac{ar^n-a}{r-1} \quad . \; . \; . \quad (1).$$

Or, by substituting l for the last term ar^{n-1}, as above found, this expression will become $S=\frac{rl-a}{r-1}$; from which two equations, if any three of the quantities a, r, n, l, S, be given, the rest may be found. Thus, from the second equation,

$$a=rl-(r-1)S\;;\; r=\frac{S-a}{S-l},\text{ and } l=\frac{(r-1)S+a}{r}.$$

In the formula (1), when $r=1$, we have $S=\frac{a-a}{1-1}=\frac{0}{0}$. Now, the value of the symbol $\frac{0}{0}$, in this particular case, shall be equal to na; because the series $a+ar+ar^2+ \; . \; . \; . \; . \; ar^{n-2}+ar^{n-1}$, for $r=1$, becomes $a+a+a+a+$, &c., and the sum of n terms of this series, is evidently equal to na; therefore $S=\frac{0}{0}=na$. Or, since $\frac{ar^n-a}{r-1}=a\,.\,\frac{r^n-1}{r-1}=a\times\frac{1-r^n}{1-r}=$ $a\,.\,[r^{n-1}+r^{n-2}+r^{n-3}\,.\,.\,+r+1]=a\times[1+r+r^2+r^3\,.\,.\,r^{n-1}]$, which, in the case of $r=1$, becomes $a\,.\,[1+1+1+$, &c.], and the sum of n terms of the series $1+1+1+$, &c. is evidently equal to n; therefore $S=a\,.\,\frac{1-r^n}{1-r}=a\,.\,\frac{0}{0}=a\,.\,(1+1+1+$, &c.) $=a\times n=an$, as before.

431. When the common factor r, in the above series, is a whole number, the terms a, ar, ar^2, ar^{n-1}, form an increasing progression; in which case n may be so taken, that the value of the sum (S) shall be greater than any assignable quantity.

432. But if r be a proper fraction, as $\frac{1}{r'}$, the series a, $\frac{a}{r'}$, $\frac{a}{r'^2}$, $\frac{a}{r'^3}$, will be a decreasing one, and the expression (Art. 430), by substituting $\frac{1}{r'}$ for r, and changing the signs of the numera-

tor and denominator, will become $\frac{ar'(1-r'^{\frac{1}{n}})}{r'-1}$; where it is plain, that the term $\frac{1}{r'^n}$ will be indefinitely small when n is indefinitely great; and consequently, by prolonging the series, S may be made to differ from $\frac{ar'}{r'-1}$ by less than any assignable quantity.

433. Whence, supposing the series to be continued indefinitely, or without end, we shall have in that case, $S=\frac{ar'}{r'-1}$; which last expression is what some call the radix, and others the limit of the series; as being of such a value, that the sum of any number of its terms, however great, can never exceed it, and yet may be made to approach nearer to it than by any given difference.

434. If the ratio, or multiplier, r, be negative, in which case the series will be of the form
$a-ar+ar^2-ar^3+ \quad . \quad . \quad . \quad . \quad . \quad \pm ar^{n-1}$, where the terms are $+$ and $-$ alternately, we shall have $S=\frac{\pm ar^n+a}{r+1}$.

And if r be a proper fraction, $\frac{1}{r'}$, as before, we shall have, for the sum of an indefinite number of terms of the series $a-\frac{a}{r'}+\frac{a}{r'^2}-\frac{a}{r'^3}+$, &c., $S=\frac{ar'}{r'+1}$.

Ex. 1. Find the sum of the series, 1, 3, 9, 27, &c. to 12 terms.

Here $\left.\begin{aligned} a&=1,\\ r&=3,\\ n&=12;\end{aligned}\right\}$ $\therefore S=\frac{ar^n-a}{r-1}=\frac{1\times3^{12}-1}{3-1}=\frac{81^3-1}{2}$
$=\frac{531441-1}{2}=\frac{531440}{2}=265720.$

Ex. 2. Find three geometric means between 2 and 32.

Here $\left.\begin{aligned} a&=2,\\ l&=32,\\ m&=3;\end{aligned}\right\}$ $\therefore \sqrt[m+1]{\frac{l}{a}}=\sqrt[4]{\frac{32}{2}}=\sqrt[4]{16}=2,$

and the means required are 4, 8, 16.

Ex. 3. The first term of a geometrical progression is 1, the ratio 2, and the number of terms 10. What is the sum of the series? Ans. 1023

Ex. 4. In a geometrical progression is given the greatest term $=1458$, the ratio $=3$, and the number of terms $=7$, to find the least term. Ans. 2.

Ex. 5. It is required to find two geometrical proportionals between 3 and 24, and four geometrical means between 3 and 96. Ans. 6 and 12; and 6, 12, 24, and 48.

Ex. 6. Find two geometric means between 4 and 256. Ans. 16, and 64.

Ex. 7. Find three geometric means between $\frac{1}{9}$ and 9. Ans. $\frac{1}{3}$, 1, 3.

Ex. 8. A gentleman who had a daughter married on New-year's day, gave the husband towards her portion 4 dollars, promising to triple that sum the first day o. every month, for nine months after the marriage; the sum paid on the first day of the ninth month was 26244 dollars. What was the lady's fortune? Ans. 39364 dollars.

Ex. 9. Find the value of $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+$ &c. *ad infinitum.* Ans. 2.

Ex. 10. Find the value of $1+\frac{3}{4}+\frac{9}{16}+\frac{27}{64}+$ &c. *ad infini-* Ans. 4.

§ III. HARMONICAL PROPORTION AND PROGRESSION.

435. Three quantities are said to be in *harmonical proportion,* when the first is to the third, as the difference between the first and second is to the difference between the second and third.

Thus, a, b, c, are harmonically proportional, when

$$a : c :: a-b : b-c, \text{ or } a : c :: b-a : c-b.$$

And c, [since $a(b-c)=c(a-b)$ or $ab=(2a-b)c$], is a third harmonical proportion to a and b, when $c=\frac{ab}{2a-b}$.

436. Four quantities are in harmonical proportion, when the first is to the fourth, as the difference between the first and second is to the difference between the third and fourth.

Thus, a, b, c, d, are in harmonical proportion, when

$$a : d :: a-b : c-d, \text{ or } a : d :: b-a : d-c.$$

And d, [since $a(c-d)=d(a-b)$ or $ac=(2a-b)d$], is a fourth harmonical proportional to a, b, c, when $d=\frac{ac}{2a-b}$.

In each of which cases, it is obvious, that twice the first term must be greater than the second, or otherwise the proportionality will not subsist.

437. Any number of quantities, $a, b, c, d, e,$ &c. are in harmonical progression, if $a : c :: a-b : b-c$; $b : d :: b-c : c-d$; $c : e :: c-d : d-e$, &c.

438. *The reciprocal of quantities in harmonical progression, are in arithmetical progression.* For, if $a, b, c, d, e,$ &c. are in harmonical progression; then, from the preceding Article, we shall have $bc+ab=2ac$; $dc+bc=2db$; $ed+cd=2ec$, &c. Now, by dividing the first of these equalities by abc; the second by bdc; the third by cde; &c., we have, $\frac{1}{a}+\frac{1}{c}=\frac{2}{b}$; $\frac{1}{b}+\frac{1}{d}=\frac{2}{c}$; $\frac{1}{c}+\frac{1}{e}=\frac{2}{d}$; &c. Therefore, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e}$, &c. are in arithmetical progression.

439. *An harmonical mean between any two quantities, is equal to twice their product divided by their sum.* For, if a, x, b, are three quantities in harmonical proportion, then, $a : b :: a-x : x-b$; $\therefore ax-ab=ab-bx$, and $x=\frac{2ab}{a+b}$.

Ex. 1. Find a third harmonical proportional to 6 and 4.

Let $x=$ the required number, then $6 : x :: 6-4 : 4-x$; $\therefore 24-6x=2x$, and $x=2$.

Ex. 2. Find an harmonical mean between 12 and 6.
Ans. 8.

Ex. 3. Find a third harmonical proportional to 234 and 144.
Ans. 104.

Ex. 4. Find a fourth harmonical proportional to 16, 8, and 3.
Ans. 2.

§ IV. PROBLEMS IN PROPORTION AND PROGRESSION.

PROB. 1. There are two numbers whose *product* is 24, and the *difference of their cubes* : *cube of their difference* :: 19 : 1. What are the numbers?

Let $x=$ the *greater* number, and $y=$ the *lesser*.

Then, $xy=24$, and $x^3-y^3 : (x-y)^3 :: 19 : 1$.

By *expansion*, $x^3-y^3 : x^3-3x^2y+3xy^2-y^3 :: 19 : 1$;

$\therefore 3x^2y-3xy^2 : (x-y)^3 :: 18 : 1$;

and, dividing by $x-y$, $3xy : (x-y)^2 :: 18 : 1$;

but $xy=24$; $\therefore 72 : (x-y)^2 :: 18 : 1$.

$$\text{Hence, } 18\,(x-y)^2=72, \text{ or } (x-y)^2=4;$$
$$\therefore x-y=2.$$
$$\text{Again, } x^2-2xy+y^2=4,$$
$$\text{and } 4xy=96,$$
$$\therefore x^2+2xy+y^2=100, \text{ and } x+y=10,$$
$$\text{but } x-y=2,$$
$$\therefore x=6, \text{ and } y=4.$$

Prob. 2. Before noon, a clock which is too fast, and points to afternoon time, is put back five hours and forty minutes; and it is observed that the time before shown is to the true time as 29 to 105. Required the true time.

Ans. 8 hours, 45 minutes

Prob. 3. Find two numbers, the greater of which shall be to the less as their sum to 42, and as their difference to 6.

Ans. 32, and 24.

Prob. 4. What two numbers are those, whose difference, sum, and product, are as the numbers 2, 3, and 5, respectively?

Ans. 10, and 2.

Prob. 5. In a court there are two square grass-plots; a side of one of which is 10 yards longer than the other; and their areas are as 25 to 9. What are the lengths of the sides?

Ans. 25, and 15 yards.

Prob. 6. There are three numbers in arithmetical progression, whose sum is 21; and the sum of the first and second is to the sum of the second and third as 3 to 4. Required the numbers.

Ans. 5, 7, 9.

Prob. 7. The arithmetical mean of two numbers exceeds the geometrical mean by 13, and the geometrical mean exceeds the harmonical mean by 12. What are the numbers?

Ans. 234, and 104.

Prob. 8. Given the sum of three numbers, in harmonical proportion, equal to 26, and their continual product $=576$; to find the numbers.

Ans. 12, 8 and 6.

Prob. 9. It is required to find six numbers in geometrical progression, such, that their sum shall be 315, and the sum of the two extremes 165.

Ans. 5, 10, 20, 40, 80, and 160.

Prob. 10. A number consisting of three digits which are in arithmetical progression, being divided by the sum of its digits, gives a quotient 48; and if 198 be subtracted from it, the digits will be inverted. Required the number.

Ans. 432.

PROB. 11. The difference between the first and second of four numbers in geometrical progression is 36, and the difference between the third and fourth is 4; What are the numbers?

Ans. 54, 18, 6, and 2.

PROB. 12. There are three numbers in geometrical progression; the sum of the first and second of which is 9, and the sum of the first and third is 15. Required the numbers.

Ans. 3, 6, 12.

PROB. 13. There are three numbers in geometrical progression, whose continued product is 64, and the sum of their cubes is 584. What are the numbers?

Ans. 2, 4, 8.

PROB. 14. There are four numbers in geometrical progression, the second of which is less than the fourth by 24; and the sum of the extremes is to the sum of the means as 7 to 3. Required the numbers.

Ans 1, 3, 9, 27.

PROB. 15. There are four numbers in arithmetical progression, whose sum is 28; and their continued product is 585. Required the numbers.

Ans. 1, 5, 9, 13.

PROB. 16. There are four numbers in arithmetical progression; the sum of the squares of the first and second is 34; and the sum of the squares of the third and fourth is 130. Required the numbers.

Ans. 3, 5, 7, 9.

CHAPTER XIV.

ON LOGARITHMS.

440. Previous to the investigation of Logarithms, it may not be improper to premise the two following propositions.

441. *Any quantity which from positive becomes negative, and reciprocally, passes through zero, or infinity.* In fact, in order that m, which is supposed to be the greater of the two quantities m and n, becomes n, it must pass through n; that is to say, the difference $m-n$ becomes nothing; therefore p, being this difference, must necessarily pass through zero, in order to become negative, or $-p$. But if p becomes $-p$, the fraction $\frac{1}{p}$ will become $-\frac{1}{p}$; and therefore it passes through $\frac{1}{0}$, or infinity.

442. It may be observed, that in Logarithms, and in some trigonometrical lines, the passage from positive to negative is made through zero; for others of these lines, the transition takes place through infinity: It is only in the first case that we may regard negative numbers as less than zero; whence there results, that the greater any number or quantity a is, when taken positively, the less is $-a$; and also, that any negative number is, *a fortiori*, less than any absolute or positive number whatever.

443. If we add successively different negative quantities to the same positive magnitude, the results shall be so much less according as the negative quantity becomes greater, abstracting from its sign. For instance, $8-1>8-2>8-3$, &c.

It is in this sense, that $0>-1>-2>-3$, &c.; and $3>0>-1>-2>-3>-4$, &c.

444. *Any quantity, which from real becomes imaginary, or reciprocally, passes through zero, or infinity.* This is what may easily be concluded from these expressions,

$$x=\sqrt{(a^2-y^2)},\ x=\frac{1}{\sqrt{(a^2-y^2)}};$$

considered in these three relations,

$$y^2<a^2,\ y^2=a^2,\ y^2>a^2.$$

§ I. THEORY OF LOGARITHMS.

445. LOGARITHMS are a set of numbers, which have been compnted and formed into tables, for the purpose of facilitating arithmetical calculations; being so contrived, that the addition and subtraction of them answer to the multiplication and division of the natural numbers, with which they are made to correspond.

446. Or, when taken in a similar, but more general sense, logarithms may be considered as the exponents of the powers, to which a given, or invariable number, must be raised, in order to produce all the common, or natural numbers. Thus, if $a^x=y$, $a^{x'}=y'$, $a^{x''}=y''$, &c.; then will the indices x, x', x'', &c. of the several powers of a, be the logarithms of the numbers y, y', y'', &c. in the scale or system, of which a is the base.

447. So that, from either of these formulæ, it appears, that the logarithm of any number, taken separately, is the index of that power of some other number, which, when it is involved in the usual way, is equal to the given number. And since the base a, in the above expressions, can be assumed of any value, greater or less than 1, it is plain that there may be an endless variety of systems of logarithms, answering to the same natural numbers.

448. Let us suppose, in the equation $a^x=y$, at first, $x=0$, we shall have $y=1$, since $a^0=1$; to $x=1$, corresponds $y=a$. Therefore, *in every system, the logarithm of unity is zero;* and also, *the base is the number whose proper logarithm, in the system to which it belongs, is unity.* These properties belong essentially to all systems of logarithms.

449. Let $+x$ be changed into $-x$ in the above equation, and we shall have

$$\frac{1}{a^x}=y:$$

Now, the exponent x augmenting continually, the fraction $\frac{1}{a^x}$, if the base a be greater than unity, will diminish, and may be made to approach continually towards 0, as its limit; to this limit corresponds a value of x greater than any assignable number whatever. Hence it follows, *that, when the base a is greater than unity, the logarithm of zero is infinitely negative.*

450. Let y and y' be the representatives of two numbers, x and x' the corresponding logarithms for the same base: we

shall have these two equations, $a^x=y$, and $a^{x'}-y'$, whose product is $a^x.a^{x'}=y.y'$, or $a^{x+x'}=yy'$, and consequently, by the definition of logarithms, $x+x'=\log.\ yy'$, or $\log.\ yy'=\log.\ y+\log.\ y'$.

And, for a like reason, if any number of the equations $a^x=y$, $a^{x'}=y'$, $a^{x''}=y''$, &c. be multiplied together, we shall have $x^{x+x'+x''+\text{etc.}}=yy'y'$, &c.; and, consequently, $x+x'+x''$, &c. $=\log.\ yy'y''$, &c.; or $\log.\ yy'y''$, &c.$=\log.\ y+\log.\ y'+\log.\ y''$, &c.

The logarithm of the product of any number of factors is, therefore, equal to the sum of the logarithms of those factors.

451. Hence, if all the factors y, y', y'', &c. are equal to each other, and the number of them be denoted by m, the preceding property will then become $\log.\ (y^m)=m$, $\log.\ y$.

Therefore, *the logarithm of the* m*th power of any number is equal to* m *times the logarithm of that number.*

452. In like manner, if the equation $a^x=y$, be divided by $a^{x'}=y'$, we shall have, from the nature of powers, $\frac{a^{x'}}{a^x}$, or $a^{x-x'}=\frac{y}{y'}$; and by the definition of logarithms, $x-x'=\log\left(\frac{y}{y'}\right)$; or $\log.\ y-\log.\ y'=\log.\left(\frac{y}{y'}\right)$.

Hence *the logarithm of a fraction, or of the quotient arising from dividing one number by another, is equal to the logarithm of the numerator* minus *the logarithm of the denominator.*

453. And if each member of the equation, $a^x=y$, be raised to the fractional power $\frac{m}{n}$, we shall have $a^{\frac{mx}{n}}=y^{\frac{m}{n}}$; and consequently, as before, $\frac{m}{n}x=\log.\ (y^{\frac{m}{n}})=\log.\ \sqrt[n]{y^m}$; or, $\log.\ y^{\frac{m}{n}}=\frac{m}{n}\log.\ y$.

Therefore, *the logarithm of a mixed root, or power, of any number, is found by multiplying the logarithm of the given number, by the numerator of the index of that power, and dividing the result by the denominator.*

454. And if the numerator m of the fractional index of the number y, be, in this case, taken equal to 1, the preceding formula will then become

$$\log.\ y^{\frac{1}{n}}=\tfrac{1}{n}\log.\ y.$$

From which it follows, *that the logarithm of the* n*th root of* *any number, is equal to the* n*th part of the logarithm of that number.*

455. Hence, besides the use of logarithms in abridging the operations of multiplication and division, they are equally applicable to the raising of powers and extracting of roots; which are performed by simply multiplying the given logarithm by the index of the power, or dividing it by the number denoting the root.

456. But, although the properties here mentioned are common to every system of logarithms, it was necessary for practical purposes to select some one of these systems from the rest, and to adapt the logarithms of all the natural numbers to that particular scale. And as 10 is the base of our present system of arithmetic, the same number has accordingly been chosen for the base of the logarithmic system now generally used.

457. So that, according to this scale, which is that of the common logarithmic tables, the numbers,

$$\text{etc. } 10^{-4},\ 10^{-3},\ 10^{-2},\ 10^{-1},\ 10^{0},\ 10^{1},\ 10^{2},\ 10^{3},\ 10^{4},\ \text{etc.; or,}$$

$$\text{etc. } \frac{1}{10000},\ \frac{1}{1000},\ \frac{1}{100},\ \frac{1}{10},\ 1,\ 10,\ 100,\ 1000,\ 10000,$$

etc., have for their logarithms,

etc. —4, —3, —2, —1, 0, 1, 2, 3, 4, etc.

which are evidently a set of numbers in arithmetical progression, answering to another set in geometrical progression; as is the case in every system of logarithms.

458. And, therefore, since the common or tabular logarithm of any number (n) is the index of that power of 10, which, when involved, is equal to the given number, it is plain, from the equation $10^x = n$, or $10^{-x} = \frac{1}{n}$, that the logarithms of all the intermediate numbers, in the above series, may be assigned by approximation, and made to occupy their proper places in the general scale.

459. It is also evident that the logarithms of 1, 10, 100, 1000, etc., being 0, 1, 2, 3, respectively, the logarithm of any number, falling between 1 and 10, will be 0, and some decimal parts; that of a number between 10 and 100, 1 and some decimal parts; of a number between 100 and 1000, 2 and some decimal parts; and so on.

460. And, for a like reason, the logarithms of $\frac{1}{10}$, $\frac{1}{100}$,

$\frac{1}{1000}$, etc. or of their equals, .1, .01, .001, etc. in the descending part of the scale, being —1, —2, —3, etc. the logarithm of any number, falling between 0 and .1, will be —1 and some positive decimal parts; that of a number between .1 and .01, —2 and some positive decimal parts; and so on.

461. Hence, as the multiplying or dividing of any number by 10, 100, 1000, etc. is performed by barely increasing or diminishing the integral part of its logarithm by 1, 2, 3, &c. it is obvious that all numbers which consist of the same figures, whether they be integral, fractional, or mixed, will have the same quantity for the decimal part of their logarithms. Thus, for instance, if i be made to denote the index, or integral part of the logarithm of any number N, and d its decimal part, we shall have log. $N=i+d$; log. $10^m \times N = (i+m)+d$; log. $\frac{N}{10^m}=(i-m)+d$; where it is plain that the decimal part of the logarithm, in each of these cases, remains the same.

462. So that in this system, the integral part of any logarithm, which is usually called its index, or characteristic, is always less by 1 than the number of integers which the natural number consists of; and for decimals, it is the number which denotes the distance of the first significant figure from the place of units. Thus, according to the logarithmic tables in common use, we have

Numbers.	*Logarithms.*
1.36820	0.1361496
335.260	2.5253817
.46521	$\bar{1}$.6676490
.06154	$\bar{2}$.7891575
&c.	&c.

where the sign — is put over the index, instead of before it, when that part of the logarithm is negative, in order to distinguish it from the decimal part, which is always to be considered as +, or affirmative.

463. Also, agreeably to what has been before observed, the logarithm of 38540 being 4.5859117, the logarithms of any other numbers, consisting of the same figures, will be as follows:

Numbers.	*Logarithms*
3854	3.5859117
385.4	2.5859117
38.54	1.5859117
3.854	0.5859117
.3854	$\bar{1}$.5859117
.03854	$\bar{2}$.5859117
.003854	$\bar{3}$.5859117

which logarithms, in this case, differ only in their indices, the decimal or positive part, being the same in them all.

464. And as the indices, or the integral parts of the logarithms of any numbers whatever, in this system, can always be thus readily found, from the simple consideration of the rule above-mentioned, they are generally omitted in the tables, being left to be supplied by the operator, as occasion requires.

465. It may here, also, be farther added, that, when the logarithm of a given number, in any particular system, is known, it will be easy to find the logarithm of the same number in any other system, by means of the equations, $a^x=n$, $e^{x'}=n$, which give

(1) . . . . $x=\log. n,\ x'=\text{l}.\ n$ . . . . (2).

Where log. denotes the logarithm of n, in the system of which a is the base, and l. its logarithm in the system of which e is the base.

466. Whence $a^x=e^{x'}$, or $a^{\frac{x}{x'}}=e$, and $e^{\frac{x'}{x}}=a$, we shall have, for the base a, $\frac{x}{x'}=\log. e$, and for the base e, $\frac{x'}{x}=l.a$; or

(3) . . . . $x=x'\log. e,\ x'=x.l.a$ . . . . . . (4).

Whence, if the values of x and x', in equations (1), (2), be substituted for x and x' in equations (3), (4), we shall have, $\log. n=\log. e\times l.n$, and $l.n=\frac{1}{\log. e}\times\log. n$; or $l.n=l.a\times\log. n$, and $\log. n=\frac{1}{l.a}\times l.n$. where log. e, or its equal $\frac{1}{l.a}$ expresses the constant ratio which the logarithms of n have to each other in the systems to which they belong.

467. But the only system of these numbers, deserving of notice, except that above described, is the one that furnishes what have been usually called hyperbolic or *Neperian* logarithms, the base of which is 2.718281828459 . . . .

468. Hence, in comparing this with the common or tabular logarithms, we shall have, by putting a in the latter of the above formulæ $=10$, the expression

$$\log.\ n=\frac{1}{l.10}\times l.n, \text{ or } l.n=l.10\times\log.\ n$$

Where log., in this case, denotes the common logarithm of the number n, and $l.$ its Neperian logarithm; the constant factor $\frac{1}{l.10}$ which is $\frac{1}{2.3025850929}$, or $\cdot4342944819$ . . being what is usually called the modulus of the common or tabular system of logarithms.

469. It may not be improper to observe, that the logarithms of negative quantities, are imaginary; as has been clearly proved, by LACROIX, after the manner of EULER, in his *Traité du Calcul Differentiel et Integral;* and also, by SUREMAIN-MISSERY in his *Théorie Purement Algébrique des Quantités Imaginaires.* See, for farther details upon the properties and calculation of logarithms, GARNIER's *d'Algebre*, or BONNYCASTLE's *Treatise on Algebra* in two vols. 8vo.

§ II. APPLICATION OF LOGARITHMS TO THE SOLUTION OF EXPONENTIAL EQUATIONS.

470. EXPONENTIAL EQUATIONS are such as contain quantities with *unknown or variable* indices: Thus, $a^x=b$, $x^x=c$, $\frac{y}{a^x}=d$, &c. are exponential equations

471. An equation involving quantities of the form x^x, where the root and the index are both variable, or unknown, seldom occur in practice, we shall only point out the method of solving equations involving quantities of the form a^x, a^{bx}, where the base a is constant or invariable.

472. It is proper to observe that an exponential of the form a^{b^x}, means, *a to the power of* b^x, and not *ab to the power of x.*

Ex. 1. *Find the value of x in the equation* $a^x=b$.

Taking the *logarithm* of the equation $a^x=b$, we have $x\times\log.\ a=\log.\ b$; $\therefore x=\frac{\log.\ b}{\log.\ a}$; thus, let $a=5$, $b=100$; then in the equation $5^x=100$,

$$x=\frac{\log.\ 100}{\log.\ 5}=\frac{2.0000000}{0.6989700}=2.864.$$

Ex. 2. *It is required to find the value of* x *in the equation* $x^{bx}=c$.

Assume $b^x=y$, then $a^y=c$, and $y\times\log. a=\log. c$; $\therefore y=\frac{\log. c}{\log. a}$. Hence $b^x=\frac{\log. c}{\log. a}$ (which let) $=d$. Take the logarithm of the equation $b^x=d$, then, by (Ex. 1), $x=\frac{\log. d}{\log. b}$.

Thus, let $a=9$, $b=3$, $c=1000$; then in the equation $9^3=1000$, $\frac{\log. c}{\log. b}=\frac{\log. 1000}{\log. 9}=3.14(=d)$; and $a=\frac{\log. d}{\log. b}=\frac{\log. 3.14}{\log. 3}=\frac{.4969296}{.4771213}=1.04$.

Ex. 3. *Make such a separation of the quantities in the equation* $(a^2-b^2)^x=a+b$, *as to show, that* $\frac{x}{1-x}=\frac{\log. (a+b)}{\log. (a-b)}$.

Taking the logarithm, we have

$x\times\log. (a^2-b^2)=\log. (a+b)$, or $x\times\log. (a+b)\times(a-b)=\log. (a+b)$;

that is, $x\times\log. (a+b)+x\times\log. (a-b)=\log. (a+b)$.

Hence $x\times\log. (a-b)=\log. (a+b)-x\times\log. (a+b)=(1-x)\log. (a+b)$; $\therefore \frac{x}{1-x}=\frac{\log. (a+b)}{\log. (a-b)}$.

Ex. 4. Given $a^x+b^y=c$, and $a^x-b^y=d$, required the values of x and y.

By *addition*, $2a^x=c+d$, or $a^x=\frac{c+d}{2}$, which put $=m$; then

$x=\frac{\log. m}{\log. a}$.

Again, by subtraction, we have $2b^y=c-d$, or $b^y=\frac{c-d}{2}$, (which let $=n$); $\therefore y=\frac{\log. n}{\log. b}$.

Ex. 5. Find the value of x in the equation $\frac{ab_x+c}{d}=e$.

Ans. $x=\frac{\log. (de-c)-\log. a}{\log. b}$.

Ex. 6. Find the value of x in the equation $a^x=\frac{\sqrt{(b^2-c^2)}}{\sqrt[4]{d\,e}}$.

Ans. $x=\frac{\frac{1}{2}\log. (b+c)+\frac{1}{2}\log. (b-c)-\frac{3}{4}\log. d-\frac{1}{4}\log. e}{\log. a}$.

Ex. 7. Find the value of x in the equation $\frac{1}{5}a^x+\frac{1}{3}=\frac{2}{15}a^x+1$.

Ans. $x=\frac{1}{\log. a}$.

Ex. 8. Given $\left.\begin{array}{l}\log. x+\log. y=\frac{5}{2}\\ \log. x-\log. y=\frac{1}{2}\end{array}\right\}$ to find the values of x and y.

Ans. $x=10\sqrt{10}$, and $y=10$.

Ex. 9. In the equation $2^x=10$, it is required to find the value of x.

Ans. $x=3.321928$, &c.

Ex. 10. Given $\sqrt[x]{729}=3$, to find the value of x.

Ans. $x=6$.

Ex. 11. Given $\sqrt[x]{57862}=8$, to find the value of x.

Ans. $x=5.2735$, &c

Ex. 12. Given $(216)^{\frac{3}{x}}=64$, to find the value of x.

Ans. $x=3.8774$, &c

Ex. 13. Given $4^{\frac{x}{3}}=4096$, to find the value of x.

Ans. $x=\frac{\log. 6}{\log. 3}=1.6309$, &c.

Ex. 14. Given $a^{x+y}=c$, and $b^{x-y}=d$, to find the values of x and y.

Ans. $=\frac{m+n}{2}$, and $y=\frac{m-n}{2}$; where $m=\frac{\log. c}{\log. a}$, and $n=\frac{\log. d}{\log. b}$.

CHAPTER XV.

ON

THE RESOLUTION OF EQUATIONS

OF THE THIRD AND HIGHER DEGREES.

§ I. THEORY AND TRANSFORMATION OF EQUATIONS.

473. In addition to what has been already said (Art. 168), it may here be observed, that the *roots* of any equation are the numbers, which, when substituted for the unknown quantity, will make both sides of the equation *identically* equal. Or, which is the same, the *roots* of any equation are the numbers, which, substituted for the unknown quantity, reduce the first member to zero, or the proposed equation to the form of $0=0$; because every equation may, designating the highest power of the unknown quantity by x^m, be exhibited under the form

$$x^m+Ax^{m-1}+Bx^{m-2}+Cx^{m-3}+\ \ldots\ Tx+V=0.\ (1),$$

A, B, C, ... T, V, being known quantities. And the *resolution of an equation* is the method of finding all the roots, which will answer the required condition.

474. This being premised, it may now be shown, *that if* a *be a root of the equation* (1), *the left-hand member of that equation will be exactly divisible by* $x-a$.

For if a be substituted for x, agreeably to the above definition, we shall necessarily have

$$a^m+Aa^{m-1}+Ba^{m-2}+Ca^{m-3}+\ \ldots\ Ta+V=0.$$

And consequently, by transposition,

$$V=-a^m-Aa^{m-1}-Ba^{m-2}-Ca^{m-3}-\ \ldots\ -Ta.$$

Whence, if this expression be substituted for V in the first equation, we shall have, by uniting the corresponding terms, and placing them all in a line,

$$(x^m-a^m)+A(x^{m-1}-a^{m-1})+B(x^{m-2}-a^{m-3})+T(x-a)=0.$$

Where, since the difference of any two equal powers of two different quantities is divisible by the difference of their roots (Art. 108), each of the quantities (x^m-a^m), $(x^{m-1}-a^{m-1})$, $(x^{m-2}-a^{m-2})$, &c. will be divisible by $x-a$. And, therefore, the whole compound expression

$$(x^m-a^m)+A(x^{m-1}-a^{m-1})+B(x^{m-2}-a^{m-2})+\text{ \&c.}=0,$$

which is equivalent to the equation first proposed, is also divisible by $x-a$; as was to be shown.

But if a be a quantity greater or less than the root, this conclusion will not take place; because, in that case, we shall not have

$$V=-a^m-Aa^{m-1}-Ba^{m-2}-Ca^{m-3}-\ \ldots\ -Ta;$$

which is an equality obviously essential to the division in question.

475. The preceding *proposition* may be demonstrated, after the manner of D'ALEMBERT, as follows: In fact, designating by X, the polynomial, which forms the first member of the equation (1); then we shall always carry on the division of X by $x-a$, till we arrive at a remainder R, independent of x, since x is only of the first degree in the divisor; so that, representing by Q the corresponding quotient, we shall have this identity,

$$X=Q(x-a)+R.$$

Now, by hypothesis, a substituted for x reduces the polynomial X to zero; and it is evident that the same substitution gives $Q(x-a)=0$; therefore we shall necessarily have $0=R$: Hence $x-a$ divides the equation (1), without a remainder.

Reciprocally, *if the first member of any equation of the form* $X=0$ *be divisible by* $x-a$, a *is a root.* In fact we have, according to this hypothesis, the identity $X=Q(x-a)$, which, for $x=a$, gives $X=0$; therefore, (Art. 473), a is a root of the proposed equation.

COR. 1. Hence we may easily conclude, that if a be not a root of the equation (1), the first member will not be divisible by $x-a$.

COR. 2. And if the first member of the equation (1), be not divisible by $x-a$, a is not a root of the proposed equation.

476. Supposing every equation to have one root, or value of the unknown quantity, it can then be shown, *that any proposed equation will have as many roots as there are units in the index of its highest term, and no more.* For let a, according to the assumption here mentioned, be a root of the equation (1),

$$x^m+Ax^{m-1}+Bx^{m-2}+Cx^{m-3}+\ \ldots\ +Tx+V=0.$$

Then, since by the last proposition this is divisible by $x-a$, it will necessarily be reduced, by actually performing the operation, to an equation of the next inferior degree, or one of the former

$$x^{m-1}+A'x^{m-2}+B'x^{m-3}+C'x^{m-4}+\ .\ .\ T'x+V'=0.$$

And as this equation, by the same hypothesis, has also a root, which may be represented by a', it will likewise be reduced, when divided by $x-a'$, to another equation one degree lower than the last; and so on.

Whence, as this process can be continued regularly in the same manner, till we arrive at a simple equation, which has only one root, it follows that the proposed equation will have m roots

$$a,\ a',\ a'',\ a''',\ .\ .\ .\ .\ .\ a^{(m-1)'};$$

and that its successive divisors, or the factors of which it is composed, will be

$$x-a,\ x-a',\ x-a'',\ x-a''',\ .\ .\ .\ .\ x-a^{(m-1)'},$$

being equal in number to the units contained in the index m of the highest term of the equation.

Cor. If the last term of an equation vanishes, as in the form $x^m+Ax^{m-1}+Bx^{m-2}+\ .\ .\ .\ .\ +Tx=0$, it is evident that $x=0$ will satisfy the proposed equation; and consequently 0 is one of its roots. And if the two last terms vanish, or the equation be of the form $x^m+Ax^{m-1}+Bx^{m-2}+\ .\ .\ .\ +Sx^2=0$, two of its roots are 0; and so on. See, for another demonstration of the preceding proposition, *Bonnycastle's Algebra*, vol. ii. 8vo.

477. Since it appears (Art. 474), that every equation, when all its terms are brought to one side, is exactly divisible by the unknown quantity in that equation *minus* either of its roots, and by no other simple factor, it is evident that the equation

$$x^m+Ax^{m-1}+Bx^{m-2}+Cx^{m-3}+\ .\ .\ Tx+V=0\ .\ (1),$$

of which $a, b, c, d, \ldots l$, are supposed to be its several roots, is composed of as many factors

$$(x-a)\ (x-b)\ (x-c)\ (x-d)\ .\ .\ (x-l)\ .\ (2),$$

as the equation has roots; and that it can have no other factor whatever of that form.

478. Whence, as these two expressions are, by hypothesis, identical, the proposed equation, by actually multiplying

the above factors, and arranging the terms according to the powers of x, will become

$$\begin{array}{l|r|r|l} x^m-a & x^{m-1}+ab & x^{m-2}-abc & x^{m-3}(abc.l)=0 \\ -b & +ac & -abd & \\ -c & +ad & -acd & \\ -d & +bc & -bcd & \\ \&c. & \&c. & \&c. & \end{array}$$

which form is general, whatever may be the different signs of the roots, or of the terms of the equation; taking a, b, c, &c. as well as A, B, C, &c. in $+$ or $-$ as they may happen to be.

479. Hence, since the two equations (1), (3), are identical, the coefficients of the like powers of x, are equal; and consequently, the following relations between the coefficients and roots will be sufficiently obvious.

I. *The sum of all the roots of any equation, having its terms arranged according to the order of the powers of the unknown quantity, is equal to the coefficient of the second term of that equation, with its sign changed.*

II. *The sum of the products of all the roots, taken two and two, is equal to the coefficient of the third term, with its proper sign;* and so on.

III. *The continued product of all the roots, is equal to the last term, taken with the same or a contrary sign, according as the equation is even or odd.*

480. It is very proper to observe, that we cannot have all at once $x=a$, $x=b$, $x=c$, &c. for the roots of any equation as in the formula (2); except when $a=b=c=d$, &c., that is, when all the roots are equal. The factors $x-a$, $x-b$, $x-c$, &c. exist in the same equation: because algebra gives, by one and the same formula, not only the solution of the particular problem from which that formula may have originated; but also the solution of all problems which have similar conditions. The different roots of the equation satisfy the respective conditions; and those roots may differ from one another by their *quantity*, and by their *mode* of existence.

481. To this we may likewise add, that, if the roots of any equation be all positive, as in formula (2), where the factors are of the form

$$(x-a)\,(x-b)\,(x-c)\,(x-d)\;\ldots\;(x-l)=0,$$

the signs of the terms will be alternately $+$ and $-$; as will readily appear from performing the operation required.

482. But if the roots be all negative, in which case the factors will be of the form

$$(x+a)\ (x+b)\ (x+c)\ (x+d)\ \ldots\ (x+l)=0,$$

the signs of all the terms will be positive; because the equation arises wholly from the multiplication of positive quantities.

Some equations have their roots in part positive, and in part negative: Thus, in the cubic equation, $(x-a)\times(x-b)\times(x+c)=0$, or $x^3+(c-a-b)x^2+(ab-ac-bc)\times x+abc=0$, there are two positive and one negative root; because, when $x-a=0$, $x=a$; $x-b=0$, $x=b$; $x+c=0$, $x=-c$.

483. *Any equation, having fractional coefficients, may be transformed into another, that shall have the coefficient of its first term unity, and those of the rest, as well as the absolute terms, whole numbers.*

For let there be taken, instead, of a general equation of this kind, the following partial example,

$$x^3+\tfrac{1}{2}x^2+\tfrac{2}{3}x+\tfrac{3}{4}=0,$$

which will be sufficient to show the method that should be followed in other cases.

Then if each of the terms be multiplied by the product of the denominators, or by their least common multiple, we shall have $12x^3+6x^2+8x+9=0$, where the coefficients and absolute term are all whole numbers.

And if $12x$, in this case, be put $=y$, or $x=\frac{y}{12}$, there will arise by substitution,

$$\frac{y^3}{12^2}+6\left(\frac{y^2}{12^2}\right)+8\left(\frac{y}{12}\right)+9=0.$$

Which last equation, when all its terms are multiplied by 12^2, gives $y^3+6y^2+96y+1296=0$; where the coefficient of the first term is unity, and those of the rest whole numbers, as was required.

So that when the value of y in this equation is known, we shall have for the proposed equation $x=\frac{y}{12}$.

484. *Any equation may be transformed into another, the roots of which shall be greater or less than those of the former by a given quantity.*

Thus, let there be taken, as before, the following general equation,

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \ldots Tx + V = 0.$$

And suppose it were required to transform it into another, whose roots shall be greater than those of the given equation by e.

Then, if y be made to represent one of these roots, we shall have, by the nature of the question,

$$y = x + e, \text{ or } x = y - e.$$

And, consequently, by substituting $y-e$ for x, in the proposed equation, there will arise

$$\left.\begin{array}{r} y^m - me \\ +A \end{array}\right| y^{m-1} + \left.\begin{array}{r} \frac{m(m-1)}{2} e^2 \\ -(m-1)Ae \\ +B \end{array}\right| y^{m-2} \text{ \&c. } \ldots = 0 \quad (4),$$

which equation will evidently fulfil the conditions required, y being here greater than x by e. And if y be taken $= x - e$, or $x = y + e$, we shall obtain, by a similar substitution, an equation whose roots are less than those of the given equation by e.

485. Whence, also, as e, in the above case, is indeterminate, this mode of substitution may be used for destroying one of the terms of the proposed equation. For putting in the above expression the coefficient $-me + A = o$, we shall have

$$e = \frac{A}{m}, \text{ and } x = y - e = y - \frac{A}{m};$$

where it is plain, *that the second term of any equation may be taken away, by substituting for the unknown quantity some other unknown quantity, together with such a part of the coefficient of the second term, taken with a contrary sign, as is denoted by the index of the highest power of the equation.*

Thus, for example, to transform the equation $x^3 - 9x^2 + 7x + 12 = 0$ into one which shall want the second term. Assume $x = y + 3$; then

$$\left.\begin{array}{rl} x^3 = & y^3 + 9y^2 + 27y + 27 \\ -9x^2 = & \quad -9y^2 - 54y - 81 \\ +7x = & \qquad\qquad +7y + 21 \\ +12 = & \qquad\qquad\qquad +12 \end{array}\right\} = 0;$$

that is, $y^3 - 20y - 21 = 0$; and if the values of y be a, b, c, the values of x are $a+3, b+3$, and $c+3$.

The third term of the proposed equation may also be taken away by means of the coefficient, or formula,

$$\frac{m(m-1)}{2}e^2 - (m-1)Ae + B = 0,$$

where the determination of e requires the solution of an equation of the second degree; and so on.

486. *Any proposed equation may be transformed into another, the roots of which shall be any multiples or parts of those of the former.*

Thus, let there be taken, as in the former propositions, the general equation

$$x^m+Ax^{m-1}+Bx^{m-2}+Cx^{m-3}+\ ..\ Tx+V=0.\ (1).$$

And, in order to convert it into another, whose roots shall be some multiple of those of the given equation; let there be put $y=ex$, or $x=\frac{y}{e}$.

Then, by substituting this value for x in the proposed equation, there will arise

$$\frac{y^m}{e^m}+A\frac{y^{m-1}}{e^{m-1}}+B\frac{y^{m-2}}{e^{m-2}}+\ ....\ T\frac{y}{e}+V=0.$$

And, consequently, if this be multiplied by e^m, we shall have

$$y^m+Aey^{m-1}+Be^2y^{m-1}+\ ....\ Te^{m-1}y+Ve^m=0,$$

which equation will evidently fulfil the conditions required, y being equal to ex.

And if y be put $=\frac{x}{c}$, or $x=ey$, we shall obtain, by a similar substitution of this value for x, and then dividing by e^m, the equation

$$y^m+\frac{A}{e}y^{m-1}+\frac{B}{e^2}y^{m-2}+\ ...\ \frac{T}{e^{m-1}}+\frac{V}{e^m}=0;$$

where the roots are equal to those of the proposed equation, divided by e.

And it may easily be proved, that if the alternate terms, *beginning with the second, be changed, the signs of all the roots are changed.*

487. For a more particular account of the general Theory and Doctrine of Equations, see BONNYCASTLE's *Algebra*, vol. ii. 8vo. BRIDGE's *Equations*, and LAGRANGE's *Traité de la Resolution des Equations Numériques;* where the intelligent reader will find a full investigation of this part of analysis.

§ II. RESOLUTION OF CUBIC EQUATIONS BY THE RULE OF CARDAN, OR OF SCIPIO FERREO.

488. *Cubic equations*, as has already been observed in

Chap. VIII., are of two kinds; that is, *pure* and *adfected*. All pure equations of the third degree are comprehended in the formula $x^3=n$, where n may be any number whatever, *positive* or *negative*, *integral* or *fractional*. And the value of x is obtained by extracting the cube root of the number n.

489. But in this manner, we obtain only one value for x; whereas every equation of the third degree has three values. In order to show how the two remaining values of x may be determined in equations of the above form, let us, for example, consider the equation $x^3-8=0$; where x is readily found $=2$. And as 2 is a root of the proposed equations, it is plain that x^3-8 must be divisible by $x-2$: therefore, this division being actually performed, the quotient will be x^2+2x+4.

Hence it follows, that the equation $x^3-8=0$, may be represented by these factors;

$$(x-2)\times(x^2+2x+4)=0.$$

490. Now the question is, to know what number we are to substitute instead of x, in order that $x^3-8=0$; and it is evident that this condition is answered by supposing the product which we have just found equal to 0: but this happens, not only when the first factor $x-2=0$, which gives $x=2$, but also when the second factor $x^2+2x+4=0$.

Let us, therefore, make $x^2+2x+4=0$; then $x=-1\pm\sqrt{-3}$. So that besides the case in which $x=2$, we find two other values of x, which will satisfy the equation $x^3-8=0$. It is true, as Euler justly observes, that these values are imaginary; but yet they deserve attention.

491. What has been just said applies in general to every pure cubic, such as $x^3=n$, and the three roots or values of x, may be found in a similar manner. To abridge the calculation, let us suppose $\sqrt[3]{n}=n'$, so that $n=n'^3$; the proposed equation will then assume this form, $x^3-n'^3=0$, which, being divided by $x-n'$, will give for the quotient $x^2+n'x+n'^2$. Consequently, the equation $x^3-n=0$, may be represented by the product $(x-n')(x^2+n'x+n'^2)=0$, which is in fact $=0$, not only when $x-n'=0$, or $x=n'$; but also when $x^2+n'x+n'^2=0$. Now this expression contains two other values of x, for it gives $x=-\frac{n'}{2}\pm\frac{n'}{2}\sqrt{-3}$; both of which answer the required condition.

492. All adfected cubic equations, after being properly reduced by the known rules, may be exhibited under the follow-

ing general forms; namely, $x^3+ax^2+bx=0$, and $x^3+a'x^2+b'x+c'=0$, where a, b, a', b', and c', may be any numbers whatever, *positive* or *negative*, *integral* or *fractional.*

493. The solution of a cubic equation, of the form $x^3+ax^2+bx=0$, is attended with no difficulty; since it may at once be put under the form $x\times(x^2+ax+b)=0$; and it is evident that the product $x\times(x^2+ax+b)$ may be $=0$, in two ways, that is, when $x=0$, or $x^2+ax+b=0$; so that nothing now remains, but to find the values of x in the quadratic equation $x^2+ax+b=0$, which are readily found to be $x=-\frac{a}{2}\pm\frac{1}{2}\sqrt{(a^2-4b)}$. Consequently, the three values of x, which answer the required condition, are $0, -\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-4b)}$, and $-\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-4b)}$.

494. An adfected cubic equation is said to be complete, when, after being properly reduced by the known rules, it is of the form $x^3+a'x^2+b'x+c'=0$. And it has already been shown, that every cubic equation of the above form, whose roots are r', r', r'', may be transformed into another *deficient in its second term*, by substituting $y-\frac{1}{3}a'$ for x in the given equation; in which case the roots of the transformed equation will be $r-\frac{1}{3}a'$ $r'-\frac{1}{3}a'$ $r''-\frac{1}{3}a'$; if, therefore, the roots of the *transformed* equation be known, the roots of the *given* equation will be known also. Hence the resolution of a cubic equation *complete* in all its terms will be effected, if we can arrive at the resolution of it in the form $x^3+ax=b$. In which a and b may be any *positive* or *negative* numbers whatever.

495. For this purpose, let there be taken $x=y+z$, and the above equation, by substitution, will become $y^3+3y^2z+3yz^2+z^3+ay+az=b$.

Or, because $3y^2z+3yz^2=3yz(y+z)$, and $ay+az=a(y+z)$, it will be $y^3+z^3+(3yz+a)(y+z)=b$.

Now, as another *unknown quantity* has been introduced into the equation, another *condition* may be annexed to its solution. Let this condition be, that $3yz+a=0$, or $z=-\frac{a}{3y}$, in which case the transformed equation becomes

$y^3+z^3=b$, or by substitution $y^3-\frac{a^3}{27y^3}=b$;

$\therefore y^6-by^3-\frac{1}{27}a^3$; which equation solved, gives

$y=\sqrt[3]{[\frac{1}{2}b+\sqrt{(\frac{1}{4}b^2+\frac{1}{27}a^3)}]}$; $\therefore$ since $z^3=b-y^3$, we have

$z=\sqrt[3]{[\frac{1}{2}b-\sqrt{(\frac{1}{4}b^2+\frac{1}{27}a^3)}]}$; and $x=y+z=\sqrt[3]{[\frac{1}{2}b+\sqrt{(\frac{1}{4}b^2+\frac{1}{27}a^3)}]}+\sqrt[3]{[\frac{1}{2}b-\sqrt{(\frac{1}{4}b^2+\frac{1}{27}a^3)}]}$ . . . (1);

where by taking a and b in $+$ or -1, as they may happen to be, we have always one root of the transformed equation; and this is the formula which is called *the Rule of Cardan.*

496. And since one value of x is now determined, the equation may be depressed to a quadratic, from which the other two roots may be readily found.

Ex. 1. Given $x^3+2x=12$, to find the values of x

Comparing this with the general equation, $x^3+ax=b$, we have $a=2$, and $b=12$; therefore, by substituting these values for a and b in the above formula (1),

$x=\sqrt[3]{[6+\sqrt{[(36+\frac{8}{27})]}}+\sqrt[3]{[6-\sqrt{(36+\frac{8}{27})}]}$

$=\sqrt[3]{(6+6.024633)}+\sqrt[3]{(6-6.024633)}$

$=\sqrt[3]{(12.024633)}+\sqrt[3]{(-.024633)}=2.29-.29=2.$

One root of the equation, therefore, is 2; divide $x^3+2x-12$ by $x-2$, and the quotient is x^2-2x+6; $\therefore x^2-2x+6=0$, whose roots are $1\pm\sqrt{-5}$. Hence, the three roots of the equation are 2, $1+\sqrt{-5}$, $1-\sqrt{-5}$, the two last of which are *imaginary.*

Ex. 2. Given $x^3-48x=128$, to find the values of x.

Here, by comparing this with the equation, (Art. 494), we have $a=-48$, and $b=128$;

$\therefore x=\sqrt[3]{[64+\sqrt{(4096-4096)}]}+\sqrt[3]{[64-\sqrt{(4096-4096)}]}$

$=\sqrt[3]{(64+0)}+\sqrt[3]{(64-0)}=4+4=8.$

One root of the equation, therefore, is 8; divide $x^3-48x-128$ by $y-8$, and the quotient is $x^2+8x+16$; $\therefore x^2+8x+16=0$, whose roots are -4 ± 0; the three roots of the proposed equation are 8, -4, -4, the two last of which are equal.

497. Hence we may infer, if a be *negative*, and $\frac{1}{27}a^3$, taken with a *positive* sign, equal to $\frac{1}{4}b^2$, or $\frac{1}{4}b^2+\frac{1}{27}a^3=0$; then two roots of the proposed equation are always equal.

498. But if a be negative, and $\frac{1}{27}a^3$, taken with a positive sign, greater than $\frac{1}{4}b^2$; then $\frac{1}{4}b^2+\frac{1}{27}a^3$ is a *negative* quantity; and consequently, $\sqrt{(\frac{1}{4}b^2+\frac{1}{27}a^3)}$ is *imaginary.*

Although the value of x cannot be obtained from CARDAN'S formula, (Art. 495), by the ordinary method, we are not, however, to conclude, that the value of x, in this case, is imaginary; since it may be proved to be a real quantity after the following manner.

499. For this purpose, let $\frac{1}{2}b$ be represented by a', and $\sqrt{(\frac{1}{4}b^2+\frac{1}{27}a^3)}$, supposed imaginary, by $b'\sqrt{-1}$; then $x=\sqrt[3]{(a'+b'\sqrt{-1})}+\sqrt[3]{(a'-b'\sqrt{-1})}$. Now, let $\sqrt[3]{(a'+b'\sqrt{-1})}$ and $\sqrt[3]{(a'-b'\sqrt{-1})}$ be expanded by means of the *binomial theorem;* and since, by adding the resulting series together,

the terms involving the imaginary quantity $\sqrt{-1}$ destroy one another, we shall have

$$x=2a'^{\frac{1}{3}}\left(1+\frac{b'^2}{9a'^2}-\frac{10b'^4}{243a'^4}+\frac{154b'^6}{6561a'^6}-, \&c.\right) \quad . \ . \ . \quad (2);$$

which is a real expression. When a' is greater than b'; the above series converges rapidly, and a few of the first terms will give a near value of the root required. But if a' is less than b', $b'\sqrt{-1}$ must be put for the first term of the *binomial*, and a' for the second: See CLAIRAUT'S *Algebra*, Vol. II.

Ex. 3. Given $x^3-6x=5.6$, to find the values of x. Comparing this with the equation $x^3+ax=b$, we have $a=-6$, and $b=5.6$; therefore,

$x=\sqrt[3]{[2.8+\sqrt{(7.84-8)}]}+\sqrt[3]{[2.8-\sqrt{(7.84-8)}]}$

$=\sqrt[3]{(2.8+.4\sqrt{-1})}+\sqrt[3]{(2.8-.4\sqrt{-1}.)}$

Now, by comparing this value of x, with $\sqrt[3]{(a'+b'\sqrt{-1})}+\sqrt[3]{(a'-b'\sqrt{-1})}$, we have $a'=2.8$, and $b'=.4$; $\therefore$ substituting these values for a' and b' in the above formula (2), $x=2\sqrt[3]{2.8}$ $\left(1+\frac{.16}{70.56}-\frac{.2560}{14936.1408}, \&c.\right)=2.82(1+.00227-.00002,$ &c.) $=2.826345$ *nearly*.

Here, three terms of the series are sufficient, on account of its converging so rapidly, to give an approximate value of x, which is exact enough for all practical purposes. And, in fact, the value may be still found more accurate by continuing the series to five or six terms.

Ex. 4. Given $z^6-3z^4-2z^2-8=0$, to find the values of z.

Let $z^2=x+1$, and the equation will be transformed into $x^3-5x=12$; $\therefore$ since $a=-5$, and $b=12$.

$x=\sqrt[3]{[6+\sqrt{(36-\frac{125}{27})}]}+\sqrt[3]{[6-\sqrt{(36-\frac{125}{27})}]}$

$=\sqrt[3]{(6+5.6009)}+\sqrt[3]{(6-5.6009)}=2.26376+.73624=3.$

And, consequently, $z^2=x+1=4$, or $z=\pm2$.

500. Two roots of the proposed equation, therefore, are 2 and -2; divide $z^6-3z^4-2z^2-8$ by z^2-4, and the quotient is z^4+z^2+2; $\therefore z^4+z^2+2=0$, whose roots are $z=\pm\sqrt{(-\frac{1}{2}\pm\frac{1}{2}\sqrt{-7})}$. Hence four roots of the proposed equation are *imaginary*.

It may be observed that, in general, all equations, as $z^{3m}+az^{2m}+bz^m+c=0$, may be reduced to one of the third degree, by putting $z^m=x-\frac{1}{3}a$.

Ex. 5. Given $x^3+30x=117$, to find the values of x.

Ans. $x=3$, or $-\frac{3}{2}\pm\frac{7}{2}\sqrt{-3}$.

Ex. 6. Given $x^3+9x=270$, to find the values of x.

Ans. $x=6$, or $-3\pm6\sqrt{-1}$.

Ex. 7. Given $x^3-36x=91$, to find the values of x.

Ans. $x=7$, or $-\frac{7}{2}+\frac{7}{2}\sqrt{-3}$.

Ex. 8. Given $x^3-6x^2+10x-8=0$, to find the values of x.
Ans. $x=4$, or $1\pm\sqrt{-1}$.

Ex. 9. Given $x^3-3x-4=0$, to find the values of x.
Ans. $x=2.2$; $1.1+\sqrt{-.63}$; $-1.1-\sqrt{-.63}$, *very nearly*.

Ex. 10. Given $x^3+24x=250$, to find the value of x.
Ans. $x=5.05$.

Ex. 11. Given $z^3-6z^2+13z-12=0$, to find the values of z.
Ans. $z=3$, or $-\frac{3}{2}\pm\frac{1}{2}\sqrt{-7}$.

Ex. 12. Given $2x^3-12x^2+36x=44$, to find the value of x.
Ans. 2.32748, &c.

§ III. RESOLUTION OF BIQUADRATIC EQUATIONS BY THE METHOD OF DES CARTES.

501. The same observation may be applied to *biquadratic* equations as was applied to *cubic* equations in (Art. 494), that, since the equation $x^4+a'x^3+b'x^2+r'x+s'=0$, may be transformed into another which shall be *deficient* in *its second term*, and whose roots shall have a given relation to the roots of the given equation, the complete solution of a biquadratic equation will be effected, if we can arrive at the solution of it in the form

$$x^4+ax^2+bx+c=0 \quad . \quad . \quad . \quad . \quad (1);$$

where a, b, c, may be any numbers whatever, *positive* or *negative*.

502. In the solution of a biquadratic equation, after the manner of *Des Cartes*, the formula x^4+ax^2+bx+c is supposed to be the product of two quadratic factors, x^2+px+q and x^2+rx+s, in which p, q, r, s, are unknown quantities. Or, which is the same, the biquadratic equation $x^4+ax^2+bx+c=0$ is considered as produced by the multiplication of the two quadratics,

$$(2) \quad . \quad . \quad . \quad . \quad x^2+px+q=0; \quad x^2+rx+s=0 \quad . \quad . \quad . \quad (3).$$

503. Hence, by the actual multiplication of the above two factors, we shall have

$$\begin{array}{l} x^4+(p+r)x^3+(s+q+pr)x^2+(ps+qr)x+qs= \\ x^4 \qquad\qquad\qquad\qquad +ax^2 \qquad\quad +bx+c. \end{array}$$

And, consequently, by equating the coefficients of the like powers of x in this last equation, we shall have the four following equations,

$$p+r=0; \quad s+q+pr=a; \quad ps+qr=b; \quad qs=c.$$

Or, if $-p$, which is the value of r in the first of these, be substituted for r in the second and third, they will become,

$$s+q=a+p^2; \quad s-q=\frac{b}{p}; \quad qs=c.$$

Whence, subtracting the square of the second of these from that of the first, and then changing the sides of the equation, we shall have

$$a^2+2ap^2+p^4-\frac{b^2}{p^2}=4qs, \text{ or } 4c.$$

And, therefore, by multiplying by p^2, and placing the terms according to the order of their powers, the result will give, $p^6+2ap^4+(a^2-4c)p^2=b^2$. . . (4).

From which last equation, if there be put $p^2=z$, we shall have, $z^3+2az^2+(a^2-4c)z=b^2$ . . . . (5).

Hence, also, since $s+q=a+p^2$, and $s-q=\frac{b}{p}$, there will arise, by addition and subtraction,

$$s=\tfrac{1}{2}a+\tfrac{1}{2}p^2+\frac{b}{2p}; \quad q=\tfrac{1}{2}a+\tfrac{1}{2}p^2-\frac{b}{2p};$$

where p being known, s and q are likewise known.

And, consequently, by extracting the roots of the two assumed quadratics, (2) and (3); or of their equals, $x^2+px+q=0$, and $x^2-px+s=0$; we shall have

$x=-\frac{1}{2}p\pm\sqrt{(\frac{1}{4}p^2-q)}$ . . . . (6);

$x=\frac{1}{2}p\pm\sqrt{(\frac{1}{4}p^2-s)}$ . . . . (7);

which expressions, when taken in + and −, give the four roots of the proposed biquadratic, as was required.

504. It may be observed, that whichever of the values of the unknown quantity, in the cubic or reduced equation (5), be used, the same values of x will be obtained.

505. To this we may further add, that when the roots of the cubic, or reduced equation (5), are *all real*, then the roots of the proposed biquadratic are *all real* also. But if only *one* root of the cubic equation (1) be *real*, and, therefore, the other *two imaginary;* then the proposed biquadratic will have *two real* and *two imaginary* roots.

Ex. 1. Given the equation $x^4-3x^2+6x+8=0$, to find its roots, or the values of x.

Comparing this equation with $x^4+ax^2+bx+c=0$, we have $a=-3$, $b=6$, and $c=8$; therefore,

$$z^3+2az^2+(a^2-4c)z-b^2=z^3-6z^2+23z-36=0.$$

Let $z=y+2$, and substitute $y+2$ for z in the latter equation; the resulting equation is $y^3-35y-98=0$. Now, by comparing this last equation with $x^3+ax=b$, we have $a=-35$, and $b=98$; therefore, (Art. 495),

$y=\sqrt[3]{[49+\frac{1}{9}\sqrt{(65856)}]}+\sqrt[3]{[49-\frac{1}{9}\sqrt{(65856)}]}$

$=\sqrt[3]{(49+28.514)}+\sqrt[3]{(49-28.514)}=\sqrt[3]{(77.514)}+\sqrt[3]{20.466)}$

$=4.264+2.736=7.$

Hence, $z=y+2=9$, and $p^2=z=9$, or $p=\pm 3$; $\therefore$ (Art. 504), taking $p=3$, $s=-\frac{3}{2}+\frac{9}{2}+1=3+1=4$, and $q=-\frac{3}{2}+\frac{9}{2}-1=2$. Consequently, by substituting these values for p, q, and s, in the equations (2), (3), we shall have

$$x^2+3x+2=0, \text{ and } x^2-3x+4=0;$$

$$\therefore x=-\tfrac{3}{2}\pm\tfrac{1}{2}, \text{ and } x=\tfrac{3}{2}\pm\tfrac{1}{2}\sqrt{-7};$$

so that the four roots of the given equation are -1, -2, $\frac{3}{2}+\frac{1}{2}\sqrt{-7}$, $\frac{3}{2}-\frac{1}{2}\sqrt{-7}$.

Ex. 2. Given $x^4-6x^2-17x+21=0$, to find the values of x.

Ans. $x=3$, or 1; or $-2\pm\sqrt{-3}$.

Ex. 3. Given the equation $x^4-4x^3-8x+32=0$, to find its roots, or the values of x. Ans. 4, or 2; or $-1\pm\sqrt{-3}$.

Ex. 4. Given the equation $x^4-6x^3+5x^2+2x-10=0$, to find its roots, or the values of x.

Ans. -1, or $+5$; or $1\pm\sqrt{-1}$.

Ex. 5. Given $x^4-9x^3+30x^2-46x+24=0$, to find the roots, or values of x. Ans. $x=1$, or 4; $2\pm\sqrt{-2}$.

Ex. 6. Given $x^4+16x^3+99x^2+228x+144=0$, to find the roots, or values of x.

Ans. $x=-1$, -3; or $-6\pm\sqrt{-12}$.

Ex. 7. What two numbers are those, whose product, multiplied by the greater, is equal to 1; and if from the square of the greater, added to six times the lesser, the cube of the lesser be subtracted, the remainder shall be 8.

Ans. $-\sqrt{2}\pm\sqrt{(1+\sqrt{2})}$, $+\sqrt{2}\pm\sqrt{(1-\sqrt{2})}$.

§ IV. RESOLUTION OF NUMERAL EQUATIONS BY THE METHOD OF DIVISORS.

506. Since the last term (V) of the equation $(A)=x^m+Ax^{m-1}+Bx^{m-2} \quad . \quad . \quad . \quad . \quad . \quad Tx+V=0$, is equal to the product of all its roots, it is evident, that if any of those roots be *whole numbers*, they will be found among the *divisors* of that term. To discover, therefore, whether any of the roots of a given equation be whole numbers, we have only to find all the divisors of its last term, and substitute each of them, first with the sign $+$ and then with the sign $-$, for x, in the given equation, such of them as reduce the equation to $0=0$, will be roots of the equation.

507. Or, if the divisors of the last term should be too numerous, the equation may be transformed into another, that shall have its last term less than that of the former; which is done by increasing or diminishing the roots by 1, or some other quantity.

Ex. 1. Given $x^3-x^2-2x+8=0$, to find the roots of the equation, or values of x.

Here the divisors of its last term, are 1, 2, 4, 8 ; substitute 1, 2, 4, 8, and -1, -2, -4, -8, for x in the given equation, and -2 will be found to be the only one of these numbers which gives the result 0 ; -2 therefore is the only integral root of the equation. Hence, $x+2$ will divide x^3-x^2-2x+8 without a remainder ; let this division be made, and the quotient being put equal to 0, we shall have $x^2-3x+4=0$, a quadratic equation which contains the other two roots. The solution of this quadratic gives $x=\frac{3}{2}\pm\frac{1}{2}\sqrt{-7}$; the three roots of the given equation, therefore, are -2, $\frac{3}{2}+\frac{1}{2}\sqrt{-7}$, $\frac{3}{2}-\frac{1}{2}\sqrt{-7}$.

508. The integral roots of any numeral equation of the kind above mentioned, may also be found, by NEWTON'S *Method of Divisors*, which is founded upon the following principles.

Let one of the roots of the equation (A)$=0$, be $-a$, or, which is the same, let the proposed equation be represented under the form $(x+a)P=0$, where the binomial $x+a$ denotes one of the divisors, or factors, of which the equation is composed, and P the product of the rest. Then, if three or more terms of the arithmetical series, 2, 1, 0, -1, -2, be successively substituted for x, the divisors of the results, thus obtained, will be

$$a+2,\ a+1,\ a,\ a-1, \text{ and } a-2.$$

And as these are also in arithmetical progression, it is plain that the roots of the given equation, when integral, will be some of the numbers in such a series.

Whence, if a progression of this kind, whose common difference is 1, can be found among the divisors of the results above mentioned, by taking one number out of each of the lines, that term of it which answers to the substitution of 0 for x, taken in $+$ or $-$, according as the series is increasing or decreasing, will generally be a root of the equation.

Ex. 2. Given $x^5+x^4-14x^3-6x^2+20x+48=0$, to find the roots of the equation, or values of x.

Num.	*Results.*	*Divisors.*	*Progress.*		
1	50	1, 2, 5, 10, 25, 50,	1	2	5
0	49	1, 2, 3, 4, 6, 8, 12, 24, 48,	2	3	4
-1	36	1, 2, 3, 4, 6, 9, 12, 18, 36,	3	4	3

Here the numbers to be tried are 2, 3, -4, all of which are found to succeed; so that the equation has three integral roots ; namely, 2, 3, -4. The equation whose roots are 2, 3, -4, is $(x-2)\,.\,(x-3)\,.\,(x+4)=x^3-x^2-14x+24=0$, let the given equation be divided by it, and the quotient is $x^2+2x+2=0$, whose roots are $-1\pm\sqrt{-1}$; the five roots of the proposed equation are, therefore, 2, 3, -4, $-1+\sqrt{-1}$, $-1-\sqrt{-1}$.

509. If the highest power of the unknown quantity has any coefficient prefixed to it, let the equation be assumed of the form $(nx+a)^p=0$, and substitute 2, 1, 0, -1, -2, successively for x, as in the former instance.

Then, as before, the divisors of the several results, arising from this substitution, will be the terms of the arithmetical series,

$$2n+a,\ n+a,\ a,\ -n+a, \text{ and } -2n+a;$$

where the common difference n must be a divisor of the first term of the equation, or otherwise the operation would not succeed.

Hence, in this instance, the progressions must be so taken out of the divisors, that their terms shall differ from each other by some aliquot part of the coefficient of the first term.

Therefore, if the terms of these series, standing opposite to 0, be divided by the common difference, the quotient thus arising, taken in + and —, according as the progression is increasing or decreasing, will generally be the roots of the equation.

It is necessary to continue the series 2, 1, 0, -1, -2, far enough to show whether the corresponding progression may not break off, after a certain number of terms; which it never can do when it contains a real rational root.

Ex. 3. Given $2x^3-3x^2+16x-24=0$, to find the roots of the equation or values of x.

Substituting 2, 1, 0, -1, -2, successively, for x, as in the former case, we shall have

Num.	*Results.*	*Divisors.*	*Prog.*
2	12	1, 2, 3, 4, 6, 12,	-1
1	-9	1, 3, 9, ...	$+1$
0	-24	1, 2, 3, 4, 6, 8, &c.	$+3$
-1	-45	1, 3, 5, 9, 15, 45,	$+5$
-2	-84	1, 2, 3, 4, 6, 7, &c.	$+7$

Where the progression is ascending, the number to be tried is, therefore, $\frac{3}{2}$, which is found to be a root of the equation.

Let the given equation be divided by $x-\frac{3}{2}$, and the quotient is $2x^2-16=0$, whose roots are $\pm2\sqrt{2}$; the three roots of the proposed equation are, therefore, $-\frac{3}{2}$, $+2\sqrt{2}$, $-2\sqrt{2}$.

Ex. 4. Given $x^4+x^3-29x^2-9x+180=0$, to find the roots of the equation. Ans. 3, 4, -3, and -5.

Ex. 5. Given $x^4-4x^3-8x+32=0$, to find the roots of the equation, or values of x.

Ans. $x=2$, or 4; or $-1\pm\sqrt{-3}$.

Ex. 6. Given $x^3-5x^2+10x-8=0$, to find the integral root of the equation. Ans. 2.

Ex. 7. Given $x^4-8x^3+x^2+82x-60=0$, to find the integral roots of the equation. Ans. 5, and -3.

Ex. 8. Given $x^5-9x^3+8x^2-72=0$, to find the roots of the equation, or values of x.

Ans. $x=-3$, or -2, or 3; or $1\pm\sqrt{-3}$.

§ V. RESOLUTION OF EQUATIONS BY NEWTON'S METHOD OF APPROXIMATION.

510. The methods laid down in the preceding section, will be found sufficient for determining the integral or rational roots of equations of all orders; but when the roots are *irrational*, recourse must be had to a different process, as they can then be obtained only by approximation; that is to say, by methods which are continually bringing us nearer to the true value, till at last the error being very small, it may be neglected.

511. Different methods of this kind have been proposed, the simplest and most useful of which, as LAGRANGE justly remarks, is that of NEWTON, first published in WALLIS's *Algebra*, and afterwards at the beginning of his *Fluxions*—or rather the improved form of it, given by RAPHSON, in his works, entitled *Analysis Æquationem Universalis.*

512. In order to investigate the above-mentioned method, let there be taken the following general equation,

$$x^m+px^{m-1}+qx^{m-2}+rx^{m-3}+\ \ldots\ sx^2+tx+u=0\ .\ (1).$$

Then, supposing a to be a near value of x, found by trial, and z to be the remaining part of the root, we shall have $x=a+z$; and, consequently, by substituting this value for x in the given equation, there will arise

$$(a+z)^m+p(a+z)^{m-1}+\ \ldots\ s(a+z)^2+t(a+z)+u=0\ ;$$

which last expression, by involving its terms, and taking the result in an inverse order, may be transformed into the equation

$$\text{P}+\text{Q}z+\text{R}z^2+\text{S}z^3+\ \ldots\ +z^m=0\ .\ .\ (2),$$

where P, Q, R, &c. are polynomials, composed of certain functions, of the known quantities, a, m, p, q, r, &c. which are derived from each other, according to a regular law.

513. Thus, by actually performing the operations above indicated, it will be found that

$$\text{P}=a^m+pa^{m-1}+qa^{m-2}+\ \ldots\ sa^2+ta+u\ ;$$

which value is obtained by barely substituting a for x in the equation first proposed.

And, by collecting the several terms of the coefficients of z, it will likewise appear, that

$$Q = ma^{m-1} + m(m-1)pa^{m-2} + \ldots + 2sa + t;$$

which last value is found by multiplying each of the terms of the former by the index of a in that term, and diminishing the same index by unity.

514. Hence, since z in equation (2) is, by hypothesis, a proper fraction, if the terms that involve its several powers z^2, z^3, z^4, &c. which are all, successively, less than z, be neglected in the transformed equation, we shall have

$$P + Qz = 0, \text{ or } z = -\frac{a^m + pa^{m-1} + \ldots\ldots + ta + u}{ma^{m-1} + (m-1)pa^{m-2} + \ldots + t}.$$

And, consequently, if the numeral value of this expression be calculated to one or two places of decimals, and put equal to b, the first approximate part of the root will be $z = b$, or $x = a + b = a'$.

Whence also, if this value of x, which is nearer its true value than the assumed number a, be substituted in the place of a in the above formula, it will become

$$z = -\frac{a'^m + pa'^{m-1} + \ldots\ldots + ta' + u}{ma'^{m-1} + (m-1)pa'^{m-2} + . + t}$$

which expression being now calculated to three or four places of decimals, and put equal to c, we shall have, for a second approximation towards the unknown part of the root,

$$z = c, \text{ or } x = a' + c = a''.$$

And, by proceeding in this manner, the approximation may be carried on to any assigned degree of exactness; observing to take the assumed root a in defect or excess, according as it approaches nearest to the root sought, and adding or subtracting the corrections b, c, &c. as the case may require.

515. A negative root of any equation may also be found in the same manner, by first changing the signs of all the alternate terms, and then taking the positive root of this equation, when determined as above, for the negative root of the proposed equation.

516. In the practical application of this rule we must endeavour to find two whole numbers, between which some one root of the given equation lies; and by substituting *each* of them for x in the given equation, and then observing which of them gives a result *most nearly* equal to 0, we shall ascertain the whole number to which x most nearly approaches; we

must then assume a equal to one of the whole numbers thus found, or to some decimal number which lies between them, according to the circumstances of the case.

517. Since any quantity, which from positive becomes negative passes through 0, if any two whole numbers n and n'; one of which, when substituted for x in the proposed equation, gives a *positive*, and the other a negative result; one root of the equation will, therefore, lie between n and n'. This, of course, goes upon the supposition that the equation contains at least one *real root.*

518. It is necessary to observe, that, when a is a *much nearer* approximation to one root of the given equation than to any other, then the foregoing method of approximation can only be applied with any degree of accuracy. To this we also farther add, that, when some of the roots are nearly equal, or differ from each other by less than unity, they may be passed over without being perceived, and by that means render the process illusory; which circumstance has been particularly noticed by LAGRANGE, who has given a new and improved method of approximation, in his *Traité de la Resolution des Equations Numériques.* See, for farther particulars relating to this, and other methods, BONNYCASTLE's *Algebra,* or BRIDGE's *Equations.*

Ex. 1. Given $x^3+2x^2-8x=24$, to find the value of x by approximation.

Here, by substituting 0, 1, 2, 3, 4, successively for x in the given equation, we find that one root of the equation lies between 3 and 4, and is evidently very nearly equal to 3. Therefore let $a=3$, and $x=a+z$.

$$\text{Then}\left\{\begin{array}{l} x^3=a^3+3a^2z+3az^2+z^3 \\ 2x^2=2a^2+4az+2z^2 \\ -8x=-8a-8z \end{array}\right\}=24.$$

And by rejecting the terms $z^3+3az^2+2z^2$, (Art. 514), as being small in comparison with z, we shall have

$$a^3+2a^2-8a+3a^2z+4az-8z=24;$$

$$\therefore z=\frac{24-a^3-2a^2+8a}{3a^2+4a-8}=\frac{3}{31}=.09;$$

and consequently $x=a+2=3.09$, *nearly*

Again, if 3.09 be substituted for a, in the last equation, we shall have

$$z=\frac{24-a^3-2a^2+8a}{3a^2+4a-8}=\frac{24-29.503629-19.0962+24.72}{28.6443+12.36-8}$$

$=.00364$; and, consequently, $x=a+z=3.09+.00364=3.09364$, for a *second approximation*

And, if the first four figures, 3.093, of this number, be substituted for a in the same equation, an approximate value of x will be obtained to six or seven places of decimals. And by proceeding in the same manner the root may be found still more correctly.

Ex. 2. Given $3x^5+4x^3-5x=140$, to find the value of x by approximation. Ans. $x=2.07264$.

Ex. 3. Given $x^4-9x^3+8x^2-3x+4=0$, to find the value of x by approximation. Ans. $x=1.114789$.

Ex. 4. Given $x^3+23.3x^2-39x-93.3=0$, to find the values of x by approximation.

Ans. $x=2.782$; or -1.36; or -24.72; *very nearly.*

Ex. 5. Find an approximate value of *one* root of the equation $x^3+x^2+x=90$. Ans. $x=4.10283$.

Ex. 6. Given $x^3+6.75x^2+4.5x-10.25=0$, to find the values of x by approximation.

Ans. $x=.90018$; or -2.023; or -5.627; *very nearly.*

CHAPTER XVI.

ON

INDETERMINATE COEFFICIENTS, VANISHING FRACTIONS, AND FIGURATE AND POLYGONAL NUMBERS.

§ I. ON INDETERMINATE COEFFICIENTS.

519. This is a species of investigation, which is frequently used for obtaining the development of certain fractional and other expressions, without having recourse to the operations of division, or the extraction of roots; the method of performing which is as follows:

RULE.

Assume a series, or other expression, with unknown coefficients, for that which is required to be found; then, having multiplied it by the denominator of the given fraction, or raised it to its proper powers, find the value of each of these coefficients, by equating the homologous terms of the two expressions, or putting such of them as have no corresponding terms, equal to 0, as the case may require.

EXAMPLE 1. Let it be required to find the development of $\frac{a}{a'+b'x^2}$, according to the above method.

Assume $\frac{a}{a'+b'x^2}=A+Bx+Cx^2+Dx^3+Ex^4$, &c.

Then, multiplying the right hand side of the equation by $a'+b'x^2$, and, transposing a, we shall have

$$\begin{array}{r|l|l|l} 0=Aa'+Ba' & x+Ca' & x^2+Da' & x^3, \&c. \\ -\ a+Ab' & +Bb' & Cb' & \end{array}$$

And by putting the first term, and the coefficients of the several powers of x, each $=0$, there will arise the following equations:

$$\left.\begin{array}{l} Aa'-\ a=0 \\ Ba'+Ab'=0 \\ Ca'+Bb'=0 \\ Da'+Cb'=0 \\ \&c. \end{array}\right| \text{ or } \left|\begin{array}{l} A=\ \frac{a}{a'} \\ B=-\frac{b'}{a'}A \\ C=-\frac{b'}{a'}B \\ D=-\frac{b'}{a'}C \\ \&c. \end{array}\right.$$

Hence, $\frac{a}{a'+b'x^2}=\frac{a}{a'}-\frac{b'}{a'}Ax-\frac{b'}{a'}Bx^2-\frac{b'}{a'}Cx^3$, &c

Where it is obvious, that each coefficient, in parting from the second inclusively, is equal to that which precedes it, multiplied by $-\frac{b'}{a'}$: which law renders it unnecessary to take a greater number of equations, or to push the calculation farther.

Ex. 2. Required the development of $\frac{a+bx}{a'+b'x+c'x^2}$, according to the same method.

Assume $\frac{a+b}{a'+b'x+c'x^2}=A+Bx+Cx^2+Dx^3$, &c.

Then multiplying the right hand side of the equation by $a'+b'x+c'x^2$, and transposing $a+bx$, we shall have

$$\begin{array}{r|l|l|l} 0=Aa'+Ba' & x+Ca' & x^2+Da' & x^3, \&c \\ -\ a+Ab' & +Bb' & +Cb' & \\ -\ b & +Ac' & +Bc' & \end{array}$$

And by putting the coefficients of the several powers of $x=0$, there will arise the equations

$Aa'-a=0$		$A=\frac{a}{a'}$
$Ba'+Ab'-b=0$	or	$B=-\frac{b'}{a'}A+\frac{b}{a'}$
$Ca'+Bb'+Ac'=0$		$C=-\frac{b'}{a'}B-\frac{c'}{a'}A$
$Da'+Cb'+Bc'=0$		$D=-\frac{b'}{a'}C-\frac{c'}{a'}B$
&c.		&c.

$$\text{Whence } \frac{a+bx}{a'+b'x+c'x^2}=$$

$$\frac{a}{a'}-\left(\frac{b'}{a'}A-\frac{b}{a'}\right)x-\left(\frac{b'}{a'}B+\frac{c'}{a'}A\right)x^2-\left(\frac{b'}{a'}C+\frac{c'}{a'}B\right)x^3, \ \&c.$$

Where each coefficient, in parting from the third inclusively, may be readily deduced from the two that precede it. So that if P, Q, R, be any three consecutive coefficients, we shall have

$$Ra'+Qb'+Pc'=0\,;\text{ or } R=-\frac{b'}{a'}Q-\frac{c'}{a'}P.$$

Ex. 3. Given $(x^2+p)^2-(qx+r)^2=x^4+ax^2+bx+c$, to find the indefinite coefficients p, q, and r.

Here, by squaring the terms on the left hand side of the equation, and collecting those that are alike, we have

$$x^4+(2p-q^2)x^2-2rqx+p^2-r^2=x^4+ax^2+bx+c.$$

And consequently, by equating the homologous terms,

$2p-q^2=a$		$2p-a=q^2$
$-2qr=b$	or	$-b=2qr$
$p^2-r^2=c$		$p^2-c=r^2$

Where it is plain, that the product of the first and third of these equations is equal to $\frac{1}{4}$ of the square of the second; or

$$2p^3-ap^2-2cp+ac=\tfrac{1}{4}b^2.$$

Hence the value of p may be found by a cubic equation, and then q and r from the former equations.

Ex. 4. It is required to convert $\dfrac{A}{b-ax}$ into a series by the above method.

$$\text{Ans. } \frac{A}{b}\left(1+\frac{ax}{b}+\frac{a^2x^2}{b^2}+\frac{a^3x^3}{b^3}+\frac{a^4x^4}{b^4}+, \&c.\right)$$

Ex. 5. It is required to convert $\frac{1+2x}{1-x-x^2}$ into a series by the same method.

Ans. $1+3x+4x^2+7x^3+11x^4+18x^5+$ &c.

Ex. 6. It is required to convert $\frac{1-x}{1-2x-3x^2}$ into a series by the same method.

Ans. $1+x+5x^2+13x^3+41x^4+121x^5+$ &c.

Ex. 7. It is required to convert $\sqrt{(1-x)}$ into a series by the same method.

Ans. $1-\frac{x}{2}-\frac{x^2}{2.4}-\frac{3x^3}{2.4.6}-\frac{3.5x^4}{2.4.6.8}-\frac{3.5.7x^5}{2.4.6.8.10}-$ &c.

§ II. ON VANISHING FRACTIONS.

520. Vanishing fractions, and other similar expressions, are such, as in certain cases, become equal to $\frac{0}{0}$; which symbol, though apparently nugatory, or of no value, must not be rejected as useless, being of frequent occurrence in several Algebraical and Fluxional investigations, where it will often from the nature of the subject, denote some fixed, or determinate quantity.

Thus, if a be made to represent the first term of any regular geometrical series, r the ratio, and n the number of terms, we shall have

$$\frac{ar^n-a}{r-1}=a+ar^2+ar^3+ar^4+\ldots\ldots ar^{n-2}+ar^{n-1}.$$

Where the left hand member of the equation is a universal expression for the sum (S) of the series, whatever may be the values of a, r, and n; as will appear by dividing the numerator by the denominator.

Let, therefore, the ratio or multiplier r, be taken $=1$; and the expression for the same will be

$$S=\frac{a-a}{1-1}=\frac{0}{0}.$$

But when $r=1$, the original series becomes of the form $S=a+a+a+a+$ &c. . . . . . to n terms; of which the sum is, evidently, $=na$; and, therefore, in this case, it follows, that $\frac{0}{0}=na$.

521. And in the same way it might be shown, that this symbol is the representative of various other quantities, accord-

ing to the nature of the expression from which it is derived; but it will be here sufficient to observe, that the true value of any fractional expression of this kind may be readily obtained as follows.

RULE.

1. If both the terms of the given fraction be rational, divide each of them by their greatest common measure; then, if the hypothesis which is found to reduce the original expression to the form $\frac{0}{0}$, be applied to the result, it will give the true value of the fraction in the state under consideration.

2. Where any part of the fraction is irrational, observe what the unknown quantity is equal to when the numerator and denominator both vanish, and put it = that quantity + or i; then, if this be substituted for the unknown quantity, and the roots of the surds be extracted, to a sufficient number of places, the result, when i is put $=0$, will give the true value of the fraction.

EXAMPLE 1. It is required to find the value of the fraction $\frac{x^2-a^2}{x-a}$, when x is equal to a.

Here, if we put $x=a$, there will arise $\frac{a^2-a^2}{a-a}=\frac{0}{0}$. But, by division, $\frac{x^2-a^2}{x-a}=x+a$; and if x be now put $=a$, we shall have $\frac{a^2-a^2}{a-a}=2a$; whence $\frac{0}{0}$, or the given fraction, in its vanishing state, is $=2a$.

Ex. 2. It is required to find the value of the expression $y=\frac{b(x-\sqrt{ax})}{x-a}$, when x is equal to a.

Here, if x be taken $=a+i$, according to the rule, we shall have $y=\frac{b(a+i-\sqrt{a^2+ai})}{i}$. And, by extracting the square root of a^2+ai, and then dividing by i,

$$y=b\left\{\tfrac{1}{2}+\frac{1}{2.4}\left(\frac{i}{a}\right)-\frac{3}{2.4.6}\left(\frac{i}{a^2}\right), \&c.\right\}$$

Whence, putting the indeterminate quantity $i=0$, there will arise

$$y=\tfrac{1}{2}b;$$

which is the true value of the expression, in the case proposed.

Ex. 3. Let there be taken, as another example of this kind, the equation

$$y=\frac{P(x-a)^m}{Q(x-a)^n};$$

where P and Q are supposed to be certain functions, or combinations of x, which do not become 0 for the same value of x.

Then taking $x=a$, the expression, according to this hypothesis, will become of the form

$$\frac{P\times 0}{Q\times 0}=\frac{0}{0}.$$

But by considering the indices m, n, of the proposed fraction, under each of the relations

$$m>n,\ m=n,\ m<n,$$

we shall have, by division, the three following results:

$$y=\frac{P(x-a)^{m-n}}{Q},\ y=\frac{P}{Q},\ y=\frac{P}{Q(x-a)^{n-m}}.$$

And consequently, by now taking $x=a$, there will arise

$$y=\frac{P\times 0}{Q},\ y=\frac{P}{Q},\ y=\frac{P}{Q\times 0}.$$

Whence, the value of the symbol $\frac{0}{0}$, in this case, will be nothing, finite, or infinite, according to the conditions above mentioned.

Ex. 4. It is required to find the value of the fraction $\frac{x-x^5}{1-x}$* when x is equal to 1. Ans. 4.

Ex. 5. It is required to find the value of the fraction $\frac{x^m-a^m}{x-a}$, when $x=a$. Ans. ma^{m-1}.

Ex. 6. It is required to find the value of the fraction $\frac{x^3-a^3}{x-a}$, when x is equal to a. Ans. $3a^2$.

* The value of this fraction was the cause of a violent controversy between *Waring* and *Powell*, in 1760, when these gentlemen were candidates for the mathematical professorship at Cambridge; Waring maintaining that the value of the fraction $\frac{x-x^5}{1-x}$ is equal to 4 when $x=1$, and Powell, (or rather Maseres, who is commonly thought to have conducted the dispute,) that it was equal to 0.

The idea of vanishing fractions first originated about the year 1702, in a contest between *Varignon* and *Rolle*, two French mathematicians of considerable eminence, concerning the principles of the Differential Calculus, of which Rolle was a strenuous opposer.

Ex. 7. It is required to find the value of $\frac{(x^2-a^2)^{\frac{3}{2}}}{(x-a)^{\frac{3}{2}}}$, when x is equal to a. Ans. $(2a)^{\frac{3}{2}}$.

Ex. 8. It is required to find the value of the expression $\frac{1-x^n}{1-x}$, when x is equal to 1. Ans. n.

Ex. 9. It is required to find the value of the expression $\frac{a\sqrt{ax-x^2}}{a-\sqrt{ax}}$, when x is equal to a. Ans. $3a$.

Ex. 10. It is required to find the value of the expression $\frac{nx^{n+1}-(n+1)x^n+1}{1-x^2}$, when x is equal to 1.

Ans. $\frac{n(n+1)}{2}$.

Ex. 11. It is required to find the value of the expression $\frac{\sqrt{x}-\sqrt{a}+\sqrt{(x-a)}}{\sqrt{(x^2-a^2)}}$, when x is equal to a. Ans. $\frac{1}{\sqrt{2a}}$.

§ III. ON FIGURATE AND POLYGONAL NUMBERS.

522. *Figurate Numbers,* are such as arise from taking the successive sums of the series of natural numbers 1, 2, 3, 4, 5, &c.; and then the successive sums of these last, and so on: and *polygonal numbers,* are those which are formed of the successive sums of the terms of any arithmetical progression beginning with unity; each of them being usually divided into orders, according to the scale of their generation, which, as far as regards those of the first class, may be shown as follows:

Order.	*Figurate Numbers.*	*Gen. Terms.*
1	1, 2, 3, 4, 5, 6, &c.	n
2	1, 3, 6, 10, 15, 21, &c.	$\frac{n(n+1)}{1 \cdot 2}$
3	1, 4, 10, 20, 35, 56, &c.	$\frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}$
4	1, 5, 15, 35, 70, 126, &c.	$\frac{n(n+1)(n+2)(n+3)}{1 \cdot 2 \cdot 3 \quad 4}$
&c.	&c.	&c.

Where it is to be observed that the general terms, here given, are so called, because if 1, 2, 3, 4, &c. be respectively substituted in each of them, for n, we shall obtain the several terms of the series.

And if, instead of the natural numbers 1, 2, 3, 4, &c. which give triangular numbers, an arithmetical series be taken, the common difference of which is 2, the sum of its successive terms will be the series of square numbers; if the common difference be 3, the series will be pentagonal numbers; if 4, hexagonal; and so on: thus,

Arith. Series.	*Ord.*	*Polygonal Numbers.*	*Gen. Terms.*
1, 2, 3, 4, &c.	1	*Tri.* 1, 3, 6, 10, &c.	$\frac{n(n+1)}{2}$
1, 3, 5, 7, &c.	2	*Sqrs.* 1, 4, 9, 16, &c.	$\frac{n(2n+0)}{2}$
1, 4, 7, 10, &c.	3	*Pent.* 1, 5, 12, 22, &c.	$\frac{n(3n-1)}{2}$
1, 5, 9, 13, &c.	4	*Hex.* 1, 6, 15, 28, &c.	$\frac{n(4n-2)}{2}$
&c.		&c.	&c.

Where the number denoting any order, is the common difference of the arithmetical series, from which the polygonal numbers, belonging to that order, are generated.

In like manner, if we take the successive sums of the several polygonals thus obtained, and then the successive sums of these last, and so on, a great variety of other orders of series of this kind may be readily obtained.

Hence, also, in general, if n be made to denote the number of terms of the series, a figurate of any m, may be expressed by the following formula.

$$\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \frac{n+(m-1)}{m}.$$

And supposing n to be the number of terms of the series, as before, a polygonal number of the order $m-2$, or one that has the number of its sides denoted by m, may be expressed by $\frac{(m-2)n^2-(m-4)n}{2}$.

So that figurate numbers, of any order, may be always determined, without computing those of the preceding orders, by taking as many factors, in the first of these formulæ, by substituting the number denoting that order for $m-2$, or the number of sides of the polygon, for m, and taking n equal to the term required.

Example 1. Required the 15th term of the second order of figurate numbers, 1, 3, 6, 10, 15, &c.

Here m being $=2$, and $n=15$, we shall have by the first formula,

$$\frac{n(n+1)}{2}=\frac{15(15+1)}{2}=\frac{15\times16}{2}=15\times8=120,$$

the term required.

Ex. 2. It is required to find the 12th term of the fifth order of polygonal numbers, being those called heptagonal, or such as would be represented by a figure of seven sides.

Here m being equal 7, and $n=12$, we shall have, by the second formula,

$$\frac{(m-2)n^2-(m-4)n}{2}=\frac{(7-2)\times144-(7-4)\times12}{2}=5\times72-3$$

$\times6=360-18=342$, the term required.

Ex. 3. It is required to find the 20th term of the 5th order of figurate numbers. Ans. 42504

Ex. 4. It is required to find the 13th term of the 9th order of figurate numbers. Ans. 293930.

Ex. 5. It is required to find the 36th term of that order of polygonal numbers, which is denoted by a figure of twenty-five sides. Ans. 14526

CHAPTER XVII.

ON

INDETERMINATE AND DIOPHANTINE ANALYSIS.

§ I. ON INDETERMINATE ANALYSIS

523. When the enunciation of a question does not furnish as many equations as there are unknown quantities to be determined, the question is said to be *indeterminate*, being usually such as admit of a great variety of solutions ; although, when the answers are required only in whole positive numbers, they are generally confined within certain limits : the determination of which forms a particular branch of Algebra, called *Indeterminate Analysis*.

To begin with one of the easiest questions; let there be required two positive integer numbers, the sum of which is equal to 10.

Let us represent them by x and y; then we have, $x+y=10$, and $x=10-y$, where y is so far only determined that it must represent an integer and positive number. We may therefore substitute for it all integer and positive numbers from 1 to infinity; but since x must likewise be a positive number, it follows, that y cannot be greater than 10; because x must be positive; and if we also reject the value $x=0$, we cannot make y greater than 9; so that only the following solutions can take place:

If $y=1, 2, 3, 4, 5, 6, 7, 8, 9$,
then $x=9, 8, 7, 6, 5, 4, 3, 2, 1$.

But the four last of these nine solutions being the same as the four first, it is evident, that the question really admits only of five different solutions.

524. As we have found no difficulty in this question, we may proceed to others, which require different considerations.

PROBLEM 1. To find the values of the unknown quantities x and y in the equation

$$ax \pm by=c, \text{ or } ax+by=c,$$

where a and b are given numbers which admit of no common divisor, except when it is, also, a divisor of c.

RULE.

525. 1. Let *wh.* denote a whole or integral number, and reduce the equation to the form $x=\frac{by+c}{a}=wh.$

2. Make $\frac{by+c}{a}=\frac{dy+e}{a}$, by throwing all whole numbers out of it, till d and e be each less than a.

3. Find the difference, or sum, of $\frac{dy+e}{a}$, or some multiple of it, and $\frac{ay}{a}$, or any other multiple of it that comes near the former, and the result will be a whole number.

4. Take this, or any multiple of it, from one of the foregoing fractions, or from any whole number which is nearly equal to it, and the result, in this case, will also be a whole number.

5. Proceed in the same manner with this last result, and so on, till the coefficient of y becomes equal to 1, or

$$\frac{y+r}{a}=wh=p.$$

6. Then will $y=ap\pm r$, where p may be any whole number whatever, that makes y positive; and as the value of y is now known, that of x may be found from the given equation.

EXAMPLE 1. Given $2x+3y=25$, to determine x and y in whole positive numbers.

Here $x=\frac{25-3y}{2}=12-y+\frac{1-y}{2}$.

Hence, since x must be a whole number, it follows that $\frac{1-y}{2}$ must also be a whole number.

Let, therefore, $\frac{1-y}{2}=wh.=p$;

then $1-y=2p$, or $y=1-2p$.

And since

$$x=12-y+\frac{1-y}{2}=12-(1-2p)+p=12+3p-1,$$

we shall have $x=11+3p$, and $y=1-2p$;
where p may be any whole number whatever, that will render the values of x and y in these two equations positive.

But it is evident, from the value of y, that p must be either 0 or negative, and, consequently, from that of x, that it must be -1, -2, or -3.

Whence, $p=0$, $p=-1$, $p=-2$, $p=-3$;

$$\text{then}\ \begin{cases} x=11,\ x=8,\ x=5,\ x=2; \\ y=\ 1,\ y=3,\ y=5,\ y=7; \end{cases}$$

which are all the answers in whole positive numbers that the question admits of.

Ex. 2. Given $21x+17y=2000$, to find all the possible values of x and y in whole numbers.

Here $x=\frac{2000-17y}{21}=95+\frac{5-17y}{21}=wh.$;

or omitting the 95, $\frac{5-17y}{21}=wh.$;

consequently, by addition, $\frac{21y}{21}+\frac{5-17y}{21}=\frac{4y+5}{21}=wh.$

Also, $\frac{4y+5}{21}\times 5=\frac{20y+25}{21}=1+\frac{4+20y}{21}=wh.$;

or, by rejecting the whole number 1, $\frac{4+20y}{21}=wh.$

And, by subtraction, $\frac{21y}{21}-\frac{4+20y}{21}=\frac{y-4}{21}=wh.=p$;

whence $y=21p+4$,

and $x=\frac{2000-17y}{21}=\frac{2000-17(21p+4)}{21}=92-17p.$

Whence, if p be put $=0$, we shall have the least value of $y=4$, and the corresponding, or greatest value, of $x=92$.

And the rest of the answers will be found by adding 21 continually to the least value of y, and subtracting 17 from the greatest value of x; which being done, we shall obtain the six following results:

$x=92$	75	58	41	24	7
$y=\ 4$	25	46	67	88	109

These being all the solutions the question admits of.

Ex. 3. Given $19x=14y-11$, to find x and y in whole numbers.

$$\text{Here } x=\frac{14y-11}{19}=wh., \text{ and } \frac{19y}{19}=wh.;$$

whence, by subtraction, $\frac{19y}{19}-\frac{14y-11}{19}=\frac{5y+11}{19}=wh.$

$$\text{Also, } \frac{5y+11}{19}\times 4=\frac{20y+44}{19}=y+2+\frac{y+6}{19}=wh.;$$

and by rejecting $y+2$, which is a whole number,

$$\frac{y+6}{19}=wh.=p; \therefore y=19p-6, \text{ and}$$

$$x=\frac{14y-11}{19}=\frac{14(19p-6)-11}{19}=\frac{266p-95}{19}=14p-5.$$

Whence, if p be taken $=1$, we shall have $x=9$ and $y=13$, for their least values; the number of solutions being obviously indefinite.

526. When there are three or more unknown quantities, and only one equation by which they can be determined, it will be proper first to find the limit of that quantity which has the greatest coefficient, and then to ascertain the different values of the rest, by separate substitutions of the several values of the former, from 1 up to the extent required, as in the following example.

Ex. 4. Given $3x+5y+7z=100$, to find all the different values of x, y, and z, in whole numbers.

Here each of the least integer values of x and y are 1, by the question; whence it follows, that

$$z=\frac{100-5-3}{7}=\frac{100-8}{7}=\frac{92}{7}=13\tfrac{1}{7}.$$

Consequently z cannot be greater than 13, which is also the limit of the number of answers; though they may be considerably less.

By proceeding, therefore, as in the former rule, we shall have

$$x=\frac{100-5y-7z}{3}=33-y-2z+\frac{1-2y-z}{3}=wh.;$$

and by rejecting $33-y-2z$,

$$\frac{1-2y-z}{3}=wh.;\text{ or }\frac{3y}{3}+\frac{1-2y-z}{3}=\frac{y+1-z}{3}=wh.=p.$$

Whence, $y=3p+z-1$; and, putting $p=0$, we shall have the least value of $y=z-1$; where z may be any number from 1 up to 13, that will answer the conditions of the question.

When, therefore, $z=1$, we have $y=0$,

$$\text{and } x=\frac{100-7}{3}=31.$$

And by taking $z=2, 3, 4, 5$, &c. the corresponding values of x and y, together with those of z, will be found to be as below.

$z=1$	2	3	4	5	6	7	8
$y=0$	1	2	3	4	5	6	7
$x=31$	27	23	19	15	11	7	3

Which are all the integer values of x, y, and z, that can be obtained from the given equation.

527. If there be three unknown quantities, and only two equations, exterminate one of these quantities in the usual way, and find the values of the other two from the resulting equation, as before; then, if the values, thus found, be separately substituted, in either of the given equations, the corresponding values of the remaining quantities will likewise be determined.

Ex. 5. Given $x-2y+z=5$, and $2x+y-z=7$, to find the values of x, y, and z.

Here, by multiplying the first of these equations by 2, and subtracting the second from the product, we shall have

$$3z-5y=3,\text{ or } z=\frac{3+5y}{3}=1+y+\frac{2y}{3}=wh.;$$

$$\text{and consequently }\frac{2y}{3},\text{ or }\frac{3y}{3}-\frac{2y}{3}=\frac{y}{3}=wh.=p$$

whence $y=3p$.

And by taking $p=0, 1, 2, 3, 4$, &c. we shall have $y=0$, 3, 6, 9, 12, 15, &c. and $z=1$, 6, 11, 16, 21, &c.

But from the first of the two given equations

$$x=5+2y-z\ ;$$

whence, by substituting the above values for y and z, the results will give

$$x=4,\ 5,\ 6,\ 7,\ 8,\ 9,\ \&c.$$

And therefore the first six values of x, y, and z, are as below:

$x=4$	5	6	7	8	9
$y=0$	3	6	9	12	15
$z=1$	6	11	16	21	26

Where the law by which they can be continued is sufficiently obvious.

Ex. 6. Given $3x=8y-16$, to find the least values of x, and in whole numbers. Ans. $x=8$, $y=5$.

Ex. 7. Given $14x=5y+7$, to find the least values of x and y in whole numbers. Ans. $x=3$, $y=7$.

Ex. 8. It is required to divide 100 into two such parts, that one of them may be divisible by 7, and the other by 11.

Ans. The only parts are 56 and 44.

Ex. 9. Given $11x+5y=254$, to find all the possible values of x and y in whole numbers.

Ans. $\begin{cases} x=19,\ 14,\ 9,\ 4, \\ y=9,\ 20,\ 31,\ 42. \end{cases}$

Ex. 10. Given $17x+19y+21z=400$, to find all the answers in whole numbers which the question admits of.

Ans. 10 different answers.

Ex. 11. Given $5x+7y+11z=224$, to find all the possible values of x, y, and z, in whole positive numbers.

Ans. The number of answers is 59.

Ex. 12. A person bought as many ducks and geese, together, as cost him 28*s.*; for the geese he paid 4*s.* 4*d.* a piece, and for the ducks 2*s.* 6*d.* a piece; what number had he of each? Ans. 3 geese and 6 ducks.

Ex. 13. How many gallons of spirits, at 12*s.*, 15*s.*, and 18*s.* a gallon, must a rectifier of compounds take to make a mixture of 1000 gallons, that shall be worth 17 shillings a gallon? Ans. $111\frac{1}{9}$ at 12*s.*, $111\frac{1}{9}$ at 15*s.*, and $777\frac{7}{9}$ at 18*s.*

PROBLEM.

528. To find such a whole number, as, being divided by other given numbers, shall leave given remainders.

RULE.

1. Call the number to be determined x, the numbers by which it is to be divided a, b, c, &c., and the given remainders f, g, h, &c.

2. Subtract each of the remainders from x, and divide the differences by a; and there will arise $\frac{x-f}{a}$, $\frac{x-g}{a}$, $\frac{x-h}{a}$, &c. = whole numbers.

3. Put the first of these fractions $\frac{x-f}{a}=p$, and substitute the value of x, as found from this equation, in the place of x in the second fraction.

4. Find the least value of p in this second fraction, by the last problem, which put $=r$, in the place x in the third fraction.

529. Find, in like manner, the least value of r, in this third fraction, which put $=s$, and substitute the value of x, in terms of s, in the fourth fraction, as before; and so on, to the last; when the value of x thus found, will give the whole number required.

Example 1. It is required to find the least whole number, which, being divided by 17, shall leave a remainder of 7, and when divided by 26, shall leave a remainder of 13.

Let $x=$ the number required.

Then $\frac{x-7}{17}$ and $\frac{x-13}{26}=$ whole numbers.

And, putting $\frac{x-7}{17}=p$, we shall have $x=17p+7$; which value of x, being substituted in the second fraction, gives $\frac{17p+7-13}{26}=\frac{17p-6}{26}=wh.$

But it is obvious that $\frac{6p}{26}-\frac{17p-6}{26}=\frac{9p+6}{26}=wh.$;

$$\text{or } \frac{9p+6}{26}\times 3=\frac{27p+18}{26}=p+\frac{p+18}{26}=wh.$$

And by rejecting p, there remains $\frac{p+18}{26}=wh.=r$;

$$\text{therefore } p=26r-18;$$

where, if r be taken $=1$, we shall have $p=8$.

And consequently $x=17p+7=17\times 8+7=143$, the number required. 31*

Ex. 2. To find a number, which, being divided by 6, shall leave the remainder 2, and when divided by 13, shall leave the remainder 3. Ans. 68.

Ex. 3. It is required to find the least whole number, which, being divided by 39, shall leave the remainder 16, and when divided by 56, the remainder shall be 27. Ans. 1147.

Ex. 4. It is required to find the least whole number, which, being divided by 11, 19, and 29, shall leave the remainders, 3, 5, and 10. Ans. 4128.

Ex. 5. It is required to find the least whole number, which, being divided by each of the nine digits, 1, 2, 3, 4, 5, 6, 7, 8, 9, shall leave no remainder. Ans. 2520.

PROBLEM.

On Compound Indeterminate Equations.

530. Equations of this kind, not higher than the second degree, which admit of answers in whole numbers, are chiefly such as consist of the products, or squares, of two unknown quantities, together with the quantities themselves; being, usually, one of the four general forms given in the following rule.

RULE.

1. If the equation be of the form $xy=ax+by+c$, we shall have, for its solution in whole numbers, $y=a+\frac{ab+c}{x-b}$; where $x-b$ must be a divisor of $ab+c$.

2. If the equation be of the form $x^2+xy=ax+by+c$, we shall have

$$y=-x+a-b+\frac{c+b(a-b)}{x-b};$$

where $x-b$ must be a divisor of $c+b(a-b)$.

3. If the equation be $x^2=y^2+ay+b$, we shall have $y=\frac{a^2-4b}{8n}+\frac{n-a}{2}$, and $x=\frac{a}{2}+y-n$; where a and n must be even numbers, and n be so taken that $8n$ may be a divisor of a^2-4b.

4. If the equation be $x^2=ay^2+by+c^2$, we shall have $y=\frac{b-2cn}{n^2-a}$, and $x=c+ny$; where n must be some whole number between $\sqrt{a}$ and $\frac{b}{2c}$.

EXAMPLE 1. Given $xy=42-2x-3y$, to find the several values of x and y in whole numbers.

Here, by the first form,

$$a=-2,\ b=-3,\ \text{and } c=42,$$

$$\text{whence } y=-2+\frac{6+42}{x+3}=-2+\frac{48}{x+3}.$$

Where it is plain, that x must be such a number, that, when added to 3, it shall be a divisor of 48. But the divisors of 48, that will give quotients greater than 2, are 16, 12, 8, 6, 4, and 2.

And consequently the integral values of the two unknown quantities are

$$x=16-3,\text{ or } 13 \mid =12-3,\text{ or } 9 \mid =8-3,\text{ or } 5 \mid =6-3,\text{ or } 3 \mid =4-3,\text{ or } 1.$$

$$y=\frac{48}{16}-2,\text{ or } 1 \mid =\frac{48}{12}-2,\text{ or } 2 \mid =\frac{48}{8}-2,\text{ or } 4 \mid =\frac{48}{6}-2,\text{ or } 6 \mid =\frac{48}{4}-2,\text{ or } 10.$$

Which are all the answers in whole positive numbers that the question admits of.

Ex. 2. Given $x^2=y^2+20y$, to find the values of x and y in whole positive numbers.

Here, by the third form, $a=20$, and $b=0$,

$$\text{whence, } y=\frac{400}{8n}+\frac{n-20}{2}=\frac{50}{n}+\frac{n}{2}-10,\ \text{and } x=10+y-n.$$

Where it is plain, that n must be some even number which is a divisor of 50.

But the only number of this kind, that will give positive results, is 2.

$$\therefore y=\frac{50}{2}+1-10=16,\ \text{and } x=10+16-2=24.$$

Ex. 3. Given $x^2=5y^2-12y+64$, to find the values of x and y in whole positive numbers.

Here, by the 4th form, $a=5$, $b=-12$, and $c=8$.

$$\text{Whence, } y=\frac{-12-16n}{n^2-5}=\frac{16(n-\frac{3}{4})}{5-n^2},\ \text{and } x=8+ny.$$

Where it is plain, that n must be less than the $\sqrt{5}$, and greater than $\frac{3}{4}$; which numbers are only 1 and 2.

$$\therefore y=\frac{-12-16}{1-5}=7 \mid =\frac{-12-32}{4-5}=44,$$

$$\text{and } x=8+1\times7=15 \mid =8+2\times44=96.$$

Ex. 4. Given $x^2+xy=2x+3y+29$, to find the values of x and y in whole positive numbers.

Ans. $\begin{cases} x=4, 5, \\ y=21, 7. \end{cases}$

Ex. 5. It is required to find two numbers, such, that their product, added to their sum, shall be 79.

Ans. $\begin{cases} 1, 3, 4, 7, \\ 39, 19, 15, 19. \end{cases}$

Ex. 6. Given $x^2+xy=4x+3y+27$, to find the several values of x and y in whole numbers.

Ans. $\begin{cases} x=4, 5, \text{and } 6, \\ y=27, 11, \text{and } 5. \end{cases}$

Ex. 7. Given $x^2=y^2+100y+1000$, to find the two last values of x and y, in whole numbers.

Ans. $x=70$, and $y=30$.

Ex. 8. Given $x^2=50y^2+100y+100$, to find the values of x and y in whole numbers.

Ans. $x=190$, and $y=40$.

§ II. ON THE DIOPHANTINE ANALYSIS.

531. The *Diophantine Analysis* relates chiefly to the finding of square, cube, and other similar numbers, or the rendering certain compound expressions free from surds; the principal methods of effecting which are comprehended in the following problems.

PROBLEM I.

532. To render surd quantities of the form $\sqrt{(a+bx+cx^2)}$ rational; or, to find such values of x as will make $a+bx+cx^2$ a square.

Case 1. When the expression is of the form $\sqrt{(a+bx)}$, that is, when $c=0$. Put $\sqrt{(a+bx)}=n$, or $a+bx=n^2$; and we shall have $x=\frac{n^2-a}{b}$; where n may be any number, either integral or fractional, that will render the value of x positive.

Example 1. It is required to find a number, such, that if it be multiplied by 5, and then added to 19, the result shall be a square.

$$\text{Let } 5x+19=n^2, \text{ or } x=\frac{n^2-19}{5};$$

where n may be any number whatever greater than $\sqrt{19}$.

Whence, if n be taken $=5, 6, 7$, respectively, we shall have

$$x=\frac{25-19}{5}=1\tfrac{1}{5}, \text{ or } \frac{36-19}{5}=3\tfrac{2}{5}, \text{ or } \frac{49-19}{4}=6\,;$$

the latter of which is the least value of x, in whole numbers, that will answer the conditions of the question; and consequently

$$5x+19=5\times6+19=30+19=49,$$

a square number, as was required.

533. Ex. 2. Find a number such, that if it be multiplied by 5, and the product increased by 2, the result shall be a square.

$$\text{Put } 5x+2=n^2, \text{ then } x=\frac{n^2-2}{5}\,;$$

if we assume $n=2$, then $x=\frac{2}{5}$; and by assuming other values for n, different values of x may be obtained.

534. Case 2. When the expression is of the form $\sqrt{(bx+cx^2)}$; that is, when $a=0$.

Put $\sqrt{(bx+cx^2)}=nx$; $\therefore bx+cx^2=n^2x^2$, then $b+cx=n^2x$; whence $x=\dfrac{b}{n^2-c}$, and whatever value may be given to n in this expression, there will result a value of x that will make $\sqrt{(bx+cx^2)}$ rational.

Example 1. It is required to find an integral number, such, that it shall be both a triangular number and a square.

It is here first to be observed, that all triangular numbers are of the form $\dfrac{x^2+x}{2}$; and therefore the question is reduced to the making $\dfrac{x^2+x}{2}$, or it equal $\dfrac{2x^2+2x}{4}$ a square. But since a square number, when multiplied, or divided, by a square number is still a square; it is the same thing as if it were required to make $2x^2+2x$ a square.

Let therefore $2x^2+2x=\dfrac{m^2x^2}{n^2}$, then dividing by x, and multiplying the result by n^2, the equation will become $2n^2x+2n^2=m^2x$; and consequently

$$x=\frac{2n^2}{m^2-2n^2}.$$

Where, if n be taken $=2$ and $m=3$, we shall have

$$x=8, \text{ and } \frac{x^2+x}{2}=\frac{64+8}{2}=\frac{72}{2}=36,$$

for the least integral triangular number that is at the same time a square.

535. Ex. 2. Find a number such, that if its half be added to double its square, the result shall be a square.

Let x denote the number, then we must have $2x^2+\frac{1}{2}x=$ a square $=n^2x^2$, or $2x+\frac{1}{2}=n^2x$; therefore, $x=\frac{1}{2n^2-4}$, n being any number whatever: if $n=2$, then $x=\frac{1}{8-4}=\frac{1}{4}$, a square number.

536. Case 3. When a is a square number, put it equal to d^2, and make $\sqrt{(d^2+bx+cx^2)}=d+nx$; then $d^2+bx+cx^2=d^2+2dnx+n^2x^2$, or $b+cx=2dn+n^2x$; and consequently, $x=\frac{2dn-b}{c-n^2}$. Or, if $b=0$, $x=\frac{2dn}{c-n^2}$.

Example 1. It is required to divide a given square number into two such parts, that each of them shall be a square number.

Let $a^2=$ the square to be divided, $x^2=$ one of its square parts, $a^2-x^2=$ the other; which is also to be a square.

Put $a^2-x^2=(nx-a)^2=n^2x^2-2anx+a^2$, and we shall have $2anx=n^2x^2+x^2$, or $n^2x+x=2an$; and consequently $x=\frac{2an}{n^2+1}$, and $nx-a=\frac{2an}{n^2+1}-a=\frac{2an^2}{n^2+1}-\frac{an^2+a}{n^2+1}=\frac{an^2-a}{n^2+1}$.

Hence, $\left(\frac{2an}{n^2+1}\right)^2$ and $\left(\frac{an^2-a}{n^2+1}\right)^2$ are the parts required; where a and n may be any numbers whatever, provided n be greater than unity.

537. Ex. 2. Find two numbers, whose sum shall be 16, and such, that the sum of their squares shall be a square.

Let $x=$ one of the numbers, then $16-x$ denotes the other, and we have to make $x^2+(x-16)^2$, or $2x^2-32x+256$, a square.

Put $2x^2-32x+256=(nx-16)^2=n^2x^2-32nx+256$;
hence, $2x^2-32x=n^2x^2-32nx$, and $2x-32=n^2x-32n$;

$$\text{consequently } x=\frac{32(n-1)}{n^2-2}.$$

If we take $n=3$, we shall have $x=9\frac{1}{7}$; therefore the two numbers are $9\frac{1}{7}$ and $6\frac{6}{7}$.

538. Case 4. When c is a square number, put it $=e^2$, and $\sqrt{(a+bx+e^2x^2)}=n+ex$; then, $a+bx+e^2x^2=n^2+2enx+e^2x^2$, or

$$a+bx=n^2+2enx;\ \therefore x=\frac{a-n^2}{2en-b}.$$

Or, if $b=0$, $x=\frac{a-n^2}{2en}$.

EXAMPLE 1. It is required to find the least integral number such, that if 4 times its square be added to 29, the result shall be a square.

This being the same thing as to make $4x^2+29$ a square; let $4x^2+29=(2x+n)^2=4x^2+4nx+n^2$. Then, $4nx+n^2=29$, or $4nx=29-n^2$; $\therefore x=\frac{29-n^2}{4n}$; where, if n be taken equal to 1, we shall have $x=\frac{29-1}{4}=\frac{28}{4}=7$, which is the only integral number that answers the conditions of the question.

539. Ex. 2. Find a number such, that if it be increased by 2 and 5 separately, the product of the sums shall be a square.

Let $x=$ the number, then we have to make $(x+2)(x+5)$, or $x^2+7x+10$, a square, which denote by $(x-n)^2$; then, $x^2+7x+10=x^2-2nx+n^2$, or

$$7x+10=-2nx+n^2;\ \therefore x=\frac{n^2-10}{7+2n}.$$

If we take $n=4$, we shall have $x=\frac{2}{5}$.

540. CASE 5. When neither a nor c are square numbers, yet if the formula can be resolved into two simple factors, (which it always can when b^2-4c is a square, but not otherwise), the irrationality of it may be taken away, by putting $\sqrt{(a+bx+cx^2)}=\sqrt{\{(d+ex)(f+gx)\}}=n(d+ex)$; in which case we shall have

$$(d+ex)(f+gx)=n^2(d+ex)^2, \text{ or } f+gx=n^2(d+ex);$$

and consequently $x=\frac{dn^2-f}{g-en^2}$.

Or, if $d=0$, $x=\frac{f}{en^2-g}$, and if $f=0$, $x=\frac{dn^2}{g-en^2}$.

The two factors above mentioned will be found by putting $a+bx+cx^2=0$; and solving this equation, we shall have

$$x=-\frac{b}{2c}+\frac{1}{2c}\sqrt{(b^2-4ac)}, \text{ and } x=-\frac{b}{2c}-\frac{1}{2c}\sqrt{(b^2-4ac)};$$

or putting $\sqrt{(b^2-4ac)}=\delta^2$, the values of x are

$$x=-\frac{b}{2c}+\frac{\delta}{2c}, \text{ and } x=-\frac{b}{2c}-\frac{\delta}{2c};$$

and, consequently, $(cx+\frac{\delta-b}{2})$, and $(x+\frac{b+\delta}{2})$, are the factors required.

Example. It is required to find such a value of x, that $\sqrt{(6+13x+6x^2)}$ shall be rational, and consequently $6+13x+6x^2$ a square.

Let $6x^2+13x+6=0$;
and solving this equation, we shall have $x=-\frac{3}{2}$, and $x=-\frac{2}{3}$: therefore the two factors are $2x+3$, and $3x+2$.

Put $(2x+3)(3x+2)=\frac{m^2}{n^2}(3x+2)^2$, or $2x+3=\frac{m^2}{n^2}(3x+2)$,

and consequently, by reduction, $x=\frac{3n^2-2m^2}{3m^2-2n^2}$.

Where it appears, that, in order to obtain a rational answer, $\frac{m^2}{n^2}$ must be less than $\frac{3}{2}$, and greater than $\frac{2}{3}$.

Whence, if $m=6$, and $n=5$, we shall have

$x=\frac{3\times25-2\times36}{3\times36-2\times25}=\frac{75-72}{108-50}=\frac{3}{58}$, the value required.

Case 6. When neither of the foregoing will apply, if the formula can be resolved into two parts, one of which is a square, and the other the product of any two simple factors.

Put $\sqrt{(a+bx+cx^2)}=\sqrt{\{(d+ex)^2+(f+gx)(h+kx)\}}=(d+ex)+n(f+gx)$; in which case we shall have $(d+ex)^2+(f+gx)(h+kx)=(d+ex)^2+2n(d+ex)(f+gx)+n^2(f+gx)^2$, or $h+kx=2n(d+ex)+n^2(f+gx)$;

$$\text{and consequently, } x=\frac{n(2d+fn)-h}{k-n(2e+gn)}.$$

$$\text{Or, if } d=0,\ x=\frac{fn^2-h}{k-n(2e+gn)}.$$

Or, if the part in this case, which is found to be a square, be a known quantity, put $\sqrt{(a+bx+cx^2)}=\sqrt{\{(d^2+(e+fx)(g+hx)=d+n(e+fx)\}}$; then we shall have

$$d^2+(e+fx)(g+hx)=d^2+2dn(e+fx)+(e+fx)^2$$
$$\text{or } g+hx=2dn+(e+fx),$$

and consequently, by transposing and uniting the different terms, $x=\frac{e+2dn-g}{h-f}$.

Example 1. It is required to find such a value of x, that $\sqrt{(13x^2+15x+7)}$ shall be rational, or $13x^2+15x+7$ a square.

Let this formula be separated into the two parts $(1-x)^2$ and $6+17x+12x^2$.

Then, since $17^2-4(6\times 12)$, which is equal to 1, is a square, the latter part may be divided in the factors $3x+2$, and $4x+3$; and consequently the original formula may be represented by

$$(1-x)^2+(2+3x)\times(3+4x).$$

Hence, putting $\sqrt{(13x^2+15x+7)}=$

$$\sqrt{\{(1-x)^2+(2+3x)\times(3+4x)\}}=(1-x)+n(2+3x),$$

we shall have $(1-x)^2+(2+3x)\times(3+4)=(1-x)^2+2n(1-x)\times(2+3)+n^2(2+3x)^2$, or

$$3+4x=2n(1-x)+n^2(2+3x);$$

and consequently, by reduction, $x=\frac{2n+2n^2-3}{4+2n-3n^2}$.

Where, taking $n=1$, we have $x=\frac{2+2-3}{4+2-3}=\frac{1}{3}$;

and $13x^2+15x+7=\frac{13}{9}+\frac{15}{3}+7=\frac{13}{9}+\frac{45}{9}+\frac{63}{9}=\frac{121}{9}$,

a square number, as required.

Ex. 2. Find a value of x, such, that $2x^2+8x+7$ shall be a square.

This expression, after a few trials, is found to be equivalent to $(x+2)^2+(x+1)\times(x+3)$, which being equated with $\{(x+2)-n(x+1)\}^2=(x^2+2)^2-2n(x+2)\times(x+1)+n^2(x+1)^2$, there results

$$x+3=-2n(x+2)+n^2(x+1);$$

$$\text{whence, } x=\frac{n^2-4n-3}{1+2n-n^2}.$$

If we take $n=3$, we shall have $x=3$, and $2x^2+8x+7=49$, a square, as was required.

PROBLEM II.

541. To render surd quantities of the form $\sqrt{(a+bx+cx^2+dx^3)}$ rational, or to find such values of x as will make $a+bx+cx^2+dx^3$ a square.

This problem is much more limited, and difficult to be resolved, than the former, there being but a few cases of it that admit of answers in rational numbers. The rules for obtaining them are of such a confined nature, that when the unknown quantity has more than one value—which, however, is not often the case—the rest can onlv be determined one at a time, by repeating the operation with the value last obtained, as often as may be found necessary.

RULE.

542. CASE 1. When $a=o$, and $b=o$, put the remaining part $\sqrt{(cx^2+dx^3)}=nx$, or $cx^2+dx^3=n^2x^2$; then we shall have

$$c+dx=n^2;\ \therefore x=\frac{n^2-c}{d}.$$

Where n may be any number whatever greater than the square root of c.

EXAMPLE 1. It is required to find such a value of x that $\sqrt{(3x^2+11x^3)}$ shall be rational, and consequently $3x^2+11x^3$ a square.

Let $\sqrt{(3x^2+11x^3)}=nx$, or $3x^2+11x^3=n^2x^2$.

Then, by dividing, we shall have $3+11x=n^2$.

And consequently $x=\frac{n^2-3}{11}$; where n may be any number, positive or negative, that is greater than $\sqrt{3}$.

Taking therefore, $n=2, 3, 4, 5$, &c. respectively, we shall have $x=\frac{1}{11}, \frac{6}{11}, \frac{13}{11}, \frac{22}{11}$, or 2, the last of which is the least integral answer which the question admits of.

Ex. 2. Find a number such, that if three times its cube be added to twice its square, the sum shall be a square.

Here we must make $3x^3+2x^2$ a square;

let n^2x^2 be the square, then $3x+2=n^2$;

$$\therefore x=\frac{n^2-3}{2}.$$

If we take $n=3$, we have $x=3$, the number required.

543. CASE 2. When a is a square number, put it equal to e^2, and make $\sqrt{(e^2+bx+cx^2+dx^3)}=e+\frac{b}{2e}x$, or $e^2+bx+cx^2+dx^3=(e+\frac{b}{2e})x=e^2+bx+\frac{b^2}{4e^2}x^2$.

Hence, $cx^2+dx^3=\frac{b^2}{4e^2}x$, and by division and reduction $x=\frac{b^2-4ce^2}{4de^2}$; or, when $c=0$, $x=\frac{b^2}{4de^2}$.

Note. The assumed root $e+\frac{b}{2e}x$ is determined by first taking it in the form $e+nx$, and then equating the second term of it, when squared with the second term of the original formula; in which case n will be found $=\frac{b}{2e}$.

EXAMPLE 1. It is required to find such a value of x, that $1+2x-x^2+x^3$ shall be a square.

Here, 1 being a square, let $1+2x-x^2+x^3=(1+x)^2=1+2x+x^2$; then, we shall have $x^3-x^2=x^2$, or $x^3=2x^2$; and consequently $x=2$, and $1+2x-x^2+x^3=1+4-4+8=9$, a square as required.

Ex. 2. Find such a value of x as will make the expression $3x^3-5x^2+6x+4$ a square.

Put $3x^3-5x^2+6x+4=(\frac{3}{2}x+2)^2=\frac{9}{4}x^2+6x+4$, then, $3x^3-5x^2=\frac{9}{4}x^2$, or $3x-5=\frac{9}{4}$; $\therefore x=\frac{29}{12}$, which being substituted in the proposed expression, makes it equal to $\left(\frac{45}{8}\right)^2$.

PROBLEM III.

544. To render surd quantities of the form $\sqrt{(a+bx+cx^2+dx^3+ex^4)}$, rational, or to find such values of x as will make $a+bx+cx^2+dx^3+ex^4$ a square.

RULE.

CASE 1. When a is a square number, put it $=f^2$, and make

$$f^2+bx+cx^2+dx^3+ex^4=\left(f+\frac{b}{2}gx+\frac{4cf^2-b^2}{8f^3}x^2\right)^2=f^2+bx+cx^2+\frac{b(4cf^2-b^2)}{8f^4}x3+\frac{(4cf^2-b^2)^2}{64f^6}x^4;$$

then since the first three terms on each side of the equation destroy each other, we shall have $\frac{(4cf^2-b^2)^2}{64f^6}x^4+\frac{b(4cf^2-b^2)}{8f^4}x^3=ex^4+dx^3$; and therefore, by division and reduction,

$$x=\frac{64df^6-8bf^2(4cf^2-b^2)}{(4cf^2-b^2)^2-64ef^6};$$

which form fails when any two of the coefficients b, c, d, are each $=0$.

EXAMPLE 1. It is required to find such a value of x, that $1-2x+3x^2-4x^3+5x^4$ shall be a square.

Here, the first term being a square number, let $1-2x+3x^2$ $4x^3+5x^4=(1-x-x^2)^2=1-2x+3x^2-2x^3+x^4$.

Then, since the first three terms on each side of the equation destroy each other, we shall have $5x^4-4x^3=x^4-2x^3$;

$\therefore x=\frac{1}{2}$, and consequently $1-2x+3x^2-4x^3+5x^4=1-1+\frac{3}{4}-\frac{1}{2}+\frac{5}{16}=\frac{9}{16}$; which is a square number, as was required.

Ex. 2. Find such a value of x that we may have $22x^4-40x^3-40x^2+64x+16$ a square. Ans. $\frac{8}{7}$.

545. Case 2. When e is a square number, put it $=g^2$, and make $g^2x^4+dx^3+cx^2+bx+a=\left(gx^2+\frac{d}{2g}x+\frac{4cg^2-d^2}{8g^3}\right)^2=g^2x^4+dx^3+cx^2+\frac{d(4cg^2-d^2)}{8g^4}x+\frac{(4cg^2-d^2)^2}{64g^6}$; then, we shall have

$$a+bx=\frac{(4cg^2-d^2)^2}{64g^6}+\frac{d(4cg^2-d^2)}{8g^4}x,$$

$$\therefore x=\frac{(4cg^2-d^2)^2-64ag^6}{64bg^6-8dg^2(4cg^2-d^2)}:$$

which form also fails in the same case as the former.

Example 1. It is required to find such a value of x, that $-2+3x-x^2-2x^3+4x^4$ shall be a square number.

Let $4x^4-2x^3-x^2+3x-2=(2x^2-\frac{1}{2}x-\frac{5}{16})^2=4x^4-2x^3-x^2+\frac{5}{16}x+\frac{25}{256}$; then we shall have $3x-2=\frac{5}{16}x+\frac{25}{256}$; $\therefore x=\frac{537}{688}$.

Ex. 2. It is required to find such a value of x, that $4x^4+4x^3+4x^2+2x-6$ shall be a square.

Put $4x^4+4x^3+4x^2+2x-6=(2x^2+x+\frac{3}{4})^2=$

$$4x^4+4x^3+4x^2+\tfrac{3}{2}x+\tfrac{9}{16},\text{ and we have}$$

$$2x-6=\tfrac{3}{2}x+\tfrac{9}{16};\ \therefore x=13\tfrac{1}{8}.$$

546. Case 3. When the first and last terms are both squares, put $a=f^2$ and $e=g^2$, and make $f^2+bx+cx^2+dx^3+g^2x^4=(f+\frac{b}{2f}x+gx^2)^2=f^2+bx+(2fg+\frac{b^2}{4f^2})^2x^2+\frac{bg}{f}x^3+g^2x^4$; then, since the second terms, as well as the first and last, on each side of the equation, destroy each other, we shall have

$$cx^2+dx^3=(2fg+\frac{b^2}{4f^2})x^2+\frac{bg}{f}x^3;$$

$$\therefore x=\frac{f^2(c-2fg)-\frac{1}{4}b^2}{f(bg-fd)}.$$

And because g is found in the original formula only in its second power, it may be taken either positively or negatively; and consequently we shall also have

$$x=\frac{\frac{1}{4}b^2-f^2(2fg+c)}{f(bg+fd)};$$

so that this mode of solution furnishes two different answers.

EXAMPLE 1. It is required to find such a value of x, as shall make $1+3x+7x^2-2x^3+4x^4$ a square.

Let $1+3x+7x^2-2x^3+4x^4=(1+\frac{3}{2}x+2x^2)^2=$

$$1+3x+\frac{25}{4}x^2+6x^3+4x^4;$$

$$\therefore 6x^3+\frac{25}{4}x^2=7x^2-2x^3, \text{ and } x=\frac{3}{32}.$$

Ex. 2. It is required to find such a value of x, as shall make $16-24x+4x^2-6x^3+x^4$ a square.

Let $x^4-6x^3+4x^2-24x+16=(x^2-3x-4)^2=$

$$x^4-6x^3+x^2+24x+16,$$

and there results

$$4x^2-24x=x^2+24x, \text{ or } 4x-24=x+24;$$

$$\therefore x=16.$$

PROBLEM IV.

547. To render surd quantities of the form $\sqrt[3]{(a+bx+cx^2+dx^3)}$ rational, or to find such values of x as will make $a+bx+cx^2+dx^3$ a cube.

CASE 1. When a is a cube number, put it $=e^3$, and take $e^3+bx+cx^2+dx^3=(e+\frac{b}{3e^2}x)^3=e^3+bx+\frac{b^2}{3e^3}x^2+\frac{b^3}{27e^6}x^3$; then we shall have $dx^3+cx^2=\frac{b^3}{27e^6}x^3+\frac{b^2}{3e^3}x^2$; or, by dividing by x^2, and reducing the terms,

$$27de^6x+27ce^6=b^3x+9b^2e^3: \text{ whence } x=\frac{9e^3(3ce^3-b^2)}{b^3-27de^6}.$$

EXAMPLE 1. It is required to find such a value of x, as will make the formula $1+x+x^2$ a cube.

Let $1+x+x^2=(1+\frac{1}{3}x)^3=1+x+\frac{1}{3}x^2+\frac{1}{27}x^3$, or

$x^2=\frac{1}{3}x^2+\frac{1}{27}x^3$; $\therefore x=18$, and consequently, $1+x+x^2=1+18+324=343=7^3$, a cube number, as was required.

Ex. 2. It is required to find such a value of x that will make the formula $2x^3+3x^2-4x+8$ a cube.

Let $2x^3+3x^2-4x+8=(-\frac{1}{3}x+2)^3=-\frac{1}{27}x^3+\frac{2}{3}x^2-4x+8$, and we have $2x^3+3x^2=-\frac{1}{27}x^3+\frac{2}{3}x^2$,

$$\text{or } 2x+3=-\frac{1}{27}x+\frac{2}{3};$$

$$\therefore x=-\tfrac{63}{55}.$$

Ex. 3. It is required to find such a value of x, as to make the formula $3x^3+2x+1$ a cube. Ans. $\frac{36}{73}$.

548. CASE 2. When d is a cube number, put it $=f^3$, and

take $a+bx+cx^2+f^3x^3=(\frac{c}{3f^2}+fx)^3=\frac{c^3}{27f^6}+\frac{c^2}{3f^3}x+cx^2+f^3x^3$; then we shall have $a+bx=\frac{c^3}{27f^6}+\frac{c^2}{3f^3}x$; $\therefore x=\frac{27af^6-c^3}{9f^3(c^2-3bf^3)}$.

Example 1. It is required to find such a value of x as will make $133+3x^2+x^3$ a cube.

Let $133+3x^2+x^3=(1+x)^3=1+3x+3x^2+x^3$; and since the two last terms of this equation destroy each other, there will remain $1+3x=133$, or $3x=133-1=132$; whence $x=\frac{132}{3}=44$, and consequently $133+3x^2+x^3=92025=(45)^3$, a cube number, as was required.

Ex. 2. It is required to find such a value of x as will make the formula $8x^3-4x^2+2x-12$ a cube.

Let $8x^3-4x^2+2x-12=(2x-\frac{1}{3})^3=8x^3-4x^2+\frac{2}{3}x+\frac{1}{27}$, and we have $2x-12=\frac{2}{3}x+\frac{1}{27}$;

$$\therefore x=\frac{325}{36}.$$

Ex. 3. It is required to find such a value of x as will make the formula x^3-3x^2+x a cube. Ans. $x=\frac{1}{3}$.

549. Case 3. When a and d are both cube numbers, let them be put $=e^3$ and f^3, and make $e^3+bx+cx^2+f^3x^3=(e+fx)^3=e^3+3fe^2x+3ef^2x^2+f^3x^3$; then, we shall have $bx+cx^2=3fe^2x+3ef^2x^2$; $\therefore x=\frac{3fe^2-b}{c-3ef^2}$; which formula may be also resolved by either of the two first cases.

Example 1. It is required to find such a value of x, that $8+28x+89x^2-125x^3$ shall be a cube.

Let $8+28x+89x^2-125x^3=(2-5x)^3=8-60x+150x^2-125x^3$; and, since the first and last terms of this equation destroy each other, there will remain $28x+89x^2=-60x+150x^2$; $\therefore 150x-89x=28+60$, or, $61x=88$, and $x=\frac{88}{61}$, the value required. And as this formula can, also, be resolved by the first or second case, other values of x may be obtained, that will equally answer the conditions of the question.

Ex. 2. It is required to find such a value of x, that the formula $8+4x+9x^2+x^3$ shall be a cube.

Let $8+4x+9x^2+x^3=(2+x)^3=8+12x+6x^2+x^3$, and we shall then have $9x^2+4x=6x^2+12x$; $\therefore x=2\frac{2}{3}$.

PROBLEM V.

On the Resolution of Double and Triple Equalities.

550. When a single formula, containing one or more unknown quantities, is to be transformed to a perfect power,

such as a square or a cube, this is called in the Diophantine Analysis, a simple equality; and when two formulæ, containing the same unknown quantity, or quantities, are each to be transformed to some perfect power, it is then called a *double equality*, and so on; the methods of resolving which, in such cases as admit of any rule, are as follows.

PROB. 1. When the unknown quantity does not exceed the first degree, as in the double equality

$$a+bx= \text{ a square, and } c+dx= \text{ a square.}$$

Let the first of these formula $a+bx=t^2$, and the second $c+dx=u^2$; then, by eliminating x from each of these equations, we shall have $bu^2+ad-bc=dt^2$, or $bdu^2+(ad-bc)d=d^2t^2$; and since the quantity on the right hand side of this last equation is now a square, it is only necessary to find such a rational value of u, as will make $bdu^2+(ad-bc)d$ a square, which being done according to one of the methods already explained, we shall have $x=\frac{u^2-c}{d}$.

EXAMPLE. It is required to find a number x, such, that $x+128$ and $x+192$ shall be both squares.

Here, let $x+128=u^2$, and $x+192=t^2$; then, by eliminating x, we shall have $u^2-128=t^2-192$; or $u^2+64=t^2$; and, as the quantity on the right-hand side of the equation is now a square, it only remains to make u^2+64 a square; for which purpose, put $u^2+64=(u+n)^2=u^2+2nu+n^2$, $2nu+n^2=64$; whence, $u=\frac{64-n^2}{2n}$; or, taking n, which is arbitrary, $=2$, we shall have $u=\frac{64-4}{4}=15$; and consequently, $x=u^2-128=225-128=97$, the number required.

551. PROB. 2. When the unknown quantity does not exceed the second degree, and is found in all the terms of the two formulæ, as in the double equality, $ax^2+bx=$ a square, and $cx^2+dx=$ a square.

Let $x=\frac{1}{y}$; then, by multiplying each of the two resulting equations by y^2, we shall have

$$a+by= \text{ a square, and } c+dy= \text{ a square;}$$

from which the value of y, and consequently that of x, may be determined, as in Problem I.

But if it were required to transform the two general expressions $a+bx+cx^2$ and $d+ex+fx^2$ into squares; the solution could only be obtained in a few particular cases, as the resulting equality would rise to the fourth power.

Example. It is required to find a number x, such, that x^2+x and x^2-x shall be both squares.

Here, let $x=\frac{1}{y}$; then the two formulæ in the question will become $\frac{1}{y^2}+\frac{1}{y}$ and $\frac{1}{y^2}-\frac{1}{y}$, or $\frac{1}{y^2}(1+y)$ and $\frac{1}{y^2}(1-y)$, which are to be squares. But since a square number, when multiplied or divided by a square number, is still a square, it is the same thing as to transform $1+y$ and $1-y$ to squares; for which purpose, let $1+y=p^2$, or $y=p^2-1$; $1-y=2-p^2$, which is also to be a square. But as neither the first nor last terms of this new formula are squares, we must, in order to succeed, find some simple number, that will answer the condition required; which, it is evident, from inspection, will be the case when $p=1$.

Let, therefore, $p=1-q$, and we shall have $1-y=2-p^2=1+2q-q^2$; or, putting $1+2q-q^2=(1-rq)^2=1-2rq+r^2q^2$; whence $2-q=-2r+r^2q$, or $q=\frac{2r+2}{r^2+1}$; and consequently,

$$x=\frac{1}{y}=\frac{1}{q^2-2q}=\frac{(1+r)^2}{4r-4r^3};$$

where, in order to make x positive, r may be taken equal to any proper fraction whatever.

Let, therefore, for the sake of greater simplicity, $r=\frac{t}{u}$, and we shall have $x=\frac{(u^2+t^2)^2}{4tu(u^2-t^2)}$; in which case, any whole numbers may be now substituted for u and t, provided u be greater than t.

If, for instance, $u=2$ and $t=1$, we shall have $x=\frac{25}{24}$; and if $u=3$ and $t=2$, $x=\frac{169}{120}$; and so on, for any other numbers.

552. Prob. 3. In the case of a triple equality, where three expressions of the former, $ax+by$, $cx+dy$, and $ex+fy$, are to be transformed to squares.

Let the first of them $ax+by=t^2$, the second $cx+dy=u^2$, d the third $ex+fy=s^2$; then, if x be eliminated from each of these equations, and afterwards y in the two resulting equations, we shall have $(af-be)u^2-(cf-de)t^2=(ad-cb)s^2$; and by putting $\frac{u}{t}=z$, or $u=tz$, there will arise

$$\frac{af-be}{ad-cb}\cdot z^2-\frac{cf-de}{ad-cb}=\frac{s^2}{t^2};$$

and since the quantity on the right-hand side of this equation, is a square, it only remains to find such a rational value of z as will make

$$\frac{af-be}{ad-cb}z^2-\frac{cf-de}{ad-cb} \text{ a square };$$

which being done by one of the methods before explained, we shall readily obtain, by means of the first two equations,

$$x=\frac{d-bz^2}{ad-cb}t^2, \text{ and } y=\frac{az^2-c}{ad-cb}t^2,$$

where t may be any number whatever.

EXAMPLE. It is required to find three numbers in arithmetical progression, such, that the sum of every two of them may be a square.

Let x, $x+y$, $x+2y=$ the three numbers; and put $2x+y=t^2$, $2x+2y=u^2$, and $2x+3y=s^2$; then, by eliminating x and y from these equations, we shall have $u^2-t^2=s^2-u^2$, or $2u^2-t^2=s^2$; and if we now put $u=tz$, then will arise $2t^2z^2-t^2=s^2$, or $2z^2-1=\frac{s^2}{t^2}$; where, $\frac{s^2}{t^2}$ being a square, it only remains to make $2z-1$ a square; what it evidently is when $z=1$. But as value would be found not to answer the conditions of the question, let $z=1-p$; $2z^2-1=2(1-p)^2-1=1-4p+2p^2$; and by putting this last expression $=(1-rp)^2$, we have $1-4p+2p^2=1-2rp+r^2p^2$, or $-4+2p=-2r+r^2p$; whence $p=\frac{2r-4}{r^2-2}$, and $z=1-\frac{2r-4}{r^2-2}=\frac{r^2-2r+2}{r^2-2}$; or, making $r=\frac{m}{n}$, $z=\frac{m^2-2mn+2n^2}{m^2-2n^2}$.

And since, by the first two equations, $y=u^2-t^2=t^2z^2-t^2=(z^2-1)t^2$, and $x=\frac{1}{2}(t^2-y^2)=\frac{1}{2}(2-z^2)t^2$; it is evident, that z must be some number greater than 1 and less than $\sqrt{2}$.

If, therefore, $m=9$ and $n=5$, we shall have $z=\frac{81-90+50}{81-50}=\frac{41}{31}$, $x=\frac{241}{312}\times\frac{t^2}{2}$, and $y=\frac{720}{312}\times t^2$; or, taking $t=2\times 31$, $x=482$, and $y=2880$. Hence,

$$x=482,\ x+y=3362, \text{ and } x+2y=6242,$$
the numbers required.

EXAMPLES FOR PRACTICE.

Ex. 1. It is required to find a number such, that $x+1$ and $x-1$ shall be both squares. Ans. $x=\frac{5}{4}$.

Ex. 2. It is required to find a number x, such, that $x+4$ and $x+7$ shall be both squares. Ans. $x=\frac{57}{16}$.

Ex. 3. It is required to find two numbers such, that if their product be added to the sum of their squares, the result shall be a square. Ans. 3 and 5.

Ex. 4. Find two numbers such, that if the square of each be added to their product, the sums shall be both squares. Ans. 9 and 16.

Ex. 5. It is required to find two whole numbers such, that the sum or difference of their squares when diminished by unity, shall be a square. Ans. 8 and 9.

Ex. 6. To find two whole numbers such, that if unity be added to each of them, and also to their halves, the sums in both cases shall be squares. Ans. 48 and 1680.

Ex. 7. It is required to find three square numbers, that shall be in arithmetical progression. Ans. 1, 25, and 49.

Ex. 8. It is required to find three square numbers that shall be in harmonical proportion. Ans. 1225, 49 and 25.

Ex. 9. To find three whole numbers such, that if to the square of each the product of the other two be added, the sums shall be squares. Ans. 9, 73 and 328.

Ex. 10. It is required to resolve 4225, which is the square of 65, into two other integral squares. Ans. 2704 and 1521.

Ex. 11. It is required to resolve 9^2+2^2, or 85 into two other integral squares. Ans. 7^2+6^2.

Ex. 12. It is required to find three square numbers, such, that their sum shall be a square. Ans. 9, 16 and $\frac{144}{25}$.

Ex. 13. To find two numbers such, that their sum shall be equal to their cubes. Ans. $\frac{5}{7}$ and $\frac{8}{7}$.

APPENDIX.

Algebraic Method of demonstrating the Propositions in the fifth book of Euclid's Elements, according to the text and arrangement in Simson's edition.

Simson's Euclid is undoubtedly a work of great merit, and is in very general use among mathematicians; but notwithstanding all the efforts of that able commentator, the fifth book still presents great difficulties to learners, and is in general less understood than any other part of the Elements of Geometry. The present essay is intended to remove these difficulties, and consequently to enable learners to understand in a sufficient degree the doctrine of proportion, previously to their entering on the sixth book of Euclid, in which that doctrine is indispensable.

I have omitted the demonstrations of several propositions, which are used by Euclid merely as lemmata, but are of no consequence in the present method of demonstration.

Instead of Euclid's definition of proportion, as given in his fifth definition of the fifth book, I make use of the common algebraic definition; but I have shown the perfect equivalence of these two definitions. This perfect reciprocity between the two definitions is a matter of great importance in the doctrine of proportion, and has not (as far as I can learn) been discussed by any preceding mathematician.

With respect to compound ratio, I have also given another definition of it instead of that given by Dr. Simson; as his definition is found exceedingly obscure by beginners, and is in my judgment one of the most objectionable things in his edition of Euclid's Elements.

The literal operations made use of in the present paper are extremely simple, and require very little previous knowledge of algebra to render them intelligible.

The algebraic signs commonly used to indicate *greater*, *equal*, *less*, *are* $\mathrel{7}$, $=$, $\angle$: thus the three expressions $a \mathrel{7} b$ $c=d$, $e \angle f$, signify that a is greater than b, that c is equal to d, and that e is less than f. The expression $c=d$ is called an equation or equality; the others $a \mathrel{7} b$, $e \angle f$, are called inequalities.

Also when four quantities are proportionals, we shall express this relation in the usual mode by points; thus,

$$A : B :: C : D$$

is to be read, A is to B as C is to D; or, A has the same ratio to B that C has to D.

THE ELEMENTS OF EUCLID, BOOK V.

Definitions.

I.

A less magnitude is said to be a *part* of a greater, when the less measures the greater, that is, when the less is contained a certain number of times exactly in the greater.

II.

A greater magnitude is said to be a *multiple* of a less, when the greater is measured by the less, that is, when the greater, contains the less a certain number of times exactly.

III.

Ratio is a mutual relation of two magnitudes of the same kind to one another in respect to quantity.

IV.

Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other.

V.

The ratio of the magnitude A to the magnitude B is the number showing how often A contains B; or, which is the same thing, it is the quotient when A is numerically divided by B, whether this quotient be integral, fractional, or surd.

Explication.

This fifth definition, with its corollaries, is used in the present essay instead of Euclid's 5th and 7th definitions: the following examples will sufficiently illustrate the definition. Let $A=20$, and $B=5$, then the ratio of A to B, or of 20 to 5, is $\frac{A}{B}$ or $\frac{20}{5}$, or 4, so that the ratio of 20 to 5 is 4. Again, let $A=5$, and $B=20$, then $\frac{A}{B}=\frac{5}{20}=\frac{1}{4}$, and therefore the ratio of 5 to 20 is $\frac{1}{4}$ Lastly, let $A=12\sqrt{2}$, and $B=4$, then $\frac{A}{B}=\frac{12\sqrt{2}}{4}=3\sqrt{2}$, and therefore the ratio of $12\sqrt{2}$ to 4 is $3\sqrt{2}$.

Corollary I. If four magnitudes, A, B, C, D, be so related that $\frac{A}{B}=\frac{C}{D}$, it is evident the ratio of A to B is the same with the ratio of C to D.

Cor. II. Any four magnitudes whatever, so related that the ratio of the first to the second is the same with the ratio of the third to the fourth, may be expressed by

$$rA,\ A,\ rB,\ B\ ;$$

the first of the four being rA, the second A, the third rB, and the fourth B ; the magnitudes A and B being any whatever, and the letter r denoting each of the two equal ratios or quotients when the first rA is divided by the second A, and the third rB divided by the fourth B.

Cor. III. When four magnitudes A, B, C, D, are so related that $\frac{A}{B}$ is greater than $\frac{C}{D}$, it is evident that the ratio of A to B is greater than the ratio of C to D ; or that the ratio of C to D is less than the ratio of A to B.

The Fifth Definition according to Euclid.

The first of four magnitudes is said to have the same ratio to the second which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth, if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth ; or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth ; or if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

Scholium. We shall demonstrate towards the close of this essay, that this definition of Euclid's and our 5th definition, acccording to the common algebraic method, are not only consistent with each other, but also perfectly equivalent, each comprehending, whatsoever is comprehended by the other.

VI.

When four magnitudes are proportionals, it is usually expressed by saying, the first is to the second as the third to the fourth.

The Seventh Definition according to Euclid.

When of the equimultiples of four magnitudes, (taken as in the fifth definition) the multiple of the first is greater than that of the second, but the multiple of the third is not greater than that of the fourth ; then the first is said to have to the second a greater ratio than the third has to the fourth ; and, on the contrary, the third is said to have to the fourth a less ratio than the first has to the second.

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VIII.

Analogy or proportion is the equality of ratios.

IX.

Omitted.

X.

When three magnitudes are proportionals, the first is said to have to the third the duplicate ratio of that which it has to the second.

XI.

When four magnitudes are continued proportional, the first is said to have to the fourth the triplicate ratio of that which it has to the second, and so on, quadruplicate, &c. increasing the denomination still by unity in any number of proportionals.

Definition A, *viz.* of compound ratio, omitted.

XII.

In proportionals, the antecedent terms are called homologous to one another, as also the consequents to one another.

XIII.

Permutando, or Alternando, by permutation, or by alternation, or alternately, are terms used, when of four proportionals it is inferred that the first is to the third as the second to the fourth.

XIV.

Invertendo by inversion, or inversely, when of four proportionals, it is inferred that the second is to the first as the fourth to the third.

XV.

Componendo, by composition, when it is inferred that the sum of the first and second is to the second as the sum of the third and fourth is to the fourth.

XVI.

Dividendo, by division, when it is inferred that the excess of the first above the second is to the second as the excess of the third above the fourth is to the fourth.

XVII.

Convertendo, by conversion, or conversely, when it is inferred that the first is to its excess above the second, as the third to its excess above the fourth.

XVIII.

Ex æquali (sc. distantia), or ex æquo, from equality of distance, when there is any number of magnitudes more than two, and as many others, so that they are proportionals when taken two and two of each rank, and it is inferred that the first is to the last of the first rank of magnitudes as the first is to the last of the others: of this there are the two following kinds, which arise from the different order in which the magnitudes are taken two and two.

XIX.

Ex æquali, from equality; this term is used simply by itself, when the first magnitude is to the second of the first rank, as the first to the second of the other rank, and the second to the third of the first rank as the second to the third of the other; and so on in order; and it is inferred that the first is to the last of the first rank as the first is to the last of the other rank.

XX.

Ex æquali, in proportione perturbata, seu inordinata, from equality in perturbate proportion: this term is used when the first is to the second of the first rank as the last but one to the last of the other rank, and the second is to the third of the first rank as the last but two to the last but one of the other rank, and so on in a cross order; and it is inferred that the first is to the last of the first rank as the first is to the last of the other rank.

XXI.

If A, B, C, D, be any number of magnitudes of the same kind, and P any other magnitude; and if we make A : B :: P : Q; and B : C :: Q : R; and C : D :: R : S; the ratio of P to S is said to be compounded of the ratios of A to B, B to C, C to D.

AXIOMS.

I. Equimultiples of the same, or of equal magnitudes, are equal.

II. These magnitudes of which the same, or equal magnitudes, are equimultiples, are equal to one another.

III. A multiple of a greater magnitude is greater than the same multiple of a less.

IV. That magnitude of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

PROPOSITIONS.

Propositions I. II. III. V. and VI. are omitted, as they do not treat of proportion, and are not wanted in the method of demonstration adopted in this essay.

PROP. IV. THEOR.

If the first of four magnitudes has the same ratio to the second which the third has to the fourth; then any equimultiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth; that is, the equimultiple of the first shall be to that of the second as the equimultiple of the third is to that of the fourth.

DEMONSTRATION.

By Cor. 2. Def. 5. let any four proportionals be represented by

$$rA, A, rB, B;$$

and m and n being any two integers greater than unity, the equimultiples of rA and rB will be

$$mrA, mrB;$$

and in like manner the equimultiples of A, B, will be nA, nB.

We are to prove that the four following quantities, mrA, nA, mrB, nB, are proportionals.

By Def. 5. the ratio of mrA to nA is $\frac{mrA}{nA}=\frac{mr}{n}$,

and the ratio of mrB to nB is $\frac{mrB}{nB}=\frac{mr}{n}$:

now these two ratios being each $=\frac{mr}{n}$,

are manifestly equal to each other, and therefore by Cor. 1. Def. 5.

$$mrA : nA :: mrB : nB. \qquad \text{Q. E. D.}$$

Cor. Likewise if the first be to the second as the third to the fourth, then also any equimultiples of the first and third shall have the same ratio to the second and fourth; and, in like manner, the first and third shall have the same ratio to any equimultiples of the second and fourth.

DEMONSTRATION.

We have first to prove that the four following,

mrA, A, mrB, B are proportionals.

The ratio of mrA to A is $\frac{mrA}{B}=mr$,

and the ratio of mrB to B is $\frac{mrB}{B}=mr$;

Therefore $mrA : A :: mrB : B$.

In like manner we prove that $rA : nA :: rB : nB$.

PROP. A. THEOR.

If the first of four magnitudes has the same ratio to the second which the third has to the fourth; then if the first be greater than the second, the third is also greater than the fourth; if equal, equal; and if less, less.

DEMONSTRATION.

By Cor. 1. Def. 5. any four proportionals may be expressed by

$$rA, A, rB, B.$$

If we have $rA > A$,	if $rA = A$,	if $rA < A$,
then by division $r > 1$,	then $r = 1$,	then $r < 1$,
and by multip. $rB > B$,	and $rB = B$,	and $rB < B$.

Q. E. D.

PROP. B. THEOR.

If four magnitudes are proportionals, they are proportionals also when taken inversely.

DEMONSTRATION.

Let rA, A, rB, B be any four proportionals, we are to prove that A, rA, B, rB will also be proportionals.

The ratio of A to rA is $\frac{A}{rA} = \frac{1}{r}$,

and the ratio of B to rB is $\frac{B}{rB} = \frac{1}{r}$;

and therefore

$$A : rA :: B : rB. \qquad \text{Q. E. D.}$$

PROP. C. THEOR.

If the first be the same multiple of the second, or the same part of it that the third is of the fourth; the first is to the second as the third is to the fourth.

DEMONSTRATION.

1. Supposing m to be any integer greater than unity, let mA the first be the same multiple of the second A, that mB the third is of the fourth B; we are to prove that mA, A, mB, B are proportionals.

The ratio of mA to A is $\frac{mA}{A} = m$,

and the ratio of mB to B is $\frac{mB}{B} = m$,

therefore $mA : A :: mB : B$.

2. The letter m still denoting an integer greater than unity, let A the first be the same part of mA the second, that B the third is of mB the fourth; then we are to show that

A, mA, B, mB are proportionals.

The ratio of A to mA is $\frac{A}{mA} = \frac{1}{m}$,

and the ratio of B to mB is $\frac{B}{mB} = \frac{1}{m}$;

therefore

$$A : mA :: B : mB. \qquad \text{Q. E. D.}$$

PROP. D. THEOR.

If the first be to the second as the third to the fourth, and if the first be a multiple, or part of the second; the third is the same multiple, or the same part of the fourth.

DEMONSTRATION.

Any four proportionals being expressed by

$$rA, A, rB, B;$$

1. Let the first rA be a multiple of A, then it is to be proved that rB is the same multiple of B.

Because rA is a multiple of A, it is evident that r is an integer greater than unity, and r being such an integer, rA, and rB are manifestly equimultiples of A and B.

2. If rA be a *part* of A, we are to show that rB is the same part of B.

Because rA is a part of A, therefore $\frac{A}{rA}=\frac{1}{r}$ must be an integer greater than unity; but $\frac{B}{rB}$, when reduced, is also equal to $\frac{1}{r}$, that is, to the same integer, and therefore rA, rB, are the same parts of A and B. Q. E. D.

PROP. VII. THEOR.

Equal magnitudes have the same ratio to the same magnitude; and the same has the same ratio to equal magnitudes.

DEMONSTRATION.

Let A and B be any two equal magnitudes, and C any other, we are to prove that A and B have each the same ratio to C, and that C has the same ratio to A and B.

Because by hypothesis $A=B$,

therefore by division $\frac{A}{C}=\frac{B}{C}$;

that is, $A : C :: B : C$.

Again, since by hypothesis $A=B$,

therefore by division $\frac{C}{A}=\frac{C}{B}$;

that is, $C : A :: C : B$. Q. E. D.

PROP. VIII. THEOR.

Of unequal magnitudes the greater has a greater ratio to the same, than the less has: and the same magnitude has a greater ratio to the less, than it has to the greater.

DEMONSTRATION.

Let A and B be two unequal magnitudes, of which A is the greater, and let C be any magnitude whatever of the same kind with A and B: it is to be shown that the ratio of A to C is greater than the ratio of B to C: and also that the ratio of C to B is greater than the ratio of C to A.

1 Because by hypothesis $A > B$,

therefore, by division $\frac{A}{C} > \frac{B}{C}$;

that is, the ratio of A to C is greater than the ratio of B to C.

2 Because by hypothesis $A \,7\, B$, therefore $B \angle A$,

and therefore by division we have $\frac{C}{B} \,7\, \frac{C}{A}$,

because the less the divisor of C is, the greater is the quotient; and therefore the ratio of C to B is greater than the ratio of C to A. Q. E. D.

PROP. IX. THEOR.

Magnitudes which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio, are equal to one another.

DEMONSTRATION.

1. Let A and B have the same ratio to C, it is to be proved that A is equal to B.

Because A and B have, by hypothesis, the same ratio to C, therefore we have the equality $\frac{A}{C} = \frac{B}{C}$, and therefore by multiplication $A = B$.

2. Because by hypothesis, C has the same ratio to A as to B, therefore we have the equality $\frac{C}{A} = \frac{C}{B}$, therefore, by dividing by C, and multiplying by A and B, we have $A = B$.

Q. E. D.

PROP. X. THEOR.

That magnitude which has a greater ratio than another has to the same magnitude, is the greater of the two: and that magnitude to which the same has a greater ratio than it has to another, is the less of the two.

DEMONSTRATION.

1. Let A have to C a greater ratio than B has to C, it is to be proved that A is greater than B.

Since the ratios of A and B to C, are $\frac{A}{C}$ and $\frac{B}{C}$,

therefore by supposition $\frac{A}{C} > \frac{B}{C}$, and therefore by multiplication $A > B$.

2. Here the ratio of C to B is greater than he ratio of C to A, and we have to prove that B is less than to A:

We have, therefore, by hypothesis $\frac{C}{B} \angle \frac{C}{A}$.

Since then C contains B oftener than C contains A, it is manifest that B must be less than A. Q. E. D.

PROP. XI. THEOR.

Ratios that are the same to the same ratio, are the same to one another.

DEMONSTRATION.

Let A be to B as C to D, and also E to F as C to D; it is to be shown that A is to B as E is to F.

Because A is to B as C to D, therefore, $\frac{A}{B}=\frac{C}{D}$,

for the same reason $\frac{E}{F}=\frac{C}{D}$; therefore

$\frac{A}{B}=\frac{E}{F}$, that is, A : B :: E : F. Q. E. D.

PROP. XII

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents.

DEMONSTRATION.

By Cor. 2. Def. 5. any number of proportionals may be expressed by rA, A; rB, B; rC, C;

Where rA, rB, rC, are the antecedents, and A, B, C, the consequents; and we are to prove that

as rA is to A, so is $rA+rB+rC$ to $A+B+C$.

The ratio of rA to A is expressed by $\frac{rA}{A}=r$, and the ratio of $rA+rB+rC$ to $A+B+C$, by $\frac{rA+rB+rC}{A+B+C}=r$; and therefore

$$rA : A :: rA+rB+rC : A+B+C.$$

Q. E. D.

PROP. XIII. THEOR.

If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth; the first shall also have to the second a greater ratio than the fifth has to the sixth.

DEMONSTRATION.

Let A, B, C, D, E, F be the first, second, third, fourth, fifth, and six magnitudes respectively.

The ratios of A to B, of C to D, and of E to F are $\frac{A}{B}, \frac{C}{D}, \frac{E}{F}$;

and since by hypothesis $\frac{A}{B}=\frac{C}{D}$,

and also $\frac{C}{D}>\frac{E}{F}$,

therefore we have $\frac{A}{B}>\frac{E}{F}$. Q. E. D.

Cor. And if the first have a greater ratio to the second than the third has to the fourth, but the third the same ratio to the fourth which the fifth has to the sixth; it may be demonstrated, in like manner, that the first has a greater ratio to the second than the fifth has to the sixth.

PROP. XIV. THEOR.

If the first has to the second the same ratio which the third has to the fourth; then, if the first be greater than the third, the second shall be greater than the fourth; if equal, equal, and if less, less.

DEMONSTRATION.

Let rA, A, rB, B, be any four proportionals.

1. Suppose $rA \urcorner rB$,

then by division $A \urcorner B$;

next, suppose $rA = rB$,

then by division $A = B$;

lastly, suppose $rA \angle rB$,

then by division $A \angle B$. Q. E. D.

PROP. XV. THEOR.

Magnitudes have the same ratio to one another which their equimultiples have.

DEMONSTRATION.

Let A, B, be any two magnitudes of the same kind; and m being any integer greater than unity, let mA, mB, be equimultiples of A, B; it is to be proved that

A, B, mA, mB, are proportionals.

The ratio of A to B is the numerical quotient $\frac{A}{B}$, and the ratio of mA to mB is $\frac{mA}{mB}$, which is reducible to $\frac{A}{B}$; therefore the two ratios $\frac{A}{B}$, $\frac{mA}{mB}$ are equal, and therefore

$$A : B :: mA : mB.$$

PROP. XVI. THEOR.

If four magnitudes of the same kind be proportionals, they shall also be proportionals when taken alternately.

DEMONSTRATION.

We may express any four proportionals by

$$rA,\ A,\ rB,\ B,$$

and we are to demonstrate that the four

$$rA,\ rB,\ A,\ B,$$

will also be proportionals.

The ratio of rA to rB is $\frac{rA}{rB}$, which, because the factor r is in both numerator and denominator, is evidently reducible to $\frac{A}{B}$: again the ratio of the third A to the fourth B is also $\frac{A}{B}$; therefore, the two ratios, viz. of rA to rB, and of A to B, being equal, we have

$$rA : rB :: A : B.$$

Q. E. D.

PROP. XVII. THEOR.

If magnitudes taken jointly be proportionals, they shall also be proportionals when taken separately; that is, if two magnitudes together have to one of them the same ratio which two others have to one of these, the remaining one of the first two shall have to the other the same ratio which the remaining one of the last two has to the other of these.

DEMONSTRATION.

By hypothesis we have $A+B : B :: C+D : D$, and we are to prove that $A : B :: C : D$.

Now the ratio of $A+B$ to B is $\frac{A+B}{B}=\frac{A}{B}+1$,

and the ratio of $C+D$ to D is $\frac{C+D}{C}=\frac{C}{D}+1$;

and since by hypothesis these two ratios are equal, therefore we have $\frac{A}{B}+1=\frac{C}{D}+1$, consequently, $\frac{A}{B}=\frac{C}{D}$; that is,

$$A : B :: C : D.$$

Q. E. D.

PROP. XVIII. THEOR.

If magnitudes taken separately be proportionals, they shall also be proportionals when taken jointly; that is, if the first be to the second as the third is to the fourth, the first and second together shall be to the second as the third and fourth together to the fourth.

DEMONSTRATION.

By hypothesis we have A : B : : C : D, and we are to demonstrate that A+B : B : : C+D : D.

Since the ratio of A to B is the same with that of C to D, therefore $\frac{A}{B}=\frac{C}{D}$,
to each side of this equation add unity, and we have

$$\frac{A}{B}+1=\frac{C}{D}+1\text{, that is, }\frac{A+B}{B}=\frac{C+D}{D};$$

and therefore A+B : B : : C+D : D. Q. E. D.

PROP. XIX. THEOR.

If a whole magnitude be to a whole, as a magnitude taken from the first is to a magnitude taken from the other, the remainder shall be to the remainder as the whole to the whole.

DEMONSTRATION.

Let A, B, be the two whole magnitudes, and C, D, the magnitudes taken from them.

So that by hypothesis A : B : : C : D,
we are to prove that A : B : : A—C : B—D.

By Prop. XVI. we have $\frac{A}{C}=\frac{B}{D}$,

consequently $\frac{A}{C}-1=\frac{B}{D}-1$, that is, $\frac{A-C}{C}=\frac{B-D}{D}$;

By this last divide the first equation,
and the equal quotients are $\frac{A}{A-C}=\frac{B}{B-D}$

and therefore by mult. and div. $\frac{A}{B}=\frac{A-C}{B-D}$,

that is, A : B : : A—C : B—D. Q. E. D.

ANOTHER DEMONSTRATION.

Since by hypothesis A : B : : C : D,
therefore by alternation, prop. XVI. A : C : : B : D,
and by division, prop. XVII. A—C : C : : B—D : D,
and by alternation, A—C : B—D : : C : D,
and therefore by prop. XI. A—C : B—D : : A : B.
Q. E. D.

ANOTHER DEMONSTRATION.

Let A+C, and B+D, be the whole magnitudes, and C, D, the magnitudes taken away, so that by hypothesis

$$A+C : B+D :: C : D.$$

And we are to show that

$$A+C : B+D :: A : B.$$

Since by hypothesis $A+C : B+D :: C : D$,
therefore by prop. XVI. $A+C : C :: B+D : D$,
consequently by prop. XVII. $A : C :: B : D$,
and therefore by prop. XVI. $A : B :: C : D$,
therefore by prop. XI. $A+C : B+D :: A : B$

Q. E. D

ANOTHER DEMONSTRATION.

Supposing r greater than unity, let rA, rB, be the two wholes, and A, C the magnitudes taken away, so that by hy pothesis, we have $rA : rB :: A : C$;

of course we have $\frac{rA}{rB}=\frac{A}{C}$, or $\frac{A}{B}=\frac{A}{C}$, whence $C=B$, and we have therefore only to show that

$$rA : rB :: rA-A : rB-B;$$

Now the ratio of rA to rB is $\frac{rA}{rB}=\frac{A}{B}$;

and the ratio of $rA-A$ to $rB-B$, is $\frac{rA-A}{rB-B}=\frac{(r-1).A}{(r-1).B}=\frac{A}{B}$, and therefore

$$rA : rB :: rA-A : rB-B.$$

Q. E. D

PROP. E. THEOR.

If four magnitudes be proportionals, they are also proportionals by conversion; that is, the first is to its excess above the second as the third is to its excess above the fourth.

DEMONSTRATION.

Let rA, A, rB, B, be the four proportionals, we have to demonstrate that

$$rA : rA-A :: rB : rB-B.$$

The ratio of rA to r A$-$A is $\frac{rA}{rA-A}=\frac{r}{r-1}$,

and the ratio of rB to $rB-B$ is $\frac{rB}{rB-B}=\frac{r}{r-1}$,

therefore $rA : rA-A :: rB : rB-B$.

Q. E. D

PROP. XX. THEOR.

If there be three magnitudes, and other three, which taken two and two have the same ratio; if the first be greater than the third, the fourth will be greater than the sixth; if equal, equal; and if less, less.

DEMONSTRATION.

Let the three first magnitudes be A, B, C,
and the other three be D, E, F;
so that by hypothesis, A is to B as D to E, and B to C as E to F; and it is to be proved that if A be greater than C, D will be greater than F; if equal, equal; and if less, less.

Because A : B : : D : E, therefore $\frac{A}{B}=\frac{D}{E}$,

and because B : C : : E : F, therefore $\frac{B}{C}=\frac{E}{F}$:

therefore by multiplication of fractions,

$$\frac{AB}{BC}=\frac{DE}{EF}, \text{ that is } \frac{A}{C}=\frac{D}{F},$$

from which it is evident that when the quotient $\frac{A}{C}$ is greater than unity, the quotient $\frac{D}{F}$ is also greater than unity; that is, if A be greater than C, D is also greater than F; in a similar manner it is shown that when A is equal to C, D is equal to F; and if less, less. Q. E. D.

PROP. XXI. THEOR.

If there be three magnitudes, and other three, which have the same ratio taken two and two, but in a cross order; if the first be greater than the third, the fourth shall also be greater than the sixth; if equal, equal; and if less, less.

DEMONSTRATION.

Let the three first magnitudes be A, B, C,
and the other three be D, E, F,
so that A is to B as E to F, and B to C as D to E; it is to be shown that if A be greater than C, D will be greater than F; if equal, equal; and if less, less.

Since A : B : : E : F, therefore we have $\frac{A}{B}=\frac{E}{F}$, and because B : C : : D : E, therefore also $\frac{B}{C}=\frac{D}{E}$; and therefore by multiplication,

$$\frac{AB}{BC}=\frac{DE}{EF}, \text{ that is, } \frac{A}{C}=\frac{D}{F};$$

from which it is manifest, that according as the quotient $\frac{A}{C}$ is greater than, equal to, or less than unity, the quotient $\frac{D}{F}$ must also be greater than, equal to, or less than unity, and therefore if A be greater than C, D will be greater than F; if equal, equal; and if less, less.

PROP. XXII. THEOR.

If there be any number of magnitudes, and as many others, which, taken two and two in order, have the same ratio; the first shall have to the last of the first rank of magnitudes, the same ratio which the first of the others has to the last.

N. B. This is usually cited by the words *ex æquali*, or *ex æquo*.

DEMONSTRATION.

Let the first rank of magnitudes be A, B, C, D,
and the second rank be E, F, G, H,
so that by hypothesis A is to B as E to F, B to C as F to G, and C to D as G to H; we are to show that A : D : : E : H.

Since A : B : : E : F, therefore we have $\frac{A}{B}=\frac{E}{F}$,

in like manner we have $\frac{B}{C}=\frac{F}{G}$,

and $\frac{C}{D}=\frac{G}{H}$,

now multiply the quotients $\frac{A}{B}$, $\frac{B}{C}$, $\frac{C}{D}$ together, and also the quotients $\frac{E}{F}$, $\frac{F}{G}$, $\frac{G}{H}$, and we have the equation $\frac{ABC}{BCD}=\frac{EFG}{FGH}$, which by reduction becomes $\frac{A}{D}=\frac{E}{H}$,

and therefore A : D : : E : H.

In like manner the truth of the proposition may be shown, whatever be the number of magnitudes.

Q. E. D.

PROP. XXIII. THEOR.

If there be any number of magnitudes, and as many others, which, taken two and two in a cross order, have the same ratio ; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last.

N. B. This is usually cited by the words *ex equali in proportione perturbata*, or *ex æquo perturbato* ; that is, *by equality in perturbate proportion.*

DEMONSTRATION.

Let the first rank of magnitudes be A, B, C, D,
and the other rank E, F, G, H,
so that, by hypothesis, A is to B as G to H ; B to C as F to G, and C to D as E to F ; we are to prove, that

$$A : D :: E : H.$$

Since A : B :: G : H, therefore $\frac{A}{B}=\frac{G}{H}$,

and because B : C :: F : G, therefore, $\frac{B}{C}=\frac{F}{G}$,

and because C : D :: E : F, therefore $\frac{C}{D}=\frac{E}{F}$,

now multiply the quotients $\frac{A}{B}$, $\frac{B}{C}$, $\frac{C}{D}$, together, and also the quotients $\frac{G}{H}$, $\frac{F}{G}$, $\frac{E}{F}$, and we have the products $\frac{ABC}{BCD}=\frac{GFE}{HGF}$,

which reduced, becomes $\frac{A}{D}=\frac{E}{H}$,

and therefore A : D :: E : H.

In like manner we may proceed for any number of magnitudes. Q. E. D.

PROP. XXIV.

If the first has to the second the same ratio which the third has to the fourth ; and the fifth to the second the same ratio which the sixth has to the fourth ; the first and fifth together shall have to the second the same ratio which the third and sixth together have to the fourth.

DEMONSTRATION.

By hypothesis we have $rA : A :: rB : B$,
and $r'A : A :: r'B : B$,
in which rA is the first, A the second, rB the third, B the fourth, $r'A$ the fifth, and $r'B$ the sixth : r' denoting each of the two equal ratios when the fifth is divided by the second. and the sixth by the fourth ; and we have to show, that

$$rA+r'A : A :: rB+r'B : B.$$

The ratio of $rA+r'A$ to A is $\frac{rA+r'A}{A}=r+r'$,

and the ratio of $rB+r'B$ to B is $\frac{rB+r'B}{B}=r+r'$;
therefore, $rA+r'A : A :: rB+r'B : B$.

Q. E. D

Cor. 1. If the same hypothesis be made as in the proposition, the excess of the first and fifth shall be to the second, as the excess of the third and sixth to the fourth.

Cor. 2. The prop. holds true of two ranks of magnitudes, whatever be their number, of which each of the first rank has to the second magnitude the same ratio which the corresponding one of the second rank has to a fourth magnitude.

PROP. XXV. THEOR.

If four magnitudes of the same kind be proportionals, the greatest and least of them together are greater than the other two together.

DEMONSTRATION.

Let the proportionals be rA, A, rB, B;
and let the first rA be the greatest: then since by hypothesis rA is the greatest, $rA > A$, therefore $r > 1$.

Again, since by hypothesis rA is the greatest, therefore $rA > rB$, and consequently $A > B$; since then r is greater than unity, and A is greater than B, it is manifest that B is the least; and we are to show that $rA+B > rB+A$

Now because $A-B=A-B$,
and $r > 1$,
therefore, by multiplication $rA-rB > A-B$;
to each side of this equation add $rB+B$,
and we shall have $rA+B > A+rB$.

A similar mode of demonstration may be adopted, whichever of the four proportionals be the greatest.

Q. E. D.

PROP. XXVI. THEOR.

If there be any number of magnitudes of the same kind, the ratio compounded of the ratios of the first to the second, of the second to the third, and so on to the last, is equal to the ratio of the first to the last.

DEMONSTRATION.

Let the magnitudes of the same kind be A, B, C, D; we are to prove that the ratio compounded of the ratios of A to B, of B to C, and of C to D, according to the definition of compound ratio, is equal to the ratio of A to D.

Take any magnitude P,
and let A be to B as P to Q, and B to C as Q to R, and C to D as R to S; then by the definition of compound ratio, the ratio of P to S is the ratio compounded of the ratios of A to B, B to C, and of C to D; and it is to be proved that the ratio of A to D is the same with P to S.

A, B, C, D,
P, Q, R, S;

Now because A, B, C, D, are several magnitudes, and P, Q, R, S, as many others, which, taken two and two in order, have the same ratio; that is, A is to B as P to Q; B to C as Q to R, and C to D as R to S; therefore *ex equali*, prop. XXII.

$$A : D :: P : S.$$

In like manner the proposition is proved for any number of magnitudes.

Q. E. D.

PROP. XXVII. THEOR.

If four magnitudes be proportionals according to the common algebraic definition, they will also be proportionals according to Euclid's definition.

DEMONSTRATION.

Let the four $rA, A, rB, B,$
be the proportionals according to our fifth definition; that is, according to the common algebraic definition; it is to be proved that the same four

$$rA, A, rB, B,$$

are proportionals by Euclid's fifth def. of the fifth book.

Let m and n be any two integers, each greater than unity, so that mrA, mrB, are any equimultiples whatever of the first and third; and nA, nB, are any whatever of the second and fourth; and the four multiples are therefore

$$mrA, nA, mrB, nB;$$

Now the thing to be proved is, that according as the multiple mrA is greater than, equal to, or less than nA; the multiple mrB will also be greater than, equal to, or less than nB.

First let $mrA > nA$,
then by division $mr > n$,
and by multiplication $mrB > nB$.
Secondly, if $mrA = nA$,
then $mr = n$,
and therefore $mrB = nB$.
Lastly, if $mrA < nA$,
then $mr < n$,
therefore $mrB < nB$. Q. E. D.

PROP. XXVIII. THEOR.

If four magnitudes be proportionals by Euclid's fifth definition, they will also be proportionals by the common algebraic definition.

DEMONSTRATION.

Let A', A, B', B, be any four magnitudes, such that m, n, being any integers greater than unity, and the equimultiples, mA', mB', being taken, and likewise the equimultiples nA, nB; making the four multiples

$$mA',\ nA,\ mB',\ nB\,;$$

the hypothesis is, that if mA' be greater than nA, mB' is also greater than nB; if equal, equal; and if less, less: and it is to be proved that

$$A' : A :: B' : B\,;$$

or, which is the same thing, that $\frac{A'}{A}=\frac{B'}{B}$.

If $\frac{A'}{A}$ be not equal to $\frac{B'}{B}$, one of these quotients must be the greater; first, let $\frac{A'}{A}$ be the greater, so that if $\frac{B'}{B}=r$, we may have $\frac{A'}{A}=r+r'$;

then the four quantities A', A, B', B,
are equal to $rA+rA$, A, rB, B.

Now, let m be such an integer greater than unity, that mr and mr' may be each greater than 2; and take n the next integer greater than mr, of course n will be less than $mr+mr'$: and the four multiples mA', nA, mB', nB,
become $mrA+mr'A$, nA, mrB, nB,

By construction $mr+mr' > n$,
and therefore $mrA+mr'A > nA$;
But by construction $mr<n$,
and therefore $mrB<nB$;
or $mB'<nB$;
thus it appears that $mA' > nA$,
but $mB'<nB$:
but, because $mA'>nA$,
therefore, by hypothesis, also $mB' > nB$;
so that mB' is both greater and less than nB, which is impossible

It is manifest therefore that $\frac{A'}{A}$ cannot be greater than $\frac{B'}{B}$;

and in like manner it is shown that $\frac{B'}{B}$ cannot be greater than $\frac{A'}{A}$; and therefore $\frac{A'}{A}=\frac{B'}{B}$,

that is $A' : A :: B' : B$. Q. E. D.

SCHOLIUM. Thus we have shown, that if four quantities be proportionals by the common algebraic definition, they will also be proportionals according to Euclid's definition ; and conversely, that if four quantities be proportionals by Euclid's definition, they will also be proportionals by the common algebraic definition ; and by a similar method of reasoning we may easily show, that when four quantities are not proportionals by one of these two definitions, they cannot be proportionals by the other definition.

Thus it appears, that the two definitions are altogether equivalent ; each comprehending, or excluding, whatever is comprehended, or excluded, by the other.

THE END