ACCOUNT NO. J13720

TITLE NO.

NAME
NATIONAL INST. EDUCATION
ROOM 230 2ND FLOOR

WASHINGTON DC

ISSN NO. | FREQ. | SORT

BINDING CLASS | CHANGE | NEW TITLE

	IN	OUT
FRONT COVER		
BACK COVER		
ADS		

TITLE PAGE

TABLE CONT

INDEX

Sew on

Tip on

BRI E

NO TRIM
☐ FRONT ☐ HEAD ☐ TAIL

SPECIAL INSTRUCTIO

TITLE

Webber

Mathematics

1808

DATE SENT

BINDERY COPY 2

BINDERY USE ONLY

5 1440

MATERIAL COLOR 783

SET OF

PRINT COLOR Gold

ROLLS

BINDERY USE ONLY

WRAP

SLOT

PICA

PRINT DATE

Bridgeport National Bindery, Inc.

"Bound to Last"

MATHEMATICS,

COMPILED

FROM THE BEST AUTHORS,

AND INTENDED TO BE THE

Text-Book

OF THE

COURSE OF PRIVATE LECTURES ON THESE SCIENCES

IN THE

UNIVERSITY AT CAMBRIDGE.

SECOND EDITION.

BY SAMUEL WEBBER, D. D. A. A. et S. P. A. SOC.

PRESIDENT OF THE UNIVERSITY AT CAMBRIDGE.

IN TWO VOLUMES—VOL. II.

CAMBRIDGE.

PRINTED AT THE UNIVERSITY PRESS,

BY WILLIAM HILLIARD.

1808.

CONTENTS OF THE SECOND VOLUME.

MENSURATION OF SOLIDS.

NAVIGATION.

CONIC SECTIONS.

DIALING.

SPHERIC GEOMETRY.

SPHERIC TRIGONOMETRY.

SPHERIC ASTRONOMY.

N. B. THE principal alterations in the second volume with respect to the first edition, beside its being made to begin with Mensuration of Solids, are the addition of Demonstrations in Notes to nearly all the Problems in Mensuration of Solids, and in Gauging, and of some Questions for practice at the end of Spheric Trigonometry; and the omission of two of the most complex and least useful Problems in Dialing.

MENSURATION

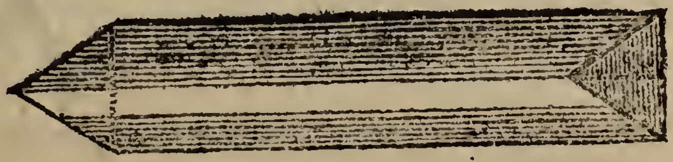
OF

SOLIDS.

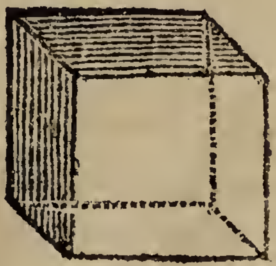
DEFINITIONS.

1. SOLIDS, or BODIES, are figures, having length, breadth, and thickness.

2. A *prism* is a solid, or body, whose ends are any plane figures, which are equal and similar; and its sides are parallelograms.

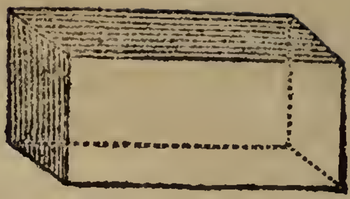


A prism is called *a triangular prism*, when its ends are triangles; *a square prism*, when its ends are squares; *a pentagonal prism*, when its ends are pentagons; and so on.

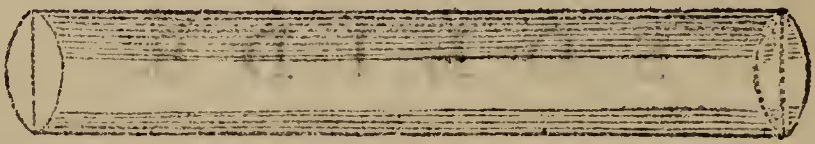


3. A *cube* is a square prism, having six sides, which are all squares. It is like a die, having its sides perpendicular to one another.

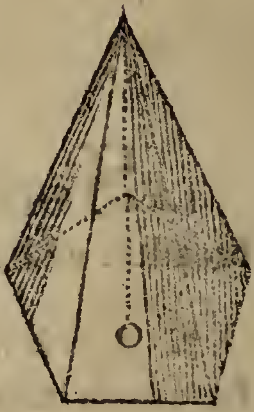
4. A *parallelopipedon* is a solid, having six rectangular sides, every opposite pair of which are equal and parallel.



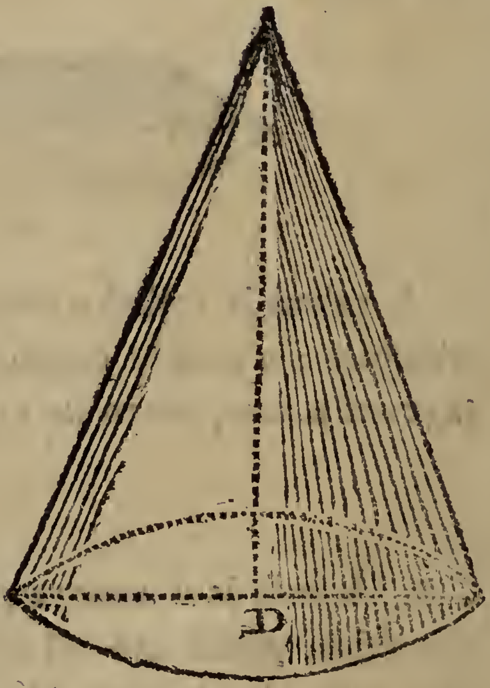
5. A *cylinder* is a round prism, having circles for its ends.



6. A *pyramid* is a solid, having any plane figure for a base, and its sides are triangles, whose vertices met in a point at the top, called the *vertex* of the pyramid.



The pyramid takes names according to the figure of its base, like the prism ; being *triangular*, or *square*, or *hexagonal*, &c.



7. A *cone* is a round pyramid, having a circular base.

8. A *sphere* is a solid, bounded by one continued convex surface, every point of which is equally distant from a point within, called the *centre*. The sphere may be conceived to be formed by the revolution of a semicircle about its diameter, which remains fixed.

9. The *axis of a solid*, is a line drawn from the middle of one end to the middle of the opposite end; as between the opposite ends of a prism. Hence the axis of a pyramid is the line from the vertex to the middle of the base, or the end, on which it is supposed to stand. And the axis of a sphere is the same as a diameter, or a line passing through the centre, and terminated by the surface on both sides.

10. When the axis is perpendicular to the base, it is a *right prism*, or *pyramid;* otherwise it is *oblique*.

11. The *height* or *altitude of a solid* is a line, drawn from its vertex, or top, perpendicular to its base. This is equal to the axis in a right prism or pyramid; but in an oblique one, the *height* is the perpendicular side of a right-angled triangle, whose hypotenuse is the axis.

12. Also a prism or pyramid is *regular* or *irregular*, as its base is a regular or irregular plane figure.

13. The *segment of a pyramid*, *sphere*, or *any other solid*, is a part, cut off the top by a plane parallel to the base of that figure.

14. A *frustum* is the part, that remains at the bottom, after the segment is cut off.

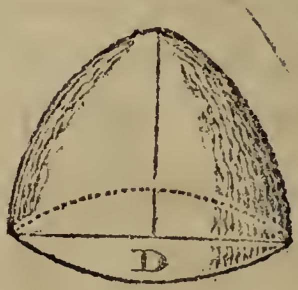
15. A *zone of a sphere* is a part, intercepted between two parallel planes; and is the difference between two segments. When the ends, or planes, are equally distant from the centre on both sides, the figure is called the *middle zone*.

16. The *sector of a sphere* is composed of a segment less than a hemisphere or half sphere, and of a cone having the same base with the segment, and its vertex in the centre of the sphere.

17. A *circular spindle* is a solid, generated by the revolution of a segment of a circle about its chord, which remains fixed.

18. A *spheroid*, or *ellipsoid*, is a solid, generated by the revolution of an ellipse about one of its axes. It is *prolate*, when the revolution is about the transverse axis; and *oblate*, when about the conjugate.

19. A *conoid* is a solid, formed by the revolution of a parabola, or hyperbola, about the axis; and is accordingly called *parabolic*, or *hyperbolic*. The parabolic conoid is also called a *paraboloid;* and the hyperbolic conoid, a *hyperboloid.*



20. A *spindle* is formed by any of the three conic sections, revolving about a double ordinate, like the circular spindle.

21. A *regular body* is a solid, contained under a certain number of equal and regular plane figures of the same sort.

22. The *faces of the solid* are the plane figures, under which it is contained. And the linear sides, or edges, of the solid are the sides of the plane faces.

23. There are only five regular bodies ; namely, first, the *tetraedron*, which is a regular pyramid, having four triangular faces ; second, the *hexaedron*, or cube, which has 6 equal square faces ; third, the *octaedron*, which has 8 triangular faces ; fourth, the *dodecaedron*, which has 12 pentagonal faces ; fifth, the *icosaedron*, which has 20 triangular faces.

NOTE. 1. If the following figures be exactly drawn on pasteboard, and the lines cut half through, so that the parts may be turned up and glued together, they will represent the five regular bodies ; namely, figure 1 the tetraedron, figure 2 the hexaedron, figure 3 the octaedron, figure 4 the dodecaedron, and figure 5 the icosaedron.

Figure 1.

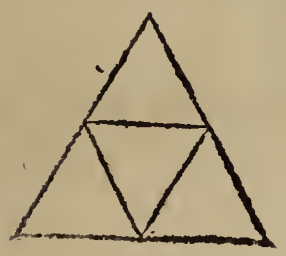


Figure 2.

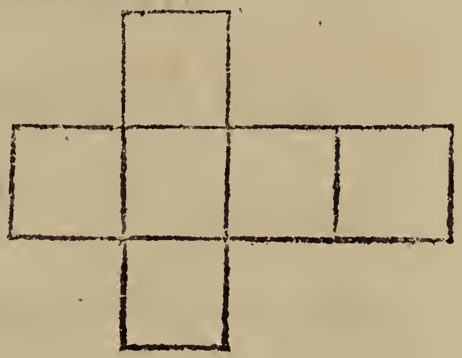


Figure 3.

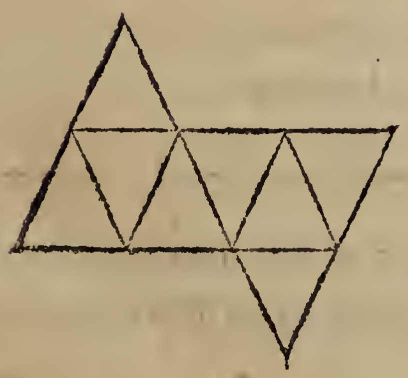


Figure 4.

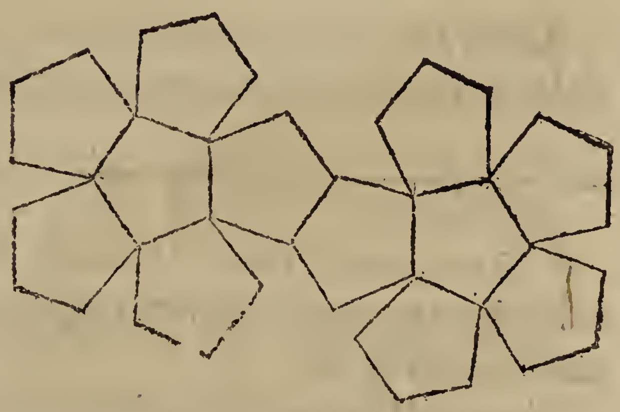
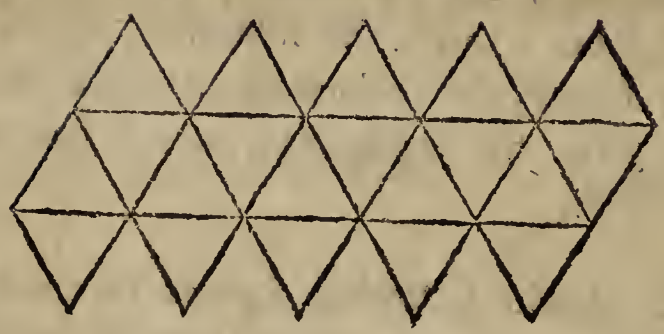


Figure 5.



NOTE 2. In cubic measure,

1728	inches make	1	foot
27	feet	1	yard
$166\frac{3}{8}$	yards	1	pole
64000	poles	1	furlong
512	furlongs . . .	1	mile.

PROBLEMS.

PROBLEM I.

To find the solidity of a cube.

RULE.*

Cube one of its sides for the content; that is, multiply the side by itself, and that product by the side again.

---

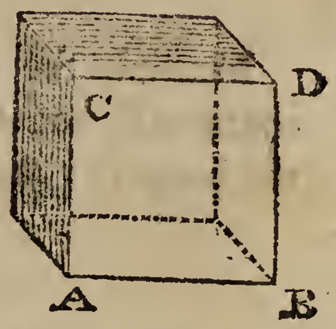
* DEMONSTRATION. Conceive the base of the cube to be divided into a number of little squares, each equal to the *superficial measuring unit.*

Then will those squares be the bases of a like number of small cubes, which are each equal to the *solid measuring unit.*

But the number of little squares, contained in the base of the cube, are equal to the square of the side of that base, as has been shown already.

EXAMPLES.

1. If the side AB, or AC, or BD, of a cube be 24 inches, what is its solidity or content?




```
   24
   24
  ---
   96
  48
  ---
  576
   24
 ----
 2304
1152
-----
13824 answer.
-----
```

2. How many solid feet are in a cube, whose side is 22 feet?

Ans. 10648.

3. How many cubic feet are in the cube, whose side is 18 inches?

Ans. $3\frac{3}{8}$.

And consequently, the number of small cubes, contained in the whole figure, must be equal to the square of the side of the base, multiplied by the height of that figure; or, which is the same thing, the square of the side of the base, multiplied by the base, is equal to the solidity. Q. E. D.

NOTE. The surface of the cube is equal to 6 times the square of its side.

PROBLEM II.

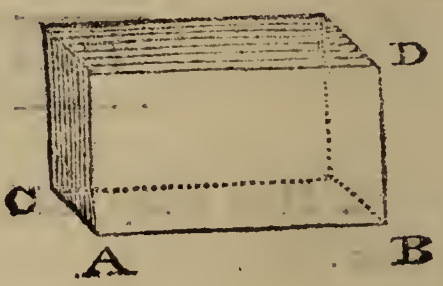
To find the solidity of a parallelopipedon.

RULE.*

Multiply the length, breadth, and depth, or altitude, all continually together for the solid content; that is, multiply the length by the breadth, and that product by the depth.

EXAMPLES.

1. Required the content of the parallelopipedon, whose length AB is 6 feet, its breadth AC $2\frac{1}{2}$ feet, and altitude BD $1\frac{3}{4}$ feet.




```
 1·75 BD
    6 AB
 ─────
10·50
  2·5 AC
 ─────
 5250
2100
 ─────
26·250 answer.
```

2. Required the content of a parallelopipedon, whose length is 10·5, breadth 4·2, and height 3·4.

Ans. 149·94.

3. How many cubic feet are in a block of marble, whose length is 3 feet 2 inches, breadth 2 feet 8 inches, and depth 2 feet 6 inches? Ans. $21\frac{1}{9}$.

---

* The reason of this rule, as well as of the following one for the solidity of the prism, is the same as that for the cube.

Note. The surface of the parallelopipedon is equal to the sum of the areas of each of its sides or ends.

PROBLEM III.

To find the solidity of a prism.

RULE.

Multiply the area of the base, or end, by the height, and it will give the content.

Which rule will give the solidity, whether the prism be triangular, or square, or pentagonal, &c. or round, as a cylinder.*

EXAMPLES.

1. What is the content of a triangular prism, whose length AC is 12 feet, and each side AB of its equilateral base $2\frac{1}{2}$ feet?

Here $\frac{5}{2}\times\frac{5}{2}=\frac{25}{4}=6\frac{1}{4}$

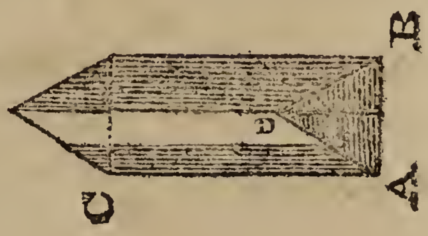
Then ·433013 tabular num.

$6\frac{1}{4}$

2·598078
108253

2·706331 area of end.
12 length.

32·475972 answer.



2. Required the solidity of a triangular prism, whose length is 10 feet, and the three sides of its triangular end, or base, are 5, 4, 3 feet.

Ans. 60.

* Note. If the areas of each of the sides and ends be calculated separately, their sum will be the surface of the prism.

3. What is the content of a hexagonal prism, the length being 8 feet, and each side of its end 1 foot 6 inches ?

Ans. 46·7653716.

4. Required the content of a cylinder, whose length is 20 feet, and circumference $5\frac{1}{2}$ feet.

Ans. 48·1459.

5. What is the content of a round pillar, whose height is 10 feet, and diameter 2 feet 3 inches ?

Ans. 63.6174.

PROBLEM IV.

To find the convex surface of a cylinder.

RULE.*

Multiply the circumference by the height of the cylinder.

NOTE. The upright surface of any prism is found in the same manner. And the solidity of a cylinder is found as the prism in the last problem.

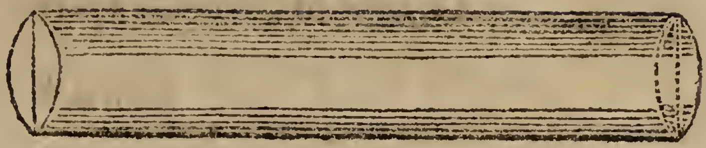
EXAMPLES.

1. What is the convex surface of a cylinder, whose length is 16 feet, and its diameter 2 feet 13 inches ?

---

* DEMONSTRATION. If the periphery of the base be conceived to move in a direction parallel to itself, it will generate the convex superficies of the cylinder ; and therefore the said periphery, being multiplied by the length of the cylinder, will be equal to that superficies. Q. E. D.

NOTE. If twice the area of either of the ends be added to the convex surface, it will give the whole surface of the cylinder.



```
3·1416
   2¼ diam.
------
 62832
  7854
------
7·0686 circumference.
    16
------
424116
70686
------
1130976 answer.
```

2. Required the convex surface of the cylinder, whose length is 20 feet, and its diameter 2 feet.

Ans. 125·664.

3. What is the convex surface of a cylinder, whose length is 18 feet 6 inches, and circumference 5 feet 4 inches?

Ans. $98\frac{2}{3}$.

PROBLEM V.

To find the convex surface of a right cone.

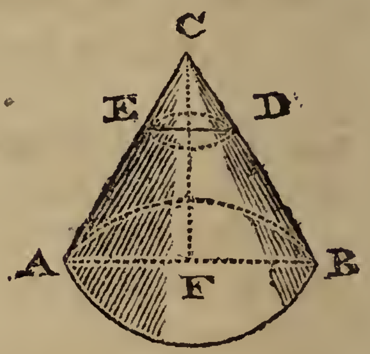
RULE.*

Multiply the circumference or perimeter of the base by the slant height, or length of the side, and half the product will be the surface.

Note. The same rule may be applied to find the surface of a right pyramid.

EXAMPLES.

1. If the diameter of the base be AB 5 feet, and the side of the cone AC 18; required the convex surface.



3·1416
5
______
15·7080 circumference.
18
______
125664
15708
______
2)282·744
______
141·372 answer.

---

* Demonstration. Let $AB=a$, $BC=b$, $3·416=p$, and $ED=y$.

then $a : b :: y : \frac{by}{a}=DC$; and py=circumference of the circle ED.—But $py\times\frac{b\dot{y}}{a}$=fluxion of the surface of CED, and its fluent$=\frac{pby^2}{2a}$; which, when $y=a$, becomes $\frac{pba}{2}$=convex surface of the whole cone. Q. E. D.

2. What is the convex surface of a cone, whose side is 20, and the circumference of its base 9? Ans. 90.

3. Required the convex surface of a cone, whose slant height is 50 feet, and the diameter of its base 8 feet 6 inches. Ans. 667·59.

PROBLEM VI.

To find the convex surface of the frustum of a right cone.

RULE.*

Multiply the sum of the perimeters of the two ends by the slant height or side of the frustum, and half the product will be the surface.

NOTE. The same rule may be used to find the surface of the frustum of a right pyramid.

DEMONSTRATION. Let the perimeter of the circle $AB=P$, that of $DC=p$, $BC=h$, and the rest as in the last Problem.

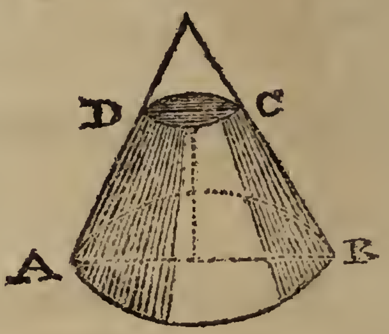
Then $P : p :: b : CE$,

and by Div. $P-p : p :: b-CE=h : CE=\frac{ph}{P-p}$, but

$\overline{P\times h+\frac{ph}{P-p}}$ = twice the convex surface of the whole cone, by the last Rule; and also $p\times\frac{ph}{P-p}$ = twice the convex surface of the part DEC. Therefore $P\times\overline{h+\frac{ph}{P-p}}-p\times\frac{ph}{P-p}=Ph+\overline{P-p}\times\frac{ph}{P-p}=Ph+hp=\overline{P+p}\times h$ = twice the convex surface of the frustum ABCD, and the half of it is $\frac{\overline{P+p}\times h}{2}$, the same as *the Rule.*

EXAMPLES.

1. If the circumferences of the two ends be 12·5 and 10·3, and the slant height AD 14; required the convex surface of the frustum ABCD.




```
   12·5
   10·3
   ----
   22·8
     14
   ----
    912
   228
  -----
2)319·2
  -----
  159·6 answer.
```

2. What is the convex surface of the frustum of a cone, the slant height of the frustum being 12·5, and the circumferences of the two ends 6 and 3·4? Ans. 90.

3. Required the convex surface of the frustum of a cone, the side of the frustum being 10 feet 6 inches, and the circumferences of the two ends 2 feet 3 inches and 5 feet 4 inches. Ans. $39\frac{13}{16}$.

PROBLEM VII.

To find the solidity of a cone, or any pyramid.

RULE.*

Multiply the area of the base by the height, and $\frac{1}{3}$ of the product will be the content.

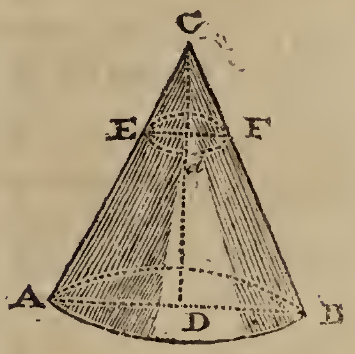
DEMONSTRATION. Let $CD=a$, $Cd=x$, and A=area of the base of the cone ACB.

EXAMPLES.

1. What is the solidity of a cone, whose height CD is $12\frac{1}{2}$ feet, and the diameter AB of the base $2\frac{1}{2}$?

Here $2\frac{1}{2}\times2\frac{1}{2}=\frac{5}{2}\times\frac{5}{2}=\frac{25}{4}=6\frac{1}{4}$

Then ˙7854
$6\frac{1}{4}$
———
4˙7124
19635
———
4˙90875 area of the base.
$12\frac{1}{2}$
———
5890500
2454375
———
3)61˙359375
———
20˙453125 answer.



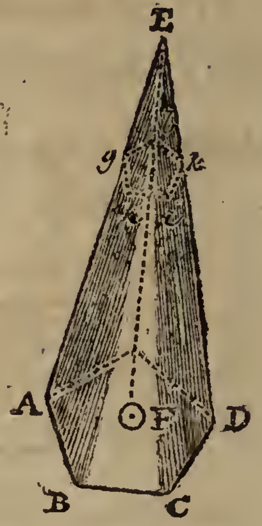
Then $a^2 : x^2 :: AB^2 : EF^2$: by similar Δs $:: A : \frac{Ax}{a}$ =area of the circle EF.

But $\frac{Ax^2}{a^2}\times\dot{x}$=fluxion of the cone ECF, and its fluent$=\frac{Ax^3}{3a}$; which, when $x=a$, becomes $\frac{Aa}{3}=A\times\frac{a}{3}$ for the solidity of the whole cone.

In the pyramid EABD, it will be $a^2 : x^2 :: EA^2 : Eg^2 :: AB^2 : gh^2$ by similar Δs $::$ A=area of base $: \frac{Ax^2}{a^2}$= area of $ghikl$.

2. What is the solid content of a pentagonal pyramid, its height being 12 feet, and each side of its base 2 feet?

1·720477	tab. area
4	square side.
6·881908	area base.
4	$\frac{1}{3}$ of height.
27·527632	answer.



3. What is the content of a cone, its height being $10\frac{1}{2}$ feet, and the circumference of its base 9 feet?

Ans. 22·56093.

4. Required the content of a triangular pyramid, its height being 14 feet 6 inches, and the 3 sides of its base 5, 6, 7.

Ans. 71·0352.

5. What is the content of a hexagonal pyramid, whose height is 6·4, and each side of its base 6 inches?

Ans. 1·38564 feet.

But $\frac{Ax}{a^2}\times\dot{x}$ = fluxion of the pyramid E g h k, and its correct fluent $=A\times\frac{a}{3}$, the same as in the cone; and the same rule is applicable, whatever be the figure of the base. Q. E. D.

PROBLEM VIII.

To find the solidity of the frustum of a cone or any pyramid.

1. *For the frustum of a cone.*

RULE.*

Divide the difference of the cubes of the diameters of the two ends by the difference of the diameters, and this quotient, being multiplied by ·7854 and again by $\frac{1}{3}$ of the height, will give the solidity.

---

* DEMONSTRATION. Let D = diameter AB, d = ED, p = ·7854, h = Ss = height of the frustum, CS = height of the cone, Cs = height above s. Then D : d : : CS : Cs,

and $D - d : d :: CS - Cs = h : \frac{dh}{D-d} = Cs$.

But $\frac{pD^2}{3} \times \overline{h + \frac{dh}{D-d}}$ = solidity of the whole cone,

and $\frac{pd^2}{3} \times \frac{dh}{D-d}$ = that of the part above ED. Therefore,

$$\frac{pD^2}{3} \times \overline{h + \frac{dh}{D-d}} - \frac{pd^2}{3} \times \frac{dh}{D-d}$$

$$= \overline{D^2 \times h + \frac{dh}{D-d} - d^2 \times \frac{dh}{D-d}} \times \frac{p}{3}$$

$$= \overline{D^2 h + \overline{D^2 - d^2} \times \frac{dh}{D-d}} \times \frac{p}{3} =$$

$\frac{D^3 - d^3}{D-d} \times \frac{hp}{3}$ = solidity of the frustum ABDE. Q. E. D.

2. *For the frustum of a pyramid.*

RULE.*

To the areas of the two ends of the frustum add the square root of their product, and this sum, being multiplied by $\frac{1}{3}$ of the height, will give the solidity.

EXAMPLES.

1. What is the content of the frustum of a cone, whose height is 20 inches, and the diameters of its two ends 28 and 20 inches ?

20	28
20	28

---

* For the frustum of a pyramid, let S = ED, $s = ed$, and m = the proper multiplier in the table of polygons ; then
$S : s :: CS : Cs$.

and $S - s : s :: CS - Cs = h : \frac{hs}{S-s}$.

But mS^2 and ms^2 are the areas of polygons, whose sides are S and s respectively. Therefore

$$\frac{mS^2}{3} \times \overline{h + \frac{hs}{S-s}} - \frac{ms^2}{3} + \frac{hs}{S-s}$$

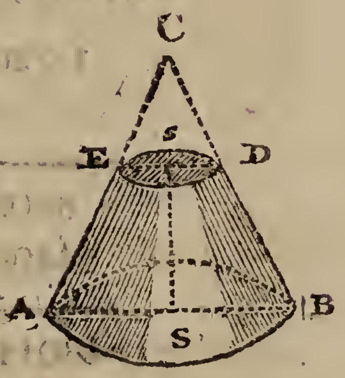
$$= \overline{mS^2 + \overline{mS^2 - ms^2} \times \frac{hs}{S-s}} \times \frac{h}{3}$$

$$= \overline{mS^2 + mSs + ms^2} \times \frac{h}{3}$$

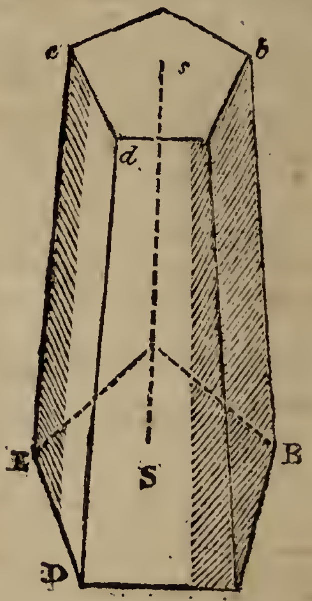
= solidity of the frustum $eEDBb$. Q. E. D.

```
 400               224
  20                56
 ----              ----
8000 = cube of DE.  784
                     28
                   ----
  28               6272
  20              1568
  --              -----
   8 = diff.      21952 = cube of AB
                   8000
                  -----
                8)13952(1744
                  8...
                  --
                  59
                  56
                  --
                   35
                   32
                   --
                    32
                    32
                    --
```

```
     1744
     ·7854
     -----
     6976
    8720
   13952
  12208
  ----------
  1369·7376
        20
  ----------
3)27394·7520
  ----------
  9131·5840   Answer.
```



2. Require the content of a pentagonal frustrum, whose height is 5 feet, each side of the base 1 foot 6 inches, and each side of the less end 6 inches.



ft.	in.	
1	6	
1	6	
	9	0
1	6	
2	3	0=2·25
0	6	
0	6	
	3=·25	

1·720477 = tabular multiplier.

2·25 = square of E D.

8602385

3440954

3440954

3·87107325 = area of E D B.

1·720477 = tabular multiplier.

·25 = square of *e d*.

8602385

3440954

·43011925 = area of *e d b*.

3·871073

430119

34839657

3871073

3871073

116132190

15484292

	1·665022047607(1·290357	
	1	3·871073
		·430119
22	66	
2	44	5·591549
249	2250	
9	2241	
25803	92204	
3	77409	
258065	1479576	
5	1290325	
2580707	18925107	
7	18064949	
2580714	860238	

```
        5·591549
          1·6666 = ⅓ of height.
        --------
        33549294
       33549294
      33549294
     33549294
     ·5591549
     ------------
     9·3188755634 Answer.
```

3. What is the solidity of the frustum of a cone, the altitude being 25, the circumference at the greater end being 20, and at the less end 10?

Ans. 464·205.

4. How many solid feet are in a piece of timber, whose bases are squares, each side of the greater end being 15 inches, and each side of the less end 6 inches; also the length, or perpendicular altitude, is 24 feet?

Ans. 19½.

5. To find the content of the frustum of a cone, the altitude being 18, the greatest diameter 8, and the least 4.

Ans. 527·7888.

6. What is the solidity of a hexagonal frustum, the height being 6 feet, the side of the greater end 18 inches, and that of the less 12 inches? Ans. 24·681722.

PROBLEM IX.

To find the solidity of a wedge.

RULE.*

To the length of the edge add twice the length of the back or base, and reserve the sum; multiply the height of the

---

* Demonstration. When the length of the base is equal to that of the edge, the wedge is evidently equal to half a prism of the same base and altitude.

According as the edge is shorter or longer than the base, the wedge is greater or less then half a prism of the same base, and altitude, and of a length equal to that of the edge, by a pyramid of the same altitude and breadth at the base with the wedge, the length of its base being equal to the difference of the lengths of the edge and base of the wedge.

Let $L = BC$ = length of the base,
$l = EF$ = length of the edge,
$b = AB$ = breadth of the base,
$h = EP$ = height of the wedge.

Then by former rules,

$$\frac{blh}{2} \pm bh \times \frac{+L \mp l}{3} = \frac{blh}{2} + bh \times \frac{L-l}{3} = bh \times \frac{3l+2L-2l}{6}$$

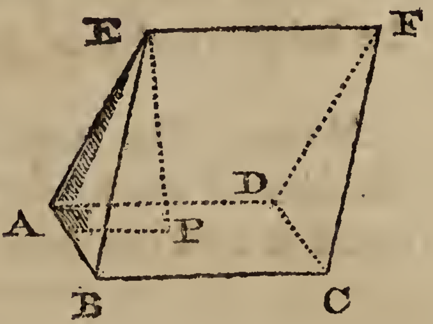
$$= bh \times \frac{2L+l}{6}. \quad \text{Q. E. D.}$$

wedge by the breadth of the base; then multiply this product by the reserved sum, and $\frac{1}{6}$ of the last product will be the content.

EXAMPLES.

1. What is the content, in feet, of a wedge, whose altitude EP is 14 inches, its edge EF 21 inches, the length of its base BC 32 inches, and its breadth AB $4\frac{1}{2}$ inches?

21	14
32	$4\frac{1}{2}$
32	56
85	7
	63
	85
	315
	504
	6,5355



1728 { 12 | 892·5 answer in cubic inches.
12 | 74·375
12 | 6·197916

·516493 answer in feet, or little more than half a cubic foot.

2. Required the content of a wedge, the length and breadth of the base being 70 and 30 inches, the length of the edge 110 inches, and the height 34·29016.

Ans. 24·8048.

PROBLEM X.

To find the solidity of a prismoid.

NOTE. A prismoid differs only from the frustum of a pyramid in not having its opposite ends similar planes.

RULE.*

Add into one sum the areas of the two ends and 4 times the middle section parallel to them; multiply this sum by the height, and $\frac{1}{6}$ of the product will be the solidity.

NOTE. The length of the middle section is equal to half the sum of the lengths of the two ends; and its breadth is equal to half the sum of the breadths of the two ends.

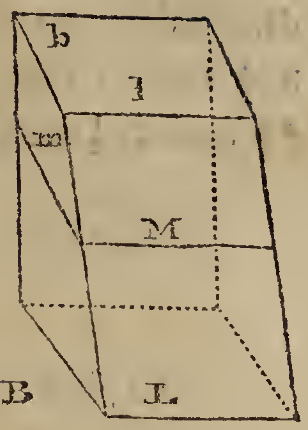
EXAMPLES.

1. How many cubic feet are there in a stone, whose ends are rectangles, the length and breadth of one being 14 and 12 inches; the corresponding sides of the other 6 and 4 inches; and the perpendicular height $30\frac{1}{2}$ feet?

---

* DEMONSTRATION. The rectangular prismoid is evidently composed of two wedges, whose heights are equal to the height of the prismoid and their bases are its two ends. Wherefore, by the last Problem, its solidity will be $= \overline{\overline{2L + l} \times B + \overline{2l + L} \times b} \times \frac{h}{6}$ (See fig.) which, since $M = \frac{L + 1}{2}$, and $m = \frac{B + b}{2}$, $= \overline{BL + bl + 4Mm} \times \frac{h}{6}$; which is *the Rule*.

This rule is applicable to any prismoid, whatever be the figures of the ends, since they may be conceived to be composed of an indefinite number of rectangular prismoids.



14
12
———
168

10
8
———
80
4
———
320
168
24
———
6)512
$85\frac{1}{3}$ mean area in inches.
$30\frac{1}{2}$ height.
———
2560
$42\frac{2}{3}$
———
144 { 12 | $2602 \cdot \dot{6}$
{ 12 | $216 \cdot \dot{8}$
$18 \cdot 074$ answer.

6
4
———
24

2. Required the content of a rectangular prismoid whose greater end measures 12 inches by 8, the less end 8 inches by 6, and the perpendicular height 5 feet.

Ans. 2·453 feet.

3. What is the content of a cart or waggon, whose inside dimensions are as follow: at the top, the length and breadth $81\frac{1}{2}$ and 55 inches; at the bottom, the length and breadth 41 and $29\frac{1}{2}$ inches; and the height $47\frac{1}{4}$ inches?

Ans. 126340·59375· cubic inches.

PROBLEM XI.

To find the convex surface of a sphere or globe.

RULE*.

Multiply its diameter by its circumference.

NOTE. in like manner, the convex surface of any zone or segment is found by multiplying its height by the whole circumference of the sphere.

EXAMPLES.

1. Required the convex superficies of a globe, whose diameter or axis is 24 inches.

---

* DEMONSTRATION. Let the diameter B G $= d$, B A $= x$, A C $= y$, B C $= z$, and 3·14159 $= p$.

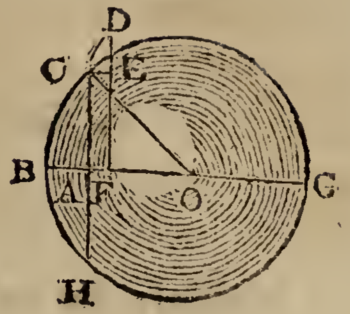
Then, by the similar triangles AOC, and CED

$$AC = y : CO = \frac{d}{2} :: CE = \dot{x} : CD = \dot{z} = \frac{d\dot{x}}{2y}.$$

But $2py\dot{z}$ is the general expression for the fluxion of any surface;

hence $2py\dot{z} = pd\dot{x}$

and the fluent of it $pdx =$ the surface of any segment of a sphere, whose height is x; and pdd=that of the whole sphere. Q. E. D.




```
 3·1416
     24 diameter.
 ------
 125664
 62832
 ------
 25·3984 circumference.
      24
 -------
 3015936
1507968
 -------
1809·5616 answer.
```

2. What is the convex surface of a sphere, whose diameter is 7 and circumference 22 ?

Ans. 154.

3. Required the area of the surface of the earth, its diamter, or axis, being $7957\frac{3}{4}$ miles, or its circumference 25000 miles ?

Ans. 198943750 square miles.

4. The axis of a sphere being 42 inches, what is the convex superficies of the segment, whose height is 9 inches ?

Ans. 1187·5248 inches.

5. Required the convex surface of a spherical zone, whose breadth or height is 2 feet, and cut from a sphere of $12\frac{1}{2}$ feet diameter.

Ans. 78·54 feet.

PROBLEM XII.

To find the solidity of a sphere or globe.

RULE.*

Multiply the cube of the axis by ˙5236.

EXAMPLES.

1. What is the solidity of the sphere, whose axis is 12?

```
    12
    12
  ----
   144
    12
  ----
  1728
 ˙5236
  ----
 10368
  5184
 3456
8640
-----------
904˙7808 answer.
```

Or thus.

```
   ˙5236
      12
  ------
  6˙2832
      12
  ------
 75˙3984
      12
  ------
904˙7808 answer.
```

---

* Demonstration. Let $AB=x$, $AC=y$, $BG=d$, and $3˙14159=p$. (See last figure).

Then $dx-x^2=y^2$;

but $py^2\dot{x}$ is the general expression for the fluxion of any solid; hence $pdx\dot{x}-px^2\dot{x}=$ fluxion of the segment CBH, whose fluent $=\frac{pdx^2}{2}-\frac{px^3}{3}=\frac{3pdx^2-2px^3}{6}=$ content of the segment CBH $=d$, being put for x, $\frac{3pd^3-2pd^3}{6}=\frac{pd^3}{6}=d^3\times{}˙5236$, which is *the Rule.*

2. To find the content of the sphere, whose axis is 2 feet 8 inches. Ans. 9·9288 feet.

3. Required the solid content of the earth, supposing its circumference to be 25000 miles.

Ans. 263858149120 miles.

PROBLEM XIII.

To find the solidity of a spherical segment.

RULE.*

To 3 times the square of the radius of its base add the square of its height; then multiply the sum by the height, and the product by ·5236.

EXAMPLES.

1. Required the content of a spherical segment, its height being 4 inches, and the radius of its base 8.

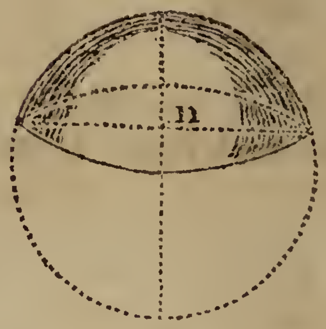
---

* Demonstration. Let r = radius of the base of the segment, h = height of the segment, other letters as in the last Problem.

Then, by Note under the last Problem, $\overline{3dh^2 - 2h^3} \times \frac{p}{6}$ = solidity of the segment.

But $\frac{r^2 + h^2}{h} = d$, by property of the circle,

hence $\overline{\frac{3r^2h^2 + 3h^4}{h} - 2h^3} \times \frac{p}{6} = \overline{3r^2 + h^2} \times \frac{ph}{6}$ = solidity of the segment, which is *the Rule*.



8	
8	
64	
3	
192	
4	'5236
4	832
16	10472
192	15708
	41888
208	435'6352 ans.
4	
832	

2. What is the solidity of the segment of a sphere, whose height is 9, and the diameter of its base 20?

Ans. 1795'4244.

3. Required the content of the spherical segment, whose height is $2\frac{1}{4}$, and the diameter of its base 8'61684.

Ans. 71'5695.

PLOBLEM XIV.

To find the solidity of a spherical zone or frustum.

RULE.*

Add together the square of the radius of each end and $\frac{1}{3}$ of the square of their distance or the height; then multiply the sum by the said height, and the product again by 1·5708.

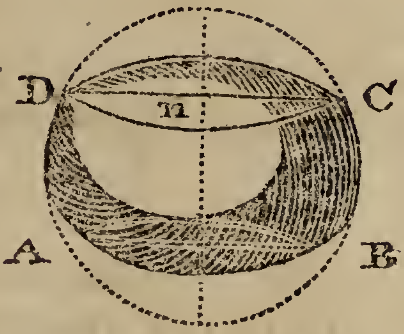
---

* DEMONSTRATION. The difference between two segments of a sphere, whose heights are H and h, and the radii of whose

EXAMPLES.

1. What is the solid content of a zone, whose greater diameter is 12 inches, the less 8, and the height 10 ?

```
  6        4              10
  6        4              10
 ---      ---            -----
 36       16            3)100
 ---      36               33⅓
          33⅓            -----
         -----
          85⅓
        1·5708
        ------
         78540
       125664
          5236
      --------
      134·0416
            10
      --------
      1340·416 answer.
      --------
```



2. Required the content of a zone, whose greater diameter is 12, less diameter 10, and height 2.

Ans. 195·8264.

---

bases are R and r, will, by the last Problem, $= \frac{h}{6} \times \overline{3R^2H + H^3 - 3r^2h - h^3}$ = zone, whose height is H—h, which, by putting a for the altitude of the frustum, and exterminating H and h by means of the two equations $\frac{R^2 + H^2}{H} = \frac{r^2 + h}{h}$, and $H - h = a$, will become $\overline{R^2 + r^2 + \frac{1}{3}a^2} + \frac{1}{2} pa = \overline{R^2 + r \ + \frac{1}{3}a^2} \times a \times 1{\cdot}5708$, which is *the Rule.*

3. What is the content of a middle zone, whose height is 8 feet, and the diameter of each end 6?

Ans. 492·2784 feet.

PROBLEM XV.

To find the surface of a circular spindle.

RULE.*

Multiply the length AB of the spindle by the radius OC of the revolving arc. Multiply also the said arc ACB by the central distance OE, or distance between the centre of the spindle and centre of the revolving arc. Subtract the latter product from the former, and multiply double the remainder by 3·1416, or the single remainder by 6·2832, for the surface.

---

* Demonstration. Put $z =$ arc Cn, $x =$ its sine nm, $r =$ radius OC, $c =$ central distance OE, and $p = 3·14159$. Then

$\sqrt{r^2 - x^2} = Om : r :: \dot{x} : \dot{z}$, by Demonstration under Problem XI, hence $\dot{z}\sqrt{r^2 - x^2} = r\dot{x}$. But $2p\dot{z} \times na = 2p\dot{z}\,\overline{\sqrt{r^2 - x^2} - c}$

$= 2p \times \overline{\dot{z}\sqrt{r^2 - x^2} - c\dot{z}}$ is the general expression of the fluxion, and therefore $2p \times \overline{r\dot{x} - c\dot{z}} =$ the fluxion of half the frustum CncP, and $2p \times \overline{rx - cz} = 2p \times \overline{r \times a\text{E} - c \times \text{C}n} =$ its surface.

When Ea = EA, the rule becomes $2p \times \overline{r \times \text{EA} - c \times \text{CA}}$, for the surface of half the spindle, and $2p \times \overline{r \times \text{AB} - c \times \text{ACB}}$ for that of the whole spindle, which is *the Rule.*

Note. The same rule will serve for any segment or zone cut off perpendicular to the chord of the revolving arc, only using the particular length of the part, and the part of the arc, which describes it, instead of the whole length and whole arc.

EXAMPLES.

1. Required the surface of a circular spindle, whose length AB is 40, and its thickness, or middle diameter CD, is 30 inches.

Here

The chord $AC = \sqrt{AE^2 + CE^2} = \sqrt{20^2 + 15^2} = 25$,

And $2CE : AC :: AC : CO = \frac{\overline{25}^2}{30} = 20\frac{5}{6}$,

Hence $OE = OC - CE = 20\frac{5}{6} - 15 = 5\frac{5}{6}$.

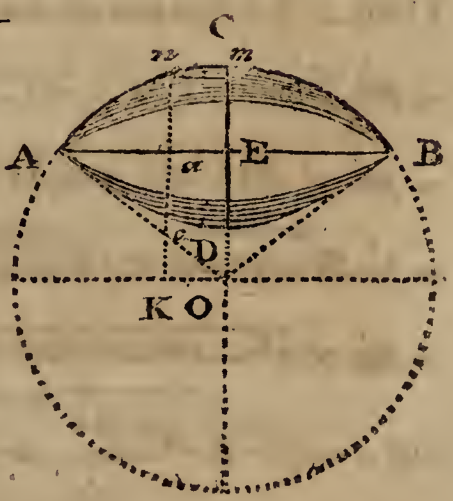
Also, by Prob. X. Rule 2, Mensuration of Superficies.

25 AC
8
———
200
40 AB
———
3)160
$53\frac{1}{3}$ arc ACB
———

Then, by the Rule,

$20\frac{5}{6}$
40
———
800
$33\frac{1}{3}$
———
$833\frac{1}{3}$
$311\frac{1}{9}$
———
$522\frac{2}{9}$, or $522\dot{2}$, or $\frac{4700}{9}$

$53\frac{1}{3}$
$5\frac{5}{6}$
———
$266\frac{2}{3}$
$44\frac{4}{9}$
———
$311\frac{1}{9}$
———



6·2832	Or thus,
	6·2832
10444	4700
156666	
4177777	439824
10444444	251328
313333333	
	9)29531·04
3281·22666	3281·226 answer nearly.

2. What is the surface of a circular spindle, whose length is 24, and thickness in the middle 18?

Ans. 1177·4485.

PROBLEM XVI.

To find the solidity of a circular spindle.

RULE.*

Multiply the central distance OE by half the area of the revolving segment ACBEA. Subtract the product from

---

* DEMONSTRATION. Put $x = Ea$, $c = OE$, &c. as in the last Demonstration. (See last figure.)

Then $\dot{s}$, the fluxion of the solid,

$$= p\dot{x} \times na^2 = p\dot{x} \times \overline{nH - c}^2$$

$$= p\dot{x} \times \overline{nH^2 - c \times \overline{2nH - c}}$$

$$= p\dot{x} \times \overline{nH^2 - c \times \overline{2na + c}}$$

$$= p\dot{x} \times \overline{r^2 - x^2 - c^2 - 2c \times na};$$

and $s = px \times \overline{r^2 - \frac{x^2}{3} - c^2 - \frac{2c}{x} \times anCE}$

$$= p \times \overline{\overline{AE^2 - \frac{x^2}{3}} \times x - 2c \times anCE}$$

$\frac{1}{3}$ of the cube of AE, half the length of the spindle. Then multiply the remainder by 12·5664, or 4 times 3·1416, for the whole content.

EXAMPLES.

1. Required the content of the circular spindle, whose length AB is 40, and middle diameter CD 30.

[See last Figure.]

By the work of the last Problem,

We have OE $= 6\frac{5}{6}$
And arc AC $= 26\frac{2}{3}$
And rad. OC $= 20\frac{5}{6}$

$533\frac{1}{3}$
$22\frac{2}{9}$

Sector OACB $555\frac{5}{9}$
AE×OE=OAB $116\frac{2}{3}$

2)$438\frac{8}{9}$

$\frac{1}{2}$ seg. ACB $219\frac{4}{9}$
OE $5\frac{5}{6}$

$1097\frac{2}{9}$
183 nearly.

$1280\frac{2}{9}$

20 half length.
20

400
20

3)8000

$2666\frac{2}{3}$
$1280\frac{2}{9}$

$1386\frac{4}{9}$

or 1386·44
4665·21 mult. invert.

138644
27739
6932
832
83
5

17423·5 answer.

= the frustum generated by *an*CE. When x = AE,

$\frac{1}{3}\overline{AE^3 - c \times ACE} \times 2p$ = the half spindle CAP,

and $\frac{1}{3}\overline{AE^3 - c \times ACE} \times 4p$ = whole spindle ACBD, which is *the Rule.*

2. What is the solidity of a circular spindle, whose length is 24, and middle diameter 18?

Ans. 3739·93.

PROBLEM XVII.

To find the solidity of the middle frustum, or zone, of a circular spindle.

RULE.*

From the square of half the length of the whole spindle take $\frac{1}{3}$ of the square of half the length of the middle frustum, and multiply the remainder by the said half length of the frustum. Multiply the central distance by the revolving area, which generates the middle frustum.—Subtract this latter product from the former; and the remainder, multiplied by 6·2832, or twice 3·1416, will give the content.

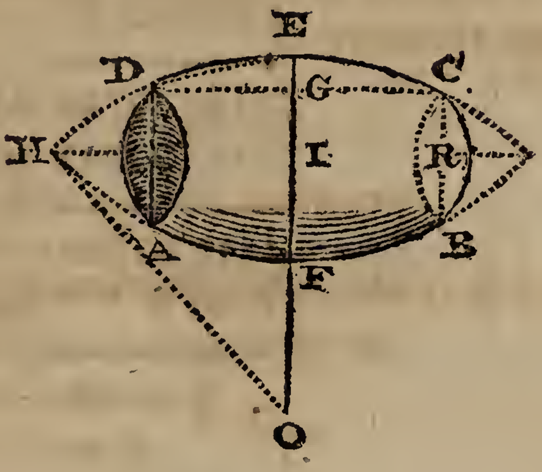
EXAMPLES.

1. Required the solidity of a frustum, whose length *mn* is 40 inches, its greatest diameter EF 32, and its least diameter AD or BC 24.

---

* Demonstration. Let $l =$ DG half the length of the zone, $L =$ HI half the length of the spindle, $c =$ FI the central distance; $a =$ the generating area of the zone. Then, by the Demonstration under the last Problem,

$\overline{\overline{L^2-\frac{1}{3}l^2}\times l-ac}\times p =$ the frustum generated by CEIR, therefore $\overline{\overline{L^2-\frac{1}{3}l^2}\times l-ac}\times 2p =$ the zone ABCD, which is the *Rule.*



Draw DG parallel to mn, then we have

$DG = \frac{1}{2} mn = 20$,

And $EG = \frac{1}{2} EF - \frac{1}{2} AD = 4$,

Chord $DE^2 = DG^2 + GE^2 = 416$,

And $DE^2 \div EG = \frac{416}{4} = 104$ the diameter of the generating circle, or the radius $OE = 52$;

Hence $OI = 52 - 16 = 36$ the central distance,

And $HI = OH^2 - OI^2 = 52^2 - 36^2 = 1408$,

$\frac{1}{3} DG^2 = \frac{1}{3}$ of $400 = 133\frac{1}{3}$

$1274\frac{2}{3}$
20

$25493\frac{1}{3}$ 1st prod.

$GE \div 2\,OE = \frac{4}{104} = \frac{1}{26} =$ ·03846 a ver. sine.

Its tabular segment ·00994

But 104^2 is 10816

43264
97344
97344

Area of seg. DECGD 107·51104

$mD \times mn = 12 \times 40$ 480·

Gener. area $mDECn$ 587·51104

OI 36

```
 352506624
176253312
---------
21150·39744  2d product.
25493·33333  1st product.
-----------
 4342·93589
 2382·6       mult. invert.
-----------
    2 60576
       8686
       3474
        130
          9
    -------
    27287·5 answer.
```

2. What is the content of the middle frustum of a circular spindle, whose length is 20, greatest diameter 18, and least diameter 8?

Ans. 3657·1613.

PROBLEM XVIII.

To find the superficies or solidity of any regular body.

RULE.*

1. Multiply the proper tabular area, taken from the following table, by the square of the linear edge of the solid, for the superficies.

2. Multiply the tabular solidity by the cube of the linear edge, for the solid content.

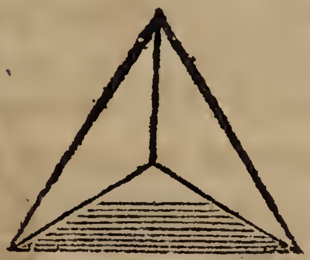
---

* DEMONSTRATION. The numbers in the Table express the surface and solidity of each body, when its edge is 1; and because in similar bodies the surfaces are as the squares of the linear edges, and the solidities as the cubes of the same; therefore the truth of the Rule is manifest.

SURFACES AND SOLIDITIES OF REGULAR BODIES.			
No. of Sides.	Names.	Surfaces.	Solidities.
4	Tetraedron	1·73205	0 11785
6	Hexaedron	6·00000	1·00000
8	Octaedron	3·46410	0·47140
12	Dodecaedron	20·64573	7·66312
20	Icosaedron	8·66025	2·18169

EXAMPLES.

1. If the linear edge of a tetraedron be 3, required its surface and solidity.



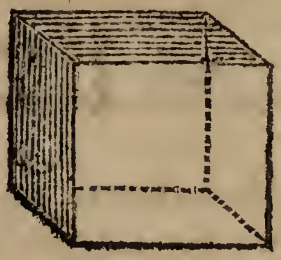
The square of 3 is 9, and the cube 27. Then,

tab. sur. 1·73205
9
Superficies 15·58845

0·11785 tab. sol.
27
82495
23570
Solidity 3·18195

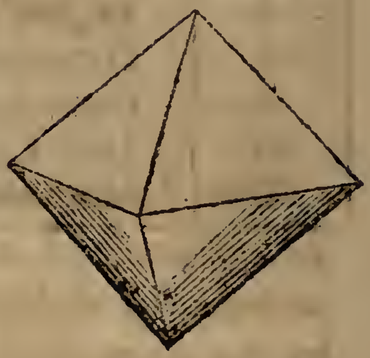
2. What is the superficies and solidity of a hexaedron, whose linear side is 2?

Ans. { Superficies 24.
Solidity 8.



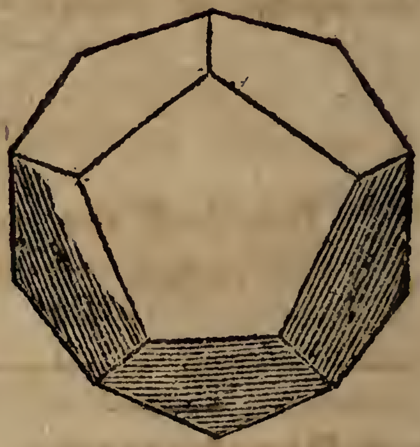
3. Required the superficies and solidity of an octaedron, whose linear side is 2.

Ans. { Superficies 13.85640.
Solidity 3.77120.



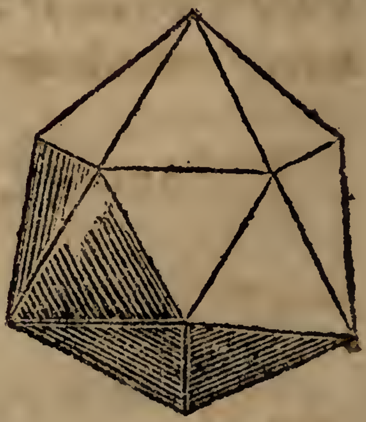
4. What is the superficies and solidity of a dodecaedron, whose linear side is 2 ?

Ans. { Superficies 82·58292.
Solidity 61·30496.



5. Required the superficies and solidity of an icosaedron, whose linear side is 2.

Ans. { Superficies 34·64100.
Solidity 17·45352.



PROBLEM XIX.

To find the surface of a cylindrical ring.

RULE.

This figure being only a cylinder bent round into a ring, its surface and solidity may be found as in the cylinder; namely, by multiplying the axis, or length of the cylinder, by the circumference of the ring, or section, for the surface; and by the area of a section, for the solidity.

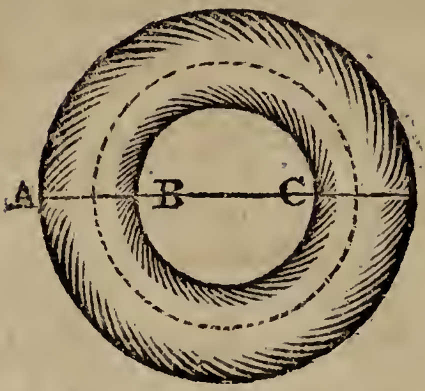
Or use the following rule for the surface:—To the thickness of the ring add the inner diameter; multiply this sum by the thickness, and the product again by 9·8696, or the square of 3·1416.*

EXAMPLES.

1. Required the superficies of a ring, whose thickness AB is 2 inches, and inner diameter BC is 12 inches.

---

* Demonstration. Let AB be any section of the ring perpendicular to its axis, then $AB \times 3{\cdot}14159$ = the circumference of that section, and $AB+BC \times 3{\cdot}14159$ = length of the axis; therefore $AB \times 3{\cdot}14159 \times \overline{AB+BC} \times 3{\cdot}14159 = \overline{AB+BC} \times AB \times \overline{3{\cdot}14159}^2 = \overline{AB+BC} \times AB \times 9.8696$ = the superficies; which is *the Rule.*



12	9·8696
2	28
14	789568
2	197392
28	276·3488 answer.

2. What is the surface of a ring, whose inner diameter is 16, and thickness 4 ? Answer 789·568.

PROBLEM XX.

To find the solidity of a cylindrical ring.

RULE*.

To the thickness of the ring add the inner diameter ; then multiply the sum by the square of the thickness, and the

---

* DEMONSTRATION. $AB^2 \times \cdot78539 = AB^2 \times \frac{3\cdot14159}{4} = \frac{1}{4} AB^2 \times 3\cdot14159$ = area of the section AB, and

$\overline{AB + BC} \times 3\cdot14159$ = length of the axis of the ring. Therefore $\overline{AB + BC} \times \frac{1}{4} AB^2 \times \overline{3\cdot14159}^2 = \overline{AB + BC} \times \frac{1}{4} AB^2 \times 9\cdot8698$. Q. E. D.

product again by 2·4674, or $\frac{1}{4}$ of the square of 3·1416, for the solidity.

EXAMPLES.

1. Required the solidity of a ring, whose thickness is 2 inches, and its inner diameter 12.

12	2·4674
2	56
14	148044
4	123370
56	138·1744 answer.

2. What is the solidity of a cylindrical ring, whose thickness is 4, and inner diameter 16?

Ans. 789·568.

PROBLEM XXI.

To find the solidity of a spheroid.

RULE*.

Square the revolving axis, multiply that square by the fixed axis, and multiply the product by ·5236, for the content.

---

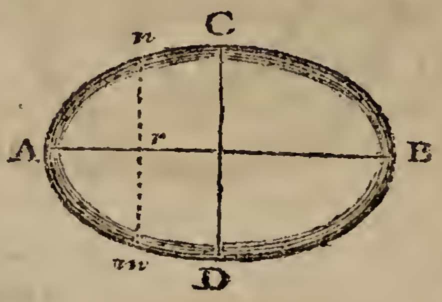
* DEMONSTRATION. Let AB $= a$, CD $= b$, $ar = x$, $rn = y$, and $p = 3\cdot14159$. Then $a^2 : b^2 :: x \times \overline{a - x} : \frac{b^2}{a^2} \times \overline{ax - x^2} = y^2$ by property of the ellipse.

Therefore the fluxion of the solid $= py^2\dot{x} = \frac{pb^2}{a^2} \times \overline{ax\dot{x} - x^2\dot{x}}$,

and its fluent $= \frac{pb^2}{a^2} \times \overline{\frac{1}{2}ax^2 - \frac{1}{3}x^3}$

EXAMPLES.

1. Required the solidity of the prolate spheroid ACBD, whose axes are AB 50, and CD 30.



30	·5236
30	45000
900	2618000
50	20944
45000	23562·0000 answer.

2. What is the content of an oblate spheroid, whose axes are 50 and 30? Ans. 39270.

3. What is the solidity of a prolate spheroid, whose axes are 9 and 7? Ans. 230·9076.

PROBLEM XXII.

To find the solidity of a segment of a spheroid.

CASE 1.

When the base is circular or parallel to the revolving axis.

= the segment nAm; which, when $x = a$, becomes

$$\frac{pb^2}{a^2} \times \overline{\tfrac{1}{2}a^3 - \tfrac{1}{3}a^3} = \frac{pab^2}{6}$$ = content of the whole spheroid.

Q. E. D.

RULE.*

Multiply the difference between triple the fixed axis and double the height of the segment by the square of the height, and the product again by ·5236.

Then, as the square of the fixed axis is to the square of the revolving axis, so is the last product to the content of the segment.

EXAMPLES.

1. Required the content of the segment of a prolate spheroid, the height AG being 5, the fixed axis AB 50, and the revolving axis CD 30.

---

* DEMONSTRATION. By the Note under the last Problem, $\frac{pb^2}{a^2} \times \overline{\frac{1}{2}ax^2 - \frac{1}{3}x^3}$ = the segment, whose height is x, the other letters representing the same quantities respectively as in the said Problem. But $\frac{pb^2}{a^2} \times \overline{\frac{1}{2}ax^2 - \frac{1}{3}x^3}$

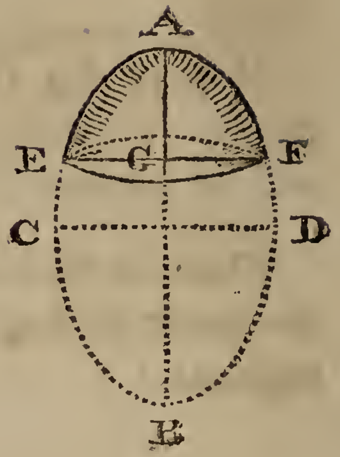
$$= \frac{pb^2}{6a^2} \times \overline{3ax^2 - 2x^3}$$

$$= \frac{b^2x^2}{a^2} \times \overline{3a - 2x} \times \cdot 5236$$

$$= \overline{3a - 2x} \times x^2 \times \cdot 5236 \times \frac{b^2}{a^2}$$

∴ as $a^2 : b^2 :: \overline{3a - 2x} \times x^2 \times \cdot 5236$: the content; which is *the Rule.*

150	·5236
10	·3500
140	2618000
25	15708
700	1832·6000
280	
3500	



Then, as 25 : 9 :: 1832·6 :
Or, as 100 : 36 :: 1832·6 : 659·736

```
           36
       ------
       109956
       54978
       ------
100)65973·6(659·736 answer.
```

2. If the axes of a prolate spheroid be 10 and 6, required the area of a segment, whose height is 1, and its base parallel to the revolving axis.

Ans. 5·277888.

3. The axes of an oblate spheroid being 50 and 30, what is the content of a segment, the height being 6, and its base parallel to the revolving axis?

Ans. 4084·07.

CASE 2.

When the base is perpendicular to the revolving axis.

RULE*.

Multiply the difference between triple the revolving axis

---

* DEMONSTRATION. Let $\frac{1}{2}$ AB $= a$, $\frac{1}{2}$ CD $= b$, $\frac{1}{2}$ the con-

and double the height of the segment by the square of the height, and the product again by ·5236.

Then, as the revolving axis is to the fixed axis,
So is the last product to the content.

EXAMPLES.

1. In the prolate spheroid ACBD, the fixed axis AB is 50, and the revolving axis CD 30; required the solidity of the segment CEF, its height CG being 6.

---

jugate to AB in the ellipse, which is parallel to the base of the segment, $= r$, CG $= x$, EG $= y$.

Then, by the nature of the ellipse, $b^2 : a^2 :: 2bx - x^2 : y^2 = \frac{a^2 \times \overline{2bx - x^2}}{b^2}$;

and $a : r :: y : \frac{ry}{a} = \frac{1}{2}$ the conjugate to the transverse EF of the base of the segment, by similar ellipses.

But the fluxion of the solid ECFG $= py\dot{x} \times \frac{ry}{a} = \frac{pry^2\dot{x}}{a}$

$= \frac{pr\dot{x}}{a} \times \frac{a^2 \times \overline{2bx - x^2}}{b^2}$; the fluent of which $= \frac{par}{b} x^2 -$

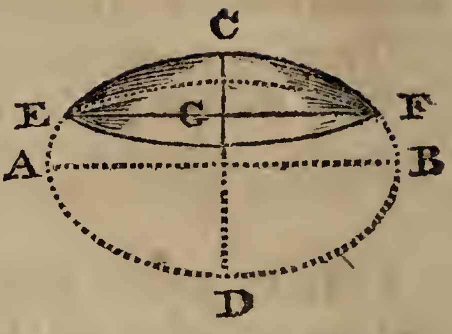
$\frac{par}{3b^2} x^3$, $=$content $= \frac{par}{3b^2} \times x^2 \times \overline{3b - x} =$, since $r = b$,

$\frac{pa}{3b} \times \overline{3b - x} \times x^2 =$, doubling a for transverse, and b for conjugate, $\frac{pa}{6b} \times \overline{3b - 2x} \times x^2$

$= \overline{3b - 2x} \times x^2 \times \cdot 5236 \times \frac{a}{b}$,

$\therefore b : a :: \overline{3b - 2x} \times x^2 \times \cdot 5236$: content of the segment, which is *the Rule*.

90	·5236
12	2808
78	41888
36	41888
468	10742
234	1470·2688
2808	



Then, as 30 : 50 :: 1470·2688 : 2450·448

5

3)7351·3440

2450·4480 answer.

2. In an oblate spheroid, whose axes are 50 and 30; required the content of the segment, whose height is 5, its base being perpendicular to the revolving axis.

Ans. 1099·56.

PROBLEM XXIII.

To find the content of the middle frustum of a spheroid.

CASE I.

When the ends are circular, or parallel to the revolving axis.

RULE.*

To double the square of the middle diameter add the square of the diameter of one end; multiply this sum by

---

* DEMONSTTATION. Let $a = \frac{1}{2}$ transverse of the generating ellipse, $b = \frac{1}{2}$ conjugate, $h =$ EA $\frac{1}{2}$ diameter of the end, $c =$ the distance of A from the centre of the spheroid or AO, $x =$ Or, $y = re$, and $p = 3 \cdot 14159$.

the length of the frustum, and the product by ·2618 for the content.

EXAMPLES.

1. Required the solidity of the middle frustum EGHF

Then $a^2 : b^2 :: a^2 - x^2 : \frac{b^2}{a^2} \times \overline{a^2 - x^2}$

$= b^2 - \frac{b^2 x^2}{a^2} = y^2$ by nature of the ellipse.

And also $a^2 : b^2 :: \overline{a^2 - c^2} : \frac{b^2 \times \overline{a^2 - c^2}}{a^2}$

$= h^2$, or $a^2 = \frac{b^2 c^2}{b^2 - h^2}$; by substituting this value of a^2 in the

former equation, we have $y^2 = b^2 - \frac{b^4 x^2 - b^2 h^2 x^2}{b^2 c^2}$

$= b^2 - \frac{b^2 x^2 - h^2 x^2}{c^2} = b^2 - \frac{x^2 \times \overline{b^2 - h^2}}{c^2}$.

Consequently the fluxion of the solid $py^2 \dot{x} = pb^2 \dot{x} -$ $\frac{px^2 \dot{x} \times \overline{b^2 - h^2}}{c^2}$; the fluent of which $= pb^2 x - \frac{px^3 \times \overline{b^2 - h^2}}{3c^2}$

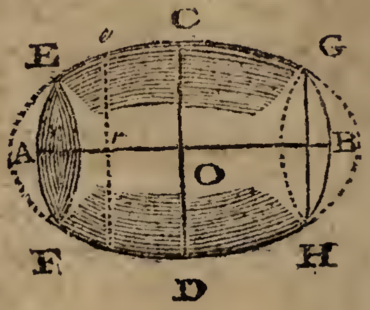
$=$, when $x = c$, $pb^2 c - \frac{pcb^2 - pch}{3}$

$= \frac{pc \times \overline{2b^2 + h^2}}{3} = \frac{pc \times \overline{8b^2 + 4h^2}}{12}$

$= \frac{pc \times \overline{2b^2 + h^2}}{3}$

$=$, putting b for the middle diameter, and h for the length of the frustum, $\overline{2b^2 + h^2} \times c \times {\cdot}2618$, which is *the Rule.*

of a spheroid, the greatest diameter CD being 30, the diameter of each end EF or GH 18, and the length AB 40.



18	30
18	30
144	900
18	2
324	1800
	324
	2124
	40
	84960
	·2618
	679680
	8496
	50976
	16992
	22242·5280 answer.

2. What is the solidity of the middle frustum of an oblate spheroid, having the diameter of each circular end 40, the middle 50, and the length 18?

Ans. 31101·84.

CASE 2.

When the ends are elliptical, or perpendicular to the revolving axis.

RULE.*

To double the product of the transverse and conjugate di-

* Demonstration. Let $a = \frac{1}{2}$ transverse AO, $b = \frac{1}{2}$ the

ameters of the middle section add the product of the transverse and conjugate of one end; multiply the sum by the length of the frustum, and the product by ·2618 for the content.

EXAMPLES.

1. In the middle frustum EFGH of an oblate spheroid, the diameters of the middle or greatest elliptic section AB

---

conjugate, $r = om$, $x = OI$, $y = IE$, and $p = 3·14159$.

Then $b^2 : a^2 :: b^2 - x^2 : \frac{a^2 \times \overline{b^2 - x^2}}{b^2} = y^2$, by the nature of the ellipse; and $a : r :: y : \frac{ry}{a} = \frac{1}{2}$ conjugate drawn through I parallel to om, by similar ellipses.

But the fluxion of the solid EABF is $py\dot{x} \times \frac{ry}{a} = \frac{pry^2\dot{x}}{a}$

$= \frac{pr\dot{x}}{a} \times \frac{a^2 \times \overline{b^2 - x^2}}{b^2}$

$= pra\dot{x} \times \frac{b^2 - x^2}{b^2}$;

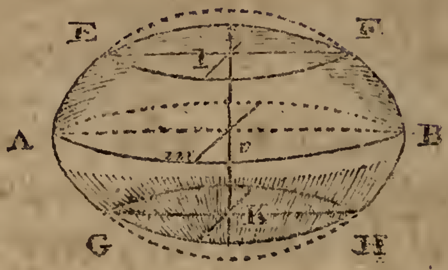
the fluent $= prax \times \frac{b^2 - \frac{1}{3}x^2}{a^2}$,

=, putting for b^2 its value $\frac{a^2 x^2}{a^2 - y^2}$, $prx \times \frac{2a^2 + y^2}{3a} = px \times \frac{2ra + \frac{ry^2}{a}}{3}$. This =, putting z for $\frac{ry}{a}$, $\frac{px}{3} \times \overline{2ra + yz}$

= the frustum ABFE, and

$\frac{p \times IK}{12} \times 2AB \times 2om + EF \times 2z =$ the middle frustum AEFB. Q. E. D.

are 50 and 30, and of one end EF or GH 40 and 24; required the content, the height IK being 9.




```
  24            50
  40            30
 ----          ----
 960           1500
                  2
               ----
               3000
                960
               ----
               3960
                  9
              -----
              35640
              ·2618
             ------
             285120
             3564
            21384
            7128
           ---------
           9330·5520 answer.
```

2. In the middle frustum of an oblate spheroid, the axes of the middle ellipse are 50 and 30, and those of each end are 30 and 18; required the content, the height being 40.

Ans. 37070·88

PROBLEM XXIV.

To find the solidity of an elliptic spindle.

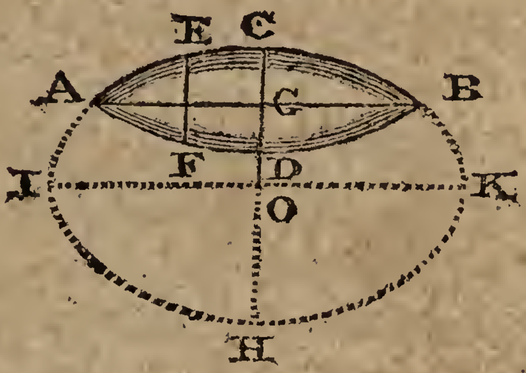
RULE.

To the square of the greatest diameter add the square of double the diameter at $\frac{1}{4}$ of the length; multiply the sum by the length, and the product by ·1309 for the solidity, very nearly.

NOTE. This rule will also serve for any other solid, formed by the revolution of any conic section.

EXAMPLE.

What is the solid content of an elliptic spindle, whose length is 20, the greatest diameter 6, and the diameter at $\frac{1}{4}$ of the length 4·74773 ?



	4·74773
	2
Double the diam.	9·49546
Ditto inverted	645949
	8545914
	379818
	85459
	4748
	380
	56

90·16375 square of double diameter.
36·00000 square of other diameter.

126·16375 sum.
20 length.

2523·27500
9031 or 1309 inverted.

2523
757
22

330·2 the solidity nearly.

PROBLEM

PROBLEM XXV.

To find the solidity of a frustum, or segment, of an elliptic spindle.

RULE.

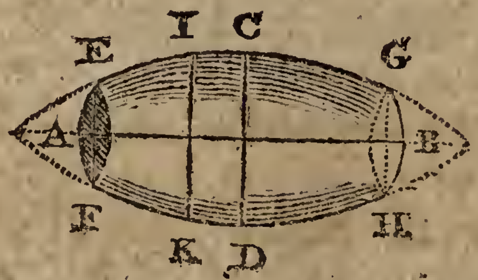
Proceed, as in the last rule, for this, or any other solid, formed by the revolution of a conic section about an axis; namely,

Add together the squares of the greatest and least diameters, and the square of double the diameter in the middle between the two; multiply the sum by the length, and the product by ·1309 for the solidity.

NOTE. For all such solids this rule is exact when the body is formed by the conic section, or a part of it, revolved about the axis of the section. And it will always be very near the truth when the figure revolves about another line to form the body.

EXAMPLES.

1. Required the content of the middle frustum EGHF of any spindle, the length AB being 40, the greatest or middle diameter CD 32, the least or diameter at either end EF or GH 24, and the diameter IK in the middle between EF and CD 30·157568.



32
32
———
64
96
———
1024
———
24
24
———
96
48
———
576
———

30'157568
2
———
60'315136 double.
51306 inverted.
———
361890
1809
60
30
———
3637'89 square of 2IK.
1024'00 square of CD.
576'00 square of EF.
———
5237'89 sum.
40 length.
———
209515·60
9031 inverted.
———
20951
6285
188
———
27424 answer.
———

2. What is the content of the segment of any spindle, the length being 10, the greatest diameter 8, and the middle diameter 6 ?

Ans. 272'272.

3. Required the solidity of the frustum of a hyperbolic conoid, the height being 12, the greatest diameter 10, the least diameter 6, and the middle diameter $8\frac{1}{2}$.

Ans. 667'59.

4. What is the content of the middle frustum of a hyperbolic spindle, the length being 20, the middle or greatest diameter 16, the diameter at each end 12, and the diameter at $\frac{1}{4}$ of the length $14\frac{1}{2}$?

Ans. 3248·938.

PROBLEM XXVI.

To find the solidity of a parabolic conoid.

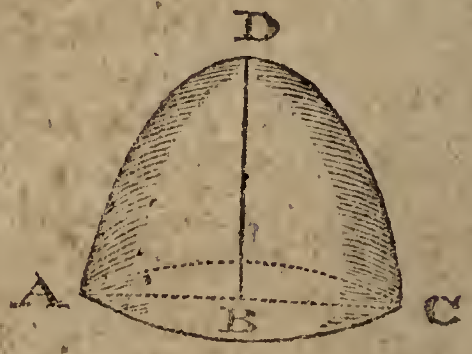
RULE.*

Multiply the square of the diameter of the base by the altitude, and the product by ·3927, for the content.

EXAMPLES.

1. Required the solidity of a paraboloid, whose height BD is 30, and the diameter of its base AC 40.

```
      40
      40
    ----
    1600
      30
   -----
   48000
   ·3927
  ------
  31416
 15708
---------
18849·6000
```



2. What is the content of a parabolic conoid, whose altitude is 42, and the diameter of its base 24 ?

Ans. 9500·1984.

PROBLEM XXVII.

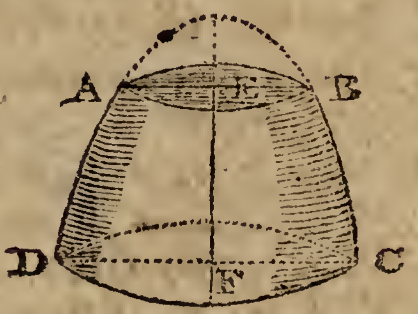
To find the solidity of the frustum of a paraboloid.

RULE.

Multiply the sum of the squares of the diameters of the two ends by the height, and the product by ·3927, for the content.

EXAMPLES.

1. Required the content of the paraboloidal frustum AB CD, the diameter AB being 20, the diameter DC 40, and the height EF $22\frac{1}{2}$.



1600	DC^2
400	AB^2
2000	
$22\frac{1}{2}$	EF
45000	
·3927	
19635000	
1·5708	
17671·5000	answer.

2. What is the content of the frustum of a paraboloid, the greatest diameter being 30, the least 24, and the altitude 9?

Ans. 5216·6266.

PROBLEM XXVIII.

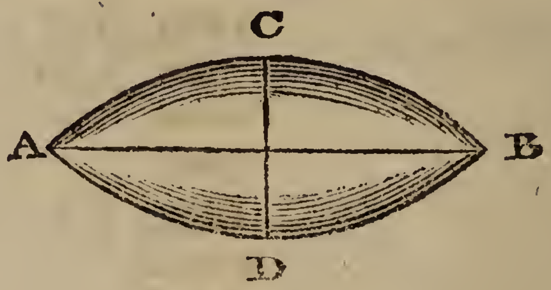
To find the solidity of a parabolic spindle.

RULE.

Multiply the square of the middle or greatest diameter by the length, and the product by ˙41888, for the content.

EXAMPLES.

1. Required the content of the parabolic spindle ACBD, whose length AB is 40, and the greatest diameter CD 16.




```
     16 CD
     16
    ----
     96
    16
    ----
    256 CD²
     40 AB
   -----
  10240
  ˙41888
 -------
 1675520
 83776
41888
--------
4289˙33120 answer.
--------
```

2. What is the solidity of a parabolic spindle, whose length is 18, and its middle diameter 6 feet?

Ans. 271˙4336.

PROBLEM XXIX.

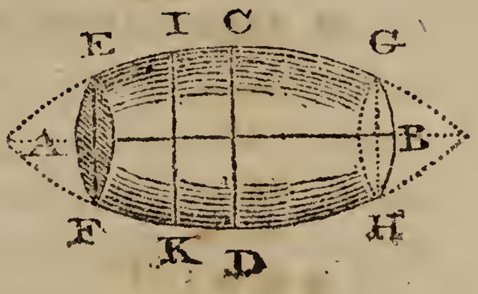
To find the solidity of the middle frustum of a parabolic spindle.

RULE.

Add together 8 times the square of the greatest diameter, 3 times the square of the least diameter, and 4 times the product of the two diameters; multiply the sum by the length, and the product by ·05236, for the solidity.

EXAMPLES.

1. Required the content of the frustum of a parabolic spindle EGHF, the length AB being 20, the greatest diameter CD 16, and the least diameter EF 12.




```
    16
    16
    ——
    96
   16
   ———
   256
     8
  ————
  2048  8CD²
   432  3EF²
   768  4CD×EF
  ————
  3248 sum
    20 AB
 —————
 64960
 ·05236
 ——————
  389760
 19488
12992
32480
——————————
3401·30560 answer.
```

2. What is the content of the frustum of a parabolic spindle, whose length is 18, greatest diameter 18, and least 10?

Ans. 3404·23776.

NOTE. The solidities of the hyperboloid and hyperbolic spindle are to be found by Rule to Prob. XXIV. And those of their frustums by Prob. XXV; where some examples of them are given.

MISCELLANEOUS QUESTIONS

IN MENSURATION OF SUPERFICIES AND SOLIDS.

1. WHAT difference is there between a floor 28 feet long by 20 broad, and two others, each of half the dimensions; and what do the three floors come to at 45s. per 100 square feet?

Ans. Diff. 280 square feet. Amount 18 guineas.

2. An elm plank is 14 feet 3 inches long, and it is desired, that just a square yard may be slit off from it; at what distance from the edge must the line be struck?

Ans. $7\frac{99}{171}$ inches.

3. A ceiling contains 114 yards 6 feet of plastering, and the room is 28 feet broad; what is the length of it?

Ans. $36\frac{6}{7}$ feet.

4. A common joist is 7 inches deep and $2\frac{1}{2}$ thick; but a scantling just as big again, that shall be three inches thick, is wanted; what will the other dimension be?

Ans. $11\frac{2}{3}$ inches.

5. A wooden trough cost 3s. 6d. for painting within at 6d. per yard; the length of it was 102 inches, and the depth 21 inches; what was the width?

Ans. $27\frac{1}{4}$ inches.

6. If a court yard be 47 feet 9 inches square, and a foot path of 4 feet wide be laid with purbeck stone along one side of it; what will the paving of the rest with flints come to, at 6d. per square yard?

Ans. 5l. 16s. $0\frac{1}{2}$d.

7. There are two columns in the ruins of Persepolis left standing upright; one is 64 feet above the plane, and the other 50; in a straight line between these stands an ancient small statue, the head of which is 97 feet from the summit of the higher, and 86 feet from the top of the lower column, the base of which measures just 76 feet to the centre of the figure's base; required the distance between the tops of the two columns.

Ans. 152 feet nearly.

8. The perambulator, or surveying wheel, is so contrived, as to turn just twice in the length of a pole, or $16\frac{1}{2}$ feet; required the diameter.

Ans. 2·626 feet.

9. In turning a one horse chaise within a ring of a certain diameter, it was observed, that the outer wheel made two revolutions while the inner made but one; the wheels were both 4 feet high; and, supposing them fixed at the statutable distance of 5 feet asunder on the axletree, what was the circumference of the track described by the outer wheel?

Ans. 63 feet nearly.

10. What is the side of that equilateral triangle, whose area cost as much for paving at 8d. sterling a foot, as the pallisading of the three sides did at a guinea a yard?

Ans. 72·746 feet.

11. A roof, which is 24 feet 8 inches by 14 feet 6 inches, is to be covered with lead, at 8lb. to the square foot; what will it come to, at 18s. per cwt.?

Ans. 22l. 19s. 10$\frac{1}{4}$d.

12. Having a rectangular marble slab, 58 inches by 27, I would have a square foot cut off parallel to the shorter edge; I would then have the like quantity divided from the remainder parallel to the longer side; and this alternately repeated, till there shall not be the quantity of a foot left; what will be the dimensions of the remaining piece?

Ans. 20·7 inches by 6·086.

13. If from a right-angled triangle, whose base is 12 and perpendicular 16 feet, a line be drawn parallel to the perpendicular cutting off a triangle, whose area is 24 square feet; required the sides of this triangle.

Ans. 6, 8, and 10.

14. If a round pillar, 7 inches over, have 4 feet of stone in it; of what diameter is the column of equal length, that contains 10 times as much?

Ans. 22·136 inches.

15. The area of an equilateral triangle, whose base falls on the diameter, and its vertex in the middle of the arc of a semicircle, is equal to 100; what is the diameter of the semicircle?

Ans. 26·32148.

16. It is required to find the thickness of the lead in a pipe of an inch and a quarter bore, which weighs 14lb. per yard in length; the cubic foot of lead weighing 11325 ounces.

Ans. ·20737 inches.

17. What is the length of a chord, which cuts off $\frac{1}{3}$ of the area of a circle, whose diameter is 289?

Ans. 278·6716.

18. What will the diameter of a globe be, when the solidity and superficial content are expressed by the same number?

Ans. 6.

19. A sack, that will hold 3 bushels of corn, is $22\frac{1}{2}$ inches broad, when empty; what will that sack contain, which, being of the same length, has twice its breadth or circumference?

Ans. 12 bushels.

20. A carpenter is to put an oaken curb to a round well, at 3d. per foot square; the breadth of the curb is to be $7\frac{1}{4}$ inches, and the diameter within $13\frac{1}{2}$ feet; what will be the expense?

Ans. 5s. $2\frac{1}{4}$d.

21. A gentleman has a garden 100 feet long and 80 feet broad, and a gravel walk is to be made of an equal width half round it; what must the breadth of the walk be to take up just half the ground?

Ans. 25·968 feet.

22. Seven men bought a grinding stone of 60 inches diameter, each paying $\frac{1}{7}$ part of the expense; what part of the diameter must each grind down for his share?

Ans. The 1st, 4·4508,
2d, 4·8400,
3d, 5·3535,
4th, 6·0765,
5th, 7·2079,
6th, 9·3935,
7th, 22·6778.

23. A maltster has a kiln, that is 16 feet 6 inches square; but he wants to pull it down and build a new

one, that will dry 3 times as much at once as the old one; what must be the length of its sides?

Ans. 28 feet 7 inches.

24. How many 3 inch cubes may be cut out of a 12 inch cube? Ans. 64.

25. What will the painting of a conical spire come to, at 8d. per yard; supposing the height to be 118 feet, and the circumference of the base 64 feet? Ans. 14l. 8$\frac{3}{4}$d.

26. The diameter of a standard corn bushel is 18$\frac{1}{2}$ inches, and its depth 8 inches; what must the diameter of that bushel be, whose depth is 7$\frac{1}{2}$ inches?

Ans. 19·1067.

27. To divide a cone into three equal parts by sections parallel to the base, and to find the altitudes of the three parts; the height of the whole cone being 20 inches.

Ans. The upper part 13·867.
The middle part 3·604.
The lower part 2·528.

28. A gentleman has a bowling green 300 feet long and 200 feet broad, which he would raise one foot higher by means of the earth to be dug out of a ditch, that goes round it; to what depth must the ditch be dug, supposing its breadth to be every where 8 feet?

Ans. 7$\frac{23}{86}$.

29. How high above the earth must a person be raised, that he may see $\frac{1}{3}$ of its surface?

Ans. To the height of the earth's diameter.

30. A cubic foot of brass is to be drawn into a wire of $\frac{1}{40}$ of an inch in diameter; what will the length of the wire be, allowing no loss in the metal?

Ans. 97784·797 yards, or 55 miles 984·797 yards.

31. If a heavy sphere, whose diameter is 4 inches, be let fall into a conical glass full of water, whose diameter is 5, and altitude 6 inches; it is required to determine how much water will run over.

Ans. 26·272 cubic inches, or nearly $\frac{35}{47}$ parts of a pint.

32. The dimensions of the sphere and cone being the same as in the last question, and the cone only $\frac{1}{5}$ full of water; required what part of the axis of the sphere is immersed in the water.

Ans. ·546 parts of an inch.

33. The cone being still the same, and $\frac{1}{5}$ full of water; required the diameter of a sphere, which shall be just covered by the water.

Ans. 2·445996.

TABLE

OF THE *AREAS* OF THE *SEGMENTS* OF A *CIRCLE*, WHOSE *DIAMETER* IS *UNITY*, AND SUPPOSED TO BE DIVIDED INTO 1000 *EQUAL PARTS*.

Height.	Area Seg.	Height.	Area Seg.	Height.	Area Seg.
·001	·000042	·029	·006527	·057	·017831
·002	·000119	·030	·006865	·058	·018296
·003	·000219	·031	·007209	·059	·018766
·004	·000337	·032	·007558	·060	·019239
·005	·000470	·033	·007913	·061	·019716
·006	·000618	·034	·008273	·062	·020196
·007	·000779	·035	·008638	·063	·020680
·008	·000951	·036	·009008	·064	·021168
·009	·001135	·037	·009383	·065	·021659
·010	·001329	·038	·009763	·066	·022154
·011	·001533	·039	·010148	·067	·022652
·012	·001746	·040	·010537	·068	·023154
·013	·001968	·041	·010931	·069	·023659
·014	·002199	·042	·011330	·070	·024168
·015	·002438	·043	·011734	·071	·024680
·016	·002685	·044	·012142	·072	·025195
·017	·002940	·045	·012554	·073	·025714
·018	·003202	·046	·012971	·074	·026236
·019	·003471	·047	·013392	·075	·026761
·020	·003748	·048	·01[illegible]818	·076	·027289
·021	·004031	·049	·014247	·077	·027821
·022	·004322	·050	·014681	·078	·028356
·023	·004618	·051	·015119	·079	·028894
·024	004921	·052	·015561	·080	·029435
·025	·005230	·053	·016007	·081	·029979
·026	·005546	·054	·016457	·082	·030526
·027	·005867	·055	·016911	·083	·031076
·028	·006194	·056	·017369	·084	·031629

Height.	Area Seg.	Height.	Area Seg.	Height.	Area Seg.
·085	·032186	·124	·056003	·163	·083320
·086	·032745	·125	·056663	·164	·084059
·087	·033307	·126	·057326	·165	·084801
·088	·033872	·127	·057991	·166	·085544
·089	·034441	·128	·058658	·167	·086289
·090	·035011	·129	·059327	·168	·087036
·091	·035585	·130	·059999	·169	·087785
·092	·036162	·131	·060672	·170	·088535
·093	·036741	·132	·061348	·171	·089287
·094	·037323	·133	·062026	·172	·090041
·095	·037909	·134	·062707	·173	·090797
·096	·038496	·135	·063389	·174	·091554
·097	·039087	·136	·064074	·175	·092313
·098	·039680	·137	·064760	·176	·093074
·099	·040276	·138	·065449	·177	·093836
·100	·040875	·139	·066140	·178	·094601
·101	·041476	·140	·066833	·179	·095366
·102	·042080	·141	·067528	·180	·096134
·103	·042687	·142	·068225	·181	·096903
·104	·043296	·143	·068924	·182	·097674
·105	·043908	·144	·069625	·183	·098447
·106	·044522	·145	·070328	·184	·099221
·107	·045139	·146	·071033	·185	·099997
·108	·045759	·147	·071741	·186	·100774
·109	·046381	·148	·072450	·187	·101553
·110	·047005	·149	·073161	·188	·102334
·111	·047632	·150	·073874	·189	·103116
·112	·048262	·151	·074589	·190	·103900
·113	·048894	·152	·075306	·191	·104685
·114	·049528	·153	·076026	·192	·105472
·115	·050165	·154	·076747	·193	·106261
·116	·050804	·155	·077469	·194	·107051
·117	·051446	·156	·078194	·195	·107842
·118	·052090	·157	·078921	·196	·108636
·119	·052736	·158	·079649	·197	·109430
·120	·053385	·159	·080380	·198	·110226
·121	·054036	·160	·081112	·199	·111024
·122	·054689	·161	·081846	·200	·111823
·123	·055345	·162	·082582	·201	·112624

Height.	Area Seg.	Height.	Area Seg.	Height.	Area Seg.
·202	·113426	·241	·145799	·280	·180019
·203	·114230	·242	·146655	·281	·180918
·204	·115035	·243	·147512	·282	·181817
·205	·115842	·244	·148371	·283	·182718
·206	·116650	·245	·149230	·284	·183619
·207	·117460	·246	·150091	·285	·184521
·208	·118271	·247	·150953	·286	·185425
·209	·119083	·248	·151816	·287	·186329
·210	·119897	·249	·152680	·288	·187234
·211	·120712	·250	·153546	·289	·188140
·212	·121529	·251	·154412	·290	·189047
·213	·122347	·252	·155280	·291	·189955
·214	·123167	·253	·156149	·292	·190864
·215	·123988	·254	·157019	·293	·191775
·216	·124810	·255	·157890	·294	·192684
·217	·125634	·256	·158762	·295	·193596
·218	·126459	·257	·159636	·296	·194509
·219	·127285	·258	·160510	·297	·195422
·220	·128113	·259	·161386	·298	·196337
·221	·128942	·260	·162263	·299	·197252
·222	·129773	·261	·163140	·300	·198168
·223	·130605	·262	·164019	·301	·199085
·224	·131438	·263	·164899	·302	·200003
·225	·132272	·264	·165780	·303	·200922
·226	·133108	·265	·166663	·304	·201841
·227	·133945	·266	·167546	·305	·202761
·228	·134784	·267	·168430	·306	·203683
·229	·135624	·268	·169315	·307	·204605
·230	·136465	·269	·170202	·308	·205527
·231	·137307	·270	·171089	·309	·206451
·232	·138150	·271	·171978	·310	·207376
·233	·138995	·272	·172867	·311	·208301
·234	·139841	·273	·173758	·312	·209227
·235	·140688	·274	·174649	·313	·210154
·236	·141537	·275	·175542	·314	·211082
·237	·142387	·276	·176435	·315	·212011
·238	·143238	·277	·177330	·316	·212940
·239	·144091	·278	·178225	·317	·213871
·240	·144944	·279	·179122	·318	·214802

Height.	Area Seg.	Height.	Area Seg.	Height.	Area Seg.
·319	·215733	·358	·252631	·397	·290432
·320	·216666	·359	·253590	·398	·291411
·321	·217599	·360	·254550	·399	·292390
·322	·218533	·361	·255510	·400	·293369
·323	·219468	·362	·256471	·401	·294349
·324	·220404	·363	·257433	·402	·295330
·325	·221340	·364	·258395	·403	·296311
·326	·222277	·365	·259357	·404	·297292
·327	·223215	·366	·260320	·405	·298273
·328	·224154	·367	·261284	·406	·299255
·329	·225093	·368	·262248	·407	·300238
·330	·226033	·369	·263213	·408	·301220
·331	·226974	·370	·264178	·409	·302203
·332	·227915	·371	·265144	·410	.303187
·333	·228858	·372	·266111	·411	·304171
·334	·229801	·373	·267078	·412	·305155
·335	·230745	·374	·268045	·413	·306140
·336	·231689	·375	·269013	·414	·307125
·337	·232634	·376	·269982	·415	·308110
·338	·233580	·377	·270951	·416	·309095
·339	·234526	·378	·271920	·417	·310081
·340	·235473	·379	·272890	·418	·311068
·341	·236421	·380	·273861	·419	·312054
·342	·237369	·381	·274832	·420	·313041
·343	·238318	.382	·275803	·421	·314029
·344	·239268	·383	·276775	·422	·315016
·345	·240218	·384	·277748	·423	·316004
·346	·241169	·385	·278721	·424	·316992
·347	·242121	·386	·279694	·425	·317981
·348	·243074	·387	·280668	·426	·318970
·349	·244026	·388	·281642	·427	·319959
·350	·244980	·389	·282617	·428	·320948
·351	·245934	·390	·283592	·429	·321938
·352	·246889	·391	·284568	·430	·322928
·353	·247845	·392	·285544	·431	·323918
·354	·248801	·393	·286521	·432	·324909
·355	·249757	·394	·287498	·433	·325900
·356	·250715	·395	·288476	·434	·326892
·357	·251673	·396	·289453	·435	·327882

Height.	Area Seg.	Height.	Area Seg.	Height.	Area Seg.
·436	·328874	·458	·350748	·480	·372704
·437	·329866	·459	·351745	·481	·373703
·438	·330858	·460	·352742	·482	·374702
·439	·331850	·461	·353739	·483	·375702
·440	·332843	·462	·354736	·484	·376702
·441	·333836	·463	·355732	·485	·377701
·442	·334829	·464	·356730	·486	·378701
·443	·335822	·465	·357727	·487	·379700
·444	·336816	·466	·358725	·488	·380700
·445	·337810	·467	·359723	·489	·381699
·446	·338804	·468	·360721	·490	·382699
·447	·339798	·469	·361719	·491	·383699
·448	·340793	·470	·362717	·492	·384699
·449	·341787	·471	·363715	·493	·385699
·450	·342782	·472	·364713	·494	·386699
·451	·343777	·473	·365712	·495	·387699
·452	·344772	·474	·366710	·496	·388699
·453	·345768	·475	·367709	·497	·389699
·454	·346764	·476	·368708	·498	·390699
·455	·347759	·477	·369707	·499	·391699
·456	·348755	·478	·370706	·500	·392699
·457	·349752	·479	·371705		

USE OF THE TABLE.

In the table, each number in the column of *area seg.* is the area of the circular segment, whose height, or the versed sine of its half arc, is the number immediately on the left of it, in the column of *heights;* the diameter of the circle being 1, and its whole area ·785398.

The use of this table is to find the area of a segment of any other circle, whatever be the diameter. See the rule, page 69.

In dividing the given height by the diameter, if the quotient do not terminate in three places of decimals without a fractional remainder, then for the area, answering to that fractional part, proportion ought to be made thus ;—having found the tabular area answering to the first three decimals of the quotient, take the difference between it and the next following tabular area, which difference multiply by the fractional remaining part of the quotient, and the product will be the corresponding proportional part to be added to the first tabular area.

EXAMPLE.

If the height of the proposed segment be $3\frac{1}{3}$, and the diameter 50 ; required the area.

Here 50)$3\frac{1}{3}$(.066$\frac{2}{3}$
Then to ·066 answers ·022154
The next area is ·022652

Their difference is 498
$\frac{2}{3}$ of which is 232
This, added to ·022154

Gives the whole tabular area ·022386
And, multiplied by 2500

Gives the area 55·965000.

When the segment to be found is greater than a semi-circle, subtract the quotient of its versed sine, divided by its diameter, from 1 ; subtract the tabular segment, which corresponds to the remainder, from ·785398 ; and multiply the remainder by the square of the diameter.

GAUGING.

GAUGING signifies the art of measuring all kinds of vessels, and determining their capacity, or the quantity of fluid or other matter they contain. The term is from a gauge, or rod; as the practitioners of the art perform the business, or make the calculations, commonly by means of instruments, called the *gauging* or *diagonal rod*, and the *gauging* or *sliding rule*. The vessels are principally pipes, tuns, barrels, rundlets, and other casks; also backs, coolers, vats, &c.

It is usual to divide casks into four cases or varieties, which are judged of from the greater or less apparent curvature of their sides; namely,

1. The middle frustum of a spheroid.

2. The middle frustum of a parabolic spindle.

3. The two equal frustums of a paraboloid.

4. The two equal frustums of a cone.

And if the content of any of these be computed in inches, by the proper rule, and this be divided by 282, 231, or 2150·4, the quotient will be the content in ale gallons, wine gallons, or malt bushels, respectively. Because

282 cubic inches make 1 ale gallon,
231 . . . . . 1 wine gallon,
2150·4 . . . . 1 malt bushel.

Or, the particular rule for each will be as in the following PROBLEMS.

PROBLEM I.

To find the content of a cask of the first form.

RULE.*

To the square of the head diameter add double the square of the bung diameter, and multiply the sum by the length of the cask. Then let the product be multiplied by ·0009¼, or divided by 1077, for ale gallons; and multiplied by ·0011⅓, or divided by 882, for wine gallons.

EXAMPLES.

1. Required the content of a spheroidal cask, whose

---

* DEMONSTRATION. Let B = the bung diameter, H = the head diameter, L = the length. Then, by Problem XXIII, Case 1, Mens. of Solids,

$\overline{2B^2+H^2} \times L \times \cdot 2618$ = Content in inches;

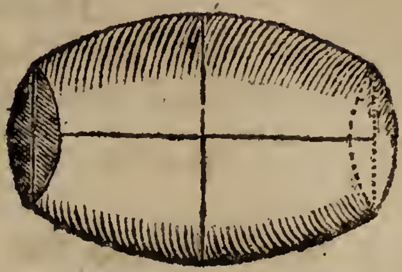
But $\dfrac{\overline{2B^2+H^2} \times L \times \cdot 2618}{282} = \dfrac{2B^2+H^2}{1077\cdot 157} \times L =$

$\overline{2B^2+H^2} \times \cdot 00092837 \times L$;

and $\dfrac{\overline{2B^2+H^2} \times L \times \cdot 2618}{231} = \dfrac{2B^2+H^2}{882\cdot 355} \times L$

$= \overline{2B^2+H^2} \times L \times \cdot 00113333$; which is *the Rule.*

length is 40, and bung and head diameters 32 and 24 inches.



24
24
———
96
48
———
576
———
32
32
———
64
96
———
1024
2
———
2048
576
———
2624
40
———
104960
·0009¼
———
944640
26240
———
970880 ale gallons.

104960
·0011⅓
———
1154560
34987
———
118·9547 wine gal.

BY THE GAUGING RULE.

40 on C being set to the ale gauge 32·82 on D, against
24 on D stands 21·3 on C,
32 on D stands 38·0 on C.
The same 38·0
———
Sum 97·3 ale gallons.

And 40 on C being set to the wine gauge 29·7 on D,
against 24 on D stands 26·1 on C,
32 on D stands 46·5 on C.
The same 46·5

Sum 119·1 wine gallons.

2. Required the content of the spheroidal cask, whose length is 20, and diameters 12 and 16 inches.

Ans. { 12·136 ale gallons. 14·869 wine gallons.

PROBLEM II.

To find the content of a cask of the second form.

RULE*.

To the square of the head diameter add double the square of the bung diameter, and from the sum take $\frac{2}{5}$ or $\frac{4}{10}$ of the square of the difference of the diameters; then multiply the remainder by the length, and the prod-

---

* DEMONSTRATION. For, by Problem XXIX, Mens. of Solids, $\overline{8B^2 + 3H^2 + 4BH} \times L \times \cdot 05236 =$ Content in inches;
But this expression

$$= \frac{8B^2 + 3H^2 + 4BH}{5} \times L \times \cdot 2618$$

$$= \overline{\frac{10B^2 + 5H^2 + 4BH}{5} - \frac{2B^2 + 2H^2}{5}} \times L \times \cdot 2618$$

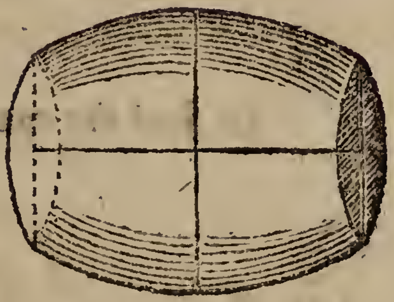
$$= \overline{2B^2 + H^2 - \tfrac{2}{5} \times \overline{B - H}^2} \times L \times \cdot 2618.$$

Hence, by proceeding as under the last Problem, the decimal multipliers are obtained from ·2618, 282, and 231. Q. E. D.

uct by ·0009¼, for ale gallons, or by ·0011⅓, for wine gallons.

EXAMPLES.

1. The length being 40, and diameters 24 and 32, required the content.



32
24
—
8
8
—
64
4
—
25·6
—

2624·0 = square of the head diameter added to twice the square of the bung diameter.
25·6
—
2598·4
40
—
103936
·0009¼
—
935424
25984
—
96·1408 ale gallons.

103936
·0011⅓
—
1143296
34645
—
117·7941 wine gallons.

BY THE GAUGING RULE.

40 on C being set to 32·82 on D, against 8 on D stands 2·4 on C; the $\frac{4}{10}$ of which is 0·96. This, taken from 97·3 in the last form, leaves 96·3 ale gallons.

And 40 on C being set to 29·7 on D, against 8 on D stands 2·9 on C; the $\frac{4}{10}$ of which is 1·16. This, taken from 119·1 in the last form, leaves 117·9 wine gallons.

2. Required the content when the length is 20, and the diameters 12 and 16.

Ans. $\begin{cases} 12·018 \text{ ale gallons.} \\ 14·724 \text{ wine gallons.} \end{cases}$

PROBLEM III.

To find the content of a cask of the third form.

RULE.*

To the square of the bung diameter add the square of the head diameter; multiply the sum by the length, and the product by ·0014, for ale gallons, or by ·0017, for wine gallons.

EXAMPLES.

1. Required the content of a cask of the third form, when the length is 40, and the diameters 24 and 32.

---

* DEMONSTRATION. By Problem XXVII Mens. of Solids, $\overline{B^2 + H^2} \times L \times ·3927 =$ Content in inches;

But $\frac{·3927}{282} = ·00139255$, &c.

and $\frac{·3927}{231} = ·0017$,

therefore $\overline{B2 + H^2} \times L \times ·0014 =$ Content in ale gallons, and $\overline{B^2 + H^2} \times L \times ·0017 =$ Content in wine gallons, which is *the Rule*.

1024 = square of bung diam.
576 = square of head diam.

1600	
40	
64000	64000
·0014	·0017
256	448
64	64
89·6 ale gal.	108·8 wine gal.

BY THE GAUGING RULE.

Set 40 on C to 26·8 on D ; then against
24 on D stands 32·0 on C,
32 on D stands 57·3 on C.

Sum 89·3 ale gal.

And set 40 on C to 24·25 on D ; then against
24 on D stands 39·1 on C,
32 on D stands 69·8 on C.

Sum 108·9 wine gal.

2. Required the content when the length is 20, and the diameters 12 and 16.

Ans. { 11·2 ale gallons.
13·6 wine gallons.

PROBLEM IV.

To find the content of a cask of the fourth form.

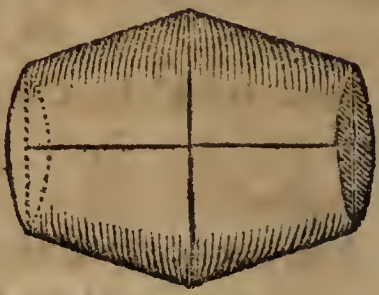
RULE.*

Add the square of the difference of the diameters to 3 times the square of their sum; then multiply the sum by the length, and the product by ·00023⅕ for ale gallons, or by ·00028⅓ for wine gallons.

EXAMPLES.

1. Required the content when the length is 40, and the diameters 24 and 32 inches.

32	32
24	24
56	8
56	8
336	64
280	9408
3136	9472
3	40
9408	378880



---

* DEMONSTRATION. By Problem VIII, Mens. of Solids,

$$\frac{B^3-H^3}{B-H} \times L \times \frac{\cdot 7854}{3} = \text{content in inches}$$

$$= \overline{B^2+BH+H^2} \times L \times \frac{\cdot 7854}{3}$$

378880	378880
·00023$\frac{1}{5}$	·00028$\frac{1}{3}$
1136640	3030040
757760	757760
75776	126293
87·90016 ale gal.	107·33933 wine gal.

BY THE SLIDING RULE.

Set 40 on C to 65·64 on D; then against
8 on D stands 0·6 on C,
56 on D stands 29·1 on C.
29·1
29·1
Sum 87·9 ale gal.

And set 40 on C to 59·41 on D; then against
8 on D stands 0·7
56 on D stands 35·6
35·6
35·6
Sum 107·5 wine gal.

$$=3\times\overline{B+H}^2+\overline{B-H}^2\times L\times\frac{\cdot7854}{12};$$

but $\frac{\cdot7854}{12\times282}=\frac{1}{4308\cdot628}=\cdot00023209$, and $\frac{\cdot7854}{12\times231}=\frac{1}{352\cdot42}$

$=\cdot0002833$; hence $\overline{B-H}^2+3\times\overline{B+H}^2\times L$

$\times\left\{\begin{matrix}\cdot00023\frac{1}{5}\\ \cdot00028\frac{1}{3}\end{matrix}\right\}$ is *the Rule* for the content in $\left\{\begin{matrix}\text{Ale}\\ \text{Wine}\end{matrix}\right\}$ Gallons.

2. What is the content of a conical cask, the length being 20, and the bung and head diameters 16 and 12 inches?

Ans. $\begin{cases} 10{\cdot}985 \text{ ale gal.} \\ 13{\cdot}416 \text{ wine gal.} \end{cases}$

PROBLEM V.

To find the content of a cask by four dimensions.

RULE.*

Add together the squares of the bung and head diameters, and the square of double the diameter taken in the

---

* DEMONSTRATION. According to *the Method of equidistant ordinates*, if the bung and head diameters, and a diameter in the middle between them, be measured in inches, then the sum of the bung, head, and 4 times the middle circle, multiplied by the length of the cask, will be nearly 6 times the content of the cask; i. e. putting δ = middle diameter,

$\overline{B^2+H^2+4\delta^2} \times {\cdot}785398 \times L = 6$ times the content in inches,

hence, $\dfrac{B^2+H^2+4\delta^2}{6} \times {\cdot}785398 \times L =$ content;

but $\dfrac{{\cdot}785398}{6 \times 282} = {\cdot}0004641$, &c, $= {\cdot}0004\frac{2}{3}$ nearly,

and $\dfrac{{\cdot}785398}{6 \times 231} = {\cdot}00056$, &c. $= {\cdot}0005\frac{2}{3}$ nearly;

therefore $\overline{B^2+H^2+4\delta^2} \times L \times \begin{Bmatrix} {\cdot}0004\frac{2}{3} \\ {\cdot}0005\frac{2}{3} \end{Bmatrix} =$ the content in

$\begin{Bmatrix} \text{ale} \\ \text{wine} \end{Bmatrix}$ gallons, nearly; which is *the Rule*.

This rule is true, not only for the four varieties of casks, but also for all casks and solids, generated by any conic section: and middle between the bung and head; then multiply the sum by the length of the cask, and the product by ·0004$\frac{2}{3}$ for ale gallons, or by ·0005$\frac{2}{3}$ for wine gallons.

EXAMPLES.

1. Required the content of any cask, whose length is 40, the bung diameter being 32, the head diameter 24, and the middle diameter between the bung and head 28$\frac{3}{4}$ inches.

28·75	24	32
2	24	32
57·5	96	64
57·5	48	96
2875	576	1024
4025		
2875		
3306·25		
1024		
576		
4906·25		
40		

196250	196250
·0004$\frac{2}{3}$	·0005$\frac{2}{3}$
785000	981250
130833	130833
91·5833 ale gal.	111·2083 wine gal.

---

although the cask be not precisely in the form of any such curve, the Rule will give the content very near the truth; so that the content of a cask of any form may, with a good degree of probability, be obtained by it to within $\frac{1}{10}$ of a gallon, if the dimensions be accurately measured.

BY THE SLIDING RULE.

Set 40 on C to 46·4 on D ; then against
24 on D stands 10·5
32 on D stands 19·0
$57\frac{1}{2}$ on D stands 62·0

Sum 91·5 ale gal.

Set 40 on C to 42·0 on D ; then against
24 on D stands 13·0
32 on D stands 23·2
$57\frac{1}{2}$ on D stands 75·0

Sum 111·2 wine gal.

2. What is the content of a cask, whose length is 20, the bung diamater being 16, the head diameter 12, and the diameter in the middle between them $14\frac{3}{8}$?

Ans. { 11·4479 ale gal.
13·9010 wine gal.

PROBLEM VI.

To find the content of any cask from three dimensions only.

RULE.*

Add into one sum 39 times the square of the bung diameter, 25 times the square of the head diameter, and 26

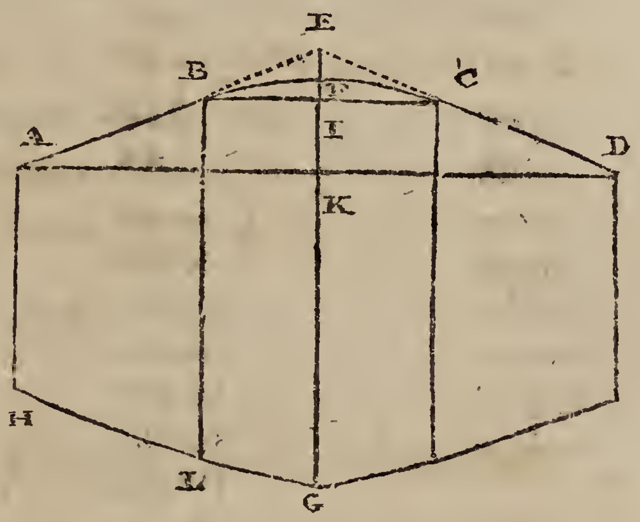
---

* DEMONSTRATION. In this Rule it is supposed, that the sides or edges of each stave of a cask for about $\frac{1}{3}$ of its length from each end, are made tapering in a straight line, and that for the middle third part they are curved, or made convex, to form the bulge or middle of the cask. Hence $\frac{1}{3}$ of a cask, at each end, is considered as the frustum of a cone ; and the middle

times the product of the two diameters; then multiply the sum by the length, and the product by $\frac{\cdot 00034}{9}$ for wine gallons, or by $\frac{\cdot 00034}{11}$, or $\cdot 00003\frac{1}{11}$, for ale gallons.

---

part as the middle frustum of a parabolic spindle, as the curve does not differ much from a parabolic curve.



Let AB and CD be the two right lined parts, and BC the parabolic part. Produce AB and DC to meet in E. Put L = AD the length of the cask, B = FG the bung diameter, and H = AH the head diameter. Then ABE is a tangent to the parabola BF, and consequently FI $= \frac{1}{2}$ EI; but BI $= \frac{1}{3}$ AK, and therefore, by similar triangles, EI $= \frac{1}{3}$ EK; consequently

$$FI = \tfrac{1}{2} EI = \tfrac{1}{6} EK = \tfrac{1}{5} FK = \frac{B-H}{10}.$$ Therefore the common

diameter $BL = FG - 2FI = B - \frac{B-H}{5} = \frac{4B+H}{5}$, for which put C.

By the Note under Problem II, putting $n = \cdot 785398$,

$$\frac{8B^2+4BC+3C^2}{15} \times \frac{Ln}{3} = \frac{328B^2+44BH+3H^2}{25 \times 45} \times Ln =$$ the content of the parabolic or middle part in inches.

And by the Note under Problem IV,

$$\frac{C^2+CH+H^2}{3} \times \frac{2Ln}{3} = \frac{160B^2+280BH+310H^2}{25 \times 45} \times Ln$$

= the content of the two conical frustums in inches. Their sum

$$= \overline{488B^2+324BH+313H^2} \times \frac{Ln}{1125} = \overline{39B^2+26BH+25H^2}$$

EXAMPLES.

1. Required the content of a cask, whose length is 40, and the bung and head diameters 32 and 24.

32	24	32
32	24	24
64	96	128
96	48	64
1024	576	768
39	25	26
9216	2880	4608
3072	1152	1536
39936	14400	19968
	39936	
	19968	
	74304	
	40	
	2972160	2972160
	·00034	$\cdot00003\frac{1}{11}$

$\times \frac{Ln}{90}$ nearly, in inches.

But $\frac{\cdot785398}{231\times90} = \frac{\cdot00034}{9}$, nearly; and since 231 : 282 :: 9 : 11, nearly, therefore $\frac{\cdot00034}{11}$ is the multiplier for ale gallons. Hence,

$$\overline{39B^2+26BH+25H^2} \times L \times \left\{ \begin{matrix} \frac{\cdot00034}{9} \\ \frac{\cdot00034}{11} \end{matrix} \right\}$$

= the content in $\left\{ \begin{matrix} \text{wine} \\ \text{ale} \end{matrix} \right\}$ gallons; which is *the Rule.*

11888640	8916480
8916480	270196
9)1010·53440	91·86676 ale gal.
112·2816 wine gal.	

2. What is the content of a cask, whose length is 20, and bung and head diameters 16 and 12?

Ans. { 11·4833 ale gal.
14·0352 wine gal. }

Note. This is the most exact rule of any, for three dimensions only; and agrees nearly with the diagonal rod.

PROBLEM VII.

To find the ullage of a cask by the sliding rule.

Note. The *ullage* of a cask is what it contains, when only partly filled. And it is considered in two positions, namely, as standing on its end with the axis perpendicular to the horizon, or as lying on its side with the axis parallel to the horizon.

RULE.

By one of the preceding problems find the whole content of the cask. Then set the length on N to 100 on SS for a segment standing, or set the bung diameter on N to 100 on SL for a segment lying; then against the wet inches on N is a number on SS or SL, to be reserved. Next, set 100 on B to the reserved number on A; then against the whole content on B will be found the ullage on A.

EXAMPLES.

1. Required the ullage answering to 10 wet inches of a standing cask, the whole content of which is 92 gallons, and length 40 inches.

Set 40 on N to 100 on SS; then against 10 on N is 23 on SS, the reserved number.

Then set 100 on B to 23 on A, and against 92 on B is 21·2 on A, the ullage required.

2. What is the ullage of a standing cask, whose whole length is 20 inches, and content 11½ gallons; the wet inches being 5?

Ans. 2·65 gal.

3. The content of a cask being 92 gallons, and the bung diameter 32, required the ullage of the segment lying, when the wet inches are 8.

16·4 gal.

PROBLEM VIII.

To ullage a standing cask by the pen.

RULE.*

Add together the square of the diameter at the surface of the liquor, the square of the diameter of the nearest end, and the square of double the diameter taken in the middle between the other two; then multiply the sum by the length between the surface and nearest end, and the pro-

---

* The reason of this *Rule* will be sufficiently apparent, if the Note, under Problem v, be fully understood.

duct by $\cdot 0004\frac{2}{3}$ for ale gallons, or by $\cdot 0005\frac{2}{3}$ for wine gallons, in the less part of the cask, whether empty or filled.

EXAMPLE.

The three diameters being 24, 27, and 29 inches, required the ullage for 10 wet inches.

24	29	54
24	29	54
96	261	216
48	58	270
576	841	2916
		841
		576
		4333
		10

43330	43330
$\cdot 0004\frac{2}{3}$	$\cdot 0005\frac{2}{3}$
173320	216650
28887	28887
20·2207 ale gal.	24·5537 wine gal.

PROBLEM IX.

To ullage a lying cask by the pen.

RULE.*

Divide the wet inches by the bung diameter; find the quotient in the column of versed sines, in the table of cir-

---

In this *Rule*, it is supposed, that the ullage, being taken in the same ratio to the whole content, which the segment of the cular segments, taking out its corresponding segment. Then multiply this segment by the whole content of the cask, and the product by $1\frac{1}{4}$ for the ullage required, nearly.

EXAMPLE.

Supposing the bung diameter 32, and content 92 ale gallons; to find the ullage for 8 wet inches.

```
32)8(·25, whose tab. seg. is ·153546
                                 92
                            -------
                             307092
                            1381914
                           --------
                          14·126232
                    ¼ is   3·531558
                           --------
                          17·657790 answer.
```

NOTE. The capacity of any vessel, having its cavity in the form of any solid, contained in MENSURATION OF SOLIDS, may be determined by means of the rule, given for the content of such solid.

---

bung circle, cut off by the surface of the liquor, has to the whole bung circle, is too small by about $\frac{1}{4}$; and Dr. Hutton observes, it is nearer to the truth than any other practical rule, that he can find.

HEIGHTS AND DISTANCES.

By the mensuration and protraction of lines and angles we determine the lengths, heights, depths, or distances, of bodies and objects. And this branch is commonly called *Heights and Distances*, or *Altimetry and Longimetry*.

Accessible lines are measured by applying to them some certain measure, as an inch, a foot, &c. a number of times; but inaccessible lines must be measured by taking angles, or by some similar method, drawn from the principles of Geometry and Trigonometry.

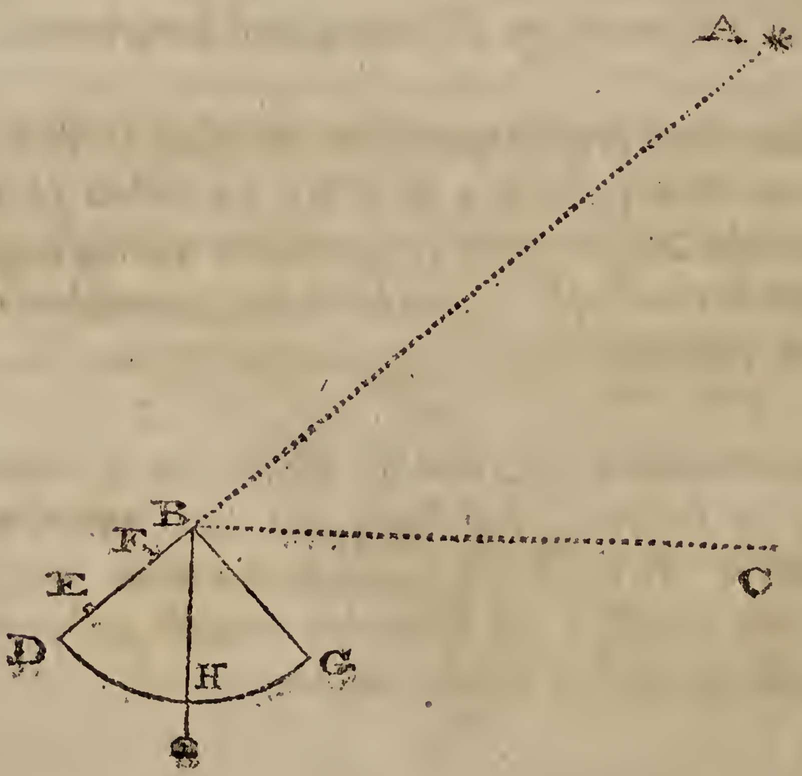
When instruments are used for taking the quantities of the angles in degrees, the lines are then calculated by Trigonometry. In the other methods the lines are calculated from the principle of similar triangles, without any regard to the quantities of the angles.

Angles of elevation, or depression, are usually taken either with a theodolite, or with a quadrant, divided into degrees and minutes, and furnished with a plummet suspended from the centre, and two sights fixed perpendicularly upon one of the radii.

PROBLEM I.

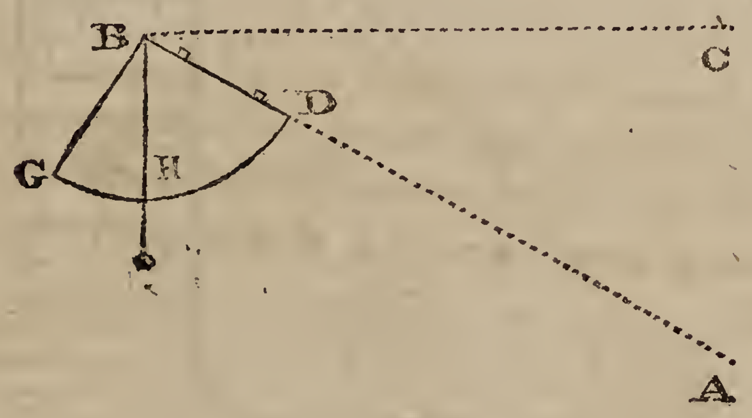
To take an angle of altitude and depression with the quadrant.

Let A be any object, as the top of a tower, hill, or other eminence; or the sun, moon, or a star; and let it be required to find the measure of the angle ABC, which a line, drawn from the object, makes with the horizontal line BC.



Fix the centre of the quadrant in the angular point, and move it round there as a centre, till with one eye at D, the other being shut, you perceive the object A through the two sights E, F; then will the arc GH of the quadrant, cut off by the plumb line BH, be the measure of the angle ABC required.

The angle ABC of depression of any object A is taken in the same manner, except that here the eye is applied to the centre, and the measure of the angle is the arc GH.



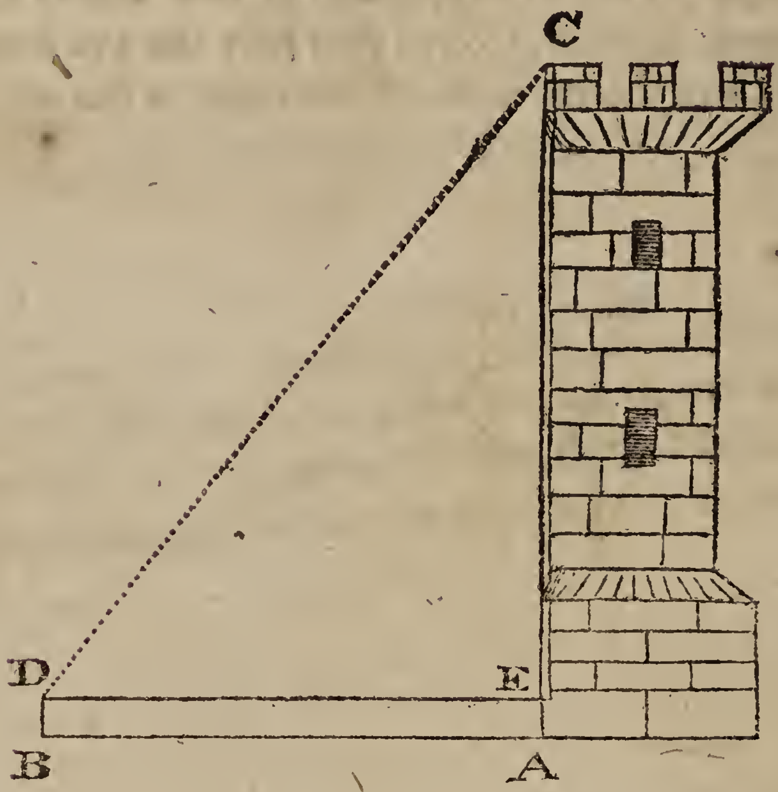
The observations with the quadrant, necessary to determine the heights and distances of objects, will be sufficiently apparent from the manner, in which the following examples are proposed; and the solutions may easily be given by any one, who understands Plane Trigonometry.

The construction of the figures to the following examples is omitted; but it is to be performed as in the problems of Trigonometry.

PROBLEM II.

To measure the height of an accessible object, standing perpendicular on a level.

EXAMPLES.

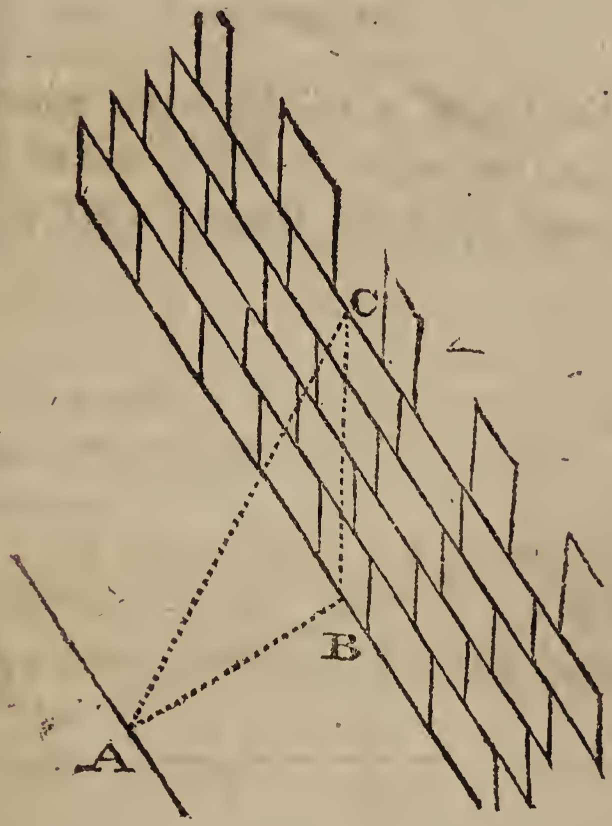


1. Having measured AB equal to 100 feet from the bottom of a tower, in a direct line on a horizontal plane, I then took the angle CDE of elevation of the top, and found it to be 47° 30′, the centre of the quadrant being fixed five feet above the ground ; required the height of the tower.

As radius	10·0000000
To t. ∠D 47° 30′	10·0379475
So is DE 100	2·0000000
To CE 109·13	2·0379475
Add AE 5	
AC 114·13	

NOTE. If you go off to such a distance from the bottom, as that the angle of elevation shall be 45°, then will the height be equal to the distance with the height of the centre of the instrument added.

2. From the edge of a ditch 18 feet wide, surrounding a fort, I took the angle of elevation of the top of the wall, and found it 62° 40′; required the height of the wall, and the length of a ladder necessary to reach from my station to the top of it.



As radius 90°	10·0000000
: tangent of 62° 40′	10·2866141
: : BA 18	1·2552725
: BC 34·8246 feet	1·5418866

As radius 90°	10·0000000
: secant of 62° 40′	10·3380298
: : BA 18	1·2552725
: AC 39·2014 feet	1·5933023

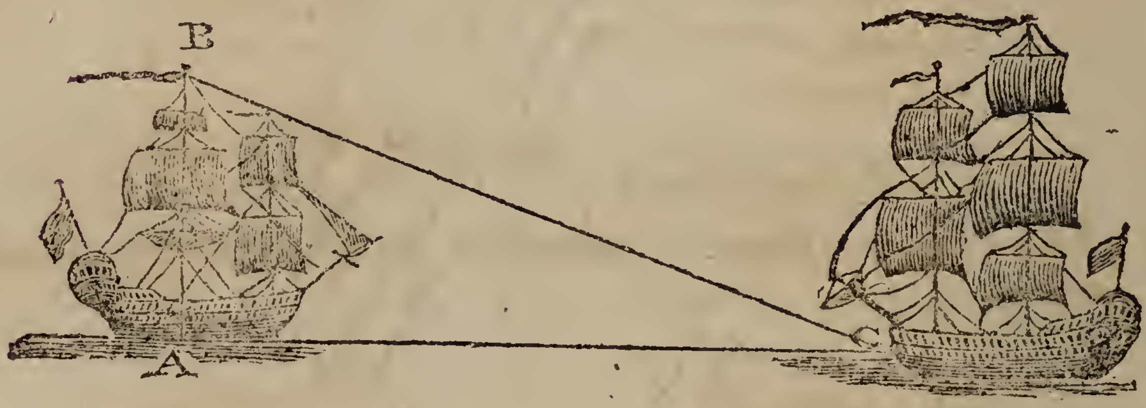
Ans. The height BC = 34·82 feet, and the length of the ladder AC = 39·2 feet.

PROBLEM III.

From a known height to find the distance of an inaccessible object on a level.

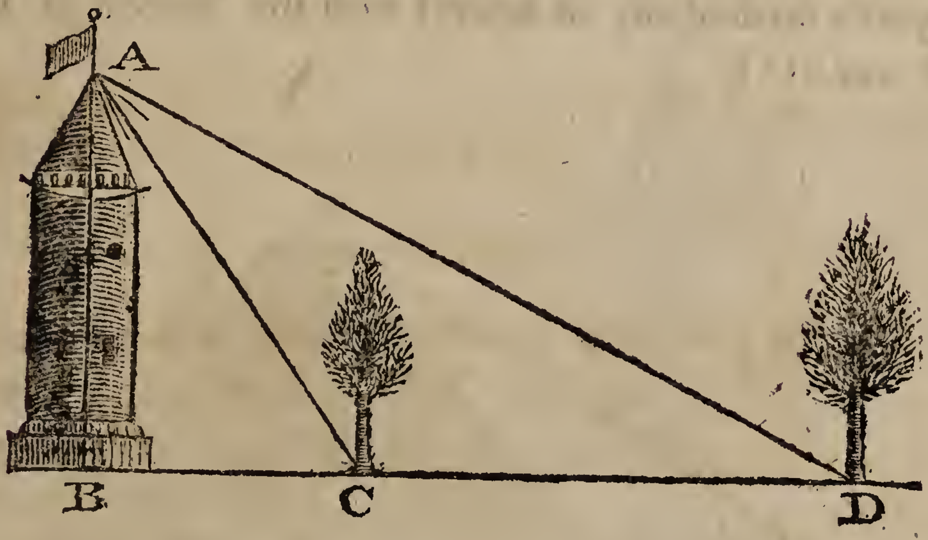
EXAMPLES.

1. From the top of a ship's mast, which was 80 feet above the water, the angle of depression of another ship's hull, at a distance upon the water, is 20° ; what is their distance ?



As radius 90°	10·0000000
: tangent of 70°	10·4389341
: : AB 80 feet	1·9030900
: AC 219·798 feet	2·3420241

2. From the top of a tower, whose height was 120 feet, I took the angle of depression of two trees, which lay in a direct line upon the same horizontal plane, with the bottom of the tower, viz. that of the nearer 57°, and that of the farther 25°½; what is the distance of each from the bottom of the tower, and their distance from each other?



As radius 90°	10·0000000
: tang. ∠ BAC 33°	9·8125174
: : 120 feet	2·0791812
: 77·9289 feet = BC	1·8916986

The distance from the bottom of the tower to the nearer tree.

Then,

As radius 90°	10·0000000
: tang. BAD 64° 30′	10·3215039
: : BA 120	2·0791812
: BD 251·5852	2·4006851

The distance of the farther tree.

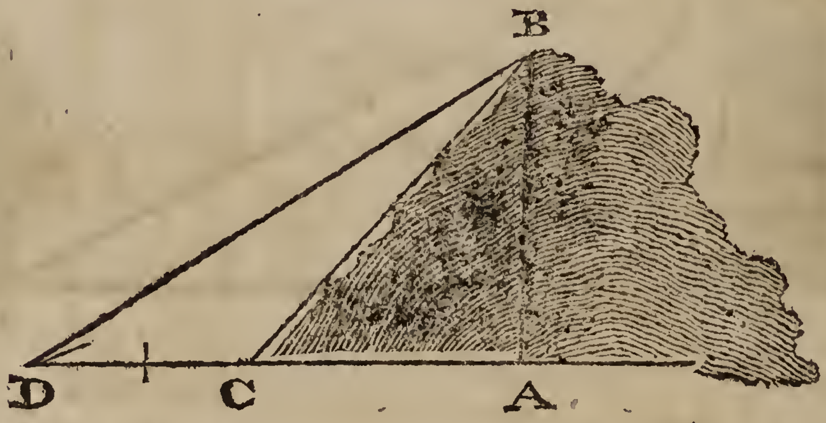
Therefore BD — BC = 251·5852 — 77·9289 = 173·656 feet = CD the distance between the two trees.

PROBLEM IV.

To find the perpendicular height of an object on a level, when the base, or lower end of the perpendicular, is inaccessible.

EXAMPLES.

1. What is the perpendicular height of a hill, whose angle of elevation, taken at the bottom of it, was 46° ; and 100 yards farther off, on a level with the bottom of it, the angle was 31° ?

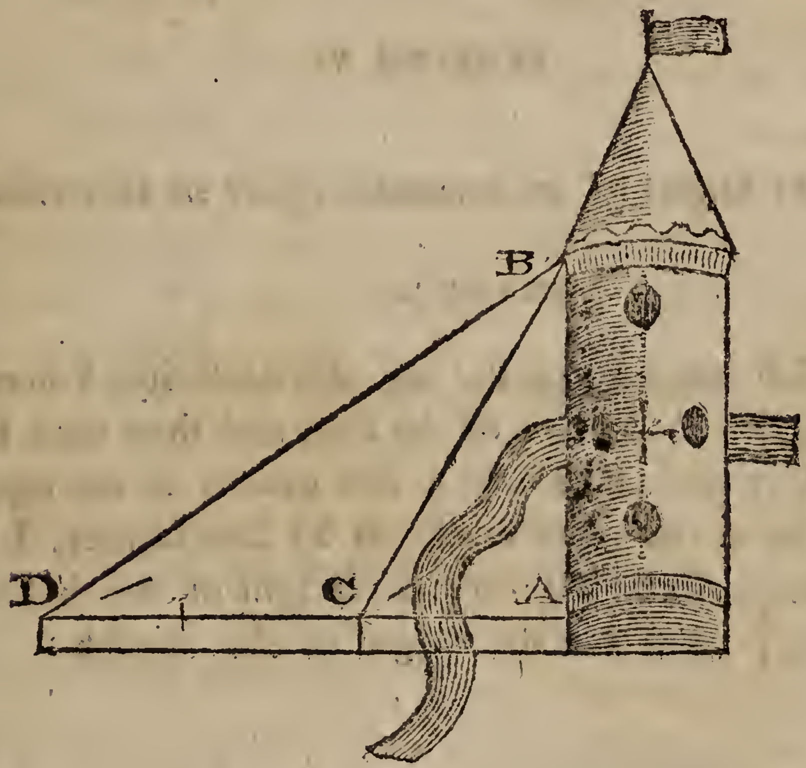


∠C	46°	Subtract
∠D	31	

∠DBC	15	9·4129962
∠D	31	9·7118393
DC	100	2·0000000
BC		2·2988431

∠A	90°	10·0000000
∠C	46	9·8569341
CB		2·2988431
AB	143·14	2·1557772

2. Wanting to know the height of an inaccessible object; at the least distance from it, upon the same horizontal plane, I took its angle of elevation equal to 58°, and going 100 yards directly farther from it, found the angle there to be only 32°: required its height, and my distance from it at the first station, the instrument being 5 feet above the ground at each observation.



∠C	58° } subtract	
∠D	32 }	
∠DBC	26°	9·6418420
∠D	32	9·7242097
DC	100	2·
BC		2·0823677
∠A	90°	10·
∠C	58	9·9284205
BC		2·0823677
AB	102·51 yards	2·0107882
	1·66, &c. yards = 5 feet	
	104·17 whole height.	

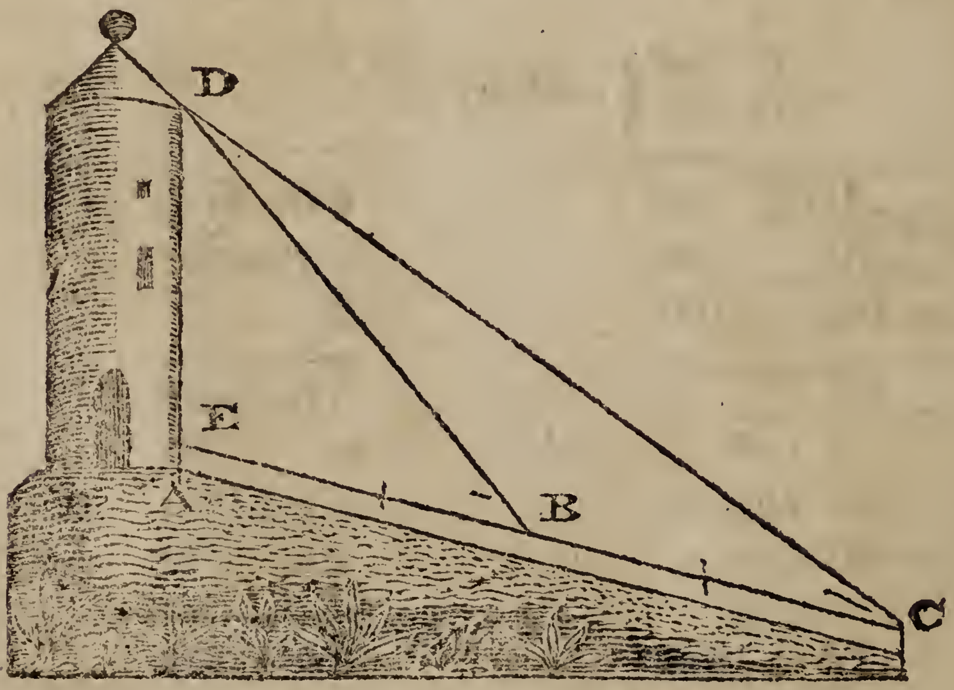
∠A	90°	10·
∠CBA	32	9·7242097
BC		2·0823677
AC	64·05 yards	1·8065774

PROBLEM V.

To find the height of an accessible object on an elevation.

EXAMPLE.

An obelisk standing on the top of a declivity, I measured from its bottom a distance of 40 feet, and then took the angle formed by the plane and a line drawn to the top, 41° ; and going on in the same direction 60 feet farther, I found the same angle to be 25° 45′, the height of the instrument being 5 feet ; what was the height of the obelisk ?



∠B	41° 00	subtract
∠C	23 45	
∠BDC	17 15	9·4720856
∠C	23 45	9·6050320
BC	60	1·7781513
BD	81·488	1·9110977

BD	81·488	
BE	40·	
Sum	121·488	2·0845333
Diff.	41·488	1·6179225
Tan. $\frac{E+EDB}{2}$	69° 30′	10·4272623
Tan. $\frac{E-EDB}{2}$	42 24½	9·9606516
Diff. =∠EDB	27 05½	

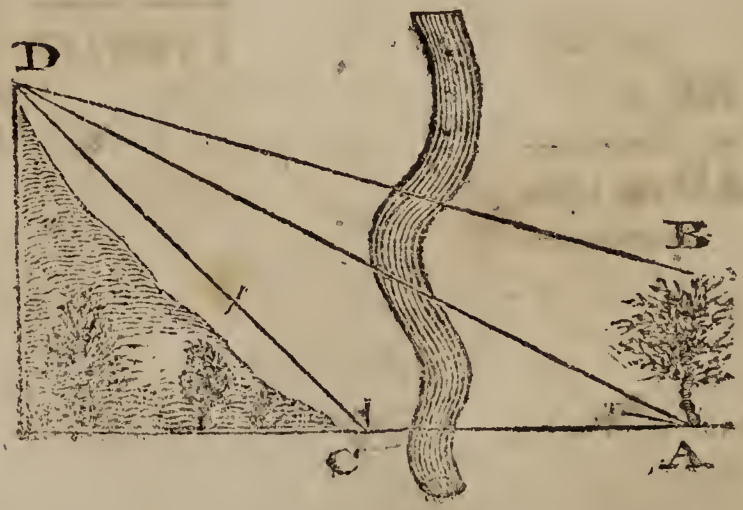
∠EDB	27° 05′½	9·6582842
∠B	41 00	9·8169429
BE	40	1·6020600
ED	57·623	1·7607187
Add AE	5	
AD	62·623.	

PROBLEM VI.

To find the height and distance of an inaccessible object by means of two stations, one on a level with the base, and the other on an elevation.

EXAMPLES.

1. Wanting to know the height and distance of an object on the other side of a river, which seemed to be upon a level with the place, where I stood, close by the side of the river ; and not having room to go backward on the same plane, on account of the immediate rise of the bank, I placed a mark where I stood, and measured in a direct line from the object up the hill, whose ascent was so regular, that I might account it for a right line, to the distance of 132 yards, where I perceived, that I was above the level of the top of the object ; I there took the angle of depression of the mark by the river's side equal 42°, of the bottom of the object equal 27°, and of its top 19° : required the height of the object and the distance of the mark from its bottom.



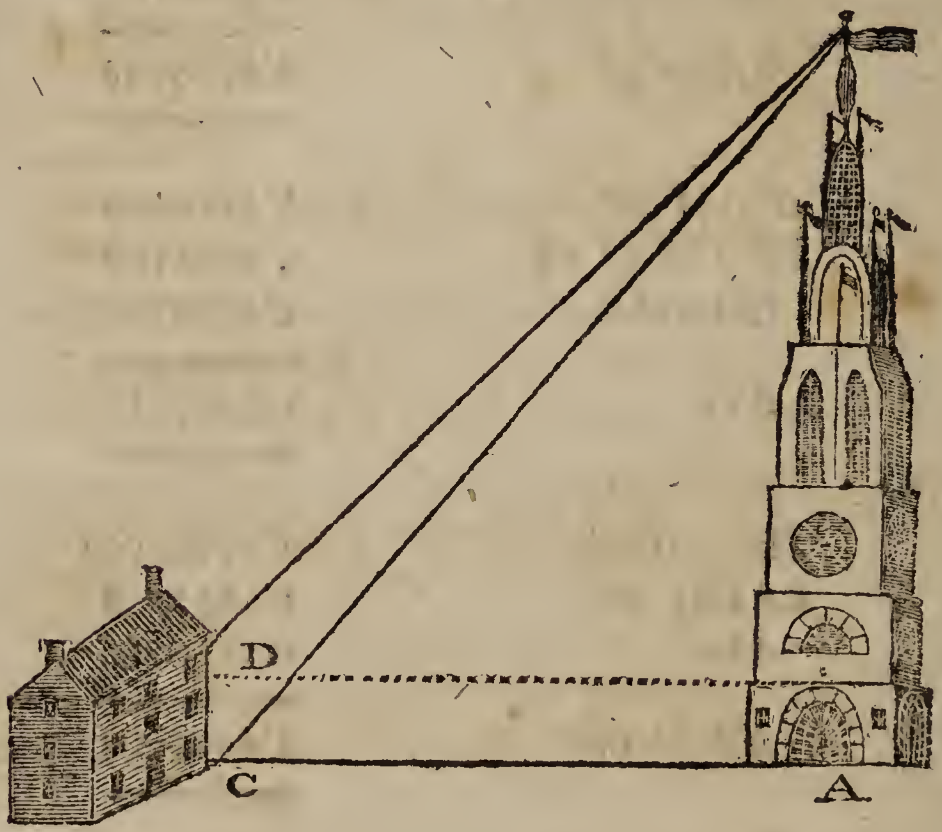
Here 42°—27°=15°=∠ADC. And 27°—19°=8°=∠ADB. Also 90°+19°=109°=∠B.

∠CAD 27°	9·6570468
∠CDA 15	9·4129962
CD 132	2·1205739
CA 75·25	1·8765233

∠CAD 27°	9·6570468
∠C 138 or 42	9·8255109
CD 132	2·1205739
AD	2·2890380

∠B 109°	9·9756701
∠ADB 8	9·1435553
AD	2·2890380
AB 28·63	1·4569232

2. From a window near the bottom of a house, which seemed to be upon a level with the bottom of a steeple, I took the angle of elevation of the top of the steeple equal to 40°, and from another window 18 feet directly above the former, the same angle was 37° 30′; what then is the height and distance of the steeple?



From ∠C	40° 00′	
Sub. ∠D	37 30	
Rem. ∠CBD	2 30	
∠DBC 2° 30		8·6396796
∠CDB 127½ or 52½		9·8994667
DC 18 feet		1·2552725
CB		2·5150596

∠A	90°	10·0000000
∠ACB	40	9·8080675
CB		25150596
AB 210·44 feet		2·3231271

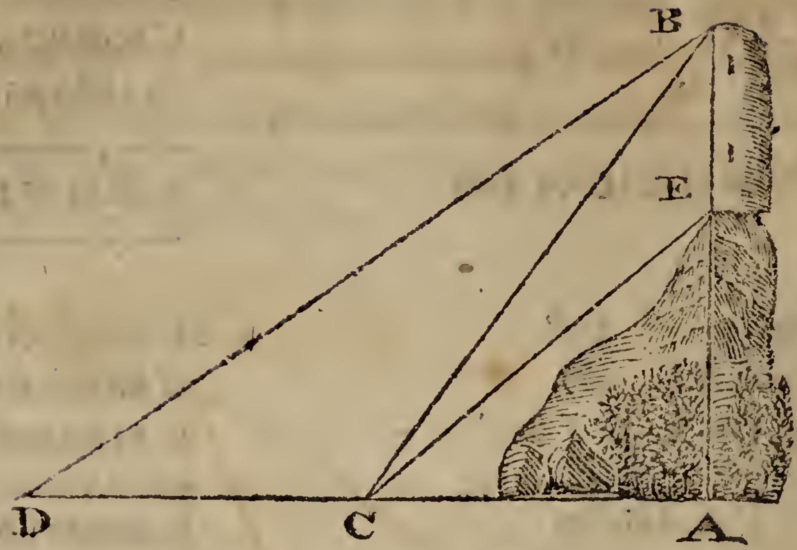
∠A	90°	10·0000000
∠CBA	50	9·8842540
CB		2·5150596
CA 250·79		2·3993136

PROBLEM VII.

To find the height of an object on an inaccessible elevation.

EXAMPLE.

Being upon a horizontal plane, and wanting to know the height of an object on the top of an inaccessible hill; I took the angle of elevation of the top of the hill equal 40°, and of the top of the object equal 51°; then, going in a direct line from it to the distance of 100 yards further, I found the angle of the top of the object to be 33° 45′: what is the object's height



∠ACB 51° 00′ } subtract
∠D 33 45

∠DBC 17 15	9·4720856
∠D 33 45	9·7447390
DC 100	2·0000000
CB	2·2726534

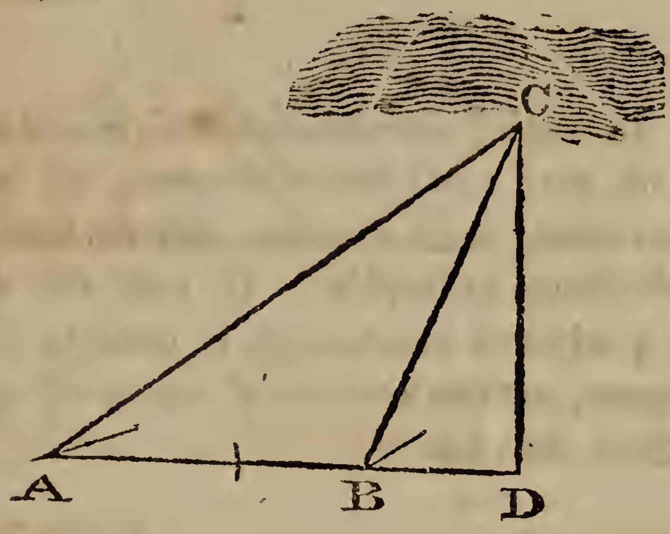
∠BEC 130°	9·8842540
∠BCE 11	9·2805988
CB	2·2726534
BE 46·66574	1·6689982

PROBLEM VIII.

To find the height of a cloud.

EXAMPLE.

What is the perpendicular height of a cloud, whose angles of elevation are 35° and 64°, taken by two observers at the same time, both on the same side of the cloud, and at the distance of 880 yards from one another, so placed that a vertical plane would pass through both their stations and the cloud; and what is its distance from the two places of observation?



∠D	90°	90° ∠D
∠B	64	35 ∠A
∠BCD	26	55 ∠ACD
		26 ∠BCD
		29 ∠ACB

∠ACB	29°	9·6855712
∠A	35	9·7585913
AB	880	2·9444827
BC	1041·125	3·0175028

∠ACB	29°	9·6855712
∠ABC	116	9·9536602
AB	880	2·9444827
AC	1631·442	3·2125717

∠D 90°	10·0000000
∠B 64	9·9536602
BC	3·0175028
DC 935·757	2·9711630

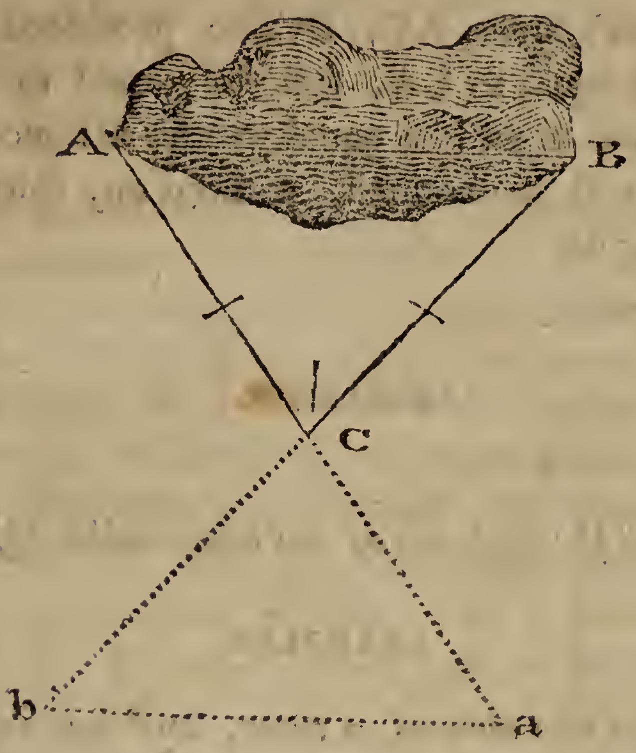
NOTE. In finding the distances of inaccessible objects, if they be of such a height as to admit of a pretty large angle of elevation, their distance may be found as in some of the foregoing examples. If not, the theodolite, or some other graduated instrument, is used to take the angles of the distance, or the horizontal angles of objects, as in some examples, that follow.

PROBLEM IX.

To find the distance of two accessible objects from each other, when one of them is inaccessible from the other in the direction of a right line.

EXAMPLE.

Suppose I wanted to know the distance of the two places A, B, to whose ends there is free access, but not to the intermediate parts, because of a hill, precipice, or water, between A and B; and that therefore I measured from A and B to any convenient place C, the distance AC equal 7·35 chains, and BC equal 8·4 chains, and found the angle ACB equal 55° 40′. What is the distance of the places A, B ?



	8·40	
	7·35	
Sum	15·75	1·1972806
Diff.	1·05	0·0211893
Tang. $\frac{A+B}{2}$	62° 10′	10·2773793
Tang. $\frac{A-B}{2}$	7 11$\frac{1}{14}$	9·1012880
∠A	69 21$\frac{1}{14}$ sum.	

∠A 69° 21$\frac{1}{14}$	9·9711982
∠C 55 40	9·9168593
CB 8·4 chains	0·9242793
AB 7·412 chains	0·8699404

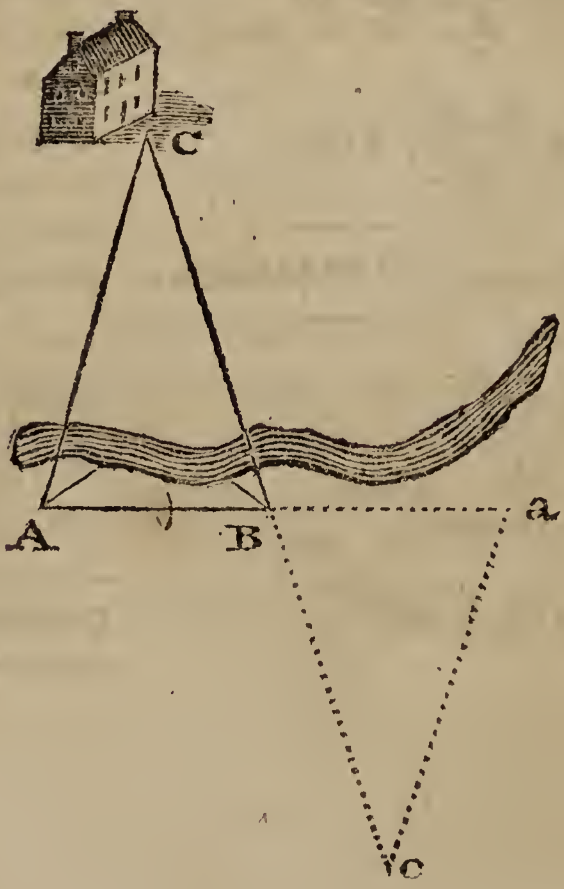
NOTE. If the lines AC, BC, be produced to *a* and *b*, till C*a*, C*b*, be equal to CA, CB, or equal to CB, CA; then the distance *ba* will be equal to the distance AB, and therefore AB will be obtained, without any calculation, by only measuring *ba*.

PROBLEM X.

To find the distance of an inaccessible object.

EXAMPLES.

1. Being on one side of a river, and wanting to know the distance to a house, which stood on the other side, I measured 200 yards in a right line by the side of the river, and found, that the two angles at each end of this line, formed by the other end and the house, were 73° 15′, and 68° 2′. What was the distance between each station and the house?

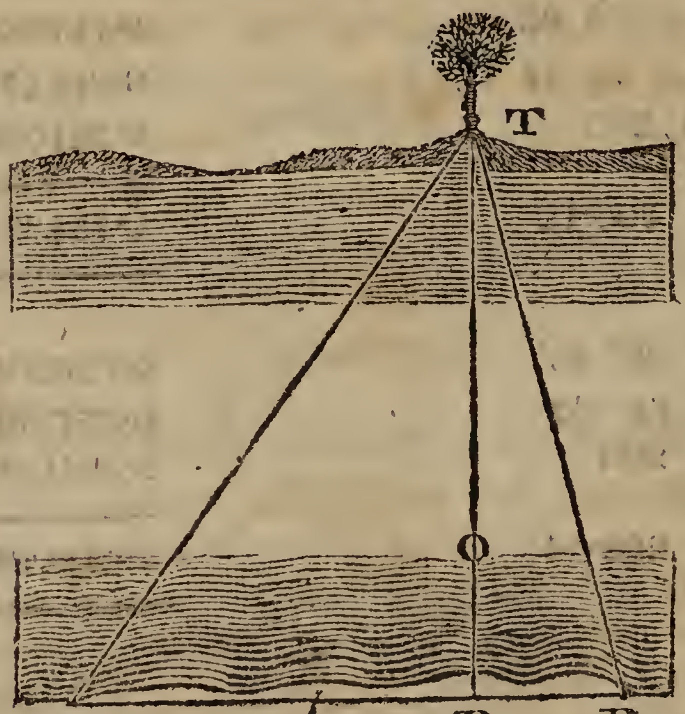


∠A	73° 15′	
∠B	68 02	
Sum	141 17	
From	180 00	
∠C	38 43	9·7962062
∠A	73 15	9·9811711
AB	200	2·3010300
BC	306·19	2·4859949

∠C	38° 43′	9·7962062
∠B	68 02	9·9672679
AB	200	2·3010300
AC	296·54	2·4720917

NOTE. If in the right line AB*a* you measure B*a* equal AB, and the line CB*c* be produced until the angle *a* be equal to the Angle A ; the distances B*c*, *ac*, will be equal to BC, AC.

2. Wanting to know the breadth of a river, I measured 100 yards in a right line close by one side of it; and at each end of this line I found the angles, subtended by the other end and a tree close by the other side of the river, to be 53° and 79° 12′. What is the perpendicular breadth?



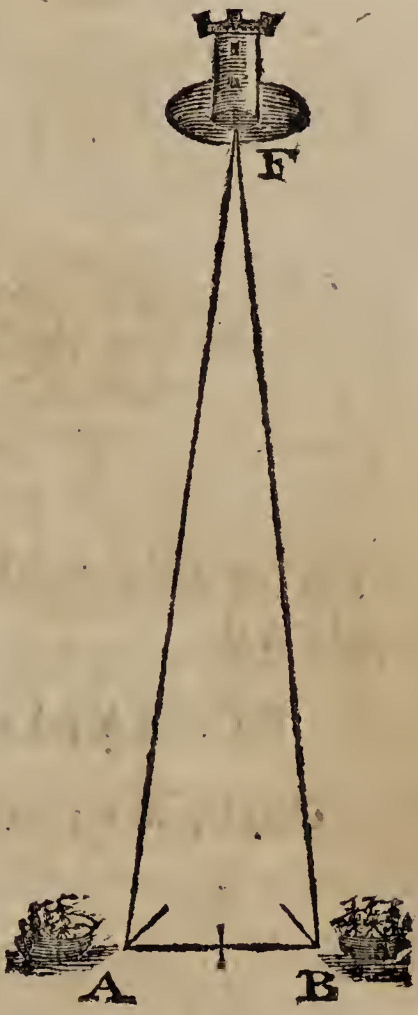
∠A	53° 00′
∠B	79 12
Sum	132 12
From	180 00

∠ATB	47 48	9·8697037
∠B	79 12	9·9922385
AB	100	2·0000000
AT		2·1225348

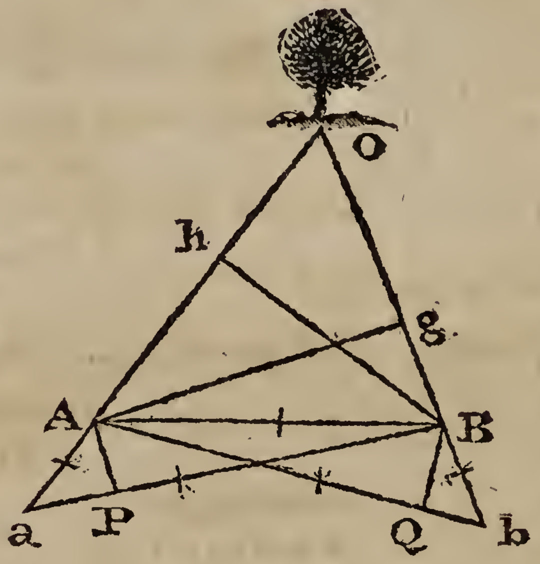
∠P	90°	10·0000000
∠A	53	9·9023486
AT		2·1225348
TP	105·89	2·0248834

3. Two ships of war, intending to cannonade a fort, are, by the shallowness of the water, kept so far from it, that they suspect their guns cannot reach it; in order therefore to measure the distance, they separate from each other half a mile, or 880 yards; then each ship observes the angle, which the other and the fort subtend, and they are found to be 85° 15′ and 83° 45′. What is the distance between each ship and the fort?

∠A	83° 45′	
∠B	85 15	
Sum	169 00	
From	180 00	
∠F	11 00	9·2805988
∠A	83 45	9·9974110
AB	880	2·9444827
BF	4584·5	3·6612949
∠F	11° 00′	9·2805988
∠B	85 15	9·9985058
AB	880	2·9444827
AF	4596·1	3·6623897



4. Suppose I want to know the breadth of a river, or my distance from an inaccessible object O, and that I have no instrument for taking angles, but only a chain or cord for measuring distances; and suppose, that from each of the two stations A, B, which are 500 yards asunder, I measure in a direct line from the object O 100 yards, viz. Aa and Bb each equal to 100 yards, and that the diagonal Ab is by measure 550 yards, and the diagonal aB 560. What then is the distance of the object from each of the stations A and B?



Let fall the perpendiculars AP, BQ. Then, in the triangle ABa

$$Ba : BA+Aa :: BA-Aa : BP-Pa,$$

That is, $560 : 600 :: 400 : 428\frac{4}{7} = BP - Pa,$

Its half $214\frac{2}{7}$

Half sum 280

BP $494\frac{2}{7}$

aP $65\frac{5}{7}$

And

$$BP = \frac{560+428\frac{4}{7}}{2} = \frac{988\frac{4}{7}}{2} = 494\frac{2}{7}$$

$$Pa = \frac{560-428\frac{4}{7}}{2} = \frac{131\frac{3}{7}}{2} = 65\frac{5}{7}$$

Then

Aa 100 : aP $65\frac{5}{7}$ :: rad. : s. ∠aAP 41° 5′
AB 500 : BP $494\frac{2}{7}$ :: rad. : s. ∠BAP 81 20

The sum	122	25
Taken from	180	00
Leaves ∠BAO	57	35

Again, in the triangle ABb,
Ab 550 : AB+Bb600 :: AB—Bb400 : AQ—QB$436\frac{4}{11}$

The half diff.	$218\frac{2}{11}$
Half sum	275
AQ	$493\frac{2}{11}$
Qb	$56\frac{9}{11}$

Then

Bb 100 : bQ $56\frac{9}{11}$ :: rad. : s. ∠bBQ 34 37′
BA 500 : AQ $493\frac{2}{11}$ :: rad. : s. ∠ABQ 80 32

The sum	115	9
Taken from	180	0
Leaves ∠ABO	64	51
Add BAO	57	35
The sum	122	26
Taken from	180	00
Leaves ∠O	57	34*

* Or the angles ABO and BAO may be otherwise found thus :

Whence

∠O 57° 34′	9·9263507
∠A 57 35	9·9264310
AB 500	2·6989700
BO 500·1	2·6990503

∠O 57° 34′	9·9263507
∠B 64 51	9·9567437
AB 500	2·6989700
AO 536·25	2·7293630

PROBLEM XI.

To find the distance of two inaccessible objects from each other.

EXAMPLES.

Wanting to know the distance between a house and a mill, which were separated from me by a river, I took another station B at the distance of 300 yards from the first station A : now, from the first station A, the angle subtended by B

Draw the perpendiculars Ag, Bh. Then, by Eucl. II. 12,

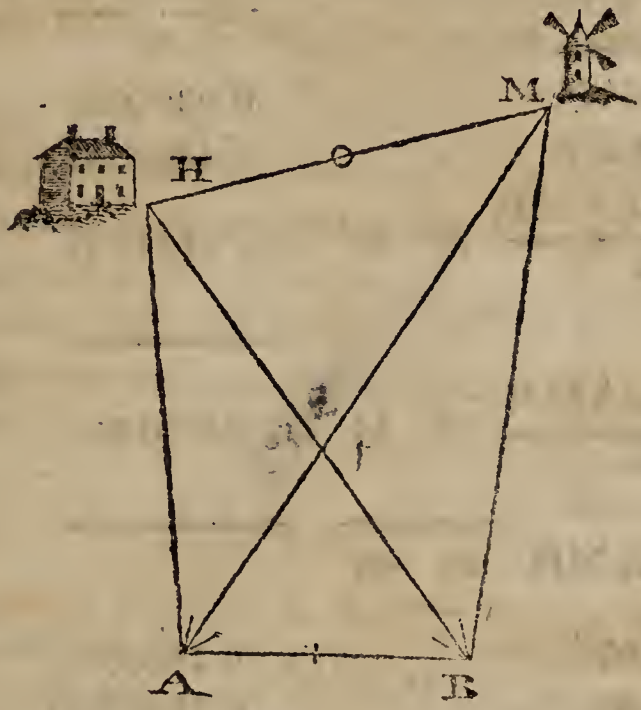
$$Bg = \frac{Ab^2 - AB^2 - bB^2}{2bB} = 212\tfrac{1}{2}.$$

And AB : Bg :: rad. : ·425 = cosine of 64° 51′ the ∠ABO.

In like manner $Ah = \frac{Ba^2 - AB^2 - Aa^2}{2aA} = 268$;

And AB : Ah :: rad. : ·536 the cosine of 57° 35′ the ∠BAO; both the same as above.

and the mill was 58° 20′, and by the mill and the house 37°; from B, the angle subtended by A and the house was 53° 30′, and by the house and the mill 45° 15′. What is the distance of the house and mill?



37° 00′	58° 20′	53° 30′
58 20	53 30	45 15
53 30	45 15	98 45 ∠ABM
148 50	157 05	
180 00	180 00	
∠AHB 31 10	22 55 AMB	

∠AHB 31 10	9·7139349
∠ABH 53 30	9·9051787
AB 300	2·4771213
AH 465·9776	2·6683651

∠AMB	22° 55′	9·5903869
∠ABM	98 45, or 81 15	9·9949158
AB 300		2·4771213
AM	761·4655	2·8816502
AH	465·9776	
Sum	1227·4431	3·0890013
Diff.	295·4879	2·4705397
Tang. $\frac{AHM+AMH}{2}$	71° 30′	10·4754801
Tang. $\frac{AHM-AMH}{2}$	35 44	9·8570185
AMH	35 46	

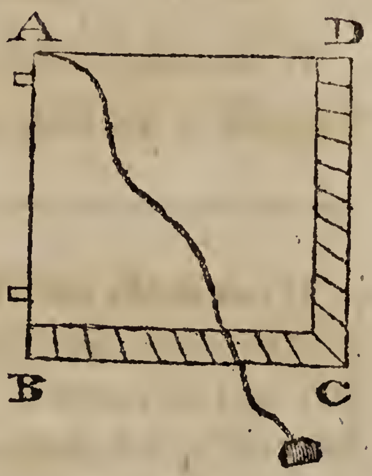
∠AMH	35° 46	9·7667739
∠HAM	37 00	9·7794630
AH		2·6683651
HM	479·7933 yards	2·6810542

Note. Much after the manner of the last examples many curious and useful problems may be resolved; for if we can determine our distance from one remote object, we can do the same for any number of objects; or if the distance between two remote objects can be determined, those between any number of objects may be determined likewise. So, we may determine the angles and sides of fields, or of very large tracts of land, and that whether we be within them, or any where without them, whence the angles can be seen; hence also ships at sea may determine their distances from known visible ports; and plans may be taken of countries, towns, harbours, fleets, fortifications, &c.*

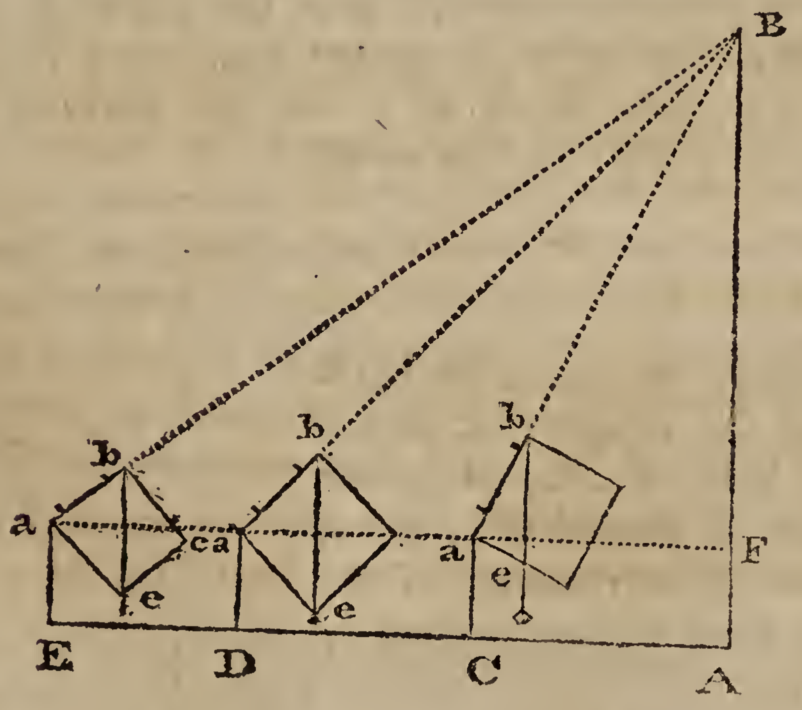
---

* Of many other methods and instruments used to find the altitudes and distances of objects some are here subjoined.

1. One very easy method is by a square, with a plummet AE suspended from one corner A, and the two sides BC, DC, meeting in the opposite angle C, divided into 10, or 100, or 1000 equal parts; and two sights on the side AB.



It is evident, that in taking any altitude AB with the square, the plummet will always cut off from the square a triangle similar to that formed by the base line *a*F, the perpendicular FB, and hypotenuse B*a*.



PROBLEM XII.

To find the distance of the most remote point, that can be seen on the earth's surface, from the top of a mountain, and its diameter.

EXAMPLE.

If the height AB of the mountain, called the peak of Teneriffe, be four miles, and the angle ABC made by a

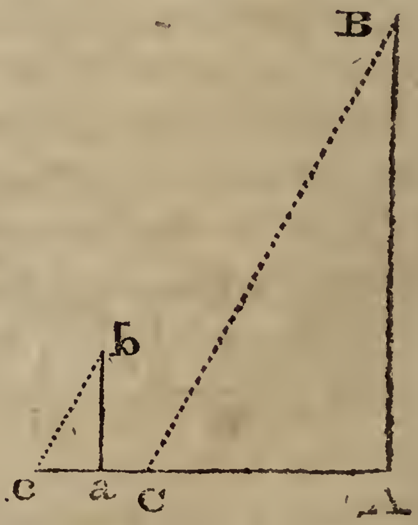
---

If the angle B*a*F be equal to 45°, then the plumb line will pass through the opposite angle of the square, and the distance DA will be equal to the altitude BF; so if DA be 60, then BF will be 60 also.

If the angle be greater than 45°, as at the station C, then the part of the side cut off *ae*, will be to the whole side *ab*, as *a*F, to FB; so if *ae* be 6 divisions, of which *ab* is 10, and the distance CA be 36 feet; then 6 : 10 : : 36 : 60 feet = the altitude BF.

If the angle be less than 45°, as at the station E then the part is cut off the other decimated side, and *bc* : *ce* : : EA : BF : so if the parts cut off be 6, and the distance 100 feet, then 10 : 6 : : 100 : 60 feet = BF.

2. Another method is by shadows, from the property of similar triangles also. For any object and a pole, set up parallel to it, are in proportion to each other, as the length of their shadows, formed by the sun, &c.

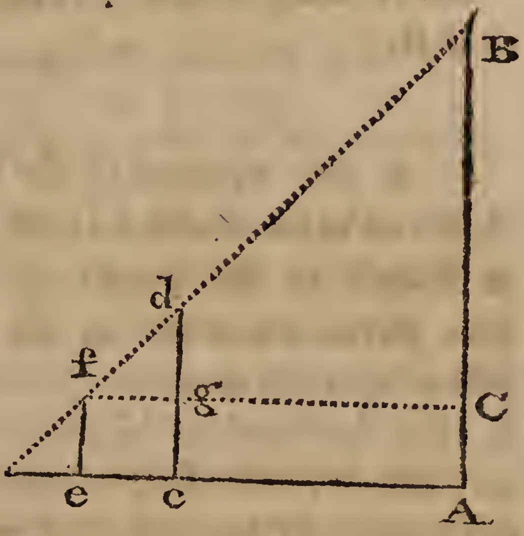


plumb line and a line BC, conceived to touch the earth in the farthest visible point C, be $87^\circ\ 25'\ 55''$; required the

---

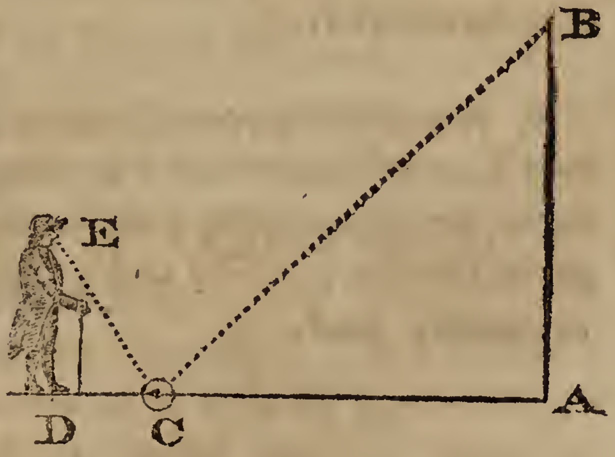
Let the height of the pole *ab* be 6 feet, the length of its shadow *ac* 4, and the shadow AC of the altitude AB 40 feet; then $ca = 4 : ab = 6 :: CA = 40 : AB = 60$.

3. Another method is by two poles, set up parallel to the object, one longer than the other; so that the observer may see the top of the object over the tops of both the poles.



Let the pole *ef* be = 4 feet, *cd* = 7 feet, their distance asunder $ec = fg = 8$ feet, and the distance *e*A of the shorter pole from the object = 160 feet. Then the triangles *fgd*, *f*CB being similar, $fg : gd :: fC : CB$, that is, $8 : 3 (= 7 - 4) :: 160 : 60$ feet = BC : hence $BC + CA = BC + fe = 60 + 4 = 64 = AB$.

4. A fourth method is by viewing the image of the top of the object, reflected from a smooth surface, as a mirror placed horizontally, or a vessel of water.



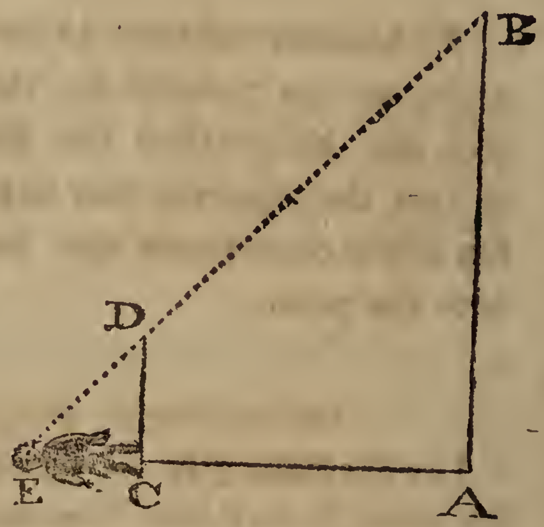
Let C be the reflecting surface, at the distance of 84 feet from the bottom of the object A B; and let a person at D, 7 feet from C, with his eye 5 feet above the ground, view the image of the utmost distance BC, that can be seen from the top of the mountain, and the diameter AE of the earth, supposing the earth to be a perfect sphere.

---

object at C ; then because the triangles CDE, CAB are similar, agreeably to a principle of Optics, we shall have

CD : DE : : CA : AB, that is, 7 : 5 : : 84 : 60 feet = AB.

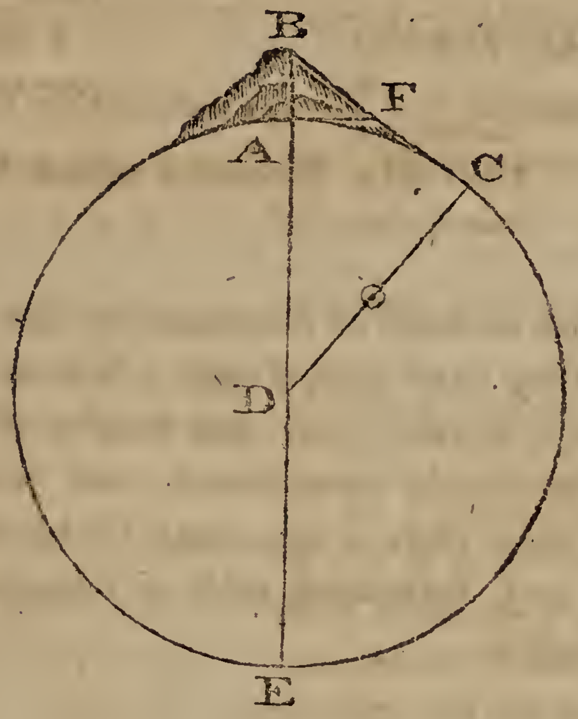
5. A fifth method is for the observer to fix a pole CD, equal in length to the height of his eye, perpendicularly at C, by trials, at such a distance from A, that having laid himself upon his back, with his feet against the bottom of the pole, he may see the tops D and B of the pole and object in the same line :



for then EA will be equal to AB, because EC is equal to CD. Or the pole may be of any length ; for the distance from the foot to the eye of the observer will be in proportion to the height of the pole, as the whole distance is to the height of the object required.

6. When the perpendicular altitude of an irregular hill or ascent is to be ascertained ; or when, in levelling, for conducting water, &c. it is required to find how much one assigned place is above another ; a level and perpendicular poles, or objects, are commonly used.

Draw AF perpendicular to AB, then, from the principles of Geometry, it is known, that CF=FA; draw also the radius CD, which will be perpendicular to CB.



Hence from	90° 00′ 00″
Take ∠B	87 25 55
∠D or AFB	2 34 5

Then as radius

90°	10·0000000
: AB 4	0·6020600
: : tang. ∠B 87° 25′ 55″	11·3482279
: AF or FC 89·184	1·9502879

And as radius

90°	10·0000000
: AB 4	0·6020600
: : sec. ∠B 87° 25′ 55″	11·3486643
: BF 89·274	1·9507243

Their sum 178·458 BC

Lastly, as radius	90°	10·0000000
To tang. ∠B		11·3482280
So BC	178·458	2·2515360
To CD	3978·909	3·5997640
	2	
	7957·818	Diameter of the Earth.

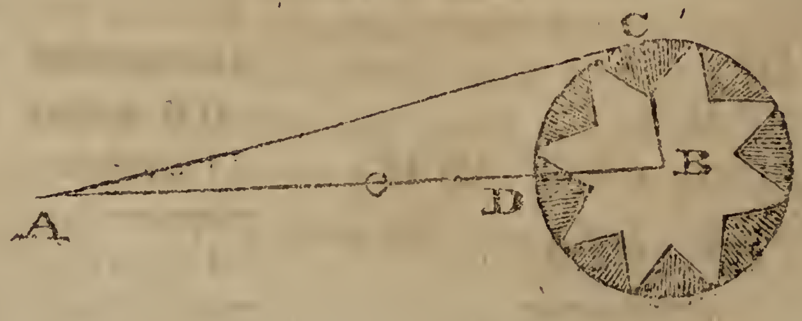
NOTE. This method of determining the magnitude of the earth is very easy and simple; but to do it with any tolerable degree of exactness, the height of the mountain must be very accurately ascertained, and the angle at the top must be taken with a quadrant divided into minutes and seconds, and furnished with a telescope, instead of the common sights.

PROBLEM XIII.

To find the ratio of the sun's diameter to its distance from the earth.

EXAMPLE.

According to Sir ISAAC NEWTON, the diameter of the sun, at the mean distance from the earth, subtends an angle of 32′ 15″. Then to how many times its diameter is its mean distance from the earth equal?



Here, in the triangle ABC, are given CB $= \frac{1}{2}$ a diameter, the angle C $= 90°$, the angle A $= 16' \; 7'' \frac{1}{2}$.

Hence, as s. ∠A	$16' 7'' \frac{1}{2}$	7·6711356
To s. ∠C 90°		10·0000000
So is CB 1 semidiameter		0·0000000
To BA 213·2379		2·3288644

That is, the mean distance of the sun's centre is 213·2379 semidiameters, or 106·6189 diameters. And if from AB be taken BD or $\frac{1}{2}$, we shall have remaining nearly 106 diameters for the distance of the surface.

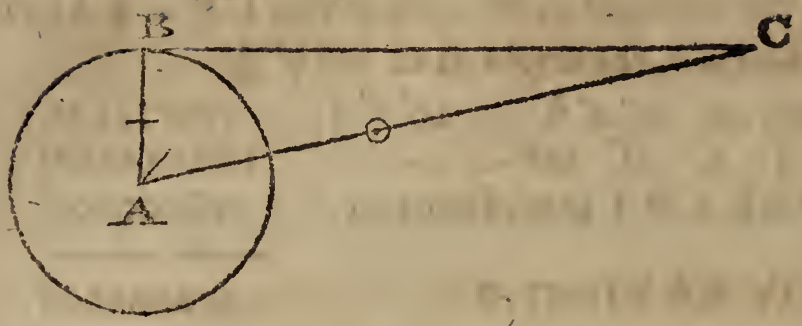
Note. After the same manner may be calculated the proportion of the diameter to the distance of any other celestial body, whose apparent diameter is large enough to be measured. And in particular, the mean distance of the moon, whose mean apparent diameter is $31' \; 16''$, will be found to be 109·95 times her diameter.

PROBLEM XIV.

To find the ratio of the earth's semidiameter to the moon's distance from the earth.

EXAMPLE.

If when the moon appears in the horizon to a spectator on the earth at her mean distance from it, her zenith distance, as calculated from astronomical tables, be $89° \; 2' \; 55''$; it is required to find how many of the earth's semidiameters the said mean distance of the moon is equal to.



Here AB = one semidiameter of the earth, and the $\angle A = 89^\circ\ 2'\ 55''$.

Hence, as s. $\angle C\ 57'\ 5''$	8·2202155
To s. $\angle B\ 90^\circ$	10·0000000
So is AB 1 semidiameter	0·0000000
To AC 60·226	1·7797845

That is, the distance of their centres is 60·226 of the earth's semidiameter's, or $30\frac{1}{9}$ diameters.

Note. Hence, and from the note to the last example, it appears, that the diameter of the earth is to that of the moon, as 109·95 to $30\frac{1}{9}$, or as 11 to 3, or as $3\frac{2}{3}$ to 1 nearly. Consequently, as $3\frac{2}{3} : 1 :: 7958$ (the earth's diameter in miles) : 2170 miles = the moon's diameter.— Likewise the surface of the earth is to that of the moon, as ($3\frac{2}{3} \times 3\frac{2}{3}$ or) $13\frac{4}{9}$ to 1 nearly; or, the earth reflects upon the moon about $13\frac{1}{2}$ times as much light, as the moon does upon the earth. Moreover, the bulk of the earth is to that of the moon, as ($3\frac{2}{3} \times 3\frac{2}{3} \times 3\frac{2}{3}$, or) 48 to 1 nearly.

MISCELLANEOUS QUESTIONS.

1. A MAYPOLE, 50 feet 11 inches high, at a certain time will cast a shadow 98 feet 6 inches long ; what then is the breadth of a river, which runs within 20 feet 6 inches of the foot of a steeple, 300 feet 8 inches high ; the steeple, at the same time, throwing the extremity of its shadow 30 feet 9 inches beyond the stream ?

Ans. 530 feet 5 inches.

2. Required the length of a shore, which being to strut 11 feet from the upright of a building, will support a jamb 23 feet 10 inches from the ground.

Ans. 26 feet 3 inches.

3. A line, 27 yards long, will exactly reach from the top of a fort to the opposite bank of a river, known to be 23 yards broad ; what is the height of the wall ?

Ans. 42 feet 5 inches.

4. Suppose the breadth of a well at the top to be 6 feet, and the angle, formed by its side and a visual diagonal line from the edge at the top to the opposite side at the bottom, 18° 30′ ; required the depth of the well.

Ans. 17·89 feet.

5. From the top of a tower 143 feet high, by the sea side, I observed, that the angle of depression of a ship's bottom, then at anchor, was 55° ; what was its distance from the bottom of the wall ? Ans. 100·13 feet.

SURVEYING.

SURVEYING is that branch of MATHEMATICS, by which we measure, plan, compute, divide, and level land.

INSTRUMENTS AND FIELD BOOK.

1. THE CHAIN.

LINES are measured with a chain, called *Gunter's chain*, four poles or rods in length. The chain consists of 100 equal links, each link being $\frac{22}{100}$ of a yard, or $\frac{66}{100}$ of a foot, or 7·92 inches long.

The length of lines, measured with a chain, are best set down in links, as integers, every chain in length being 100 links; and not in chains and decimals.

2. THE PLANE TABLE.

This instrument consists of a plane rectangular board, of any convenient size, the centre of which, when used, is fixed by means of screws to a three-legged stand, having a ball and socket, or joint, at the top, by means of which, when the legs are fixed on the ground, the table is inclined in any direction.

To the table belong

1. A frame of wood, made to fit round its edges, and to be taken off, for the convenience of putting a sheet of paper upon the table. One side of this frame is usually divided into equal parts, for drawing lines across the table, parallel or perpendicular to the sides; and the other side of the frame is divided into 360 degrees from a centre, which is in the middle of the table; by means of which the table is to be used as a theodolite, &c.

2. A needle and compass, screwed into the side of the table, to point out the directions, and to be a check upon the sights.

3. An index, which is a brass two foot scale, with either a small telescope, or open sights, erected perpendicularly upon the ends. These sights, and one edge of the index, are in the same plane, and that edge is called the *fiducial edge* of the index.

When this instrument is to be used, take a sheet of paper, which will cover it, and wet it to make it expand; then spread it flat on the table, pressing down the frame upon the edges, to stretch it and keep it fixed there; and when the paper is become dry, it will be smooth and even. On this paper is to be drawn the plan.

Then, begin at the part of the ground where you think it most expedient, and make a point on a convenient part of the paper or table, to represent that point of the ground; then fix in that point one leg of the compasses, or a fine steel pin, and apply to it the fiducial edge of the index, moving it round till through the sights you perceive some remarkable object, as the corner of a field, &c.

and from the station point draw a line with a point of the compasses along the fiducial edge of the index; then set another object or corner, and draw its line; do the same by another, and so on, till as many objects are set, as may be thought advantageous. Then measure from the station toward as many of the objects as may be thought necessary, and no more, taking the requisite offsets to corners or crooks in the hedges, &c. and lay the measures down on their respective lines on the table. Then, at any convenient place, the distance of which has been measured, fix the table in the same position, and set the objects, which appear thence, &c. as before; and thus continue till the work be finished, measuring such lines as are necessary, and determining as many, as may be, by intersecting lines of direction, drawn from different stations.

When one paper is full, and there is occasion for more, draw a line in any manner through the farthest point of the last station line, to which the work can be conveniently laid down; take the sheet off the table, and fix another on, drawing a line upon it, in the part most convenient for the rest of the work; fold or cut the full sheet by the line drawn on it, apply the edge to the line on the new sheet, and, as they lie in that position, continue the last station line upon the new sheet; and then proceed with the work from the point where it terminated on the former sheet. And thus as many sheets, as are necessary, may be used in succession.

When the work is done, the aforesaid lines are to be accurately joined together, as when the operation was continued from one sheet to another, in order to form an entire rough plan.

But it is to be noted, that if the said joining lines, on two sheets, have not the same inclination to the side of the table, the needle will not point to the original degree, when the table is rectified; and if the needle be required to respect still the same degree of the compass, the easiest way of drawing the lines, in the same position, is to draw them both parallel to the same sides of the table, by means of the equal divisions marked on the other two sides.

3. THE THEODOLITE.

The theodolite is a brazen circular ring, divided into 360 degrees, and having an index with sights, or a telescope, placed on the centre, about which the index is moveable ; also a compass fixed to the centre, to point out courses and check the sights ; the whole being fixed by the centre on a stand of a convenient height for use.

In using this instrument, an exact account, or field book, of all measures and things, necessary to be remarked in the plan, must be kept, and from this the plan is afterward to be drawn.

Begin at such part of the ground, and measure in such directions, as you judge most convenient ; taking angles or directions to objects, and measuring such distances, as appear necessary, under the same restrictions, as in the use of the plane table. And it is safest to fix the theodolite in the original position at every station by means of fore and back objects, and the compass, exactly as in using the plane table ; registering the number of degrees, cut off by the index when directed to each object ; and, at any station, the index being placed at the same degree, as when the direction toward that station was taken from the last preceding one, to fix the theodolite there in the original position.

The best method of laying down the aforesaid lines of direction is to describe a pretty large circle, quarter it, and lay on it the several number of degrees, cut off by the index in each direction, drawing lines from the centre to all the marked points in the circle. Then, by means of a parallel rule, draw, from station to station, lines parallel to the lines, drawn from the centre to the respective points in the circumference.

4. THE CROSS.

The croos consists of two pair of sights, set at right angles to each other on a staff with a sharp point at the bottom, to be stuck in the ground.

The cross is very useful in measuring small pieces of ground of an irregular form ; and it is applied in the following manner. Measure a base or chief line, usually in the longest direction of the piece, from corner to corner ; and while measuring it, find the places, where perpendiculars would fall on this line from the several corners and bends in the boundary of the piece, with the cross, fixing it by trial on such parts of the line, that through one pair of the sights both ends of the line may appear, and through the other pair the said bends or corners ; and then measure the lengths of the perpendiculars.

5. OTHER INSTRUMENTS.

Beside the aforesaid instruments, which are most commonly used, there are some others ; as the *Circumferentor*, which resembles the theodolite in shape and use ; and the *Semicircle*, for taking angles, &c.

The *Perambulator* is used for measuring roads, and other great distances on a level ground, and by the sides of rivers. It has a wheel of $8\frac{1}{4}$ feet, or half a pole, in circumference, on which the machine turns ; and the distance measured is pointed out by an index, which is moved round by clock work.

Levels, with telescopic or other sights, are used to find the level of places, or how much one place is higher or lower than another. And *station poles*, or *staves*, having *vanes*, that slide on them, are used with levels. And in measuring any sloping or oblique line, either ascending or descending, a small pocket level may be used to shew how many links are to be deducted from each chain, to reduce it to the true horizontal distance. But KING's *Surveying Quadrant* may be used with much more advantage in connexion with the chain for the purpose of reducing oblique to horizontal distances ; and it shows also the perpendicular ascent or descent of an oblique chain.

A *pocket Sextant* is sometimes used in taking offsets.

An *offset staff* is a very useful and necessary instrument, for measuring the offsets and other short distances. It is ten links in length, being divided and marked at each of the ten links.

Ten small arrows, or rods of iron or wood, are used to mark the end of every chain's length on lines. And sometimes *pickets*, or *staves with flags*, are set up as marks or objects of direction.

Various *scales* are also used in protracting and measuring on the plan or paper; such as *plane scales*, *line of chords*, *protractor*, *compasses*, *reducing scale*, *parallel* and *perpendicular rules*, &c. Of plane scales there should be several sizes, as 1 chain in an inch, 1 chain in $\frac{3}{4}$ of an inch, 1 chain in $\frac{1}{2}$ an inch, &c. And the best of them for use are those, that are laid on the edges of the ivory scale, by which distances may be marked off without compasses.

6. THE FIELD BOOK.

In surveying with the plane table, a field book is not used, as every thing is drawn on the table when it is measured. But in surveying with the theodolite, &c.

some kind of a field book must be used, in which a register, or account of whatever is done and occurs relative to the survey, is to be entered.

This book each surveyor forms and rules for himself, as he thinks best. The following is a specimen of a form much used. It is ruled into three columns; the middle or principal column is for the stations, angles, bearings, distances measured, &c. and those on the right and left are for the offsets on the right and left, which are set against their corresponding distances in the middle column; as also for such remarks, as may occur, and be proper to be noted in drawing the plan, &c.

Here ⊙ 1 is the first station, where the angle or bearing is 105° 25′. On the left, at the distance of 73 links in the principal line, is an offset of 92; and at 610, an offset of 24 to a cross hedge. On the right, at 0 or the beginning, an offset 25 to the corner of the field; at 248 Brown's boundary hedge commences; at 610 an offset 35; and at 954, the end of the first line, the 0 denotes its terminating in the hedge. And so on, for the other stations.

A line is drawn under the work at the end of every station line, to prevent confusion.

FORM OF THIS FIELD BOOK.

Offsets and Remarks on the left.	Stations, Bearings, and Distances.	Offsets and Remarks on the right.
	⊙ 1	
	105° 25′	
	00	25 corner
92	73	
	248	Brown's hedge
cross a hedge 24	610	35
	954	00
	⊙ 2	
	53° 10′	
	00	00
house corner 51	25	21
	120	29 a tree
34	734	40 a stile
	⊙ 3	
	67° 20′	
	61	35
a brook 30	248	
	639	16 a spring
foot path 16	810	
cross hedge 18	973	20 a pond

But some skilful Surveyors now make use of a different method for the field book; namely, beginning at the bottom of the page they write upward, sketching a neat boundary on each hand as they pass along. An example of this method will be given under the Problem, which requires the survey of a large estate.

In smaller surveys and measurements, a good way of setting down the work is to draw, by the eye, on a piece of paper a figure resembling that, which is to be measured; and to write the dimensions, as they are found, against the corresponding parts of the figure. And this method may be practised to a considerable extent, even in the larger surveys.

PRACTICE OF SURVEYING.

THIS part contains the several operations proper to be performed in the field, or the modes of measuring with all the instruments, and in all situations.

PROBLEM I.

To measure a line or distance.

To measure a line on the ground with the chain; being provided with a chain and ten small arrows, or rods, that one one may be stuck into the ground, as a mark, at the end of every chain, two persons carry the chain, one at each end of it, and all the ten arrows are taken by one of them, who is to go foremost, and is called the *leader*; the other being called the *follower*, for distinction.

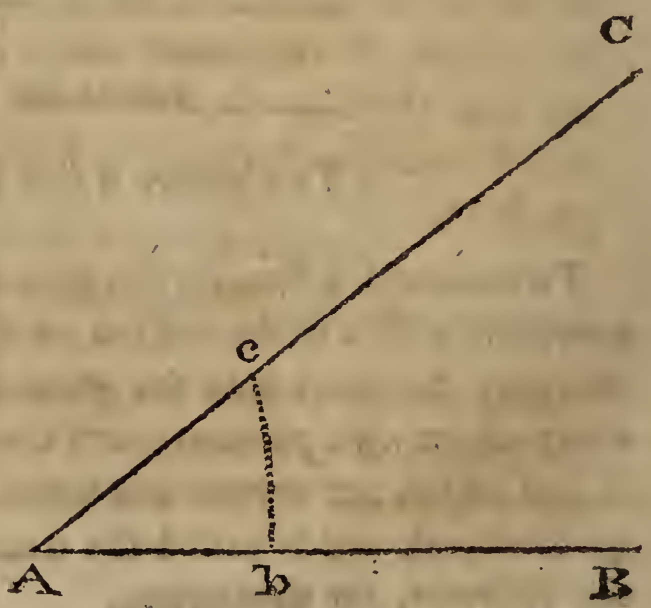
A picket, or station staff, being set up in the direction of the line to be measured, if no mark appear in that direction; they measure in a straight line toward it, the leader fixing in the ground an arrow at the end of every chain, and the follower always taking it up, till all the ten arrows are used, and in the hands of the follower. They are then all returned to the leader, to be used again. And thus the arrows are changed from one to the other, at every ten chains' length, till the whole line is finished; then the number of changes of the arrows shows the number of tens, to which the follower adds the arrows he holds in his hand, and the number of links of another chain from him to the mark or end of the line. So, if there be three changes of the arrows, the follower hold six, and the end of the line cut off 45 links more, the whole length of the line is 3645 links.

When the line is on a declivity, ascending or descending, at every chain's length apply a small pocket level, or King's quadrant, to the chain, that it may show how many links the slope is longer than the corresponding level line, and correct the length.

PROBLEM II.

To take angles and bearings.

Let B and C be two objects, or two pickets set up perpendicular; and let it be required to take their bearings, or the angle formed between them at any station A.



1. *With the Plane Table.*

The table being covered with a paper, and fixed on its stand ; plant it at the station A, and fix a fine pin, or a point of the compasses, in a proper part of the paper, to represent the point A. Close by the side of this pin lay the fiducial edge of the index, and turn it about, still touching the pin, till one object B can be seen through the sights ; then by the fiducial edge of the index draw a line. In the same manner draw another line in the direction of the other object C, and it is done.

2. *With the Theodolite, &c.*

Direct the fixed sights along one of the lines, as AB, by turning the instrument about till you see the mark B through the sights ; and there screw the instrument fast. Then turn the moveable index about till, through its sights, you see the other mark C. Then the degrees cut by the index, upon the graduated limb or ring of the instrument, shows the quantity of the angle.

3. *With the Magnetic Needle and Compass.*

Turn the instrument, or compass, so, that the north end of the needle point to the flower-de-luce. Then direct the sights to one mark, as B, and note the degrees cut by the needle. Next direct the sights to the other mark C, and note the degrees cut by the needle.—Then their sum, or difference, as the case is, will give the quantity of the angle BAC.

4. *By Measurment with the Chain, &c.*

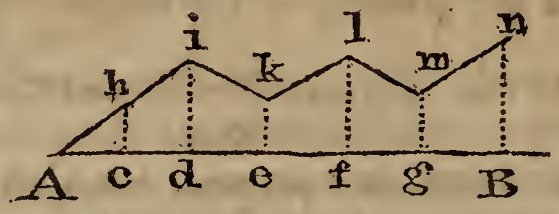
Measure one chain's length, or any other distance, in both directions, as to *b* and *c*. Then measure the distance *bc*, and it is done. This is easily transferred to paper, by mak-

making a triangle A*bc* with these three lengths, and then measuring the angle A by a line of chords, or the graduated arc of an instrument.

PROBLEM III.

To measure the offsets.

A*hiklmn* being a crooked hedge, or river, &c. from A measure in a straight direction along the side of it to B. And in measuring along this line AB, observe when you are directly opposite to any bends or corners of the hedge, as at *c*, *d*, *e*, &c. and thence measure the perpendicular offsets, *ch*, *di*, &c. with the offset staff, if they are not very long, otherwise with the chain itself; and the work is done. And the register, or field book, may be as follows.



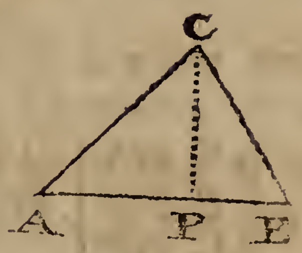
Offsets left.		Base line AB.	
	0	0	A
ch	62	45	A*c*
di	84	220	A*d*
ek	70	340	A*e*
fl	98	510	A*f*
gm	57	634	A*g*
Bn	91	785	AB

PROBLEM IV.

To Survey a triangular field.

1. *By the Chain.*

AP	794
AB	1321
PC	826



Having set up marks at the corners, which is to be done in all cases where there are not marks ; measure with the chain from A to P, where a perpendicular would fall from the angle C, and set up a mark at P, noting the distance AP. Then complete the distance AB, by measuring from P to B. Having set down this measure, return to P, and measure the perpendicular PC. And thus, having the base and perpendicular, the area is easily found from them ; and having also the place P of the perpendicular, the triangle is easily constructed.

Or, measure all the three sides with the chain, and note them ; from these the content is easily found, or the figure constructed.

2. *By taking one or more of the angles.*

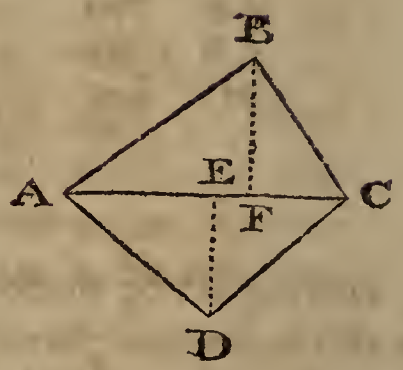
Measure two sides AB, AC, and the angle A between them. Or, measure one side AB, and the two adjacent angles A and B. From either of these sets of measures the figure is easily planned ; and then by measuring the perpendicular CP on the plan, and multiplying it by half AB, the content is obtained.

PROBLEM V.

To measure a four-sided field.

1. *By the Chain.*

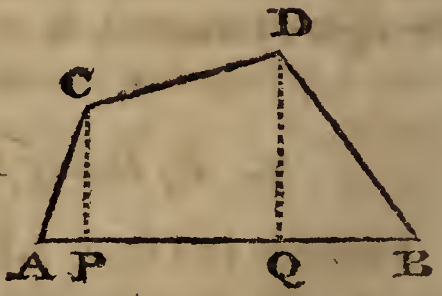
AE	214	210	DE
AF	362	306	BF
AC	592		



Measure along either of the diagonals, as AC; and either the two perpendiculars DE, BF, as in the last Problem; or else the sides AB, BC, CD, DA. From either of these sets of measures the figure may be planned and computed, as before directed.

Otherwise by the Chain.

AP	110	352	PC
AQ	745	595	QD
AB	1110		



Measure, on the longest side, the distances AP, AQ, AB; and the perpendiculars PC, QD.

2. *By taking one or more of the angles.*

Measure the diagonal AC, and the angles CAB, CAD, ACB, ACD. Or measure the four sides, and any one of the angles, as BAD.

[See last Figure but one.]

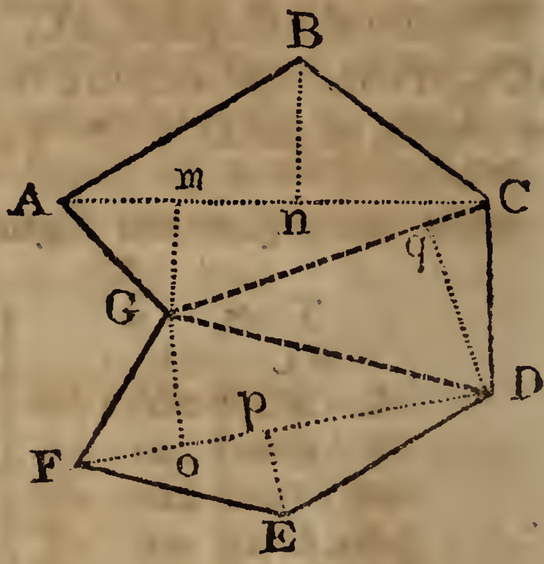
Thus,		Or thus,	
AC	591	AB	486
CAB	37° 20′	BC	394
CAD	41 15	CD	410
ACB	72 25	DA	462
ACD	54 40	BAD	78° 35′

PROBLEM VI.

To survey a field by a chain only.

Having set up marks, where necessary, at the corners of the proposed field ABCDEFG; walk over the ground, and consider how it can best be divided into triangles and trapeziums; and measure them separately, as in the two last Problems. And in this way it will be proper to divide it into as few separate triangles, and as many trapeziums, as may be, by drawing diagonals from corner to corner; and so, that all the perpendiculars may fall within the figure. Thus, the following figure is divided into the two trapeziums ABCG, GDEF, and the triangle GCD. Then, in the first, beginning at A, measure the diagonal AC, and the two perpendiculars G*m*, B*n*. Then the base GC, and the perpendicular DE. Lastly, the diagonal DF, and the two perpendiculars *p*E, OG. All which measures write against the corresponding parts of a rough figure, drawn to resemble the figure to be surveyed, or set them down in any other form you choose.

A*m*	135	130	*m*G
A*n*	410	180	*n*B
AC	550		
C*q*	152	230	*q*D
CG	440		
FO	206	120	OG
FP	288	80	*p*E
FD	520		



Or thus,

Measures all the sides AB, BC, CD, DE, EF, FG, GA; and the diagonals AC, CG, GD, DF.

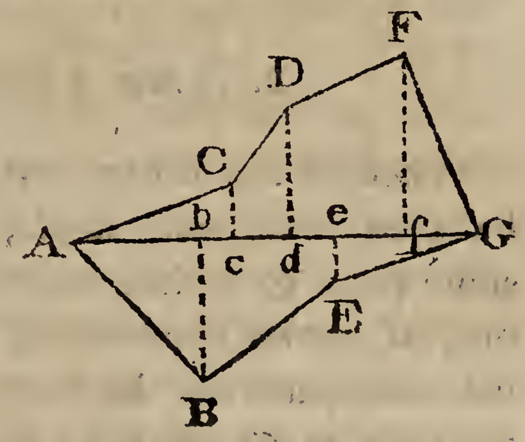
OTHERWISE.

Many pieces of land may be very well surveyed, by measuring any base line, either within or without them, together with the perpendiculars let fall upon it from every corner of them. For they are by these means divided into several triangles and trapezoids, all whose parallel sides are perpendicular to the base line; and the sum of these triangles and trapeziums will be equal to the figure proposed, if the base line fall within it; if not, the sum of the parts, which are without, being taken from the sum of the whole, which are both within and without, will leave the area of the figure proposed.

In pieces, which are not very large, it will be sufficiently exact to find the points in the base line, where the several perpendiculars will fall, by means of the *cross*, and measure thence to the corners for the lengths of the perpendiculars.—And it will be most convenient to draw the line so, that all the perpendiculars may fall within the figure.

Thus, in the following figure, beginning at A, and measuring along the line AG, we find the distances and perpendiculars, on the right and left, to be as follow.

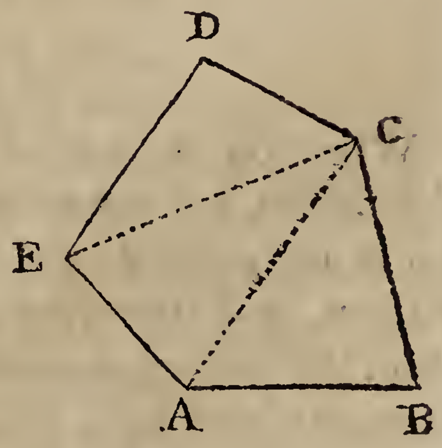
A*b*	315	350	*b*B
A*c*	440	70	*c*C
A*d*	585	320	*d*D
A*e*	610	50	*e*E
A*f*	990	470	*f*F
AG	1020	0	



PROBLEM VII.

To survey a field with a Plane Table.

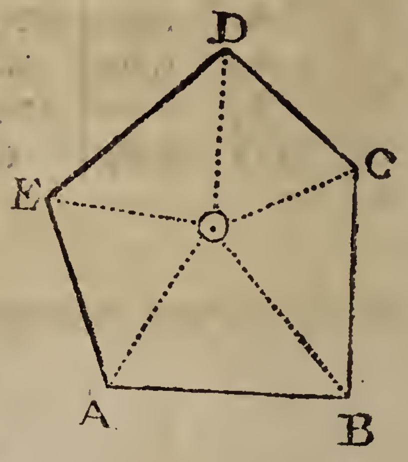
1. *From one station.*



Plant the table at any angle, as C, whence all the other angles, or marks set up, can be seen; and turn it about till the needle point to the flower-de-luce, and there screw it fast. Make a point for C on the paper, on the table, and lay the edge of the index to C, turning it about C till, through the sights, you see the mark D;

and by the edge of the index draw a dry or obscure line; then measure the distance CD, and lay that distance down on the line CD. Then turn the index about the point C till the mark E be seen through the sights, by which draw a line, and measure the distance to E, laying it on the line from C to E. In like manner, determine the positions of CA and CB, by turning the sights successively to A and B; and lay the lengths of those lines down. Then connect the points with the boundaries of the field, by drawing the black lines CD, DE, EA, AB, BC.

2. *From a station within the field.*



When all the other parts cannot be seen from one angle, choose some place ⊙ within, or even without, if more convenient, whence the other parts can be seen. Plant the table at ⊙, then fix it with the needle north, and mark the point ⊙ on it. Apply the index successively to ⊙, turning it round with the sights to each angle A, B, C, D, E, drawing dry lines to them by the edge of the index, then measuring the distances ⊙ A, ⊙ B, &c, and laying them down upon those lines. Lastly, draw the boundaries AB, BC, CD, DE, EA.

3. *By going round the field.*

When on account of a wood, or water, or some other obstruction, you cannot measure lines across the field; begin at any point A, and measure round it, either within or without, and draw the directions of all the sides, thus. Plant the table at A, turn it with the needle to the north, or flower-de-luce, fix it and mark the point A. Apply the index to A, turning it till you can see the point E, and there draw a line ; and then the point B, and there draw a line ; then measure these lines, and lay them down from A to E and B. Next move the table to B, lay the index along the line AB, and turn the table about till you can see the mark A, and screw fast the table ; in which position also, the needle will again point to the flower-de-luce, as it will indeed at every station, when the table is in the right position. Here turn the index about B till, through the sights, you see the mark C ; there draw a line, measure BC, and lay the distance on that line after you have set down the table at C. Turn it then again to its proper position, and in like manner find the next line CD. And so on, quite round by E to A. Then the proof of the work will be the joining at A. For if the work be all right, the last direction EA, on the ground, will pass exactly through the point A, on the paper ; and the measured distance will also reach exactly to A. If these do not nearly or quite coincide, there is some error, and the work must be examined.

PROBLEM VIII.

To survey a field with the Theodolite, &c.

1. *From one point or station.*

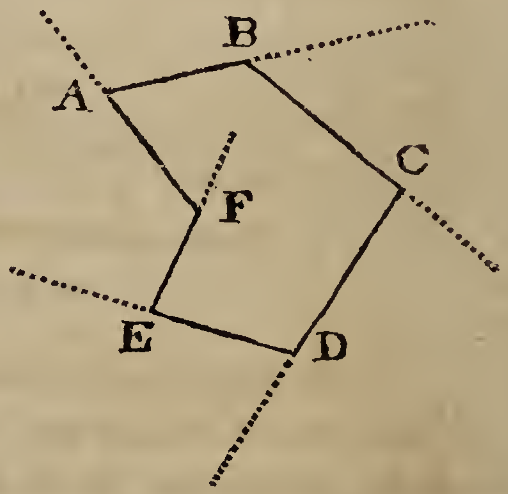
When all the angles can be seen from one point, as the angle C (first figure to the last Prob.) ; place the instrument at C, and turn it about till, through the fixed sights, you see the mark B, and there fix it. Then turn the moveable index about till the mark A is seen through the sights, and note the degrees cut on the instrument. Next turn the index successively to E and D, noting the degrees cut off at each; and all the angles BCA, BCE, BCD, are found. Lastly, measure the line CB, CA, CE, CD; and enter the measures in a field book, or rather against the corresponding parts of a rough figure, drawn by guess to resemble the field.

2. *From a point within or without.*

Plant the instrument at ⊙, (last figure) and turn it about till the fixed sights point to any object, as A; and there screw it fast. Then turn the moveable index round till the sights point successively to the other points E, D, C, B, noting the degrees cut off at each of them; and all the angles round the point ⊙ are found. Lastly, measure the distances ⊙ A, ⊙ B, ⊙ C, ⊙ D, ⊙ E, noting them down as before, and the work is done.

3. *By going round the field.*

By measuring round, either within or without the field, proceed thus:



Having set up marks at B, C, &c. near the corners as usual, plant the instrument at any point A, and turn it till the fixed index be in the direction A B, and there screw it fast; then turn the moveable index to the direction AF, and the degrees cut off will be the angle A. Measure the line AB, and plant the instrument at B, and there in the same manner observe the angle B. Measure BC, and observe the angle C. Measure the distance CD, and take the angle D. Measure DE, and take the angle E. Measure EF, and take the angle F. And lastly, measure the distance FA.

To prove the work ; add all the inward angles A, B, C, &c. together, and when the work is right, their sum will be equal to twice as many right angles, as the figure has sides, wanting four right angles. And when there is an angle, as F, that bends inward, and you measure the external angle, which is less than two right angles, subtract it from four right angles, or 360 degrees, to find the internal angle greater than a semicircle or 180 degrees.

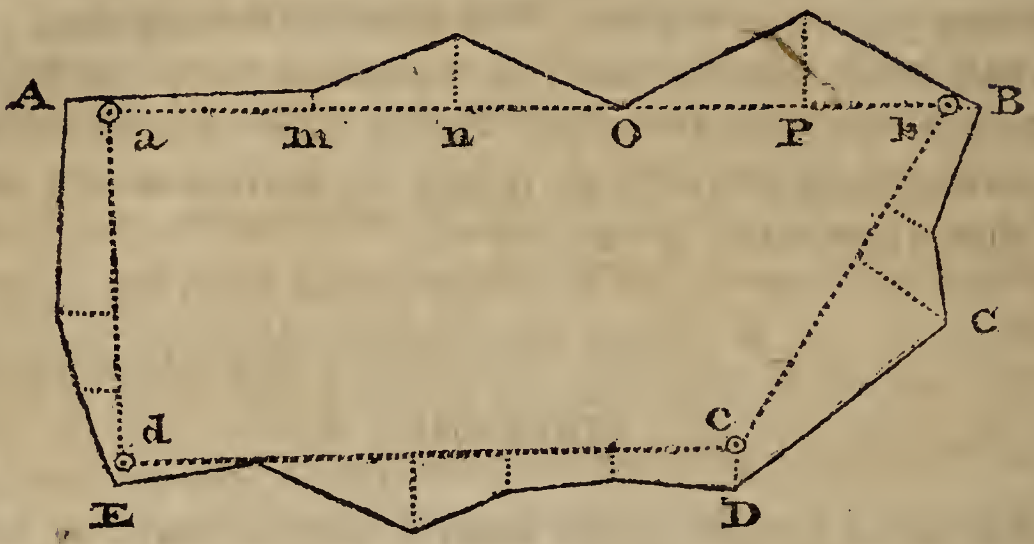
OTHERWISE.

Instead of observing the internal angles, you may take the external angles, formed without the figure by producing the sides beyond the angular points. And in this case, when the work is right, their sum will be equal to 360 degrees. But when one of them, as F, runs inward, subtract it from the sum of the rest, to leave 360 degrees.

PROBLEM IX.

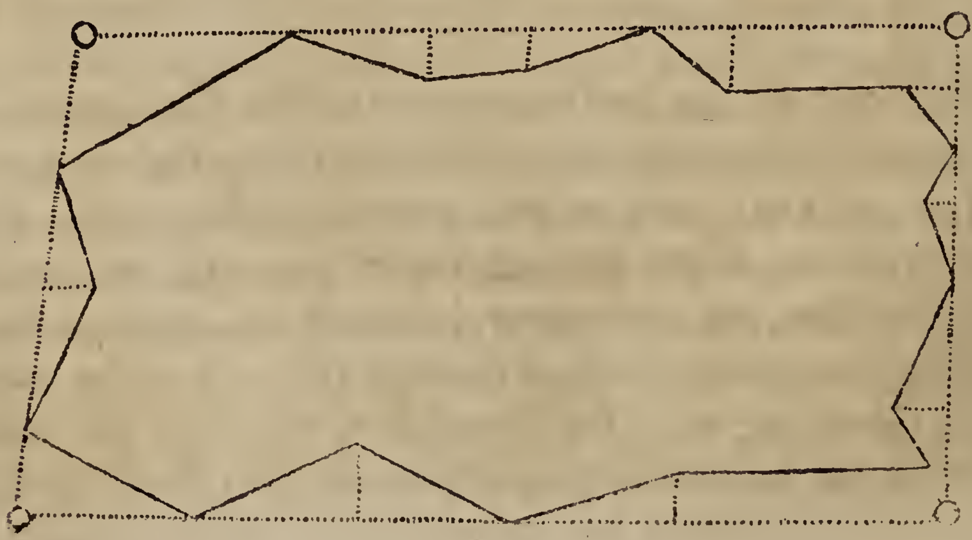
To survey a field with crooked hedges, &c.

With any of the instruments measure the lengths and positions of imaginary lines, running as near the sides of the field as you can ; and, as you pass along these lines, measure the offsets, in the manner before taught ; and you will have the plan, on the paper in using the plane table, by drawing the crooked hedges through the ends of the offsets ; but in surveying with the theodolite, or other instrument, set down the measures properly in the field book, and plan them after returning from the field, by laying down all the lines and angles.



So in surveying the piece ABCDE, set up marks *a*, *b*, *c*, *d*, dividing it into as few sides as may be. Then begin at any station *a*, and measure the lines *ab*, *bc*, *cd*, *da*, and take their positions, or the angles *a*, *b*, *c*, *d*; and going along the lines, measure all the offsets, as *m*, *n*, *o*, *p*, &c. along every stationary line.

And this is done either within the field, or without, as may be most convenient. When there are obstructions within, as wood, water, hills, &c. then measure without, as in the following figure.



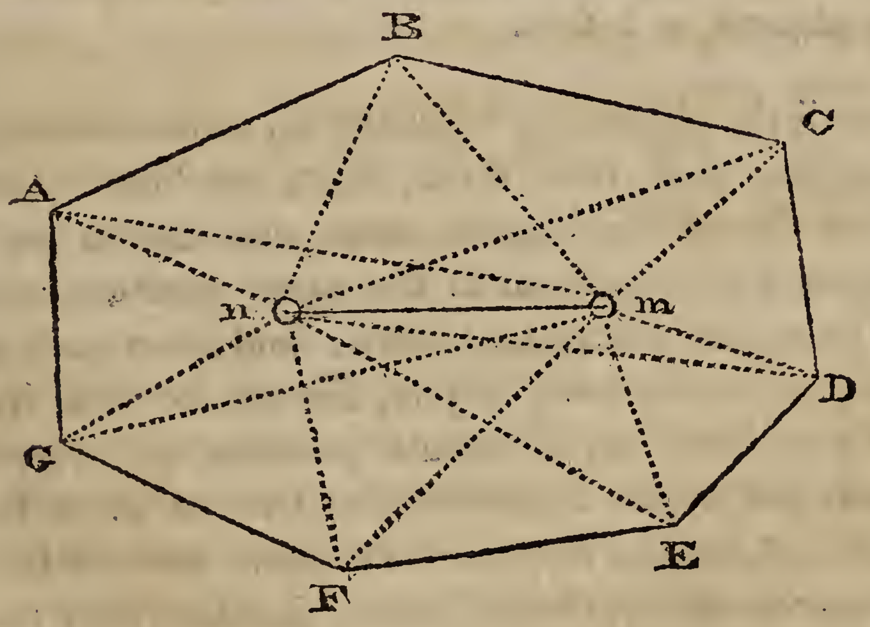
PROBLEM X.

To survey a field by two stations.

This is performed by choosing two stations, whence all the marks and objects can be seen, then measuring the distance between the stations, and at each station taking the angles, formed by each object and the direction of the stationary line or distance.

The two stations may be taken either within the bounds, or in one of the sides, or in the direction of two of the objects, or at a distance and without the bounds of the objects, or tract to be surveyed.

In this manner, not only grounds may be surveyed without even entering them, but a map of the principle parts of a country, or the chief places of a town, or the survey of any part of a river, coast, or any inaccessible objects, may be taken by means of two stations on two towers, two hills, or such like.



When the plane table is used; plant it at one station *m*, draw a line *mn* on it, along which lay the edge of the index, and turn the table about till the sights point directly to the other station; and there screw it fast. Then turn the sights round *m* successively to all the objects A, B, C, &c. drawing a dry line by the edge of the index at each, as *m*A, *m*B, *m*C, &c. Then measure the distance to the other station, there plant the table, and lay that distance down on the station line from *m* to *n*. Next lay the index by the line *nm*, and turn the table about till the sights point to the other station *m*, and there screw it fast. Then direct the sights successively to all the objects A, B, C, &c. as before, drawing lines each time, as *n*A, *n*B, *n*C, &c. and their intersection with the former lines will give the places of all the objects, or corners, A, B, C, &c.

When the theodolite, or any other instrument for taking angles, is used; proceed in the same way, measuring the station distance *mn*, planting the instrument first at one station, and then at the other; then placing the fixed sights in the direction *mn*, and directing the moveable sights to every object, noting the degrees cut off at each time.—Then, these observations being planned, the intersections of the lines will give the objects, as before.

When all the objects, to be surveyed, cannot be seen from two stations, then take three, four, or more stations; measuring always the distance from one station to another; placing the instrument in the same position at every station, by means described before; and from each station observing or setting every object, that can be seen from it, by taking its direction, or angular position, till every object be determined by the intersection of two or more lines of direction. And thus may very extensive surveys be taken, as of large commons, rivers, coasts, counties, hilly grounds, and such like.

PROBLEM XI.

To survey a large estate.

If the estate be very large, and contain a great number of fields, it will not well answer the purpose to survey all the fields singly, and then put them together; or to take all the angles and boundaries, that inclose it. For in these cases, any small errors will be so multiplied, as to render it very much distorted.

1. Walk over the estate two or three times in order to get a perfect idea of it, and till you can carry the map of it tolerably well in your head. And to assist the memory and guide you, draw an eye draught of it, or at least of the principal parts of it, on paper, setting the names within the fields in the draught.

2. Choose two or more eminent places in the estate for stations, whence you can see all the principal parts of it; and let these stations be as far distant from one another as possible.

3. Take such angles between the stations, as you think necessary, and measure the distances from station to station, always in a right line; these things must be done till you get as many angles and lines, as are sufficient for determining all the points of station. And in measuring any of these station distances, mark accurately where these lines meet with any hedges, ditches, roads, lanes, paths, rivulets, &c. and where any remarkable object is placed, by measuring its distance from the station line, and where a perpendicular from it cuts that line. And thus, as you go along any main station line, take offsets to the ends of all hedges, and to any pond, house, mill, bridge, &c. omitting nothing, that is remarkable, and noting down the whole.

4. As to the inner parts of the estate, they must be determined in like manner, by new station lines; for after the main stations are determined, and every thing adjoining to them, then the estate must be subdivided into two or three parts by new station lines; taking inner stations at proper places, where you can have the best view. Measure these station lines as you did the first, and all their intersections with hedges, and all offsets to such objects as appear. Then proceed to survey the adjoining fields, by taking the angles, that the sides make with the station line at the intersections, and measuring the distances to each corner from the intersections. For the station lines will be the bases to all the future operations; the situation of all parts being entirely dependent on them; and therefore they should be taken as long as possible; and it is best, that they should run along some of the hedges or boundaries of one or more fields, or pass through some of their angles. All things being determined for these stations, you must take more inner stations, and continue to divide and subdivide till at last you come to single fields; repeating the same work for the inner stations, as for the outer ones, till all be done; and close the work as often as you can, and in as few lines as possible.

5. An estate may be so situated, that the whole cannot be surveyed together, as one part of it may not be visible from another. In this case, you may divide it into three or four parts, and survey the parts separately, as if they were lands belonging to different persons; and at last join them together.

6. As it is necessary to protract or lay down the work, as you proceed in it, you must have a scale of a proper length to do it by. To get such a scale, measure the whole length of the estate in chains; then consider how many inches long the map is to be; and hence it will be

known how many chains must be contained in an inch; then make the scale accordingly, or choose one already made.

A NEW METHOD.

In the former method of measuring a large estate, the accuracy of it depends on the correctness of the instruments, used in taking the angles. To avoid the errors incident to such a multitude of angles, other methods have of late years been used by some skilful surveyors; the most practical, expeditious, and correct, seems to be the following.

Choose two or more eminences, as grand stations, and measure a principal base line from one station to the other, noting every hedge, brook, or other remarkable object, as you pass by it; measuring also short perpendicular lines to the bends of such hedges, as may be near. From the extremities of this base line, or from any convenient parts of the same, go off with other lines to some remarkable object, situated toward the sides of the estate, without regarding the angles they make with the base line or with one another; still remembering to note every hedge, brook, or other object you pass by. These lines, when laid down by intersections, will form with the base line a grand triangle on the estate; several of which, if need be, being thus laid down, you may proceed to form other smaller triangles and trapezoids on the sides of the former; and so on, till you finish with the enclosures individually.

The grand triangle being completed, and laid down in a rough plan, the parts, exterior as well as interior, are to be completed by smaller triangles and trapezoids.

In countries, where the lands are inclosed with high hedges, and where many lanes pass through an estate, a theodolite may be used to advantage in measuring the angles; by which means a kind of skeleton of the estate may be obtained, and the lane lines serve as the bases of such triangles and trapezoids, as are necessary to fill up the interior parts.

The field book is ruled into three columns. In the middle column are set down the distances on the chain line, at which any mark, offset, or other observation is made; and in the right and left columns are entered the offsets and observations, made on the right and left of the chain line respectively.

It is of great advantage, both for brevity and perspicuity, to begin at the bottom of the leaf and write upward; denoting the crossing of fences by lines, drawn across the middle column, or, for greater distinctness, only by parts of such lines on the right and left, opposite to the figures; and the corners of fields, and other remarkable turns in the fences, to which offsets are taken by lines joining in the manner the fences do, as will be best seen by comparing the following field book with the plan annexed.

The letter in the left hand corner at the beginning of every line is the mark, or place, *from* which the measure is taken; and that in the right-hand corner at the end is the mark, *to* which the measure is taken; but when it is not convenient to go exactly from a mark, the place, from which we measure, is described *such a distance from one mark toward another;* and when we do not measure *to* a mark, the exact place is ascertained by saying, *turn to the right* or *left*, *such a distance to such a mark;* it being always understood, that those distances are taken in the chain line.

The characters used are ⌈ for *turn to the right*, ⌉ for *turn to the left*, and Λ, placed over an offset, to show, that it is not taken at right angles with the chain line, but in the line with some straight fence, being chiefly used when crossing their directions ; and this is a better way of obtaining their true places than by offsets at right angles.

When a line is measured, whose position is determined either by former work, as in the case of producing a given line, or measuring from one known place or mark to another, or by itself, as in the third side of a triangle, it is called a *fast line*, and a double line is drawn across the book at the conclusion of it ; but if its position be not determined, as in the second side of a triangle, it is called a *loose line*, and a single line is drawn across the book. When a line becomes determined in position, and is afterward continued, a double line is drawn half across the book.

When a loose line is measured, it becomes absolutely necessary to measure some line, that will determine its position. Thus, the first line *ah*, being the bass of a triangle, is always determined ; but the position of the second side *hj* does not become determined till the third side *jb* is measured ; then the triangle may be constructed, and the position of both is determined.

At the beginning of a line, to fix a loose line to the mark or place, from which we measure, the sign of turning to the right or left must be added, as at *j* in the third line ; otherwise a stranger, when laying down the work, may as easily construct the triangle *hjb* on the wrong as on the right side of the line *ah* ; but this error cannot happen, if the said sign be carefully observed.

In choosing a line to fix a loose one care must be taken, that it do not make a very acute or obtuse angle. For in the triangle *p*B*r*, the angle at B being very obtuse, a small deviation from truth at *p* or *r* would make the error at B, when the triangle is constructed, very considerable ; but in constructing the triangle *p*B*q*, a small deviation is of much less consequence.

Where the phrase *leave off* is written in the field book; it is to signify, that taking offsets is thence discontinued; and of course something is wanting between that and the next offset.

The field book for this method and the plan drawn from it follow.

FIELD BOOK.

	3074	to *b*
	2494	
	2100	*l*
0	2072	
54	1730	
80	1530	—
	1420	*k*
56 + 30	1170	—
52	620	
j 32	280	40
	2574	*j*
—	2494	—
	2000	44
	1880	50
	1840	
50	1794	*i*
34 + 50	1464	
76	1328	
96	1240	
52+34	1130	
34	860	
h 66	190	
	4450	*h*
	3570	*g*
	2620	*f*
—	2590	—
	2210	
	2080	*e*
	1574	*d*
	1550	
	1510	*c*
	990	*b*
a —	806	

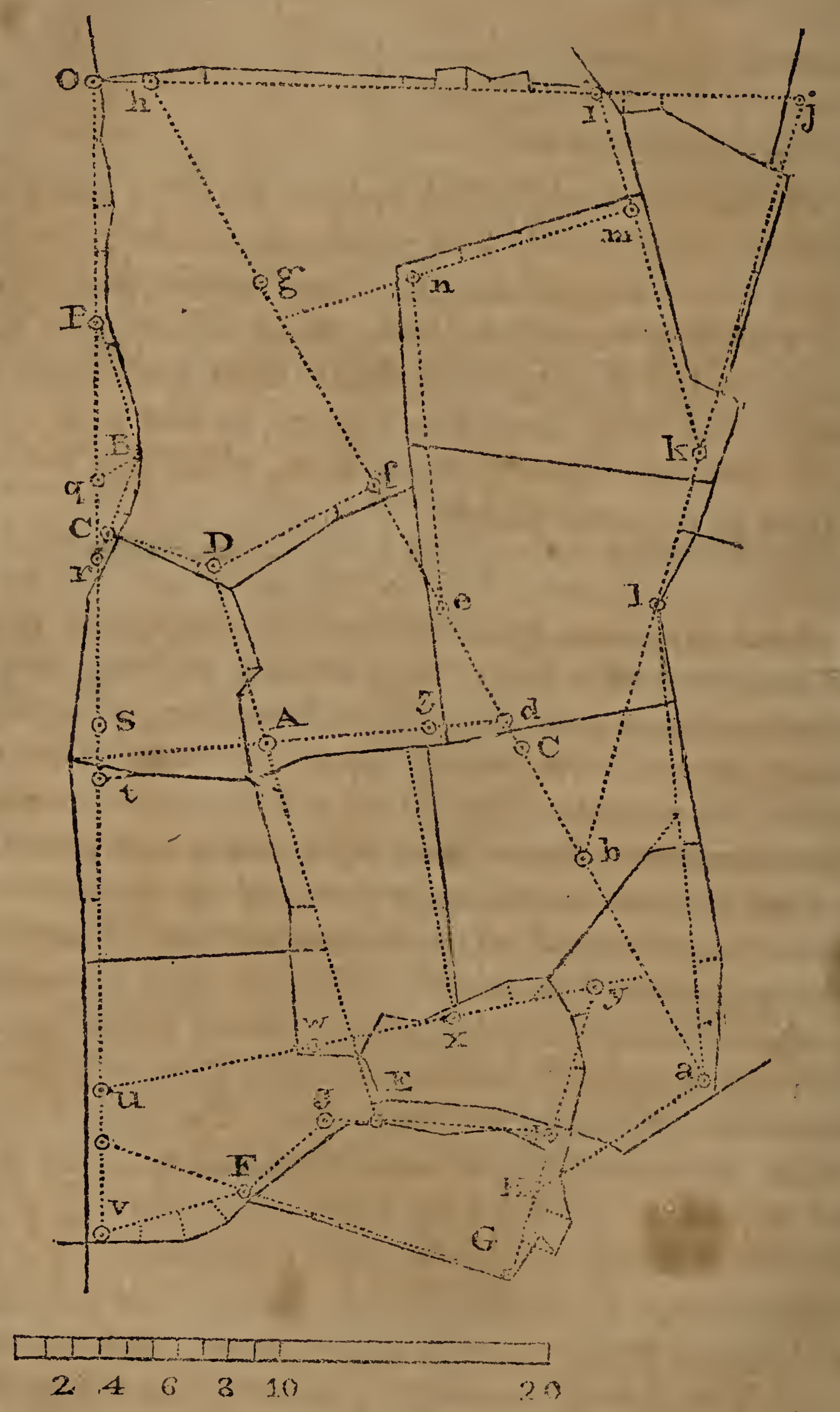
	0	1750	—
		1600	44 to S.
	—	1020	—
		940	A
	44	666	
	—70	310	z
d	—60	236	—
		1310	56. to *e*
		836	A 56....
n	——	684	A 50—
		1480	90 to *g*
		960	A 24
		930	*n*
		700	48
m		400	50
		1430	to *i*
		1290	40
	—	1004	36—
		980	*m*
		610	24
k		280	32
		1794	to *l*
	—	1464	22—
		1050	
	30	920	32—
		650	60
		350	48
a		0	14....

		2148		480 to *b*
		1950	*y*	
		1836	—	
	1. 28.	1724		
	60	1600		
	30	1480	*x*	
	0	1320		
	50	1110		
		1080	—	
		990	*w*	
u	—	750	50	
	—	4440	36	
		4420	*v*	
		3884	*u*	
	—	3380	60	
		2992	90	
		2692	*t*	
	120	2624		
		2592	—	
		2500	*s*	
		2070	56	leave off
		1900		
		1840	*r*	
	60	1770		
		1320	*q*	
		808	*p*	
leave off	40	650		
	80	360		
O	20	170		

	—	2280	—
		2270	E
	20	2230	
	56	2050	
		2030	
		1940	180 to u
		1552	180
		1380	96
		950	110
	—	860	54
A			
		768	to A
		526	70
		496	
	70	460	
	40	124	
D		100	
		644	to f
D		488	32
		455	D
		400	76
C	—	48	10
		600	to r
		432	C
	56	160	
B	44	36	
B		152	to q
		480	B
p	24	160	

	220	O
h produce by *i* —	190	46
	580	to *v*
40	500	
76	300	
F 76	100	
	360	to F
J 20	150	
	954	J
15	850	
	730	to E
30 —	490	
	340	60
0	280	
I 20	170	50
	744	to H
—	672	0
a 70	450	0
	1160	to *y*
32	1000	
—	890	—
	780	32
	590	40
	570	I
—	530	40
	376	H
	256	150 leave off
	190	64
G	144	130
	1676	G
	1676	30
	896	24
	632	
	620	50
180 from *u* toward *v*	588	F

PLAN FROM THE FIELD BOOK.



PROBLEM XII.

To survey a county, or large tract of land.

1. Choose two, three, or four eminent places for stations; such as the tops of high hills or mountains, towers, or church steeples, which may be seen from one another, and from which most of the towns, and other places of note, may also be seen; and so as to be as far distant from one another as possible. On these places raise beacons, or long poles with flags of different colors flying at them, which may be visible from all the other stations.

2. At all the places, which you would set down in the map, plant long poles with flags of several colors on them, to distinguish the places from one another; fixing them on the tops of church steeples, or the tops of houses, or in the centres of less towns. These marks being set up at a convenient number of places, and being such as may be seen from both stations; go to one of these stations, and with an instrument to take angles, standing at that station take all the angles between the other station and each of these marks. Then go to the other station, and take all the angles between the first station and each of the former marks, setting them down with the others, each against the corresponding one of the same color. You may also, if you can, take the angles at some third station, which may serve to prove the work, if the three lines intersect in the point, where any mark stands. The marks must stand till the observations are finished at both stations; and then they must be taken down, and set up at new places. And the same operations must be performed, at both stations, for these new places; and the like for others. The instrument for taking angles must be an exceedingly good one, having telescopic

sights and a good length of radius. A circumferentor is reckoned a good instrument for this purpose.

3. And though it be not absolutely necessary to measure any distance, because a stationary line being laid down from any scale, all the other lines will be proportional to it; yet it is better to measure some of the lines to ascertain the distances of places in miles, and to know how many geometrical miles there are in any length; and thence to make a scale to measure any distance in miles. In measuring any distance, it will not be exact enough to go along the high roads, which scarcely ever lie in a right line between the stations, or can with ease and accuracy be reduced to one. But the better way is to measure in a right line with a chain, between station and station, over hills and dales, or level fields, and all obstacles. Only in case of water, woods, towns, rocks, banks, &c. where one cannot pass, such parts of the line must be measured by the methods of inaccessible distances; and beside, allowing for ascents and descents, when they occur. A good compass, that shows the bearing of the two stations, will always direct you to go straight, when you do not see the two stations; and in the progress, if you can go straight, offsets may be taken to any remarkable places, and the intersection of the stationary line with all roads, rivers, &c. noted.

4. From all the stations, and in the whole progress, be very particular in observing sea coasts, rivers' mouths, towns, castles, houses, churches, windmills, watermills, trees, rocks, sands, roads, bridges, fords, ferries, woods, hills, mountains, rills, brooks, parks, beacons, sluices, flood gates, locks, &c. and in general all things, that are remarkable.

5. After you have done with the first and main station lines, which command the whole county, take inner stations at some places already determined; which will divide the whole into several partitions; and from these stations determine the pla-

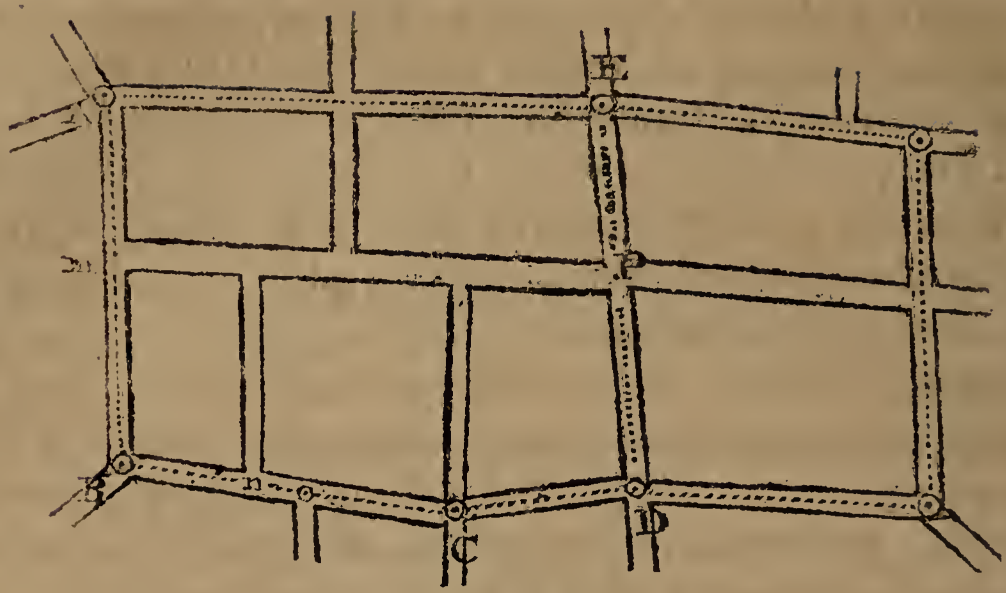
ces of as many of the remaining towns as you can. And if any remain in that part, take more stations at some places already determined, from which you can determine the rest. And thus go through all the parts of the county, taking station after station, till all, that is wanted, be determined. And in general the station distances must always pass through such remarkable points, as have been determimed before by the former stations.

PROBLEM XIII.

To survey a town or city.

This may be done with any of the instruments for taking angles, but best with the plane table, where every minute part is drawn while in sight. It is convenient also to have a chain 50 feet long, divided into 50 links, and an offset staff 10 feet long.

Begin at the meeting of two or more of the principal streets, through which you can have the longest prospects, to get the longest station lines. There having fixed the instrument, draw lines of direction along those streets, using two men as marks, or poles set in wooden pedestals, or perhaps some remarkable places in the houses at the farther ends, as windows, doors, corners, &c. Measure these lines with the chain, taking offsets with the staff at all corners of streets, bendings or windings, and to all remarkable things, as churches, markets, halls, colleges, eminent houses, &c. Then remove the instrument to another station along one of these lines; and there repeat the same process. And so on, till the whole be finished.



Thus, fix the instrument at A, and draw lines in the direction of all the streets meeting there ; and measure AB, noting the street on the left at *m*. At the second station B, draw the directions of the streets meeting there ; measure from B to C, noting the places of the streets at *n* and o, as you pass by them. At the third station C take the directions of all the streets meeting there, and measure CD ; at D do the same, and measure DE, noting the place of the cross streets at P. And in this manner go through all the principal streets. This done, proceed to the smaller and intermediate streets ; and lastly to the lanes, alleys, courts, yards, and every part, that it may be thought proper to represent in the plan.

PLANNING AND COMPUTING.

PROBLEM I.

To plan.

If the survey have been taken with a plane table, you have a rough plan of it already on the paper, which covered the table. But if the survey have been made with any other instrument, a plan of it is to be drawn from the measures, that were taken in the survey, and first of all a rough plan on paper.

To do this, you must have a set of proper instruments for laying down both lines and angles ; as scales of various sizes, the more of them and the more accurate, the better ; scales of chords, protractors, perpendicular and parallel rules, &c. Diagonal scales are best for lines, because they extend to three figures, or chains and links, which are hundredth parts of chains. But in using the diagonal scale, a pair of compasses must be employed to take off the lengths of the principal lines very accurately. But a scale with a thin divided edge is much readier for laying down the perpendicular offsets to crooked hedges, and for marking the places of those offects upon the station line ; which is done at only one application of the edge of the scale to that line, and then pricking off all at once the distances along it. Angles are to be laid down either with a good scale of chords, which is perhaps the most accurate way ; or with a large protractor, which is much readier when many angles are to be laid down at one point, as they are pricked off all at once round the edge of the protractor.

In general, all the lines and angles must be laid down on the plan in the same order, in which they were measured in the field, and in which they were written in the field book ; the angles for the position of lines being first, then the lengths of the lines, with the places of the offsets, and then the lengths of the offsets themselves, all with dry or obscure lines ; then a black line, drawn through the extremities of all the offsets, will be the hedge or bounding line of the field, &c. After the principal bounds and lines are laid down, and made to fit or close properly, proceed next to the smaller objects, till you have entered every thing, that ought to appear in the plan, as houses, brooks, trees, hills, gates, stiles, roads, lanes, mills, bridges, woodlands, &c.

The north side of a map or plan is commonly placed uppermost, and the meridian drawn on some part, with the compass or flower-de-luce pointing northward. Also in a vacant part, a scale of equal parts or chains is drawn, and the title of the map in conspicuous characters, and embellished with a compartment. Hills are shadowed, to be distinguished in the map. Color the hedges with different colors ; represent hilly grounds by broken hills and valleys ; you may draw single dotted lines for foot paths, and double ones for horse or carriage roads. Write the name of each field and remarkable place within it, and, if you choose, its content in acres, roods, and perches.

In a very large estate, or a county, draw vertical and horizontal lines through the map, denoting the spaces between them by letters, placed at the top, bottom, and sides, for readily finding any field, or other object, mentioned in a table.

In mapping counties and estates, that have uneven grounds of hills and valleys, reduce all oblique lines, measured up and down hills, to horizontal straight lines, if that was not done in the field before they were entered in the field book, by making a proper allowance. For which purpose there is a small table, engraven on some of the instruments for surveying.

PROBLEM II.

To compute the contents.

An *acre* of land is equal to ten square chains ; that is, ten chains in length and one chain in breadth. Or, it is 220×22, or 4840 square yards. Or, it is 40×4, or 160 square poles. Or, it is 1000×100, or 100000 square links. These being all the same quantity.

Also an acre is divided into four parts, called *roods*, and a rood into 40 parts, called *perches*, which are square poles, or the square of a pole of $5\frac{1}{2}$ yards long, or the square of $\frac{1}{4}$ of a chain, or of 25 links, which is 625 square links. So that the divisions of land measure will be thus:

625 square links = 1 pole or perch

40 perches = 1 rood

4 roods = 1 acre.

To find the content:

1. Compute the contents of the figures, whether triangles or trapeziums, &c. by the proper rules for the several figures, given in Mensuration; the factors being expressed in links; the product, being square links, is reduced to acres by cutting off five figures on the right, for decimals; then bring these decimals to roods and perches, by multiplying first by 4 and then by 40.

2. In small and separate pieces, it is usual to cast up their contents from the measures of the lines, taken in surveying them, without making a correct plan of them.

Thus, in the triangle, Prob. IV. of the Practice of Surveying, where we had AP=794, and

```
          AB=1321
          PC= 826
          -------
             7926
            2642
           10568
          -------
        2)10·91146
           5·45573
                 4
          -------
           1·82292
                40
          -------
          32·91680   Ans. 5 acres, 1 rood, 33
          -------         perches, nearly.
```

Or the first example to Prob. V. of the same part, thus:

```
   AE   214 | 210   ED
   AF   362 | 306   FB
   AC   592 | ---
            | 516 sum of perp.
              592 AC
              ----
             1032
            4644
           2580
          -------
        2)3·05472
          1·52736
                4
          -------
          2·10944
               40
          -------
          4·37760    Ans. 1ac. 2r. 4p.
          -------
```

Or the second example to the same Prob. thus:

```
  AP   110 | 352  PC
  AQ   745 | 595  QD
  AB  1110 |

  PC    352    PC  352      QD  595
  AP    110    QD  595      QB  365
  ---------    -------      -------
2APC  38720    Sum 947         2975
  ---------    PQ  635        3570
                -------      1785
                  4735      --------
                 2841       217175=2QDB
                5682        601345=2PCDQ
                -------      38720=2APC
      2PCDQ     601345      --------
                -------  2)8·57240= double the
                            4·2862     whole
                                 4
                            ------
                            1·1448
                                40
                            ------
Ans. 4ac. 1r. 5p.           5·7920
                            ------
```

3. In pieces, bounded by very crooked and winding hedges, measured by offsets, all the parts between the offsets are most acurately measured separately by small trapezoids. Thus, for the example to Prob. III. where

A*c*	45	62	*ch*
A*d*	220	84	*di*
A*e*	340	70	*ek*
A*f*	510	98	*fl*
A*g*	634	57	*gm*
AB	785	91	B*n*

Then

```
Ac  45        ch  62        di  84
ch  62        di  84        ek  70
  ----          ----          ----
    90           146           154
   270       cd  175       de  120
  ----          ----          ----
  2790           730         18480
  ----          1022          ----
                 146
                ----
                25550
                ----

ek  70        fl  98        gm  57
fl  98        gm  57        Bn  91
  ----          ----          ----
   168           155           148
cf 170       fg  124       gB  151
  ----          ----          ----
 11760           620           148
 168            310           740
  ----          155           148
 28560          ----          ----
  ----         19220         22348
                ----          ----
```

2790
25550
18480
28560
19220
22348

2)1·16948
·58474
4

2·33896
40

13·55840 Content 2r. 13p.

4. Sometimes such pieces, as that above, are computed by finding a mean breadth, by dividing the sum of the offsets by the number of them, accounting that for one of them where the boundary meets the station line, as A; and then multiplying the length AB by that mean breadth.

Thus:

00	785	AB
62	66	mean breadth
84		
70	4710	
98	4710	
57		
91	·51810	Content 2r. 2p. by this method, which is 10 perches too little. And it is commonly in some degree erroneous.
	4	
7)462		
66	2·07240	
	40	
	2·89600	

5. But in large pieces, and whole estates, consisting of many fields, it is the common practice to make a rough plan of the whole, and from it to compute the contents, quite independently of the measures of the lines and angles, that were taken in the field. For then new lines are drawn in the fields in the plan, so as to divide them into trapeziums and triangles, the bases and perpendiculars of which are measured on the plan by means of the scale, from which it was drawn, and then multiplied together for the contents. In this way the work is very expeditiously done, and sufficiently correct; for such dimensions are taken, as afford the most easy method of calculation; and, among a number of parts, thus taken and applied to a scale, it is likely that some of the parts will be taken somewhat too little, and others too great; so that they will, upon the whole, in all probability, very nearly balance one another. After all the fields and particular parts are thus computed separately, and added together, calculate the whole estate independently of the fields, by dividing it into large and arbitrary triangles and trapeziums, and add these also together. Then if this sum be equal to the former, or nearly so, the work is right; but if the sums have any considerable difference, it is wrong, and they must be examined, and recomputed till they nearly agree.

A specimen of the division of a large tract into trapeziums and triangles may be seen in Prob. VI. of the Practice of Surveying.

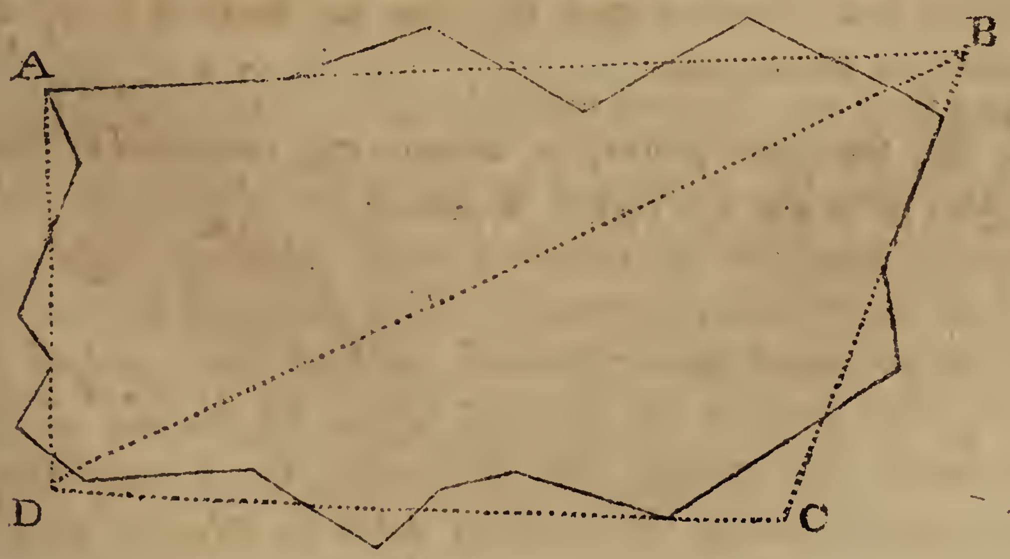
6. But the chief secret, in computing, consists in finding the contents of pieces bounded by curved, or very irregular lines, or in reducing such crooked sides of fields or boundaries to straight lines, that shall inclose the same or an equal area with these crooked sides, and so obtaining the area of the curved figure by means of the right-lined one, which will commonly be a trapezium. Now this reducing the crooked sides to straight ones is

very easily and accurately performed, thus :—Apply the straight edge of a thin, clear piece of lantern horn to the crooked line, which is to be reduced in such a manner, that the small parts, cut off from the crooked figure by it, may be equal to those, which are taken in ; which equality of the parts included and excluded, you will presently be able to judge of very nicely by a little practice ; then with a pencil draw a line by the straight edge of the horn. Do the same by the other sides of the field or figure. So shall you have a straight-sided figure equal to the curved one ; the content of which, being computed as before directed, will be the content of the curved figure proposed.

Or, instead of the straight edge of the horn, a horse hair may be applied across the crooked sides in the same manner ; and the easiest way of using the hair is to string a small slender bow with it, either of wire, cane, whale bone, or such like slender, elastic matter ; for, the bow keeping it always stretched, it can be easily and neatly applied with one hand, while the other is at liberty to make two marks by the side of it, by which the straight line may be drawn.

EXAMPLE.

Let it be required to find the content of the same figure as in Prob. IX, of the Practice of Surveying, to a scale of 4 chains to an inch.



Draw the four dotted straight lines AB, BC, CD, DA, cutting off equal quantities on both sides of them, which they do as nearly as the eye can judge: so is the crooked figure reduced to an equivalent right-lined one of four sides, ABCD. Then draw the diagonal BD, which, by applying a proper scale to it, is found to be 1256. Also the perpendicular, or nearest distance from A to this diagonal, is 456; and the distance of C from it 428.

```
Then          456
              428
             ----
              884
             1256
            -----
             5024
           10048
          10048
         --------
       2)11·10304
          5·55152
                4
         --------
          2·20608
               40
         --------
          8·24320  Content 5ac. 2r. 8p.
         --------
```

And thus the contents of the trapezium, and consequently of the irregular figure, to which it is equal, is easily found to be 5 acres, 2 roods, 8 perches.

PROBLEM III.

To transfer a plan to another paper, &c.

After the rough plan is completed, and a fair one is wanted; this may be obtained, either on paper or vellum, by any of the following methods.

FIRST METHOD.

Lay the rough plan on the clean paper, and keep them always pressed flat and close together by weights laid on them. Then, with the point of a fine pin or pricker, prick through all the corners of the plan to be copied. Separate them and connect the pricked points on the clean paper with lines, and it is done. This method is only to be practised in plans of such figures, as are small and tolerably regular, or bounded by right lines.

SECOND METHOD.

Rub the back of the rough plan over with black lead powder; and lay the said black part upon the clean paper, on which the plan is to be copied, and in the proper position. Then, with the blunt point of some hard substance, as brass, &c. trace over the lines of the whole plan; pressing the tracer so much, that the black lead under the lines may be transferred to the clean paper; after which take off the rough plan, and trace over the leaden marks with common ink, or with Indian ink, &c. Or, instead of blacking the rough plan, you may keep constantly a blacked paper to lay between the plans.

THIRD METHOD.

Another way of copying plans is by means of squares. This is performed by dividing both ends and sides of the plan, which is to be copied, into any convenient number of equal parts, and connecting the corresponding points of division with lines, which will divide the plan into a number of small squares. Then divide the paper, upon which the plan is to be copied, into the same number of squares, each equal to one of the former, when the plan is to be copied of the same size, but greater or less in the proportion, in which the plan is to be increased or diminished, when of a different size. Lastly, copy into the clean squares the parts contained in the corresponding squares of the other plan; and you will have the copy either of the same size, or greater or less in the required proportion.

FOURTH METHOD.

A fourth way is by the instrument, called a *Pantograph*, which also copies the plan in any size required.

FIFTH METHOD.

But the neatest method of any is this. Procure a copying frame or glass, made in this manner; namely, a large square of the best window glass, set in a broad frame of wood, which can be raised up to any angle, when the lower side of it rests on a table. Set this frame at any angle before you, facing a strong light; fix the plan and clean paper together with pins in the edges, to keep them together; the clean paper being laid uppermost, and on the face of the plan to be copied. Lay them with the back of the plan on the glass, namely, the part you intend to copy first, and by means of the light shining through the papers you will very distinctly perceive every line of the plan through the clean paper. In this state then trace all the lines on the paper with a pencil. Having drawn that part, which covers the glass, slide another part over the glass, and copy it in the same manner. And then another part; and so on, till the whole be copied.

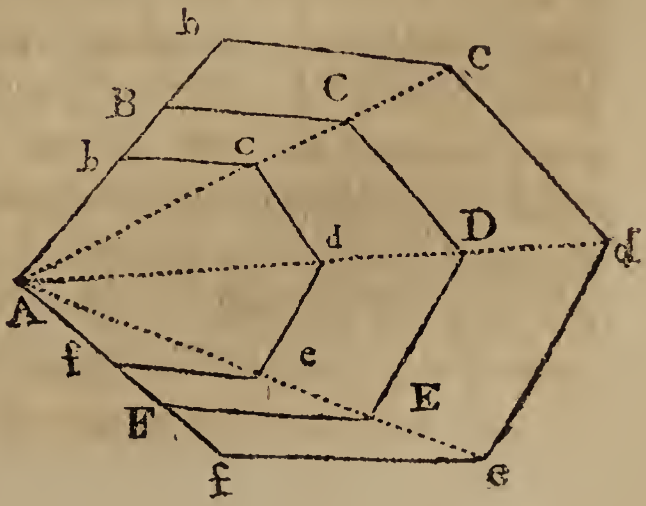
Then separate them, and trace all the pencil lines over with a fine pen and Indian ink, or with common ink.

And thus you may copy the finest plan, without injuring it in the least.

When the lines, &c. are copied on the clean paper or vellum, the next business is to write such names, remarks, or explanations, as may be judged necessary; drawing the scale for taking the lengths of any parts, a flower-de-luce to point out the direction, and the proper title, ornamented with a compartment; and illustrating or coloring every part in such manner, as shall seem most natural; as shading rivers or brooks with crooked lines, drawing the representation of trees, bushes, hills, woods, hedges, houses, gates, roads, &c. in their proper places; running a single dotted line for a foot path, and a double one for a carriage road; and representing either the bases or the elevations of buildings, &c.

PROBLEM IV.

To change a figure from one scale to another.



From one angle A draw lines AC, AD, AE, &c. to all the other angles of the given figure; then augment or diminish one side AB, till A*b* be to AB in the given proportion of the scales; and, by means of a parallel rule, draw *bc* parallel to BC, and meeting AC in c, and in the same manner *cd* parallel to CD, *de* parallel to DE, *ef* parallel to EF; so shall A*bcdef*A be the figure required.

DIVISION OF LAND.

IN the division of commons, after the whole is surveyed and computed, and the proper quantities to be allowed for roads, &c. deducted, divide the net quantity remaining among the several proprietors by the rule of Fellowship, in proportion to the real value of their estates, and you will thereby obtain their proportional quantities of the land. But as this division supposes the land, which is to be divided, to be all of equal goodness, you must observe, that if the part, in which any one's share is to be marked off, be better or worse than the general mean quality of the land, then you must diminish or augment the quantity of his share in the same proportion.*

* Or, which comes to the same thing, divide the ground among the claimants in the direct ratio of the value of their claims, and the inverse ratio of the quality of the ground allotted to each; that is, in proportion to the quotients arising from the division of the value of each person's estate by the number, which expresses the quality of the ground in his share.

But these regular methods cannot be always put in practice; so that, in the division of commons, the usual way is to measure separately all the land, that is of different values, and add into two sums the contents and the values; then, by the first part of

PROBLEM I.

It is required to divide any given quantity of ground, or its value, into any given number or parts, and in proportion to any given numbers.

Divide the given piece, or its value, as in the rule of Fellowship, by dividing the whole content or value by the sum of the numbers expressing the proportions of the several shares, and multiplying the quotient severally by the said proportional numbers for the respective shares required, when the land is all of the same quality. But if the shares be of different qualities, then divide the numbers expressing the proportions of values of the shares by the numbers, which express the qualities of the land in each share; and use the quotients instead of the former proportional numbers.

EXAMPLES.

1. If the total value of a common be 2500l. it is required to determine the values of the shares of the three claimants A, B, C, whose estates are respectively of these values, 10000, 15000, and 25000 pounds.

The estates being in proportion as the numbers 2, 3, 5, whose sum is 10, we shall have $2500 \div 10 = 250$; which being severally multipled by 2, 3, 5, the products 500, 750, 1250, are the values of the shares required.

---

the following Problem I, the value of every claimant's share is computed by dividing the whole value among them in proportion to their estates; and, lastly, by Problem II, a quantity is laid out for each person, that shall be of the value of his share before found.

2. It is required to divide 300 acres of land among A, B, C, D, E, F, G, and H, whose claims upon it are respectively in proportion as the numbers 1, 2, 3, 5, 8, 10, 15, 20.

The sum of these proportional numbers is 64, by which 300 being divided, the quotient is 4ac. 2r. 30p. which being multiplied by each of the nmubers 1, 2, 3, 5, &c. we have for the several shares as follow :

	Ac.	R.	P.
A =	4	2	30
B =	9	1	20
C =	14	0	10
D =	23	1	30
E =	37	2	00
F =	46	3	20
G =	70	1	10
H =	93	3	00
Sum =	300	0	00

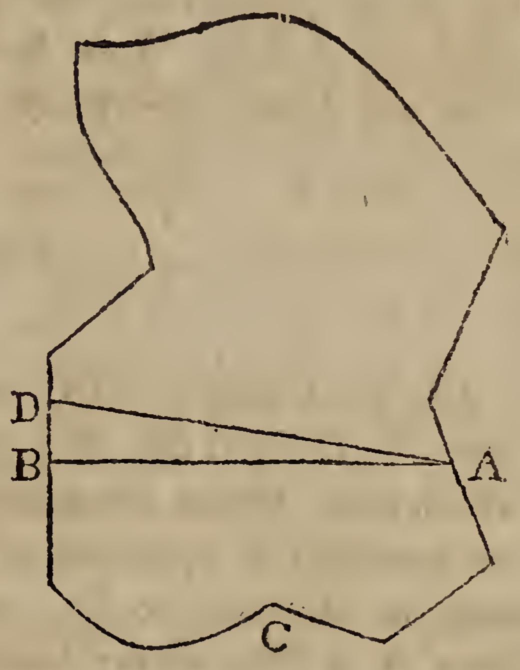
3. It is required to divide 780 acres among A, B, and C, whose estates are 1000, 3000, and 4000 pounds a year ; the ground in their shares being worth 5, 8, and 10 shillings the acre respectively.

Here their claims are as 1, 3, 4 ; and the qualities of their land are as 5, 8, 10 ; therefore their quantities must be as $\frac{1}{5}$, $\frac{3}{8}$, $\frac{2}{5}$, or, by reduction, as 8, 15, 16. Now the sum of these numbers is 39 ; by which the 780 acres being divided, the quotient is 20 ; which being multiplied severally by the three numbers 8, 15, 16, the three products are 160, 300, 320, for the shares of A, B, C, respectively.

PROBLEM II.

To cut off from a plan a given number of acres, &c. by a line drawn from any point in the side of it.

Let A be the given point in the annexed plan, from which a line is to be drawn cutting off, suppose, 5ac. 2r. 14p.



Draw AB cutting off the part ABC as nearly as can be judged equal to the quantity proposed; and suppose the true quantity of ABC, when calculated, be only 4ac. 3r. 20p. which is less than 5ac. 2r. 14p. the true quantity, by 2r. 34p. or 71250 square links. Then measure AB, which suppose = 1234 links, and divide 71250 by 617 the half of it, and the quotient 115 links will be the altitude of the triangle to be added, and its base is AB. Therefore, if on the centre B, with the radius 115, an arc be described; and a line be drawn parallel to AB, touching the arc, and cutting BD in D; and if AD be drawn, it will be the line cutting off the required quantity ADCA.

NOTE. If the first piece had been too much, then D must have been set below B.

In this manner the several shares of commons, to be divided, may be laid down on the plan, and transferred thence to the ground itself.

Also for the greater ease and perfection in this business, the following Problems may be added.

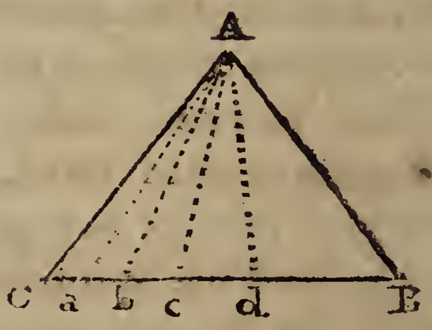
PROBLEM III.

From an ngle, in a given triangle, to draw lines to the opposite side, dividing the triangle into any number of parts, which shall be in any assigned proportion to each other.

Divide the base into the same number of parts, and in the same proportion, by Prob. I; then from the several points of division draw lines to the proportional angle, and they will divide the triangle as required.*

EXAMPLE.

Let the triangle ABC, of 20 acres, be divided into five parts, which shall be in proportion to the numbers 1, 2, 3, 5, 9; the lines of division to be drawn from A to CB, whose length is 1600 links.



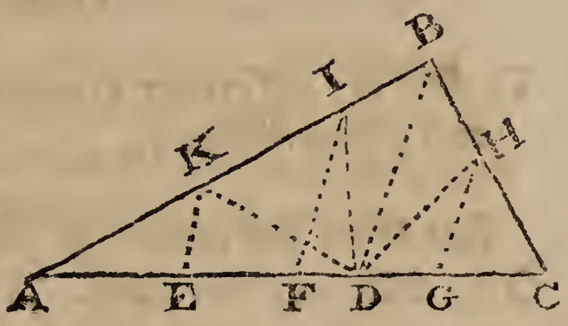
Here $1+2+3+5+9=20$, and $1600 \div 20 = 80$; which being multiplied by each of the proportional numbers, we have 80, 160, 240, 400, and 720. Therefore make Ca=80, ab=160, bc=240, cd=400, and dB =720; then, by drawing the lines Aa, Ab, Ac, Ad, the triangle is divided as required.

---

* Demonstration. For the several parts are triangles of the same altitude, and which therefore are as their bases, which bases are taken in the assigned proportion.

PROBLEM IV.

From any point, in one side of a given triangle, to draw lines to the other two sides, dividing the triangle into any number of parts, which shall be in any assigned ratio.



From the given point D draw DB to the angle, opposite the side AC, in which the point is taken; then divide the same side AC into as many parts AE, EF, FG, GC, and in the same proportion with the required parts of the triangle, as in the last Problem; and from the points of division draw lines EK, FI, GH, parallel to the line BD, and meeting the other sides of the triangle in K, I, H; lastly, draw KD, ID, HD; so shall ADK, KDI, IDHB, HDC, be the parts required.*

An example of this is performed like that of the last Problem.

---

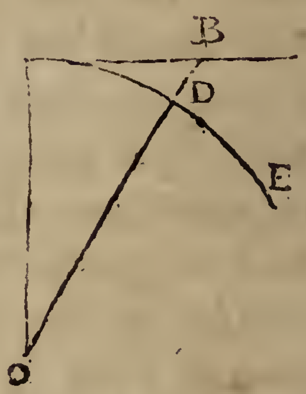
* Demonstration. The triangles ADK, KDI, IDB, being of the same height, are as their bases AK, KI, IB; which, by means of the parallels EK, FI, DB, are as AE, EF, FD; in like manner, the triangles CDH, HDB are to each other as CG, GD: but the two triangles IDB, BDH, having the same base BD, are to each other as the distances of I and H from BD, or as FD to DG; consequently the parts DAK, DKI, DIBH, DHC, are to each other as AE, EF, FG, GC.

LEVELLING.

By Levelling we find how much one point on the surface of the earth is higher or lower than another; or the difference in their distances from the centre of the earth.

PROBLEM I.

To find the difference between the true and apparent levels at a given distance.



Let O be the centre of the earth, OA, OD, any two radii, AB a tangent to the surface ADE at A. Now, AB, which is the line of sight in the level when properly adjusted, is called the *line of apparent level.* But a line, as AD, equally distant from the centre of the earth in all its points, is called the *line of true level.* Hence DB is the height of the apparent above the true level at the distance AD or AB, being the difference of these heights above the centre of the earth. It is therefore evident, that the difference of the true and apparent height is always equal to the difference of the radius of the earth and the secant of the arc of distance.

By a property of the circle, we have $\overline{2OD+DB}\times DB=AB^2$; or because the diameter of the earth is so great with respect to the line BD at all distances, to which an operation of levelling commonly extends, that 2OD may safely be taken for 2OD+DB, without any sensible error; $2OD\times DB \doteq AB^2$, or $DB=\dfrac{AB^2}{2OD}=\dfrac{AD^2}{2OD}$ nearly.

That is, the difference between the true and apparent level is equal to the square of the distance between the places, divided by the diameter of the earth ; and consequently it is always proportional to the square of the distance.

Now suppose, for example, we want to know the difference between the true and apparent level at the distance of an English mile, or 1760 yards : the square of this number is 3097600, which, being divided by the diameter of the earth 13953280, expressed in the same measure, gives 0·222 of a yard, nearly ; which being multiplied by 36, the number of inches in a yard, the product is 7·992, or nearly 8 inches, for the said difference.

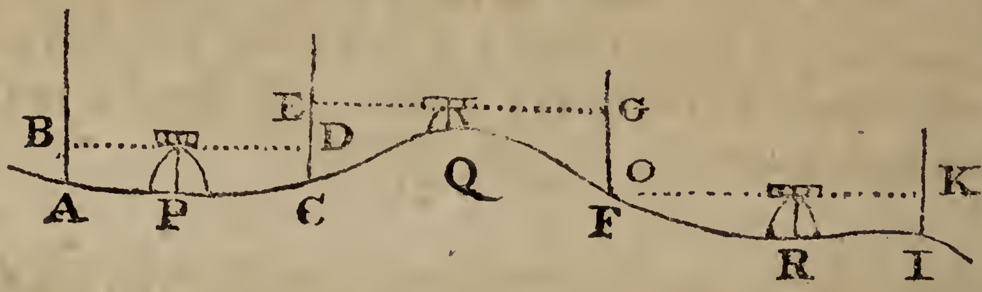
Note 1. As the correction for reducing the apparent to the true level at the distance of 1 mile is very nearly 8 inches ; therefore, as the square of 1 mile : 8 inches :: the square of any other distance : the correction for curvature.

Note 2. A table of corrections for given distances may be easily computed by means of logarithms. For the logarithm of the diameter of the earth 13953280, expressed in yards, is 7·14468, (more figures being unnecessary) which being constantly subtracted from double the logarithms of the distances, the differences will be the logarithms of the corrections, expressed in decimal parts of a yard ; which, being multiplied by 36, the number of inches in a yard, gives the corrections in inches and decimals of an inch.

PROBLEM II.

To find the level of two places ; or the ascent or decent from one to the other.

This is best done with a spirit level, having telescopic sights ; which may be set horizontally by screws, that raise or lower the ends of the tube ; and station staves with sliding vanes.



To take a level from A to I :—let one assistant stand at A and another at C, and at a convenient place, as P, between A and C, place the level, and set it horizontally by means of the screws. Let the assistant at A hold the staff upright in his hand at A, while from P you look through the sights toward A, and direct him to slip the vane up or down till the white line at B be level with the sights ; then note AB. Direct the other assistant to stand at C, and to hold the staff upright, then turning the instrument round at P, and looking toward C, direct him to raise or lower the vane till, through the sights, you can see the white line at D ; then note the height above the ground, CD. And the difference of the heights, AB and CD, is the ascent or decent with respect to the apparent level, if there be no other station.

But, if there be many stations, make a table with two columns, one for the back stations, and the other for the fore stations. Now to proceed, set down the two heights AB, CD ; and let the assistant at A go to F with his staff, and remove the instrument to Q, and level it ; then direct the sights toward C, and let the assistant at C slip his vane up or down till the white line appear through the sights, as at E ; and note CE. Turning the instrument about, look toward F till, through the sights, you see the white line at G, and note FG ; in the mean time let the assistant at C remove to I, and placing the instrument at R, direct it backward toward F and forward toward K. And so proceed, from station to station, to the end. And set down the back heights AB, CE, FO, and the fore heights CD, FG, IK, in their proper columns. Then add the columns, and take the difference of the sums ; and if the fore heights exceed, the line is a descent ; but if the back heights exceed, it is an ascent. If they be equal, it is an apparent level.

Stations.	Back Heights.		Fore Heights.	
	ft.	in.	ft.	in.
1	4	0	2	6
2	5	4	7	3
3	0	0	3	0
	9	4	12	9
			3	5

Hence, as the fore heights are greater, there is a descent, below the apparent level, of 3 feet 5 inches from A to I.

But to reduce this to the difference of true heights, the distances from each station of the level, both to the back and fore stations of the staves, must be measured, and the heights corrected, by the last Problem, when necessary ; then, the columns of corrected heights being added, the difference of their sums will be the difference of true heights.

NOTE 1. The operation may also be performed by placing the Level first at one place, as A, and then successively at other convenient stations, taking fore observations only, till a height at the other, as I, is found.

NOTE 2. If the distances of the instrument from the back station staff be every where equal to its distances from the corresponding fore station staff, there will be no need ef correction for curvature.

NOTE 3. The velocity of running water depends on the fall. Where the fall is only 3 or 4 inches in a mile, the velocity is very small. Some canals have been cut with a fall from 4 to 6 inches.

NAVIGATION.

NAVIGATION teaches to conduct a ship on the sea from one port or place to another.

In Navigation there are four principal things; two of which being given, the rest are thence determined.

1. Difference of latitude.

2. Difference of longitude.

3. Distance, or length of the run.

4. Course, or rhumb line, on which the ship sails.

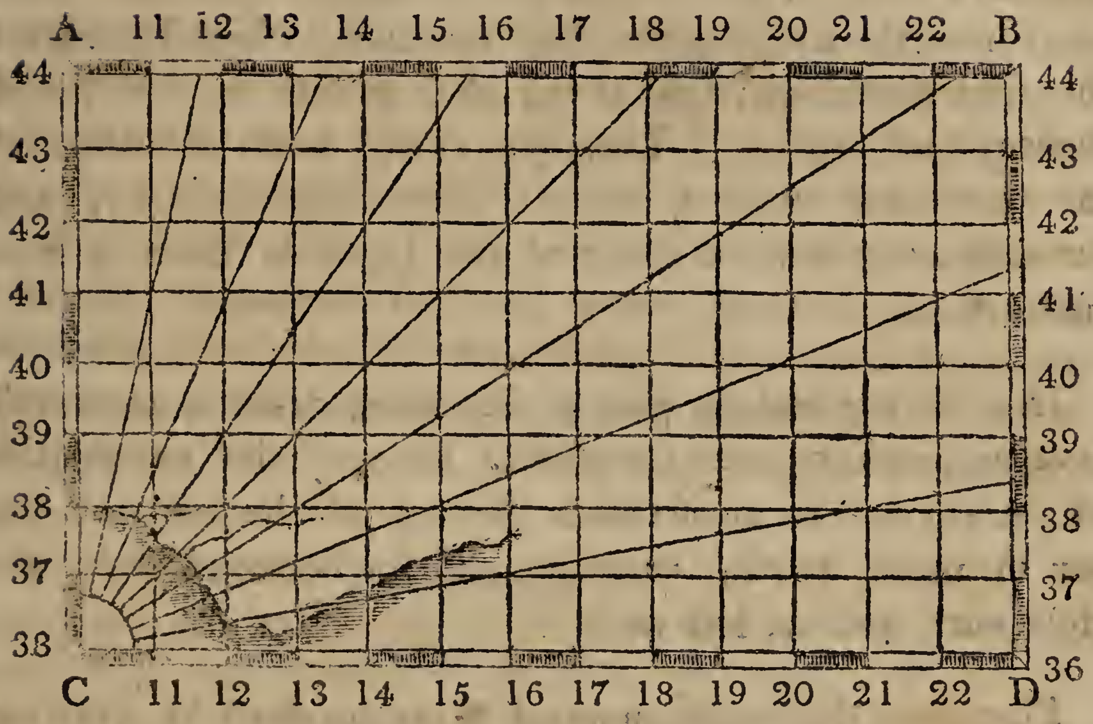
The distance is measured by the log line; and the rhumb is shown by the compass.

PLANE CHART.

It is absolutely necessary to a traveller, that he should be acquainted with the situation of those places, to which he intends to go. And as the situations of places on the earth are known from their latitudes and longitudes, a sufficiently correct and copious table of latitudes and longitudes, or an equivalent to such a table, is one of the helps, which a navigator ought not to be without. A table of this kind is susceptible of the advantage of denoting the situation of places to any required degree of accuracy; but, at the same time, it must be confessed, that names and numbers convey a very imperfect notion of these situations to the imagination; and this purpose is more effectually answered by the use of *maps* or *charts*, which, in general, are drawings or pictures of the face of the earth and sea, as they would appear to an eye at a sufficient distance. Those may be called *true charts*, which are either globes or delineations according to the rules of perspective; but neither of them is used at sea, because of the few straight lines they contain. The *charts*, used at sea, are either plane charts, or Mercator's charts. In the *plane chart* the meridians and parallels of latitude are right lines, at right angles to each other; consequently, it has all its meridians parallel to each other, and all its parallels of latitude equal. It is therefore useful only in small spaces of the earth's surface, which do not much differ from planes; such, for example, as a ship's run in a day, or the extent of a bay or harbor. This kind of chart may be used for several degrees of latitude on each side of the equator, because the meridians are there almost parallel. *Mercator's chart* is similar to the plane chart, excepting that the degrees of the meridian are not equal, but enlarged toward the poles; by which contrivance it acquires many valuable properties. It will be farther considered in another place.

PROBLEM.

To construct a plane chart.



Draw two right lines parallel to each other across the paper, one at the top, and the other at the bottom, AB, CD. At right angles to these, draw two other parallel lines, one near the right, and the other near the left extremity of the paper, AC, BD. Divide the two first lines, each into a like number of equal parts, within the points of intersection, formed by the crossing of the other two right lines. And divide the other two right lines in like manner into parts, each equal to a part of the first lines. Then the two first lines will represent the extreme parallels of latitude, and the two latter the extreme meridians of the intended chart; and the subdivisions, on each, will represent degrees, miles, leagues, or any other measure, which may best suit the purpose of the designer. If one of the parallels be supposed to represent the equator, the divisions of the meridians must be numbered thence; but if not, the divisions must be reckoned from the latitude of that parallel, which is nearest the equator. And so likewise, if one of the meridians be supposed to represent the first meridian, the divisions of the parallels must be reckoned thence ; but if not, the divisions must be reckoned from the longitude of that meridian, which is nearest the first meridian ; that is to say, as far as 180°, and thence back again. Through every tenth division of the meridians draw a parallel across the chart ; and through every tenth division of the parallels draw a meridian.*

In some convenient part of the chart draw a mariner's compass, and continue the rhumb lines to the extremities of the chart. In most charts the top of the book or paper is made north, consequently the bottom south, the right east, and the left west.

Places are delineated or marked on the chart by drawing a parallel through the latitude of the place, on each graduated meridian ; and a meridian through the longitude of the place, on each graduated parallel ; the point of intersection of the parallel and meridian, thus drawn, will give the required situation of the place. Coasts are laid down by marking a sufficient number of points from the known latitudes and longitudes of places on the coast, and the coast itself is drawn by the hand through those points. In all sea charts, the line of the coast is gradually shaded off on the land side, to denote the rise of the land above the water. This may be done either with the pen or Indian ink.

On this chart, the situation of a place may be marked, if the bearing and distance from a given place in the chart be known. For, draw a line from the given place, paral-

* In the figure, which extends from 10° to 23° of longitude, and from 36° to 44° of north latitude, a meridian is drawn to each degree of longitude, and a parallel to each degree of latitude.

lel to that rhumb of the compass, which denotes the bearing, and on this line set off the distance in parts of the graduated parallel or meridian. The extreme point of the distance marks the place required.

The operator will soon observe, that places may be delineated on the chart by help of the parallels and meridians already drawn, without having recource to a multiplicity of other lines.

NOTE. Computation is so much more accurate than delineation, and the helps of tables are so many, that charts are seldom used but in coasting navigation.

PLANE SAILING.

PLANE SAILING is the method of deducing a ship's place, or of determining things relating to the navigation of a ship, by the principles of the plane chart.

The *course* is the acute angle, formed between the line described, or proposed to be described, by a ship under way, and the meridian, from which she is departing.

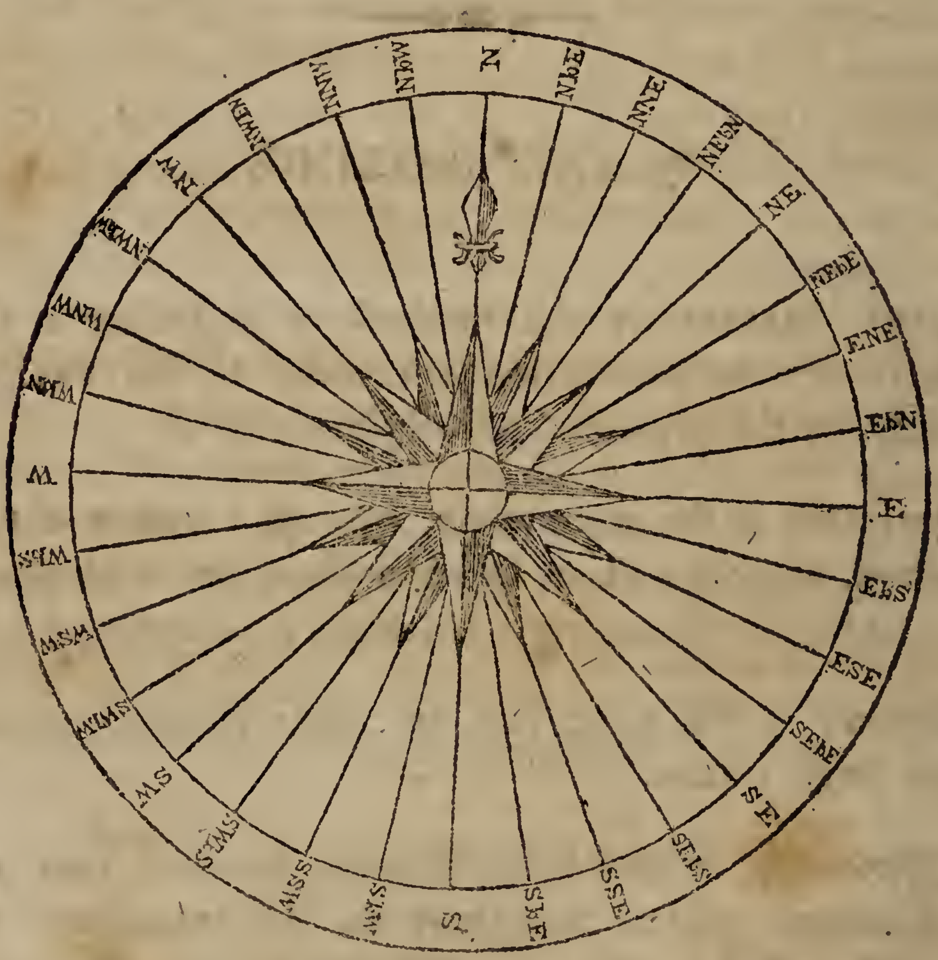
Distance is the right line, or rather rhumb, described on any single course.

Difference of latitude is the distance between two parallels of latitude, and is measured on the meridian. It is frequently called *northing*, or *southing*.

Departure is the distance on the plane chart between two meridians, and is measured on a parallel of latitude. It is frequently called *easting*, or *westing*.

Difference of longitude is the distance between two meridians, and is measured on the equator. On the plane chart it is the same as departure.

It is absolutely necessary, that the navigator should be able to box the compass, as it is called ; that is, to repeat the names of the points in order from memory, either backward or forward, and to tell readily the distance of any point of the compass from the meridian, either in points, or degrees and minutes. For this purpose, the figure of the compass is here drawn. The points being 32, each is equal to 11° 15′.



The following rules do not require any explanation.

1. If the latitudes of two places be given, their difference of latitude may be found by taking the difference of the two latitudes, if of the same name; or their sum, if of contrary names.

2. If one of two latitudes be given, together with the difference, the other may be found by taking the sum of the given latitudes, if of the same name; or their difference, if of contrary names; and the latitude found will be of the same name with the greater of those given.

3. If the longitudes of two places be given, their difference of longitude may be found by taking the difference of the two longitudes, if of the same name; or their sum, if of contrray names. But if this sum exceed 180°, its supplement is the difference of longitude.

4. If one of two longitudes be given, together with the difference, the other may be found by taking the sum of the given longitudes, if of the same name; or their difference, if of contrary names; and the longitude found will be of the same name with the greater of those given.

The two last rules are applicable likewise to departure.

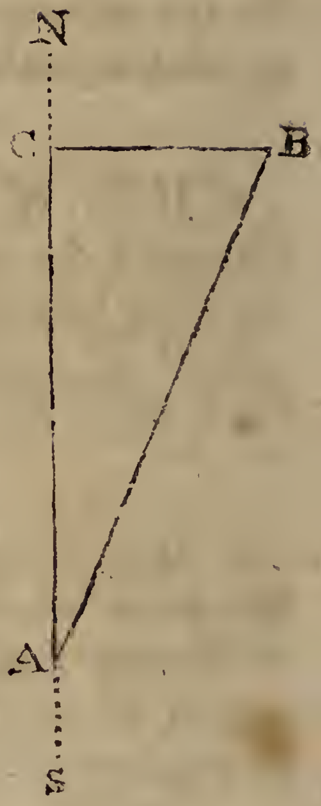
EXAMPLES

IN PLANE SAILING ON A SINGLE COURSE.

1. A ship, in latitude 37° 10′ N sails N N E 80 miles; it is required to find her difference of latitude and departure.

CONSTRUCTION.

Draw an unlimited line NS, to represent the meridian sailed from, the upper part N being understood to be north. From any assumed point A, in NS, draw the line AB, making the angle CAB=2 points, or 22° 30′, the angle of the course NNE the line AB being drawn upward, because the course is northerly, and to the right of NS, because it is likewise easterly ; make AB=80 miles, or the distance ; and let fall the perpendicular BC from B on the line NS.



The line AC will be the differense of latitude, and CB the departure.

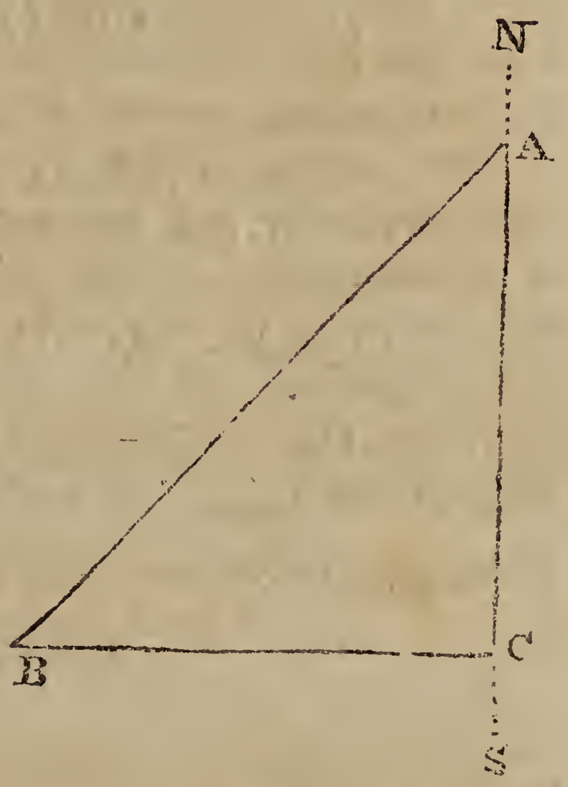
COMPUTATION.

Radius	10·00000
: Distance AB=80	1·90309
:: Sin. Co. ∠A=22° 30′	9·58284
: Departure CB=30 6	1·48593

Radius	10·00000
: Distance AB=80	1·90309
:: Cos. Co. ∠B=67° 30′	9·9656[illegible]
: Diff. lat. AC=73 9	1·86807

2. A ship* takes her departure from the Bill of Portland, bearing N E *b* E 8 leagues; it is required to find her difference of latitude and departure at that time thence.

CONSTRUCTION.



Because the land bears N E *b* E 8 leagues, it is evident, that the ship is exactly in the same situation, as if she had sailed that distance thence on the opposite side of the compass, namely, S W *b* W. Therefore, draw the meridian line NS, and from the assumed point A draw AB=24 miles, the distance, making the angle BAC=5 points, or 56° 15′, the line AB being drawn downward and to the left, because the course is in the S W quarter. Let fall the perpendicular BC on AC.

The line AC will be the difference of latitude, and BC the departure.

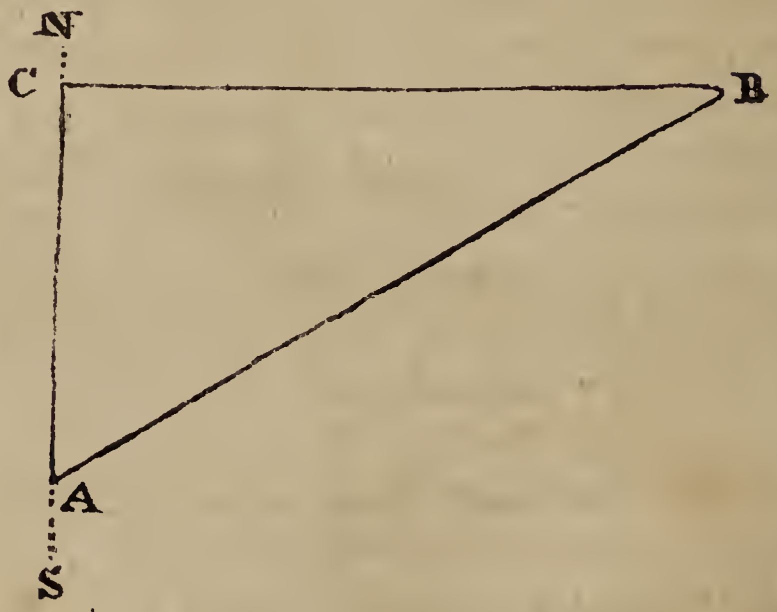
* When the bearing and distance of any land is taken for the purpose of settling the ship's place, to begin the reckoning, the ship is said to take her departure thence.

COMPUTATION.

Radius	10·00000
: Distance AB=24	1·38021
: : Sin. Co. ∠A=56° 15′	9·91985
: Departure CB=19 9	1·30006
Radius	10·00000
: Distance AB=24	1·38021
: : Cos. Co. ∠B=33° 45′	9·74474
: Diff. lat. AC=13 3	1·12495

3. A ship in latitude 23° 7′ S sails E N E till she comes to the latitude of 22° 10′ S. Her distance and departure are required.

CONSTRUCTION.



The difference between 23° 7′ and 22° 10′, that is, 57m is the difference of latitude, which is northing, because the course is northerly. Draw, therefore, the meridian NS, and from any point A in it set off AC = 57 miles to the north. From A draw AB in the NE quarter, unlimited toward B, making the angle BAC=6 points, or 67° 30′ = to the given course. And at C erect the perpendicular CB, intersecting AB in B.

The line AB will be the distance, and CB the departure.

COMPUTATION.

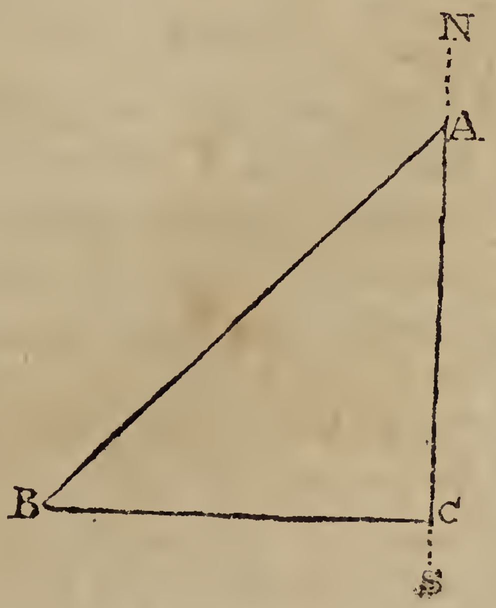
Cos. Co. ∠B=22° 30′	9·58284
: Difference of latitude AC=57	1·75587
: : Radius	10·00000
: Distance AB=149	2·17303

Radius	10·00000
: Distance AB=149	2·17303
: : Sin. Co. ∠A=67° 30′	9·96561
: Departure CB=137·6	2·13864

4. A ship sails from a port in latitude 40° 10′ N to another port, which lies in 38° 52′ N latitude, and 79 miles to the westward of the first. The ship's course and distance are required.

CONSTRUCTION.

The difference between 40° 10′ and 38° 52′, that is, 78m is the difference of latitude, which is southing, because the ship sails from a northerly port to one more southerly. Draw, therefore, the meridian NS, and from any point in it A, set off AC = 78 miles to the south. From C, the latitude come to, draw to the



westward the perpendicular CB, which will represent the parallel of latitude, and on it set off CB = 79 miles, the departure. Join AB.

The angle A is the course, and the line AB is the distance.

COMPUTATION.

Difference of latitude AC=78	1·89209
: Radius	10·00000
:: Departure CB=79	1·89763
: Tang. Co. $\angle$A=45° 22′	10·00554

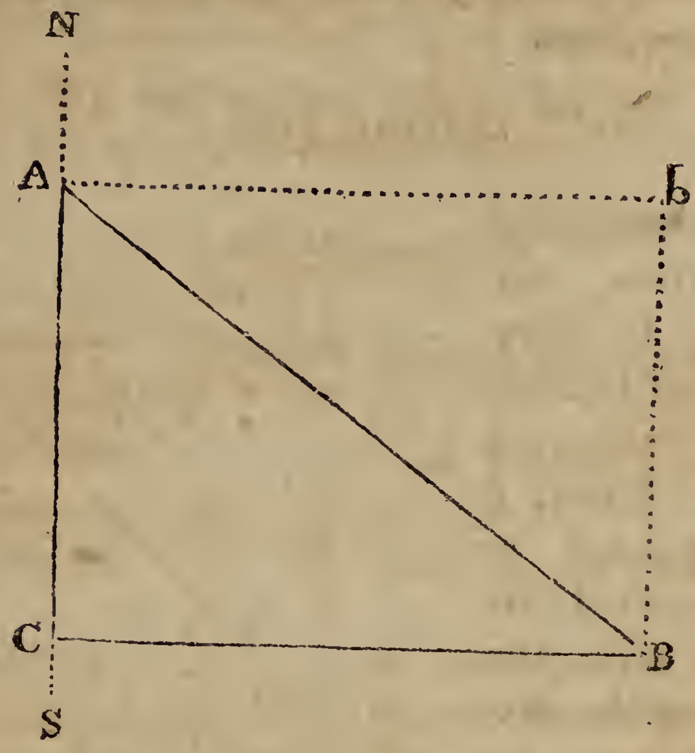
Sin. Co. $\angle$A=45° 22′	9·85225
: Departure CB=79	1·89763
:: Radius	10·00000
: Distance AB=110·6	2·04538

5. A ship in latitude 30° 5′ N sails SE*b*E till her departure is 46 miles. What is her present latitude, and what distance has she run?

CONSTRUCTION.

Draw the meridian NS, and from any point A in the same, draw the parallel of latitude A*b* to the eastward = 46 miles, or the easting. From A draw AB in the SE quarter, unlimited toward B; and making the angle BAC =56° 15′, or the course. From *b* draw *b*B parallel to AC, and intersecting AB in B. Draw likewise BC parallel to *b*A, and intersecting AC in C.

The line AB will be the distance, and AC the difference of latitude.



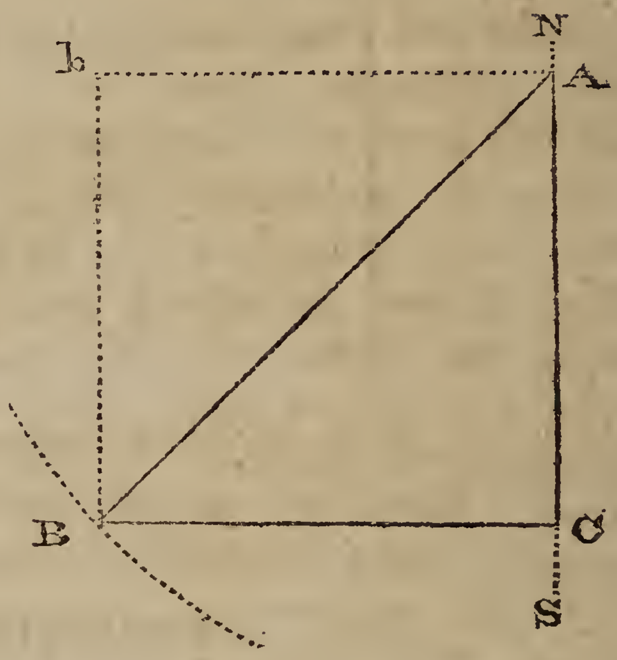
COMPUTATION.

Sin. Co. ∠A=56° 15′	9·91985
: Departure CB=46	1·66276
: :Radius	10·00000
: Distance AB=55·3	1·74291
Sin. Co. ∠A=56° 15′ ar. co.	0·08015
: Departure CB=46	1·66276
: : Cos. Co. 33° 45′	9·74474
: Difference of latitude=30·7	1·48765

Now the difference of latitude is southing, and therefore of a different name from the given latitude. Their difference, or 30° 5′—31′=29° 34′, is the present latitude, which is north, because the greater of the two latitudes is north.

6. A ship sails on a southerly course 118 miles, and makes 83 miles westing. Her course and difference of latitude are required.

CONSTRUCTION.



Draw the meridian NS, and from any point A in it draw the parallel of latitude A*b* to the westward =83 miles. From *b* draw *b*B parallel to AC, and unlimited toward B. With the extent 118, or the distance from A, describe an arc cutting *b*B in B. Join AB, and draw BC parallel to *b*A, intersecting the meridian in C.

The angle BAC will be the course, and AC the difference of latitude.

COMPUTATION.

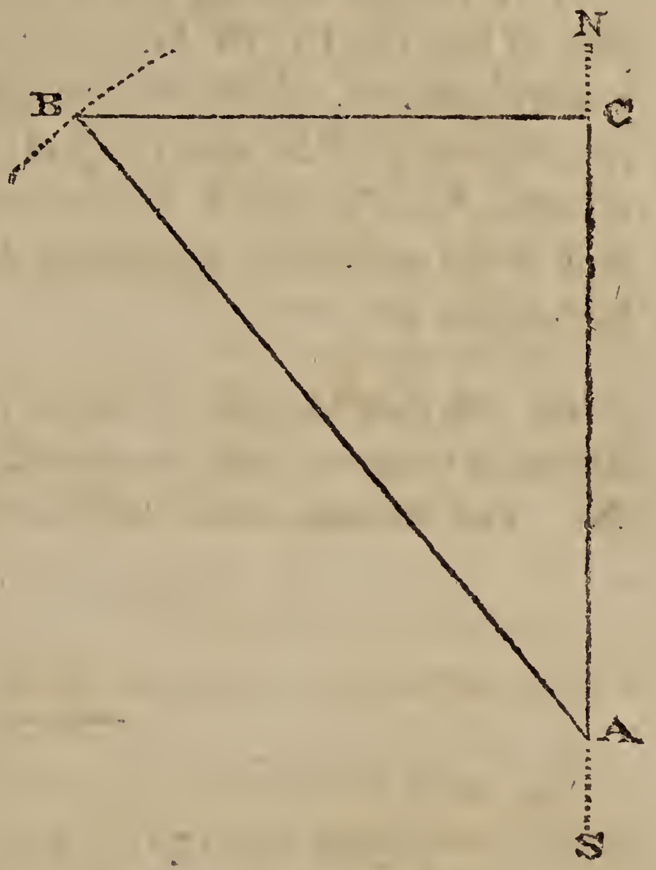
Distance AB=118	2·07188
: Radius	10·00000
: : Departure CB=83	1·91908
: Sin. Co. ∠A=44° 42′	9·84720

Radius	10·00000
: Distance AB=118	2·07188
: : Cos. Course 45° 18′	9·85137
: Difference of latitude 83·8	1·92325

7. A ship from latitude 1° 4′ S sails between the N and W 162 miles, and then finds her latitude by observation to be 52′ N. It is required to find her course and departure.

CONSTRUCTION.

Draw the meridian NS, and upon it set off AC = 1° 56′, or 116 miles, the difference of latitude.—From C, the latitude come to, draw the parallel of latitude CB, unlimited to the westward. From A, the latitude sailed from, with the extent AB =162, the distance, describe an arc intersecting the parallel in B; and join AB.



The angle BAC will be the course, and CB the departure.

COMPUTATION.

Distance AB=162	2·20951
: Radius	10·00000
:: Difference of latitude AC=116	2·06456
: Cosine Course 45° 45′	9·85505
Radius	10·00000
: Distance AB=162	2·20951
:: Sine Course 44° 15′	9·84372
: Departure 113	2·05323

8. There are two islands, A and B, and the island B bears WNW of A. Now a ship, after running 30 miles due west from A, observed the island B to bear N. What is the distance of the two islands?

Ans. 32·5 miles, or 11 leagues, nearly.

9. A frigate sails on a due south course 62 miles from a port in latitude 18° 10′ N, at which time she speaks with a merchantman, who had observed a privateer of the enemy, cruizing in the same latitude, viz. 18° 10′ N. The merchantman's course thence was ESE. How far distant is the privateer, supposing her not to have changed her station? Ans. 162 miles.

10. A ship in sight of Cape St. Vincents bearing W*b*S distant 9 leagues, finds her latitude by observation 36° 46′ N. The latitude of the cape is required.

Ans. 36° 41′ N.

OBLIQUE PLANE SAILING.

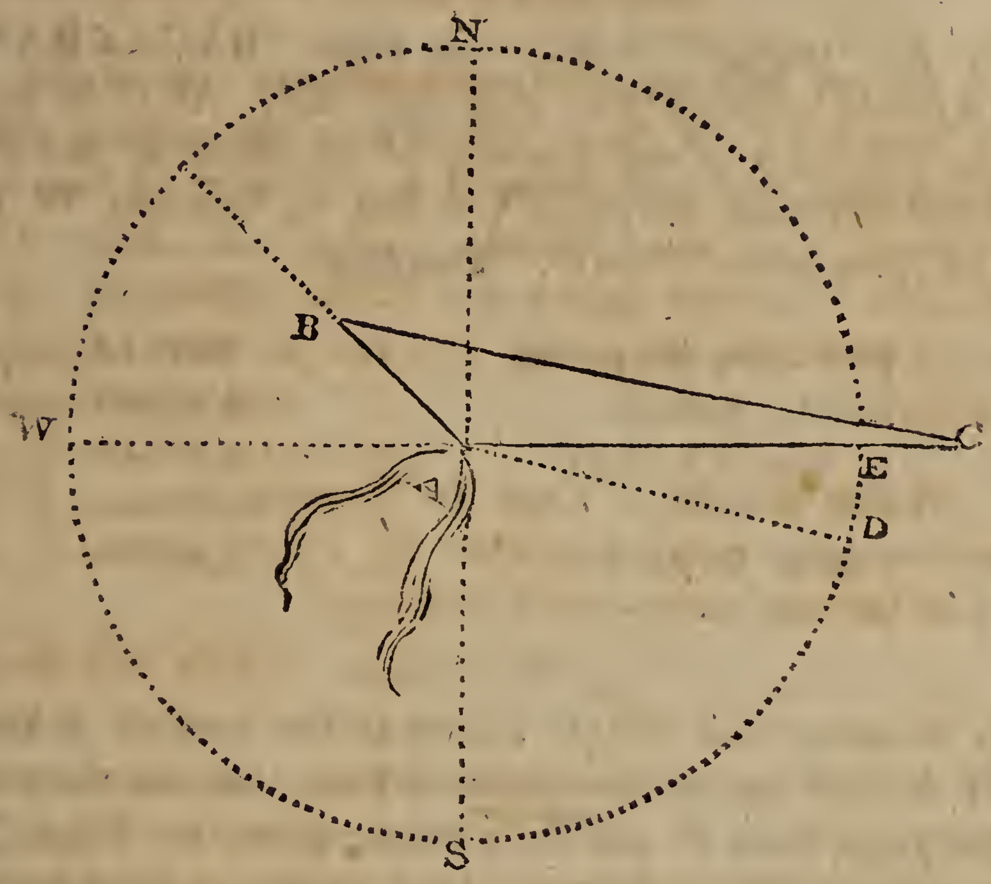
THOUGH almost every thing, which occurs in the practice of navigation, may be solved by the use and application of right-angled triangles; yet there are many instances, especially in coasting or surveying harbors, which are much more readily and elegantly solved by the help of oblique-angled triangles, and some, which can be solved no other way. This method of solution is usually termed OBLIQUE SAILING.

EXAMPLES.

1. A ship sailing on a NW course, at the rate of 5 knots an hour, observes a head land bearing E and four

hours afterward the same head land bore S 78° E. What was its distance at the last time it was set ?

CONSTRUCTION.



With the radius or chord of 60°, describe the circle NESW, representing the horizon ; through the centre of which draw the meridian NS, and the parallel of latitude WE. From A draw AB=20 miles on a NW course because the 5 knots or miles per hour, for 4 hours, make that quantity. Draw AC on an E course, unlimited toward C. Draw AD likewise on a S 78° E course by setting off from S the arc SD = 78°. And parallel to AD, through B, draw BC, which will make the same angles with NS toward the same part, and consequently will be on the same rhumb as AD. Prolong BC till it meet AC in C.

The side BC of the triangle ABC is the distance required.

COMPUTATION.

In the triangle ABC, the angle BAC=∠BAN+∠NAE=45°+90°=135°. And the angle ACB is equal to the difference of the bearings or positions of the lines CA and CB from the point C; that is, between W and N 78 W or 90°—78°=12°. Therefore,

Sine ∠C=12° ar. co.	0·68213
: Side AB=20	1·30103
:: Sine ∠A=135°	9·84948
: Side BC=68	1·83264

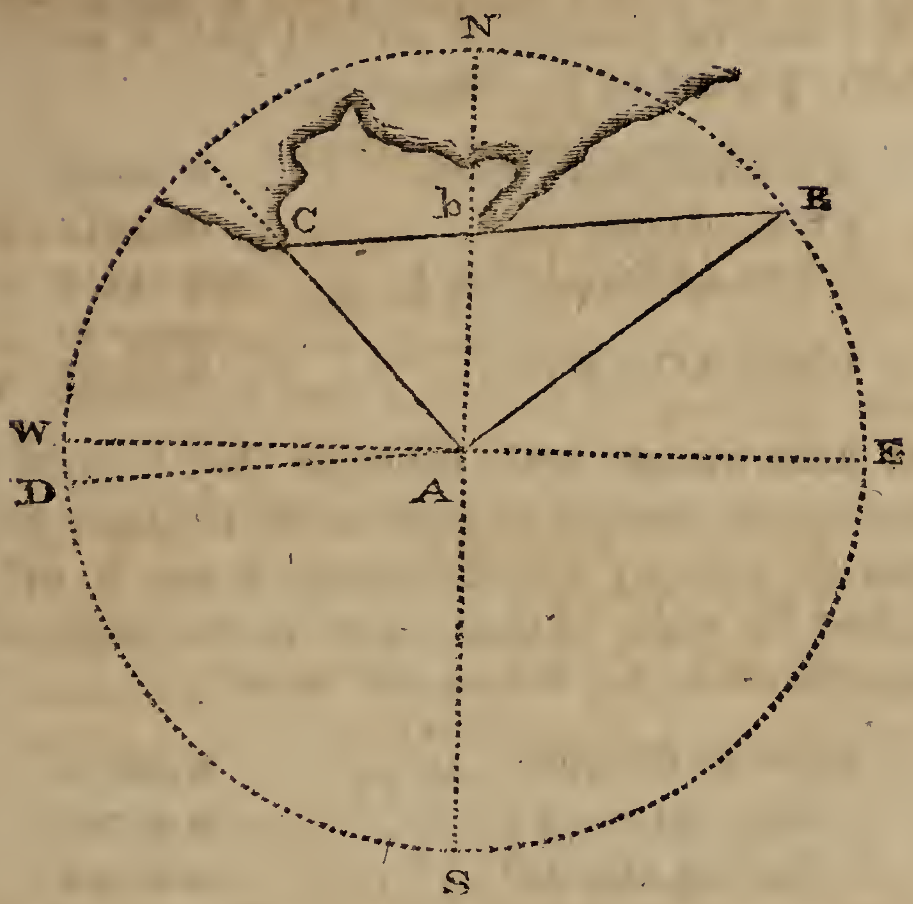
2. Running on a NE½E course at the rate of 6 knots, at 11 A. M.* the easterly point of Praya Bay, on the island of St. Jago, bore N and the westerly point, or Point Tubaron, bore NW. At noon both points in one† bore S 84° W. It is required to determine how wide the entrance of the bay is.

---

* A. M. signifies *Ante Meridiem*, or before noon; P. M. *Post Meridiem*, or afternoon. Seamen read them by naming the letters as they stand.

† When two objects, as, for example, pilots' marks, points of land, &c. are seen on the same rhumb or bearing, so that the remoter is immediately behind the nearer object, they are said to be *in one*.

CONSTRUCTION.



Describe the horizon with the meridian and parallel of latitude, as in the last example. From A draw AB NE$\frac{1}{2}$E= 6 miles. Draw likewise A*b* north, unlimited toward *b*, AC NW unlimited toward C, and AD S 84 W. Parallel to AD through B draw BC, intersecting A*b* and AC in *b* and C.

The line *b*C is the width of the bay, or distance of its extreme points, as required.

COMPUTATION.

In the triangle ABC, the angle ACB is equal to the difference of the bearings or positions of the lines CA and CB from the point C; that is, between SE and N 84° E = 51°. And the angle ABC is equal to the difference of the bearings or positions of the lines BA and BC from the point B; that is, between S 50° 7′½ and S 84° W, or 84°—50° 7′½ = 33° 52′½. Therefore,

Sine∠ACB=51° ar. co.	0·10950
: Side AB=6	0·77815
: : Sine ∠ABC=33° 52′½	9·74615
: Side AC=4·3	0·63380

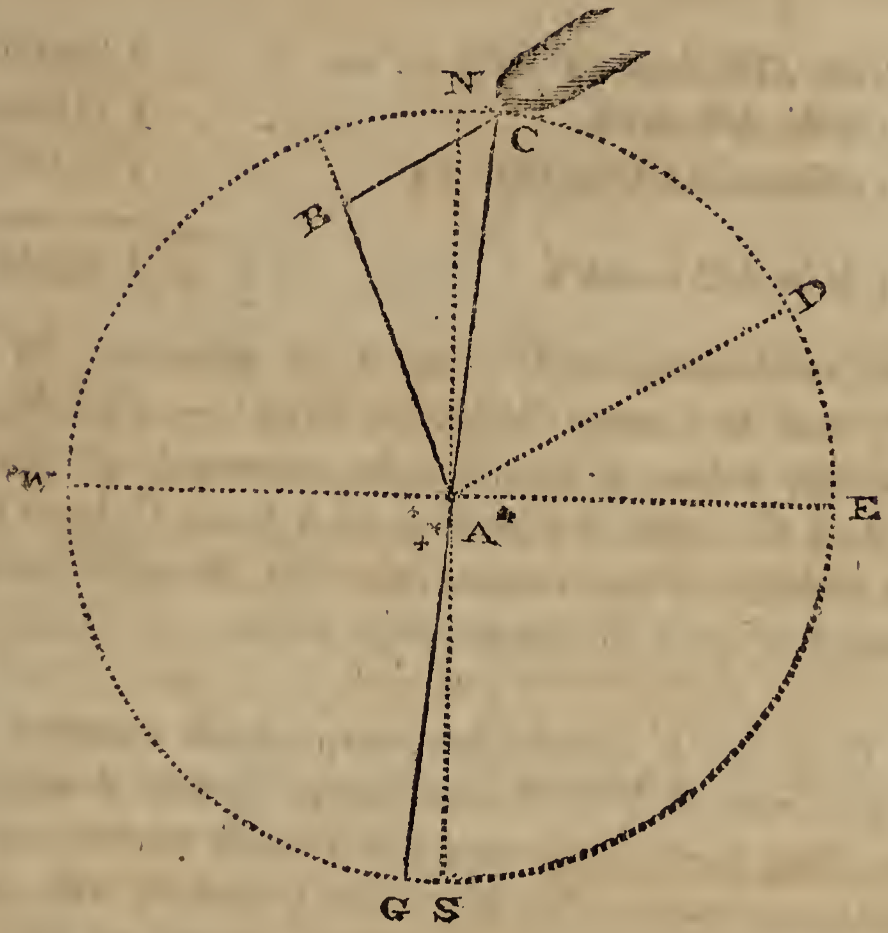
And in the triangle A*b*C, the angle A*b*C is equal to the difference of the bearings or positions of the lines *b*A and *b*C from the point *b*; that is, between S and S 84° W= 84°. And the angle *b*AC is equal to the angle of the bearing of C from A; that is, NW or 45°.

Sine ∠A*b*C=84° ar. co.	0·00239
: Side AC=4·3	0·63380
:: Sine ∠*b*AC=45°	9·84948
: Side *b*C= 3·06	0·48567

3. A ship at sea, sailing NNW at the rate of 6½ knots, observed at day break, that she had just passed near a dangerous reef of rocks. Continuing her course for 8 hours, she then made the land bearing NE*b*E distant 10 leagues. It is required to find the bearing and distance of the reef of rocks from this land.

CONSTRUCTION.

Describe the horizon with its meridian and parallel, as before. Make AB=52 miles on a NNW course. Draw AD NE*b*E and through B draw BC parallel to AD. Upon BC from B set off 30 miles toward the same side of the line AB as AD is drawn. Join AC, which continue southward toward G.



The angle SAG is the bearing, and the line CA is the distance of the reef of rocks from the land.

COMPUTATION.

In the triangle ABC, the sides AB, BC, are known, and the angle at B is the difference of the bearings of the lines BA and BC from B; that is, between NE*b*E and SSE = 9 points of the compass, or 101° 15′. Therefore,

The sum of the sides = AB+BC = 82 ar. co.		8·08619
: Their difference = AB—BC = 22		1·34242
: : Tang. ½ sum ∠s not included = 39° 22′½		9·91416
: Tang. ½ their difference = 12 25		9·34277

½ sum + ½ difference = 51 47½ = the greater ∠.

And ∠BCA is the greater ∠, because it is opposite to AB the greater side. And the less ∠BAC will be = 26° 57′½.

Sine ∠BCA = 51° 47′½ ar. co.	0·10470
: Side AB = 52	1·71600
:: Sine ∠ABC = 101° 15′	9·99157
: Side AC = 64·9	1·81227

Now, the bearing of C from A is measured by angle NAC, and the angle NAC = ∠BAC— ∠BAN = 26° 57′½ — 22° 30′ = 4° 27′½ to the eastward of north, or N 4° 27′½ E. And the bearing of A from C is on the opposite rhumb. That is to say, the reef of rocks bears from the land S 4° 27′½ W distant 64·9 miles.

4. There is a certain harbour, whose extreme points bear ESE and WNW of each other, distant 4 miles, and in the offing to the southward lies a small island, on which a light house is erected. Now, the boatmen, who are employed to convey provisions and necessaries to the island, reckon it to be 5 miles distant from the westerly point of the harbour, and 7 miles from the easterly point. What are the bearings of the island from each point, supposing these estimated distances to be accurate?

CONSTRUCTION.

After describing the horizon with its meridian and parallel, draw AB = 4 miles WNW, and the extremities of AB will represent the two points. From the easterly point A, with the extent 7 miles, describe an arc, and from the westerly point B, with the extent 5 miles, describe another arc, cutting the former in C, which will represent the place of the island. Through A draw AD parallel to BC, and on the same side of AB.

The angle SAC is the bearing of the island from the easterly point A, and the angle SAD is the bearing of the island from the westerly point B.

COMPUTATION.

In the triangle ABC, the sides only are given. Therefore,

The base AC = 7 ar. co.	9·15490
: S. of the legs AB+BC = 9	0·95424
:: Diff. legs BC—AB = 1	0·00000
: Diff. Seg. CG—AG = 1·29	0·10914

$\frac{1}{2}$ base = 3·5
+ $\frac{1}{2}$ diff. seg. = ·64

Greater segment 4·14 = CG.

In the right-angled triangle CBG

Side CB	=5	0·68997
: Radus		10·00000
: : Side CG	=4·14	0·61700
: Sine ∠CBG=55° 54′		9·91803
Its comp.	=34 06 =	∠BCG.

The angle C, in the triangle ABC, being thus found, the other angles are obtained thus:

Side AB=4 ar. co.	9·39794
: Sine ∠C=34° 6′	9·74868
: : Side BC=5	0·69897
: Sine ∠A=44° 29′	9·84559

From 180°
Take 78 35′=∠C+∠A=34° 6′+44° 29′

Rem. 101 25=∠ABC.

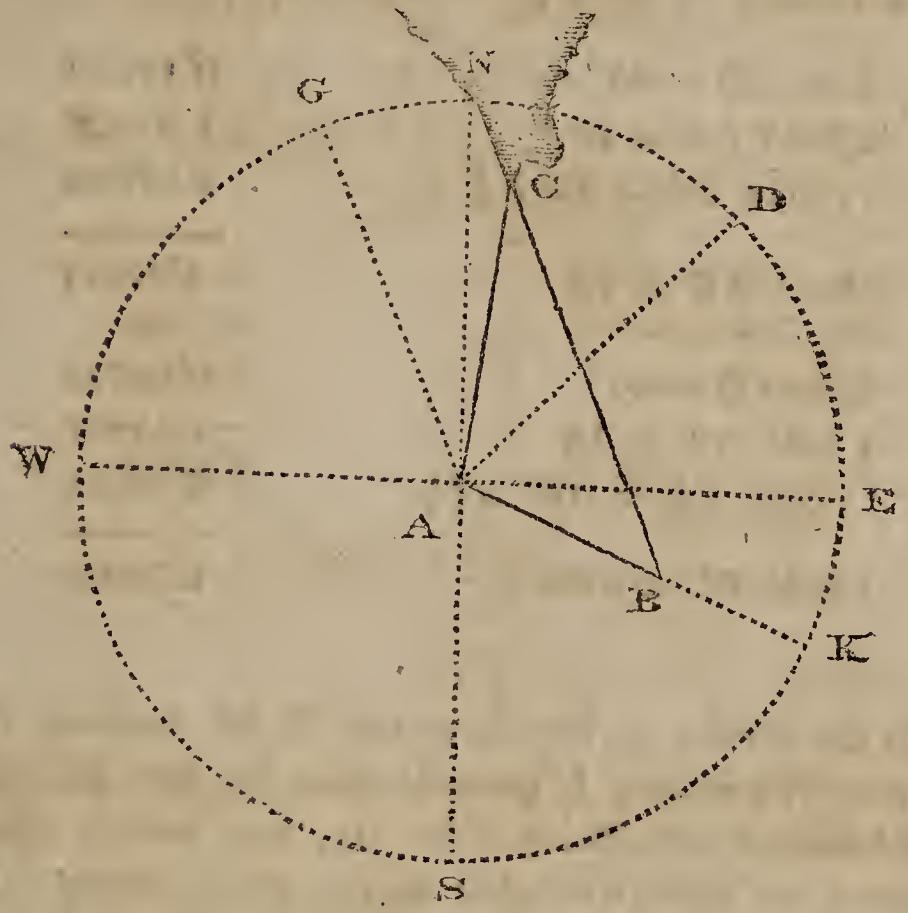
From a proper consideration of these angles the bearings may be found.

For the ∠SAC, or the bearing of C from A, is =∠ SAB—∠CAB, and the ∠SAB=90°+∠WAB=90° +22° 30′=112° 30′, and the ∠CAB=44° 29′. Therefore 112° 30′—44° 29′=68° 1′=∠SAC.

Again, the line BA lies in an ESE direction from B, and the line BC is 101° 25′ to the left of BA. If, therefore, from ESE or S 67° 30′ E there be reckoned 101° 25′ to the left, the difference 33° 55′, to the westward of south, will be the bearing of C from B. That is to say, the Island bears S 68° 1′ W from A, and S 33° 55′ W from B.

5. The wind being at NE, a ship is bound to a port bearing N½E 18 leagues distant, and can make her course good within 6 points of the wind. How much must she run on each tack, to reach the harbor?

CONSTRUCTION.



Construct the horizon with its meridian and parallel of latitude. Draw AD NE, and on each side of D in the circumference of the circle set off DG and DK, each equal to 6 points, and draw the lines AG, AK. Draw AC = 18 leagues N½ E, and parallel to (either KA or) GA, through C draw CB, meeting the other line KA in B.

The line AB will be the distance on the larboard, and BC the distance on the starboard tack.*

---

* A ship is said to be on her *starboard tack*, when she is as near the wind as she will lie, and the wind is on the right side of the ship; but in like circumstances, with the wind on her left side, she is said to be on the *larboard tack*.

COMPUTATION.

In the triangle ABC, is given the side AC and the angles. For the angle CAB = ∠CAD + ∠DAK = 39° 22′½ + 67° 30′ = 106° 52′½. The angle CAB = supplement ∠CBK = supplement ∠GAK = 45°. And the angle ACB = ∠CAG = 28° 7′½. Therefore,

Sine ∠B = 45° ar. co.	0·15052
: Side AC = 18	1·25527
: : Sine ∠C = 28° 7′½	9·67338
: Side AB = 12	1·07917

Sine ∠B = 45°	0·15052
: Side AC = 18	1·25527
: : Sine ∠A = 106° 52′½	9·98088
: Side BC = 24·4	1·38667

6. In the Straits of Sunda, at two P. M. steering SE*b*S, at the rate of 5 knots, I passed close by the SE of the small islands off Hog Point. At six, not having changed our course, we came to anchor on the Java shore. Upon setting the said island from this anchoring place, it appears to bear due N, its distance by the chart being 22 miles. It follows hence, that our course has been affected by a current. The velocity and direction of the current are required.

CONSTRUCTION.

Construct the horizon with its meridian and parallel. From A draw AB = 20 miles SE*b*S, which will represent the ship's track through the water, and would have been her real course and distance, if the water had remained without any progressive motion. Draw AC = 22 miles S, and C will represent the ship's real place ; join BC. It follows therefore, that while the ship has been describing the line AB, with respect to the water, the water itself has moved through a space equal to the distance BC in the direction of that line. Draw AD parallel to BC, and on the same side of AB as BC is drawn.

The line BC will be the space passed through by the current in four hours, and the angle CAD will be its direction or course.

COMPUTATION.

In the triangle ABC are given the sides AB, AC, and the included angle $A=33^\circ\ 45'$. Therefore,

Sum of sides AB+AC=42 ar. co.	8·37675
: Difference of sides AC—AB=2	0·30103
:: Tang. $\frac{1}{2}$ sum ∠s not included=$73^\circ\ 7'\frac{1}{2}$	10·51800
: Tang. $\frac{1}{2}$ their difference = 8 55	9·19578

$\frac{1}{2}$ sum + $\frac{1}{2}$ difference = 82 $2\frac{1}{2}$ = greater ∠ = ∠B, because opposite to the greater side.

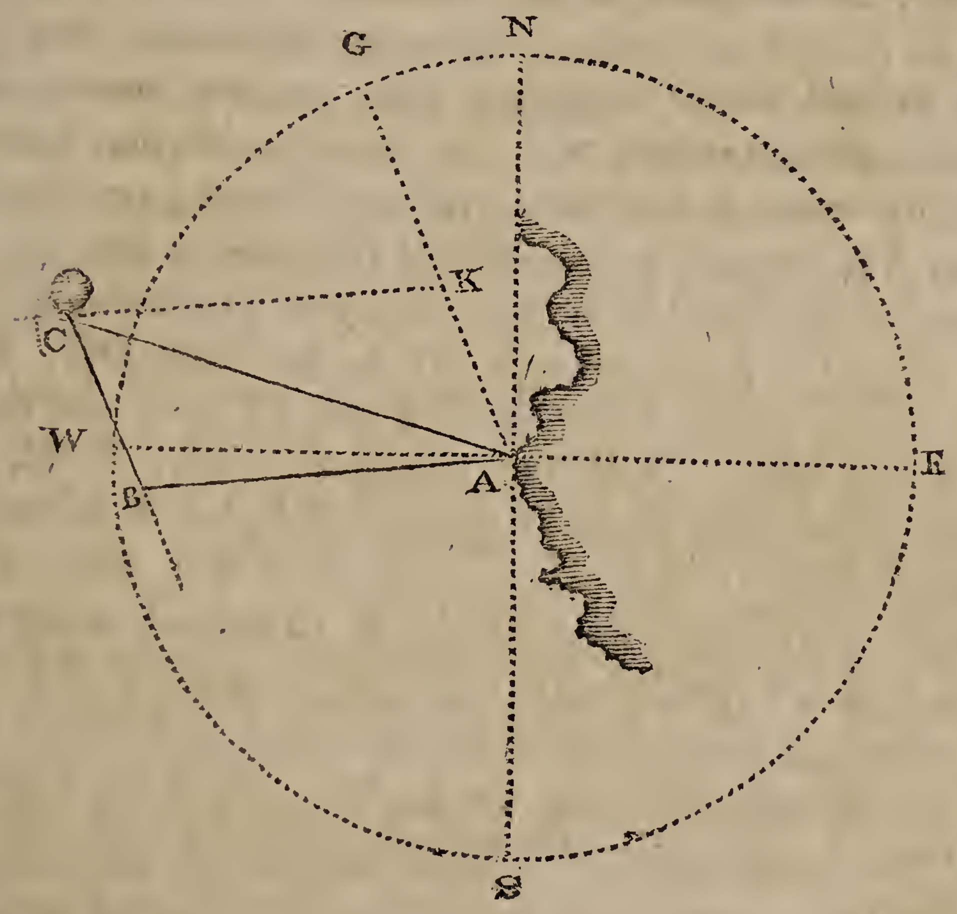
Sine ∠B=$82^\circ\ 2'\frac{1}{2}$ ar. co.	0·00420
: Side AC= 22	1·34242
:: Sine ∠A= $33^\circ\ 45'$	9·74474
: Side BC= 12·3	1·09136

The bearing of the line BC may be gathered from its position with respect to BA, which lies N $33^\circ\ 45'$ W from B, and BC is $82^\circ\ 2'\frac{1}{2}$ to the left of BA; that is, BC is $115^\circ\ 47'\frac{1}{2}$ from the north, or $64^\circ\ 12'\frac{1}{2}$ from the south. Now, the distance BC= 12 miles, passed by the current in four hours, gives 3 miles to one hour. Consequently the current sets S $64^\circ\ 12'\frac{1}{2}$ W, or nearly WSW$\frac{1}{4}$S, at the rate of 3 miles an hour.

From the consideration of this example it is shown, that a ship under way in a current will, at the end of any given time, be found at the same place, as if she had been in still water, after describing like courses and distances, together with another course and distance answering to the direction and velocity of the current. For the ship, under the action of the current, arrives at the same place C, as if she had been in still water, by sailing through AB and BC; and it is evidently of no importance, in this case, provided the time continues unaltered, whether AB be made good on a single course, or on a number of courses.

7. A sloop is bound from the main land of Africa to an island bearing W*b*N, distant 22 leagues, a current setting NNW $2\frac{1}{2}$ miles an hour. What is the course to arrive at the island in the shortest time, supposing the sloop to sail at the rate of 6 knots an hour; and likewise what time will she take?

CONSTRUCTCON.



Construct the horizon with its meridian and parallel of latitude. Draw AC=22 leagues = 66 miles W*b*N, and draw AG NNW. Upon AG set off AK=25 miles, or any multiple of $2\frac{1}{2}$ miles; and from K, with KD = 60 miles, or a like multiple of 6 miles, describe an arc cutting AC, prolonged if necessary, in D. Join KD. From C draw CB parallel to KA, and from A draw AB parallel to KD and meeting CB in B.

The angle SAB is the course, and AB the distance of the run, corresponding with the time. But if the velocity of the vessel be not greater than that of the current, that is, if KD be not greater than KA, the arc will not intersect AC, and it will be impossible for her to reach the island.

COMPUTATION.

The sloop will arrive in the shortest time at the island by sailing in a right line. And, by the remark on the sixth example, her situation at the end of the time will be the same, as if she had described her apparent course and distance in still water, together with another course and distance corresponding with the motion of the current. Now, it is evident, that the apparent distance, or distance by log, AB, must be to the drift of the current BC, as the velocity of the vessel is to the velocity of the current, that is, as 6 to $2\frac{1}{2}$. In the triangle DKA, the sides DK, KA, are respectively like multiples of 6 and $2\frac{1}{2}$, and are therefore in that ratio. And the triangles DKA, ABC, are similar, because the alternate angles KDA, CAB, are equal. Therefore KD : AK :: 6 : $2\frac{1}{2}$ :: AB : BC. Consequently AB is the line, which, if described apparently by the vessel, would bring her to C by reason of the drift BC. Now to find the direction and length of this line AB.

In the triangle AKD are given AK and KD, together with the $\angle A = \angle NAC - \angle NAG = 78^\circ\ 45' - 22^\circ\ 30' = 56^\circ\ 15'$. Therefore,

Side DK	=66 ar. co.	8·18046
: Sine ∠A	=56° 15′	9·91985
: : Side AK	=25	1·39794
: Sine ∠KDA	= 18° 21′	9·49825

In the similar triangles AKD, CBA, the $\angle KAD = \angle ACB = 56^\circ\ 15'$, and the $\angle KDA = \angle CAB = 18^\circ\ 21'$.

Therefore, in the triangle ABC is given the side AC=66 miles, and two angles. Consequently the ∠B is likewise known, being = 105° 24′. Whence,

Sine ∠B =105° 24′ ar. co.	0·01588
: Side AC= 66	1·81954
:: Sine ∠C=56° 15′	9·91985
: Side AB=56·9	1·75527

The angle NAB is equal to ∠NAC+∠CAB=78° 45′ +18° 21′=97° 6′, and the angle SAB, or the apparent course, is equal to the supplement of ∠NAB, which is 82° 54′. And the apparent distance, or distance by log, =56·9, or 57 miles, at the rate of 6 knots an hour, gives the time 9½ hours. That is to say, the sloop will reach the island directly in 9½ hours, by steering S 82° 54′ W, or nearly W½S.

TRAVERSE SAILING.

THE application of plane trigonometry to the solution of nautical questions has been exemplified. But the solution by logarithms is thought too operose for daily use, and it is seldom that mariners avail themselves of the logarithmic lines on Gunter's Scale. It is now a long time since tables of difference of latitude and departure, commonly called *Traverse Tables*, have been calculated; by the help of which any person, though totally ignorant of the trigonometrical proportions, may readily find the parts of any right-angled triangle, provided two of them exclusive of the right angle be given. For, since the distance sailed on any course is the hypotenuse of a right-angled triangle, of which the difference of latitude and departure are the legs, a table of difference of latitude and departure to a sufficient number of distances regularly increasing, and to every course in the quadrant consisting of whole degrees, will not only be useful for working questions in plane sailing by inspection, but likewise may serve to give a nearly accurate solution to many other questions, in which right-angled triangles are concerned.

There are three columns in each set of columns, entitled at top *Distance*, *Latitude*, *Departure* ; but *Distance*, *Departure*, *Latitude*, at bottom. At the top and bottom are numbers denoting the course, either in points of the compass or degrees ; the titles at the top of the columns answer to the course at top ; and the titles at the bottom, to the course at bottom. The difference of latitude and departure, answering to any given course and distance, are found opposite the distance and under or over the course. Therefore, if any two of the four things, viz. course, distance, difference of latitude, and departure, be given, and those two be found together in the table, the other two may be found together in their respective places.

When a ship sails on several different courses, these courses and distances, considered together, are called a *Traverse*.

In working a traverse it is obvious, if a ship, on several courses, make her differences of latitude all the same way, the sum of all the differences of latitude will be the difference of latitude made good ; but, if on some courses she make northing, and on others southing, the difference between the total northing and total southing will be the difference of latitude made good, which will be of the same name with the greater.

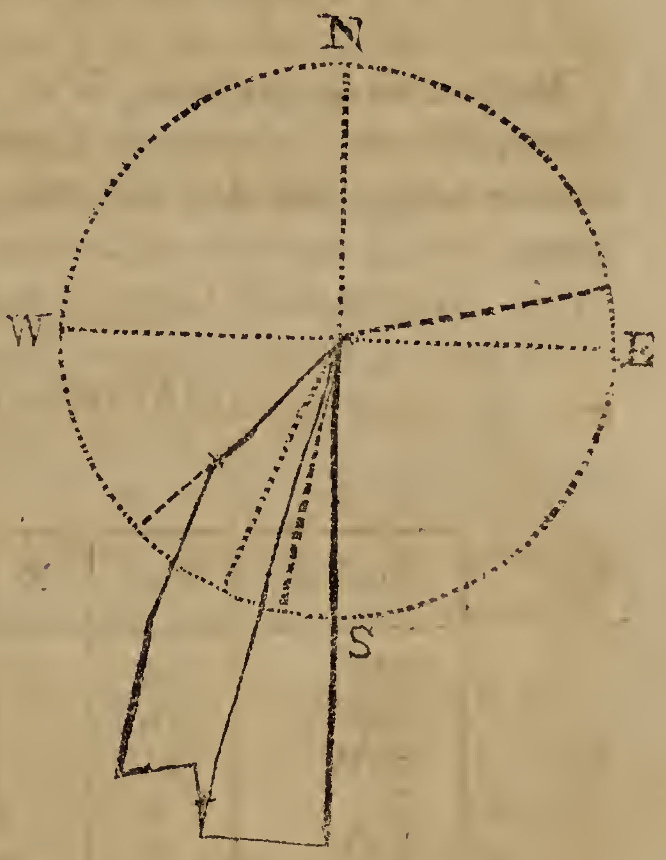
And the like is true with respect to departure, or easting and westing.

EXAMPLES

of traverses, worked by the table of difference of latitude and departure.

1. A ship sails on the following courses, viz. SW 25 miles, SSW 18 miles, S*b*W 25 miles, E 4 miles, E*b*N 7 miles, S 10 miles. It is required to find the course, distance, difference of latitude, and departure made good.

CONSTRUCTION.



Construct the horizon with its meridian and parallel. Set off the first course and distance from the centre of the horizon. Draw the rhumb of the second course from the centre of the horizon; and, parallel to this rhumb, from the outer extremity of the first distance set off the second distance. Draw the rhumb of the third course from the centre of the horizon; and, parallel to this rhumb, from the outer extremity of the second distance set off the third distance. And proceed, in this manner, till every course and distance of the traverse be drawn. From the centre of the horizon to the outer extremity of the last distance draw a right line. This right line will be the distance made good, and the acute angle, formed between it and the meridian, will be the course made good. From the outer extremity of the last distance draw a parallel of latitude, falling on the meridian, prolonged if necessary. The east and west line, comprehended between the meridian and the outer extremity of the last distance, will be the departure; and the north and south line, comprehended between the first and last parallels of latitude, will be the difference of latitude.

As the construction of every traverse is performed in the same manner, they will, for brevity, be omitted in future.

COMPUTATION BY THE TABLE.

Make a table consisting of six columns, and title them course, distance, northing, southing, easting, westing, each column being intended to contain that, which its title denotes.

TRAVERSE TABLE.

Course	Dist.	N	S	E	W
SW	25		17·7		17·7
SSW	18		16·6		6·9
S*b*W	25		24·5		4·9
E	4			4	
E*b*N	7	1·4		6·9	
S	10		10		
		1·4	68·8	10·9	29·5
			1·4		10·9
S. 15°W	70		67·4		18·6

The courses of the traverse must be entered in the first column, and their respective distances opposite to them in the second. Seek the first course SW, or 4 points, in the table of difference of latitude and departure, and against the distance 25 will be found 17·7 difference of latitude, and 17·7 departure, which enter in the columns of southing and westing, because the course is in the SW quarter. In like manner with the course SSW, or two points, against distance 18 is found 16·6 difference of latitude, and 6·9 departure, which are likewise southing and westing. S*b*W, or one point, distance 25 gives 24·5 southing, and 4·9 westing. E distance 4 miles, need not be sought out, as the distance is all easting, and must, therefore, be entered in the E column. E*b*N, or 7 points, distance 7 gives 1·4 northing, and 6·9 easting. And S distance 10 miles is all southing. The difference between the total southing and the total northing is 67·4 southing; and the difference between the total westing and the total easting is 18·6 westing. With these numbers, in the table of difference of latitude and departure, the distance is found 70, and, at the top of the page, 15 degrees for the course, which is in the SW quarter, the course being taken from the top and not the bottom of the page, because the titles of lat. dep. at the top agree with the nature of numbers.

The course therefore is S 15 W distance 70 miles, difference of latitude 67·4 miles N, departure 18·6 W.

2. A ship in 17° 12′ N latitude, bound to a port in 18° 40′ N latitude, and 220 miles to the westward, sails N W *b* W 73 miles, W N W 40, S S W 18. What is her present latitude and departure made good, and what are the bearings and distance of the port, to which she is bound?

COMPUTATION.

Latitude sailed from	17° 12′	N
Difference of latitude made good	39	N
Ship's latitude	17 51	N
Latitude bound to	18 40	N
Difference of latitude to be made	49	N
Whole departure	220	W
Departure made good	105	W
Departure to be made	115	W

TRAVERSE TABLE.

Courses	Dist.	N	S	E	W
NW*b*W	73	40·6			60·7
WNW	40	15·3			37·0
SSW	18		16·6		6·9
		55·9			104·6
		16·6			
N 21° W	112	39.3			

With 49 miles northing, and 115 miles westing, seek for course and distance in the table of difference of latitude and departure, which being found N 67 W distance 126, are the required bearing and distance of the port, according to the principles of plane sailing.

3. A ship sails on the following courses, SE 40 miles, NE 28 miles, SW*b*W 52 miles, NW*b*W 30 miles, SSE 36 miles, SE*b*E 58 miles. Required her course, distance, difference of latitude, and departure made good.

TRAVERSE TABLE.

Courses	Dist.	N	S	E	W
SE	40		28·3	28·3	
NE	28	19·8		19·8	
SW*b*W	52		28·9		43·2
NW*b*W	30	16·7			24·9
SSE	36		33·3	13·8	
SE*b*E	58		32·2	48·2	
		36·5	122·7	110·1	68·1
			36·5	68·1	
S 26° E	96		86·2	42·0	

PARALLEL SAILING.

IT has already been observed, that sailing on the principle of the plane chart is too erroneous, when applied to the surface of a sphere, to be used in any but small distances, or between the tropics, where the meridians have but little convergency, and the rhumb lines do not widely differ from portions of great circles. In all sea reckonings, the principles of the plane chart are supposed to be exact enough for the distance of a day's run; and, at the end of every 24 hours, the ship's place is determined in latitude and longitude, by applying the difference of latitude, found by these principles, to the latitude, and the difference of longitude to the longitude of the place, from which the ship sails. This difference of longitude is found, either by parallel sailing, middle latitude sailing, or Mercator's sailing.

Parallel sailing is the method of finding the difference of longitude made by a ship, when her course and distance, on a known parallel of latitude, are given; and the contrary.

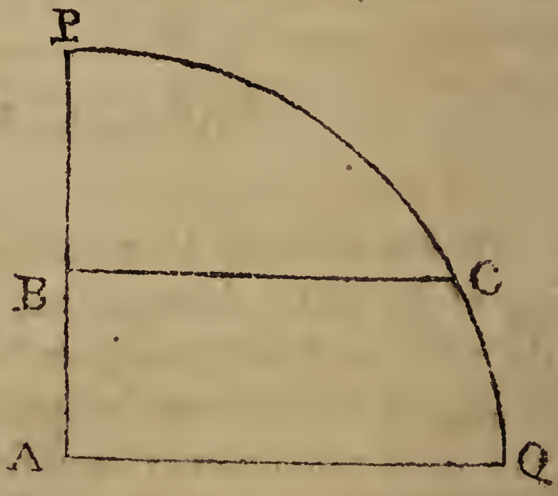
The computations in parallel sailing depend on the following

THEOREM.

On the globe, the difference of longitude between any two meridians is to the distance between those meridians, measured on any parallel of latitude, as radius to the cosine of the latitude.*

---

* Demonstration.



Let PCQ be a quadrant of the meridian, AQ the radius of the equator, BC the radius of a parallel of latitude. Then, if the quadrant PCQ be turned round its axes PA, the points Q and C will describe the circumferences of two circles, which will be to each other in the ratio of their radii AQ, BC.

But AQ, the radius of the equator, is the sine of the arc PQ, a quadrant, and BC, the radius of the parallel of latitude, is the sine of the arc PC, the complement of the latitude of the

Cor. 1. The length of a degree of longitude in any parallel is as the cosine of the latitude.

Cor. 2. As the cosine of any parallel is to the cosine of any other parallel, so is the length of any arc of the former to the length of the corresponding arc of the latter.

Cor. 3. From this theorem is derived the method of constructing the *line of longitude*, in the Plane Scale.*— [See Plane Scale.]

EXAMPLES.

1. A ship in latitude 44° 12′ N sails E 79 miles. Required her difference of longitude.

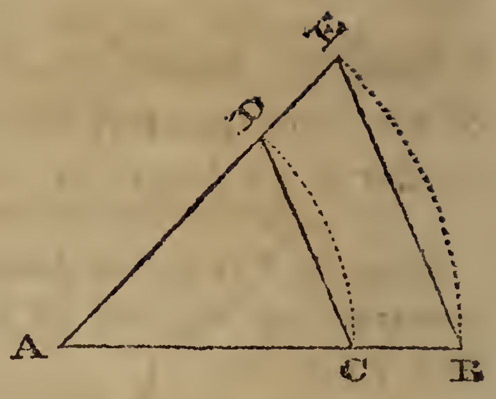
---

place C. Therefore, as the circumference of the equator : the circumference of the parallel : : radius : the cosine of the latitude.

But similar parts are as the wholes. Therefore, the difference of longitude between any two meridians is to the distance between those meridians, measured on any parallel of latitude, as radius to the cosine of the latitnde. Which was to be proved.

* The truth of the process may be easily shown. For, since AF, equal to radius CE, is to the cosines A50, A40, A30, &c. as 60 to 50, 40, 30, &c. respectively, an equatorial degree, or 60 miles, must have the same ratios to the degrees of the parallels, of whose latitudes the lines are the cosines. But E50, E40, E30, &c. are the chords of those latitudes. Therefore, the latitude being taken on the chord line, the corresponding number on the line of longitude shows the length of a degree on that parallel, in 60th parts of an equatorial degree, or in nautical miles.

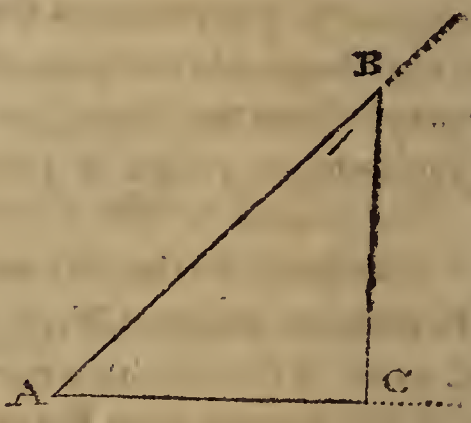
CONSTRUCTION.



Make AC = cosine 44° 12′, and AB = radius or sine 90°. From the centre A describe the arcs CD, BE, passing through CB. On the arc CD, from C set off 79 miles as a chord; and through D, its other extremity, draw AD, prolonged till it meet the arc BE in E. Join BE, and the right line BE is the difference of longitude.

This construction is a geometrical solution of the proportion in the theorem, taken in an inverted order. That is, in the similar triangles ACD, ABE, cosine latitude AC : radius AB :: departure, or meridian distance CD : difference of longitude.

ANOTHER CONSTRUCTION.



Make an angle CAB of as many degrees and minutes as the latitude. From the angular point A set off AC = 79 miles = the departure or meridional distance. From C erect the perpendicular CB, meeting AB in B. AB will be the difference of longitude required.

This construction exhibits the same proportion as the foregoing. For, in the right-angled triangle ABC, cosine latitude : radius :: departure AC : difference of longitude AB.

COMPUTATION.

Cosine latitude 45° 48′	9·85546
: Radius	10·
:: Departure 79	1·89763
: Difference of longitude 110	2·04217
Or 1° 50′.	

SOLUTION BY THE LINE OF LONGITUDE.

Opposite to $44\frac{1}{4}$ = the latitude on the line of chords, stands 43 on the line of longitude, which is, therefore, the number of miles in a degree of longitude, in that latitude. Whence 43 : 60 :: 79 : 110 miles, or the difference of longitude.

SOLUTION BY THE TRAVERSE TABLE.

With the colatitude as a course, and the meridional distance as departure, find the corresponding distance. This distance is the difference of longitude. That is, with 46° at the bottom of the page, seek 79 in the column of departure, and the distance 110 is the difference of longitude.

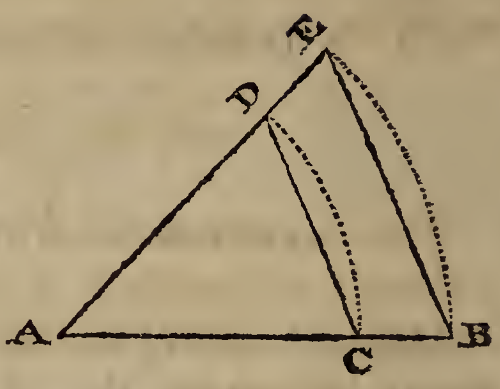
The reason of this operation is clear from considering, that in the right-angled triangle ABC, the angle B, or the colatitude, is opposed to the departure AC, exactly the same as if it were an angle of course, and that the difference of longitude AB is the side opposite to the right angle, in like manner as the distance in plane sailing.

2. A ship in latitude 46° 5′ N is bound to Gibraltar, which lies on the same parallel, and in 4° 46′ W longitude. By an eclipse of the moon, she finds her present longitude to be 13° 8′ W. What is her meridional distance ?

CONSTRUCTION.

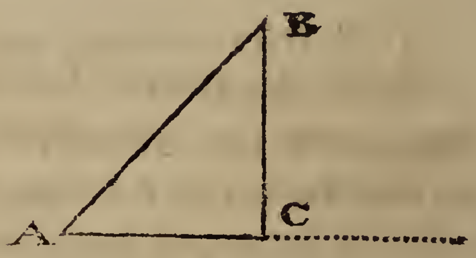
The difference of longitude, or difference between 4° 46′ and 13° 8′, is 8° 22′, or 502 miles.

With the radius AB, or sine 90°, describe the arc BE, on which set off BE= 502 miles. With the cosine of 46° 5′= AC, describe the arc CD. Join AE, intersecting CD in D. Draw the right line CD, and it will be the meridional distance.



OR OTHERWISE.

Make an angle CAB, whose measuring arc = 46° 5′ = the latitude. On one of the legs set off AB = 502 = difference of longitude, and from its outer extremity let fall the perpendicular BC. AC will be the meridional distance.



COMPUTATION.

Radius		10·00000
: Cosine latitude 43° 55′		9·84112
:: Diff. long.	502 0	2·70070
: Departure 348		2·54182

SOLUTION BY THE LINE OF LONGITUDE.

Opposite to 46° = the latitude on the line of chords, stands $41\frac{3}{4}$, the number of miles in a degree of longitude in that latitude. Therefore, 60 : $41\frac{3}{4}$ :: 502 : $348\frac{1}{2}$ miles, the meridional distance or departure.

BY THE TRAVERSE TABLE.

With the colatitude = 44° as a course, and half the difference of longitude = 251 as distance, is found 174 departure, which doubled = 348, the meridional distance or departure.

MIDDLE LATITUDE SAILING.

Middle latitude sailing is a method of finding a ship's place on the globe, by applying the principles of parallel sailing to a course made good on an oblique rhumb.

When a ship sails on an oblique rhumb, that is, on a course between the meridian and parallel, she alters at the same time both her latitude and meridional distance. But the departure, found by plane sailing, will not be her meridional distance, either at the latitude sailed from or come to. For, at the greater latitude it will be too great, because the meridians converge toward the poles ; and for a contrary reason, it will be too small at the less latitude. Whence it follows, that the departure is the true meridional distance, measured on a parallel, which lies between the two extreme parallels, namely, that sailed from and that come to. In middle latitude sailing, the departure is taken for the meridional distance, measured on the parallel of the middle latitude ; and, though these be not strictly equal, yet the error, which arises from this method, is of no consequence in a day's run, except in very high latitudes.

The middle latitude is found by taking half the sum of the two latitudes, if of the same name ; or half their difference, if of contrary names ; and the questions are solved by the help of the following proportions.

1. Cosine middle latitude
   : Radius
   :: Departure
   : Difference of longitude.

This proportion is the inverse of the theorem of parallel sailing.

2. Cosine middle latitude
   : Tangent of course
   :: Difference of latitude
   : Difference of longitude.*

For examples in middle Latitude Sailing, see the examples in Mercator's Sailing.

---

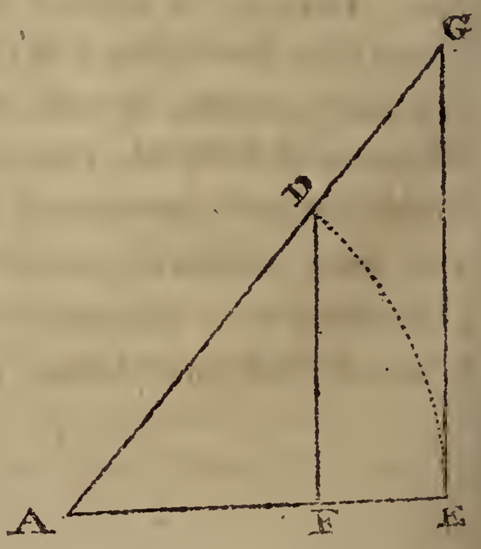
* This proportion is deduced from the foregoing, as is here demonstrated.

DEMONSTRATION.

Cosine middle latitude : radius :: departure : diff. long. And diff. lat. : radius :: departure : tang. of course, by plane sailing.

Therefore, cosine middle lat. : tang. of course :: diff. lat. : difference of longitude.

Which was to be proved.



MERCATOR'S SAILING.

Mercator's sailing is the method of finding a ship's place on the globe by Mercator's chart, or equivalent tables.

In Mercator's chart the meridians are drawn parallel to each other, as in the plane chart, and therefore the meridional distace is every where the same. But, in order to compensate for this error, parallels of latitude, which are equidistant on the globe, are not so on the chart, but are more distant the higher their latitudes. Or, in other words, the degrees of the meridian are not all equal, but increase in length the more remote they are from the equator. And this is so provided, that any very minute part of the artificial meridian bears the same proportion to a like part of the parallel of its latitude, as do the like parts of both on the globe itself.*

* This method of constructing a chart of the world, whose meridians and parallels are right lines, is hinted at in the writings of Ptolemy. Gerard Mercator first published one of these charts, in 1556, with the theory of which he did not appear to be acquainted, for the parts of his meridian were not increased in the true ratio. In the year 1599 Mr. Edward Wright published his *Correction of Errors in Navigation*, in which the theory is demonstrated, and the method of computation by a table of meridional parts explained. And Dr. Halley, in the Philosophical Transactions, first demonstrated, that the artificial meridian line is a scale of logarithmic tangents of half the colatitudes, beginning with radius.

THEOREM I.

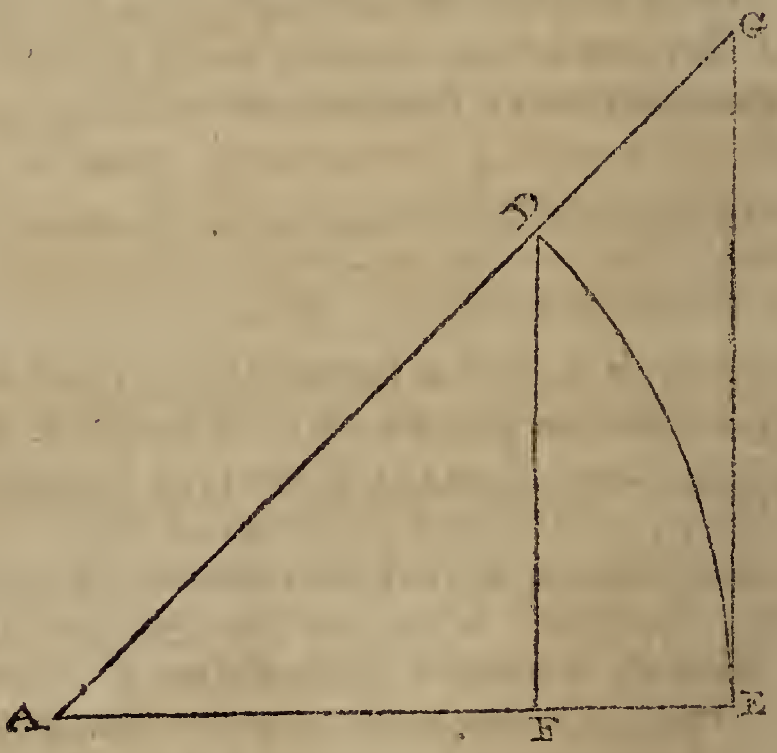
Radius
: Cosine of the latitude
:: Secant of the latitude
: Radius.*

THEOREM II.

The distance of any parallel of latitude from the equator, on Mercator's chart, is as the sum of the secants of all the arcs of latitude, beginning at 0, and increasing arithmetically, by an indefinitely small common difference, till the last arc be that of the given latitude.†

---

* Demonstration.



Let ED represent an arc of latitude. Then AF will be its cosine, and AG its secant. The triangles ADF, AGE, are similar, for they are right-angled, and have a common angle at A.

Therefore AD = radius : AF = cosine of lat. :: AG = secant of lat. : AE = radius. Which was to be proved.

† Demonstration. Radius : cosine lat. :: part of the equator : like part of the parallel.

Cor. The nautical meridian may be divided practically, by assuming arcs of latitude, whose common difference is known and determined ; and the graduations will be more accurate the smaller the common difference. Thus,

---

And radius : cosine lat. :: part of meridian : like part of parallel, because the meridian and equator on the globe are equal.

Again, radius : cosine lat. : : secant lat. : radius, by Theorem I.

Therefore, secant lat. : radius :: part of meridian : like part of parallel.

And on Mercator's chart, secant lat. : radius : : part of meridian lying in that latitude, (which part must consequently be indefinitely small) : like part of parallel.

$$\text{Whence } \frac{\text{secant lat.}}{\text{part of Mer.}} = \frac{\text{radius}}{\text{part of parallel}}.$$

But radius is a constant quantity, and so likewise is the part of the parallel, because all the parallels of latitude are equal on Mercator's chart. Assume, therefore, any number of arcs of latitude increasing arithmetically from 0, by an indefinitely small common difference, and call the parts of Mercator's meridian, corresponding successively with the differences of latitude, by the letters *a*, *b*, *c*, &c. Then,

$$\frac{\text{Secant lat. 1.}}{\text{part of Mer. } a} = \frac{\text{secant lat. 2.}}{\text{part of Mer. } b} = \frac{\text{secant lat. 3}}{\text{part of Mer. } c}, \text{ \&c.}$$

because they are respectively equal to the constant quantity $\frac{\text{radius}}{\text{part of parallel}}$.

Whence sec. lat. 1 : part of Mer. *a* : : secant lat. 2 : part of Mer. *b* : : secant lat. 3 : part of Mer. *c*, &c.

And secant lat. 1 : part of Mer. *a* : : secant lat. 1 + secant lat. 2 + secant lat. 3, &c. : parts of Mer. $a+b+c$, &c.

if the common difference be taken = 1 minute, it will be secant 1m. : meridional part 1m. : : secant 1m. + secant 2m. + secant 3m. &c. : meridional parts 1+2+3m. &c.

And $\frac{\text{Mer. pt. 1m.} \times \text{sec. 1m.} + \text{sec. 2m.} + \text{sec. 3m. \&c.}}{\text{secant 1 min.}} =$

meridional parts 1+2+3m. &c. That is to say, the sum, or aggregate of the secants of all the arcs, arithmetically increasing by the difference of 1 minute to any given latitude, being divided by the secant of 1 minute, will give the length of the nautical meridian to that latitude, in parts equal to the first meridional parts, namely, in equatorial miles, because the first meridional part, contained between the equator and the latitude of 1 minute, does not exceed an equatorial mile by any quantity, which need be considered in practice. By this method Mr. WRIGHT calculated the table of meridional parts.

SCOLIUM. The line, marked mer. on Gunter's scale, is a nautical meridian, adapted to the scale of equal parts, or equatorial degrees, which is under it, and marked EP. By the help of these lines, and the instructions already given concerning the plane chart, it is easy to construct a Mercator's chart, whose degrees shall be of the particular magnitude exhibited on the scale. But the most convenient method of making a chart, whose degrees shall be of a required magnitude, is to graduate the equator into equal parts or degrees as required, and from it, as a scale, take the distances of the several parallels of latitude from the

---

Now the sum of all the parts of the artificial meridian, regularly taken, will be equal to the distance from the equator on Mercator's chart of the parallel of latitude, expressed by the highest secant.

Therefore, &c.

Which was to be proved.

equator, which set off on the meridian. These distances are found in a table of meridional parts to each latitude, where it is to be noted, that if the equator be graduated into minutes, the meridional parts are to be taken as in the table; but if it be divided only into degrees, the meridional parts must previously be reduced to equatorial degrees by dividing by 60.

THEOREM III.

The proper difference of latitude
: the meridional difference of latitude on Mercator's chart
:: the departure
: the difference of longitude.*

---

* Demonstration. Let there be taken an indefinitely small part of an oblique rhumb on the globe; then, radius : cosine lat. :: difference of longitude : departure.

And secant lat. : radius :: diff. long. : departure.

Or, departure : radius :: diff. long. : sec. lat.

Now, any definite part of the rhumb may be conceived to be composed of a number of such indefinitely small parts, crossing a like number of parallels of latitude.

Therefore, the sum of all the departures, or the whole departure answering to the definite part of the rhumb : the like multiple of radius :: the sum of all the differences of longitude corresponding with the departures, or the whole difference of longitude made : the aggregate of the secants of all the latitudes.

That is to say, the departure : difference of longitude :: the multiple of radius answering to the number of parallels of lat. crossed : the aggregate of all the secants of those latitudes.

Again, let the same indefinitely small part of the same oblique rhumb be taken, and, by Theorem II. secant lat. : radius :: indefinitely small part of Mercator's meridian in that latitnde, or the meridional difference of latitude : like part of the parallel, or proper difference of latitude, because the parallels on the chart are equal to the meridians on the globe.

Cor. Hence it follows, that the rhumbs are represented on Mercator's chart by right lines.*

Or, proper difference of latitude : radius :: meridional difference of latitude : secant of latitude.

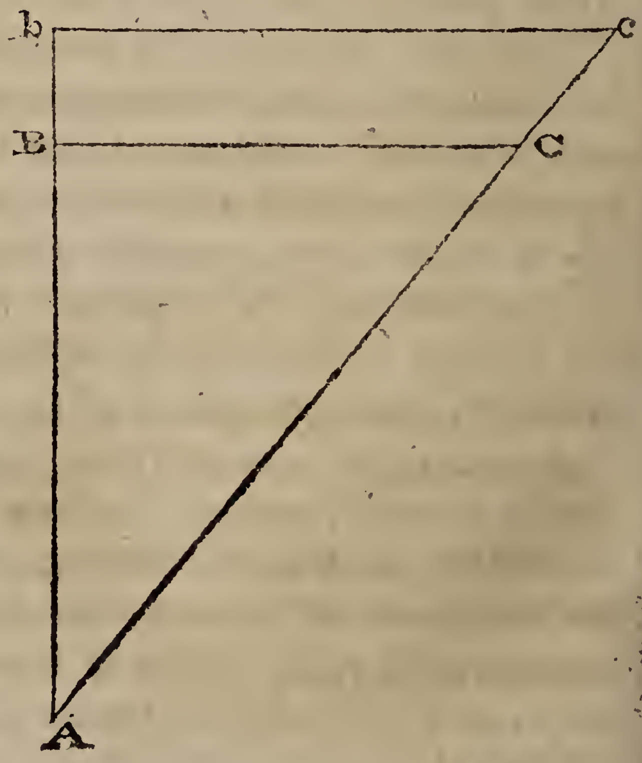
And if the same definite part of the rhumb be taken as before, the number of indefinitely small parts, which may be conceived to make the whole difference of latitude, will also be equal to the number of parallels crossed.

Therefore, by the argument already used, the proper difference of latitade : Mer. diff. lat. :: the multiple of radius answering to the number of parallels crossed : the aggregate of all the secants of those latitudes.

But it has been proved, that the departure : difference of longitude :: the multiple of radius answering to the number of parallels crossed : the aggregate of all the secants of those latitudes.

Whence, proper difference of latitude : Mer. diff. lat :: departure : difference of longitude.

Which was to be proved.



* For, if AB represent the proper difference of latitude, and BC the departure made by running on the rhumb AC; and if A*b* be the meridional difference of latitude, and *bc* the parallel of latitude, at which a ship has arrived; in Mercator's chart, the line AC prolonged will cut the parallel in *c*, and *bc* will be the difference of longitude. For the tri-

THEOREM IV.

The meridional difference of latitude

: The difference of longitude

:: Radius

: The tangent of the course.*

---

angles ABC, A*bc*, are similar ; being both right-angled, and having a common angle at A. Whence AB, or the proper difference of latitude : A*b*, or the Mer. diff. lat. :: BC, or the departure : *bc*, or the difference of longitude, which agrees with the theorem. And in like manner, every departure whatever, which can be made by sailing on the rhumb AC, will have the corresponding difference of longitude terminate in the same right line continued. It is therefore easily inferred, that lines on Mercator's chart, which cross the meridians at the same latitudes, as the rhumbs do the meridians on the globe, are right lines. That is to say, the rhumbs are represented, on Mercator's chart, by right lines. Which was to be shown.

* Demonstration. For, proper difference of latitnde : meridional difference of latitude : departure : difference of longitude. Theorem III.

Or, proper difference of latitude : departure :: meridional difference of latitude : difference of longitude.

Now, proper difference of latitude : departure :: radius : tangent course ; according to plane sailing.

Therefore, meridional difference of latitude : difference of longitude :: radius : tangent course.

Which was to be proved.

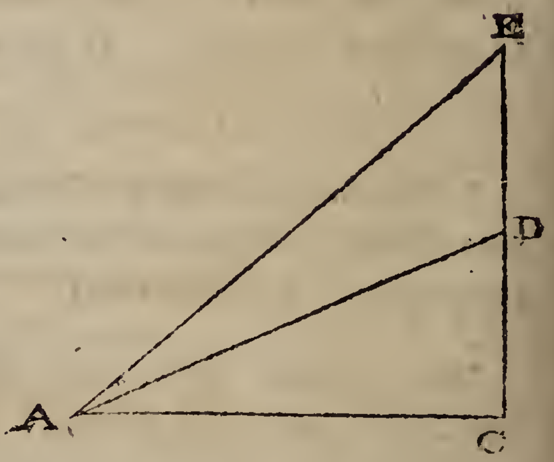
EXAMPLES.

1. It is required to find the course and distance from the land's end to the island of Bermudas.

SOLUTION BY MIDDLE LATITUDE SAILING.

Land's end in lat.	50° 06′ N,	and long.	6° 00′ W
Bermudas lat.	31 20 N,	long.	64 48 W
Diff.	18 46 or 1126m. S		58 48, or 3528m. W
Sum	81 26		
½Sum	40 43 is middle lat.		
	49 17 comiddle lat.		

CONSTRUCTION.



Make an angle CAE of as many degrees and minutes as the middle latitude, viz. 40° 43′.— From the angular point A set off AE=3528m. = the difference of longitude, and from its outer extremity E let fall the perpendicular EC. AC will be the departure or meridional distance.

From C on CE, prolonged if necessary, set off CD= 1126 miles = the difference of latitude, and join AD.

The angle ADC, which is opposite to the departure, will be the course, and AD will be the distance required.

The triangle AEC is constructed on the principles of parallel sailing, already explained, and the triangle ADC is constructed on the principles of plane sailing; difference of latitude and departure being given, to find the rest.

It is obvious, that the other method of constructing questions in parallel sailing may be applied instead of that, which is here used; but it is presumed to be unnecessary to enter into a detail on that head.

COMPUTATION.

To find the departure by inverting the first proportion.

Radius		10·00000
: Cosine mid. lat.	49° 17′	9·87964
:: Difference of long.	3528	3·54753
: Departure	2674	3·42717

To find the course by the second proportion.

Difference of latitude	1126 ar. co.	6·94846
: Difference of long.	3528	3·54753
:: Sine comid. lat.	49° 17′	9·87964
: Tangent course	67 10	10·37543

To find the distance by plane sailing.

Sine course	67° 10′	9·96456
: Departure	2674	3·42717
:: Radius		10·00000
: Distance	2901	3·46261

BY THE TRAVERSE TABLE.

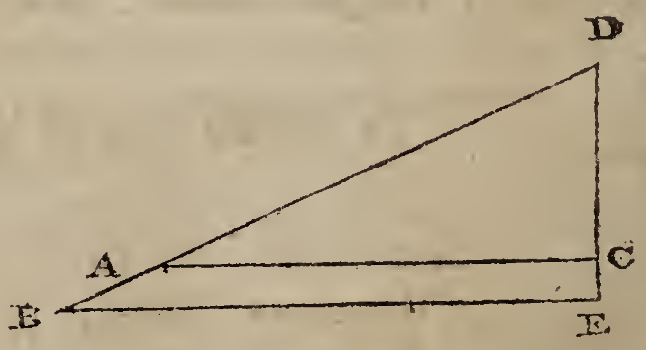
With the comiddle latitude as course, and difference of longitude as distance, find the corresponding departure, which will be the true departure. In the present example, the numbers being so high, a proportional part of each must be taken, suppose $\frac{1}{12}$; that is to say, with the comiddle latitude 49°, and $\frac{1}{12}$ of 3528=294 as distance, the departure is found 221·9. And with this departure, and $\frac{1}{12}$ of 1126 the difference of latitude, viz. 93·8, the course 67°, and distance 2413, are found; which distance, being multiplied by 12, gives 2892 for the true distance.

It may be observed, that the remainders, which are lost in dividing large numbers, render the solution by the traverse table less accurate in such cases, as require that operation.

SOLUTION BY MERCATOR'S SAILING.

Land's end is in lat.	50° 06′ N	Meridional parts	3484
Bermudas	31 20 N	Meridional parts	1981
		Meridional difference of latitude	1503

CONSTRUCTION.



Make BE = 3528 the difference of longitude, and erect the perpendicular ED=1503 the meridional difference of latitude. Join BD. From D set off DC=1126, the proper difference of latitude; and through C draw CA parallel to EB, and cutting DB in A.

The angle ADC will be the course, the line AD the distance, and AC the departure.

COMPUTATION.

To find the course by Theorem IV.

Mer. diff. lat. 1503	3·17696
: Diff. long. 3528	3·54753
:: Radius	10·00000
: Tangent course 66° 55′	10·37057

To find the distance by plane sailing.

Cosine course 23° 5′	9·59336
: Difference of latitude 1126	3·05154
:: Radius	10·00000
: Distance 2872	3·45818

SOLUTION BY THE TRAVERSE TABLE.

With the meridional difference of latitude, or a proportional part of the same, as difference of latitude ; and the difference of longitude, or a like part, as departure ; seek in the traverse table for the correspondent course.

And under the same course with the proper difference of latitude, will be found the true distance and departure.

From the different results of these two methods it appears, that middle latitude sailing is improper to be applied, when the latitudes differ considerably.

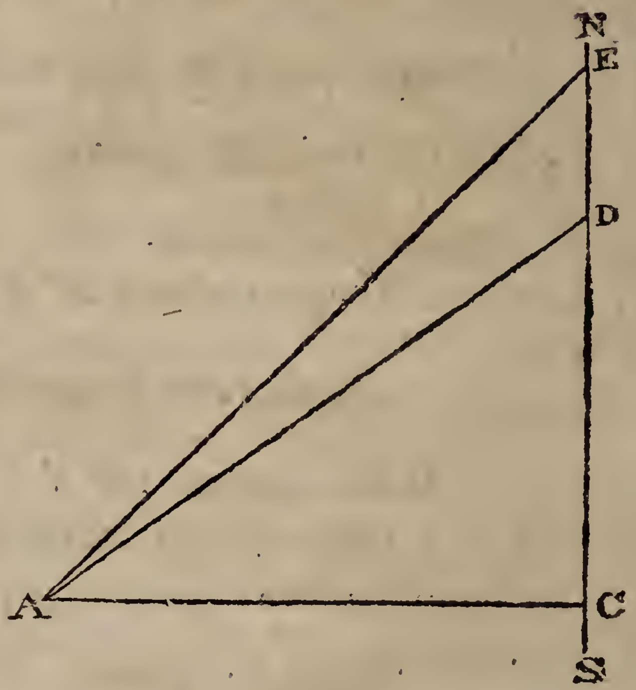
2. A ship in latitude 47° 23′ N, and longitude 10° 17′ W, sails SW*b*W 126 miles. Her present latitude and longitude are required.

SOLUTION BY MIDDLE LATITUDE SAILING.

CONSTRUCTION.

Draw the meridian NS, and from D draw the line DA = 126m. the distance in the SW quarter, making an angle with NS equal to the course, or 56° 15′. From A let fall the perpendicular AC upon the meridian NS.

DC will be the difference of latitude, and AC the departure.



With the difference of latitude, when found, and the latitude, from which she sailed, find the present latitude, and thence the middle latitude.

From A draw AE, making the angle CAE = the middle latitude, and intersecting NS in E. The said line AE will be the difference of longitude.

COMPUTATION.

To find the difference of latitude by plane sailing.

Radius	10·00000
: Distance 126m.	2·10037
:: Cosine course 33° 45′	9·74474
: Difference of latitude 70	1·84511
Or 1° 10′	

Latitude sailed from	47° 23′ N
Difference of latitude	1 10 S
Present latitude	46 13 N
Sum of latitudes	93 36
Half or middle latitude	46 48 N
Comiddle latitude	43 12

To find the difference of longitude

S. comiddle latitude	43° 12′ ar. co.	0·16459
: Tangent	56 15	10·17511
:: Difference of latitude	70	1·84511
: Difference of longitude	153 Or 2° 33′	2·18481

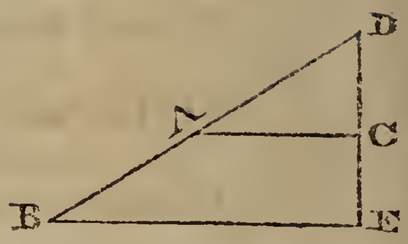
Longitude sailed from	10 17 W
Difference of longitude	2 33 W
Present longitude	12 50 W

BY THE TRAVERSE TABLE.

With course 5 points, and distance 126, are found difference of latitude 70, and departure 104·8. And with comiddle latitude 43° as course, and departure 104·8, is found distance 153, which is the difference of longitude.

SOLUTION BY MERCATOR'S SAILING.

CONSTRUCTION.



Construct the triangle DAC as before, and find the present latitude. From the two latitudes and their correspondent meridional parts, find the meridional difference of latitude, which set off from D southward to E. Draw EB to the westward of E and parallel to CA. Continue DA till it intersect EB in B.

EB will be the difference of longitude.

COMPUTATION.

The present latitude is found by plane sailing, as has already been done.

Lat. sailed from 47° 23′ N	Mer. parts	3237
Lat. come to 46 13 N	Mer. parts	3134
Meridional difference of latitude		103

To find the difference of longitude by Theorem IV.

Radius		10·00000
: Tangent course	56° 15′	10·17511
:: Mer. diff. lat.	103	2·01284
: Difference of longitude	154	2·18795

BY THE TRAVERSE TABLE.

Find the proper difference of latitude and departure as before, and under the same course with Mer. diff. lat. instead of difference of latitude, find the correspondent departure, which will be the difference of longitude. That is, with the course 5 points, against 103 as difference of latitude, is found 154 departure, which is the difference of longitude.

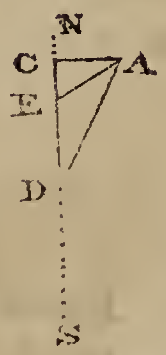
3. A ship in sight of the Peak of Teneriffe, bearing NNE, finds her latitude by observation 27° 5′ N. The ship's longitude is required to be found.

SOLUTION BY MIDDLE LATITUDE SAILING.

Teneriffe latitude	28° 25′ N	Long. 16° 25′ W
Ship's latitude	27 5 N	
Difference of lat.	1 20 or 80 miles S from the land.	
Sum of latitudes	55 30	
Middle latitude	27 45	
Comiddle lat.	62 15	

CONSTRUCTION.

Draw the meridian NS, and from a point D in the same draw DA NNE. Make DC to the northward = 80 miles, the difference of latitude, and draw the parallel of latitude CA. From A draw AE, making the angle CAE = the middle latitude, and intersecting NS in E. AE will be the difference of longitude.



COMPUTATION.

To find the departure by plane sailing.

Cosine course 67° 30′ ar. co.		0·03438
: Diff. lat. 80m.		1·90309
:: Sine course 22° 30′		9·58284
: Departure	33	1·52031

To find the difference of longitude.

Cosine middle latitude	62° 15	9·94693
: Radius		10·00000
:: Departure 33		1·52031
: Difference of longitude	37·5	1·57338

Longitude of Teneriffe	16° 25′ W	
Diff. long.	37 W	from Teneriffe.
Ship's longitude	17 02 W.	

BY THE TRAVERSE TABLE.

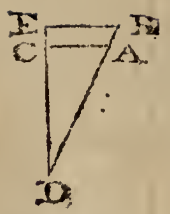
With the course two points and difference of latitude 80m. is found departure 33m. and with the comiddle latitude 62° as course, and the same departure is found distance 37, which is the difference of longitude.

SOLUTION BY MERCATOR'S SAILING.

Latitude of Teneriffe	28° 25′ N	Mer. parts	1780
Ship's latitude	27 05 N	Mer. parts	1689
		Mer. diff. lat.	91

CONSTRUCTION.

Construct the triangle DAC, as before. rom D set off DE=91, the meridional difference of latitude, Draw EB to the eastward of E, and parallel to CA. Continue DA till it intersect EB in B. EB will be the difference of longitude.



COMPUTATION.

To find the longitude.

Radius	10·00000
: Tangent course 22° 30′	9·61722
Mer. diff. lat. 91	1·95904
: Diff. long. 37·7	1·57626

BY THE TRAVERSE TABLE.

With course 2 points and meridional difference of latitude = 91, instead of proper difference of latitude, the departure =37·9 is found, which is the difference of longitude.

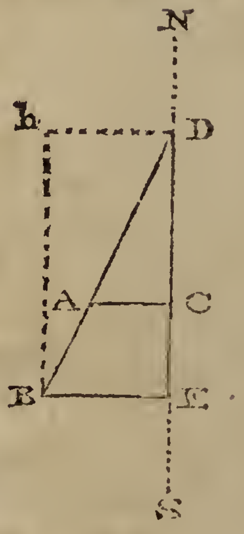
4. A ship, after taking her departure from Cape Clear, sails SSW, and finds by observation of a lunar eclipse, that her longitude is 10° 54′ W. What distance has she sailed, and also what is her present latitude ?

This question does not admit of a direct solution by middle latitude sailing.

SOLUTION BY MERCATOR'S SAILING.

Cape Clear lat. 51° 12′ N	Long.	9° 45′ W
	Ship's long.	10 54 W
	Diff. long.	1 09 W
	or 69m. W	

CONSTRUCTION.



Draw the meridian NS, and from a point D in the same, draw DB SSW. From the same point D draw D*b* W = 69 miles, the difference of longitude. Parallel to DS draw *b*B intersecting DB in B, and parallel to *b*D draw BE intersecting DS in E. DE will be the meridional difference of latitude.

Subtract the number of miles in the meridional difference of latitude from the meridional parts corresponding with 51° 12′, the latitude, from which she sailed ; the remainder will be the meridional parts corresponding with the latitude, to which she is come. Whence both latitudes being known, the proper difference of latitude may be found.

Set off the proper difference of latitude from D to C, and draw the parallel CA intersecting DB in A. CA will be the departure, and DA the distance.

COMPUTATION.

To find the meridional difference of latitude by inverting the proportion in Theorem IV.

Tangent of course 22° 30′	9·61722
: Radius	10·00000
:: Difference of long. 69m.	1·83885
: Mer. diff. lat. 166·6	2·22163

Cape Clear lat.	51° 12′ N	Mer. parts	3588
Subtract mer. diff. lat.			167
Present lat.	49° 25′	Mer. parts	3421
Proper diff. lat.	1 47 or 107m.		

To find the distance.

Cosine course	67° 30′	9·96561
: Proper diff. lat.	107	2·02938
:: Radius		10·00000
: Distance	115·8	2·06377

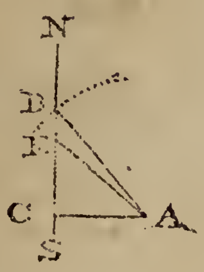
BY THE TRAVERSE TABLE.

With the course 22° 30′, or 2 points, and difference of longitude 69 as departure, is found the difference of latitude 167, which is the meridional difference of latitude. Hence find the proper difference of latitude, as has already been shown; and under the same course, and against the proper difference of latitude, the distance 116 will be found.

5. A ship in latitude 38° 10′ N sails 80m. on a south-easterly course, and makes 54m. easting. What course has she steered, and what difference of latitude and longitude has she made?

SOLUTION BY MIDDLE LATITUDE SAILING.

CONSTRUCTION.



Draw the meridian NS, and from a point C in the same draw the parallel of latitude CA to the eastward = 54 miles, the easting. From A, with the distance 80 miles, describe an arc cutting NS in D to the northward of C, and join DA.

The angle CDA will be the course, the line DA the distance, and DC the southing or difference of latitude. Whence the present latitude, and consequently the middle latitude, may be found.

From A draw AE, making the angle CAE = the middle latitude, and intersecting NS in E. The line AE will be the difference of longitude.

COMPUTATION.

To find the course by plane sailing.

Distance 80	1·90309
: Radius	10·00000
:: Departure 54	1·73239
: Sine course 42° 27′	9·82930
Its comp. 47 33	

To find the difference of latitude by plane sailing.

Radius	10.00000
: Distance 80	1·90309
:: Cosine course 47° 33′	9·86798
: Difference of latitude 59	1·77107

Latitude sailed from	38° 10′ N
Present latitude	37 11 N
Sum	75 21
Middle latitude	37 40
Comiddle latitude	52 20

To find the difference of longitude.

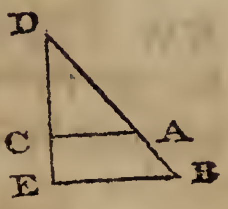
Cosine middle latitude 52° 20′	9·89849
: Departure 54	1·73239
:: Radius	10·00000
: Difference of longitude 68·2	1·83390
Or 1° 8′ E.	

BY THE TRAVERSE TABLE.

With distance 80 and departure 53·5 (the nearest in the table to 54) are found course 42°, and difference of latitude 59·4. And with comiddle latitude 52 as course, and departure 53·6 is found distance 68, which is the difference of longitude.

SOLUTION BY MERCATOR'S SAILING.

CONSTRUCTION.



Construct the triangle DAC as before, and find the present latitude. From the two latitudes and their correspondent meridional parts, find the meridional difference of latitude, which set off from D southward to E. Draw EB to the eastward, and parallel to CA. Continue DA till it intersect EB in B.

EB will be the difference of longitude.

COMPUTATION.

The present latitude and course are found by plane sailing, as has already been done.

Lat. sailed from	38° 10′ N	Mer. parts	2481
Lat. come to	37 11 N	Mer. parts	2406
		Mer. diff. lat.	75

To find the difference of longitude.

Proper diff. lat.	59	ar. co.	8·22893
: Mer. diff. lat.	75		1·87506
:: Departure	54		1·73239
: Diff. long.	68·6		1·83638

BY THE TRAVERSE TABLE.

With distance 80, and departure 53·5, are found course 42° and difference of latitude 59·4. And under the same course with meridional difference of latitude 75, instead of proper difference of latitude, is found departure 68·6, or difference of longitude.

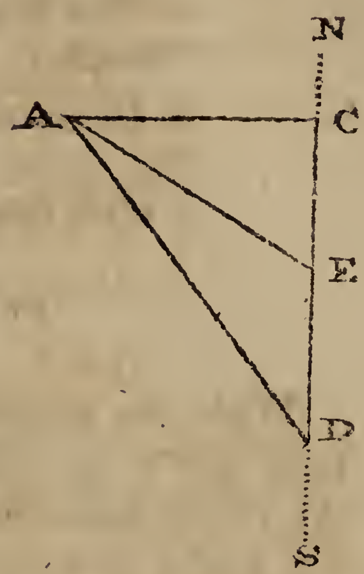
6. A ship in latitude 29° 40′ S sails 250 miles in the NW quarter, and then finds her latitude by observation to be 26° 13′ S. Her course, departure, and difference of longitude are required to be found.

SOLUTION BY MIDDLE LATITUDE SAILING.

Latitude sailed from	29° 40′ S
Latitude come to	26 13 S
Sum	55 53
Middle latitude	27 56
Comiddle latitude	62 04
Difference of latitude	3 27 or 207m. northing.

CONSTRUCTION.

Draw the meridian NS, and from a point D in the same, set off to the northward DC=207 miles, the difference of latitude. From C, the latitude come to, draw the parallel of latitude CA, unlimited toward A westward. From D, with the distance 250m. describe an arc cutting CA in A, and join DA.



From A draw the line AE, making the angle CAE=27° 56′, the middle latitude, and intersecting NS in E.

The angle ADN will be the course, the line AC the departure, and the line AE the difference of longitude.

COMPUTATION.

To find the course by plane sailing.

Distance 250	2·39794
: Radius	10·00000
:: Diff. lat. 207	2·31597
: Cos. course 55° 54′	9·91803

Its comp. 34 06 or the angle of course.

To find the departure by plane sailing.

Radius	10·00000
: Distance 250	2·39794
:: Sine course 34° 6′	9·74868
: Departure 140	2·14662

To find the difference of longitude.

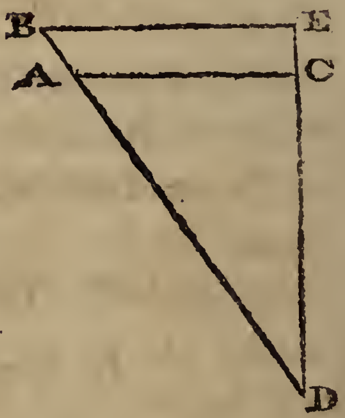
Cosine mid. lat. 62·04	9·94620
: Radius	10·00000
:: Departure 140	2·14662
: Diff. long. 158·6	2·20042

BY THE TRAVERSE TABLE.

With the distance 250, and nearest difference of latitude 207·3 is found course 34°, and departure 139·8. And with the comiddle latitude 62° as course, and departure 139·5, is found distance 158, which is the difference of longitude.

SOLUTION BY MERCATOR'S SAILING.

CONSTRUCTION.



Construct the triangle DAC, as before. Find the meridional difference of latitude, and set it off northward from D to E. Draw EB to the westward of E, and parallel to CA. Continue DA till it intersect EB in B.

EB will be the difference of longitude.

COMPUTATION.

The course and departure are found by plane sailing, as before.

Lat. sailed from	29° 40′ S	Mer. parts 1865
Lat. come to	26 13 S	Mer. parts 1631
		Mer. diff. lat. 234

To find the difference of longitude.

Radius	10·00000
: Tangent course 34° 6′	9·83062
:: Mer. diff. lat. 234	2·36921
: Diff. long. 158·4	2·19983

BY THE TRAVERSE TABLE.

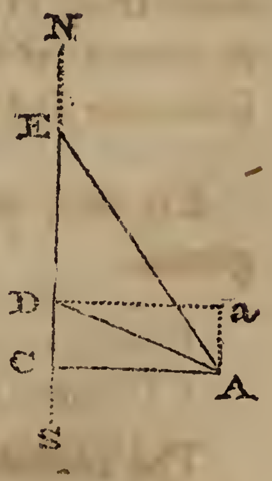
Course and departure being found as before, the meridional difference of latitude 234, taken as difference of latitude under the same course, gives departure or difference of longitude 158·3.

7. From a port in latitude 53° 7′ N a ship sails ESE, till her departure becomes 112 miles. It is required to determine the distance sailed, and the difference of longitude made.

SOLUTION BY MIDDLE LATITUDE SAILING.

CONSTRUCTION.

Draw the merdian NS, and from a point D in the same draw AD on a SSE course. Draw the parallel of latitude D*a* to the eastward = 112 miles, the departure. Draw likewise the line *a*A parallel to NS, and intersecting DA in A. Through A draw the parallel AC intersecting NS in C. DC will be the difference of latitude, and DA the distance.



Find the latitude, to which she is come, and thence the middle latitude.

From A draw the line AE, making the angle CAE = the middle latitude, and the said line will be the difference of longitude.

COMPUTATION.

To find the distance by plane sailing.

Sine course 67° 30′	9·96561
: Departure 112	2·04922
: : Radius	10·00000
: Distance 121·2	2·08361

To find the difference of latitude by plane sailing.

Radius	10·00000
: Distance 121·2	2·08361
: : Cosine course 22° 30′	9·58284
: Difference of latitude 46·4	1·66645

Latitude sailed from	53°	07′	N
Difference of latitude		46	S
Latitude come to	52	21	N
Sum	105	28	
Middle latitude	52	44	
Comiddle latitude	37	16	

To find the difference of longitude.

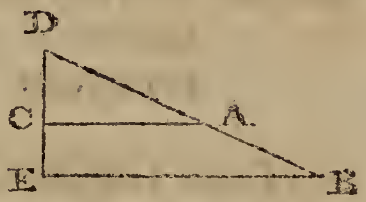
Cosine middle latitude	37° 17′ ar. co.	0·21770
: Tangent course	67 30	10·38277
: : Difference of latitude	46·4	1·66645
: Difference of longitude	184·9	2·26692

BY THE TRAVERSE TABLE.

With course 6 points, and distance 112 miles, is found difference of latitude 46·3, and distance 121. And with comiddle latitude 37° as course, and the same departure, is found 186 = the difference of longitude in the column of distance.

SOLUTION BY MERCATOR'S SAILING.

CONSTRUCTION.



Construct the triangle DAC, as before, and find the present latitude. From the two latitudes and their corresponding meridional parts find the meridional difference of latitude, which set off to the southward from D to E. Draw EB to the eastward from E, and parallel to CA. Continue DA till it intersect EB in B.

EB will be the difference of longitude.

COMPUTATION.

The distance and difference of latitude are found by plane sailing, as has already been done; and thence the two latitudes.

Lat. from which she sailed	53° 07′ N	Mer. parts	3775
Lat. to whch she is come	52 21 N	Mer. parts	3699
		Mer. diff. lat.	76

To find the difference of longitude.

Proper diff. lat.	46·4	ar. co.	8·33355
: Mer. diff. lat.	76		1·88081
:: Departure	112		2·04922
: Diff. long.	183·5		2·26358

BY THE TRAVERSE TABLE.

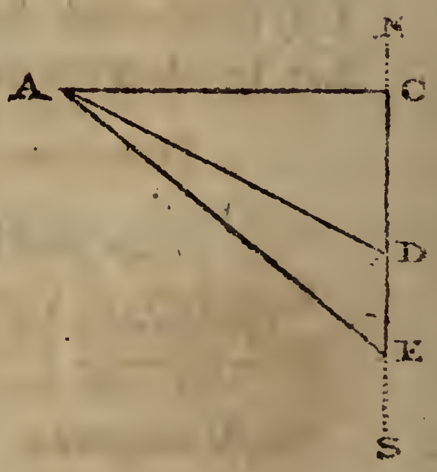
The difference of latitude and distance being found, as before, under the course 6 points, the difference of longitude 183·9 is found in the departure column; under the same course, against 76, the meridional difference of latitude, taken as difference of latitude.

8. A ship in latitude 36° 20′ S by observation, meets another ship, that had made 210 miles easting or departure from Cape Lagulhas. What is her present longitude, and also the bearing and distance of the Cape?

SOLUTION BY MIDDLE LATITUDE SAILING.

Cape Lagulhas lat.	34° 44′ S	Long. 20° 32′ E
Ship's latitude	36 30	
Difference of latitude	1 36 or 96m. N from the ship.	
Sum of latitudes	71 04	
Middle latitude	35 32	
Comiddle latitude	54 28	

CONSTRUCTION.



Draw the meridian NS, and from a point D in the same set off 96 miles to the northward to C. From C draw the parallel CA to the westward =210 miles, the departure. Join DA.

From A draw AE, making the angle CAE=35° 32′, the middle latitude, and intersecting NS in E.

The line AE will be the difference of longitude, the angle CDA the course or bearing of the Cape, and the line DA the distance.

COMPUTATION.

To find the course or bearing by plane sailing.

Difference of latitude 96	1·98227
: Radius	10·00000
:: Departure 210	2·32222
: Tangent course 65° 26′	10·33995

To find the distance by plane sailing.

Sine course 65° 26′	9·95879
: Departure 210	2·32222
:: Radius	10·00000
: Distance 230·9	2·36343

To find the difference of longitude.

Cosine middle latitude 54° 28′	9·91050
: Radius	10·00000
:: Departure 210	2·32222
: Difference of longitude 258·1 Or 4° 18′	2·41172

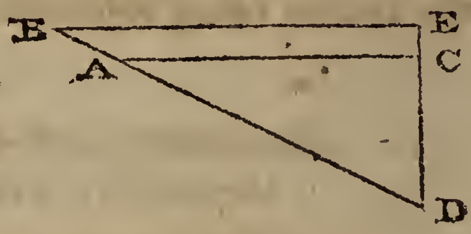
Longitude of Cape Lagulhas	20° 32′ E
Difference of longitude E from the Cape	4 18
Ship's longitude	24 50 E

BY THE TRAVERSE TABLE.

Seek difference of latitude 96, and departure 210, in the table ; the nearest found are 97·6 and 209·4, which correspond with 65° course, and 231 distance. Again, with comiddle latitude 54° as course, and 210 departure, is found 260 difference of longitude in the column of distance

SOLUTION BY MERCATOR'S SAILING.

CONSTRUCTION.



Construct the triangle DAC, as before, and set off the meridional difference of latitude to the northward from D to E. Draw EB to the westward from E, and parallel to CA. Continue DA till it intersect EB in B.

EB will be the difference of longitude.

COMPUTATION.

The bearing and distance are found by plane sailing, as before.

Cape Lagulhas, lat.	34° 44′ S	Mer. parts	2225
Ship's latitude	36 20 S	Mer. parts	2343
		Mer. diff. lat.	118

To find the difference of longitude.

Proper diff. lat.	96	ar. co.	8·01773
: Mer. diff. lat.	118		2·07188
:: Departure	210		2·32222
: Diff. long.	258·1		2·41183

BY THE TRAVERSE TABLE.

The bearing and distance being found, as already shown; under the same course with 118 as difference of latitude, is found departure or difference of longitude 254. The difference in these results is owing to the odd minutes in the course being rejected.

EXAMPLE OF A TRAVERSE.

Suppose a ship to sail from latitude 43° 25′ N on the following courses, viz. SW*b*S 63 miles, SSW$\frac{1}{2}$W 45 miles, S*b*E 54 miles, and SW*b*W 74 miles. Required the latitude arrived at, and the difference of longitude made good.

SOLUTION BY MIDDLE LATITUDE SAILING.

The difference of latitude and difference of longitude, corresponding to each course and distance, are found to be as in the following table.

Courses	Dist.	N	S	E	W
SW*b*S	63		52·4		47·85
SSW½W	45		39·7		28·62
S*b*E	54		53·0	13·73	
SW*b*W	74		41·1		81·08
			186 2	13·75	157·55
					13·75
					143·80

Therefore the latitude arrived at is 40° 19′ N and the difference of longitude made good is 143·80′ = 2° 23′ 48″ westward.

SOLUTION BY MERCATOR'S SAILING.

The difference of latitude and the difference of longitude, corresponding to each course and distance, are found to be as in the following table.

Courses	Dist.	N	S	E	W
SW*b*S	63		52·4		47·44
SSW½W	45		39·7		28·85
S*b*E	54		53·	13·92	
SW*b*W	74		41·1		80·61
			186·2	13·92	156·90
					13·92
					142·98

NOTE. The book belonging to a ship, in which are entered the courses, distances, winds, &c. from which the daily computation of the ship's place at sea is made, is called the *Log Book*; the account or register itself, *the log*; and the computations from the log, *the dead reckoning*, or *account*. Since this reckoning is liable to many errors, arising in the measures of distance, determinations of course, effects of tides and currents, and the estimation of lee-way, &c. it is very important to procure corrections frequently, by making astronomical observations suitable for determining the latitude and longitude. Meridian altitudes of the sun, or, when an observation of this kind cannot be obtained, two other altitudes of the same luminary with the intermediate time, for the latitude; and distances of the moon from the sun and certain fixed stars, for the longitude, may be considered as observations best adapted to the general practice of mariners.

Began Conic Sections
October 13th 1816

CONIC SECTIONS.

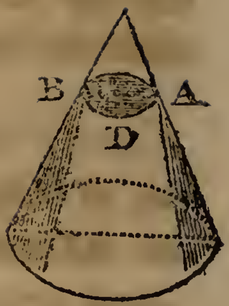
DEFINITIONS.

1. CONIC SECTIONS are the figures, made by the mutual intersection of a cone and a plane.

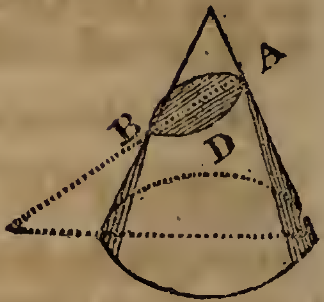
2. According to the different positions of the cutting plane, there arise five different figures or sections; namely, a triangle, a circle, an ellipse, a parabola, and a hyperbola: only the *three last* of which are peculiarly called *conic sections*.

3. If the cutting plane pass through the vertex of the cone, and any part of the base, the section will evidently be a *triangle*; as VAB.

4. If the plane cut the cone parallel to the base, or make no angle with it, the section will be a circle ; as ABD.



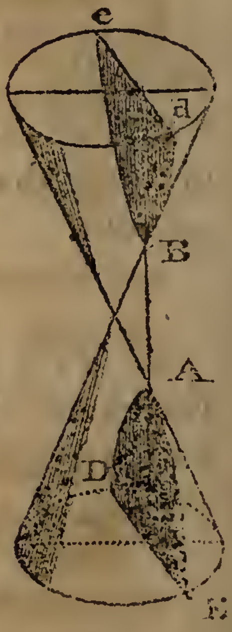
5. The section DAB is an *ellipse*, when the cone is cut obliquely through both sides, or when the plane, thus cutting it, is inclined to the base in a less angle than the side of the cone is.



6. The section is a *parabola*, when the cone is cut by a plane parallel to the side ; or when the cutting plane and the side of the cone make equal angles with the base ; as ADE.

7. The section is a *hyperbola*, when the cutting plane makes a greater angle with the base than the side of the cone makes ; as ADE.

8. And if all the sides of the cone be continued through the vertex, forming an opposite equal cone, and the plane be also continued to cut the opposite cone, this latter section will be the *opposite hyperbola* to the former ; as *Bae*.



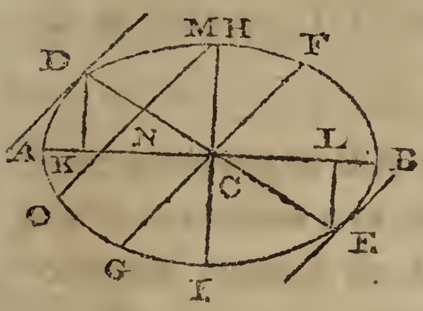
9. The *vertices* of any section are the points, where the cutting plane meets the opposite sides of the cone, or the sides of the vertical triangular section ; as A and B.

Hence the ellipse and the opposite hyperbolas have each two vertices ; but the parabola only one, unless we consider the other as at an infinite distance.

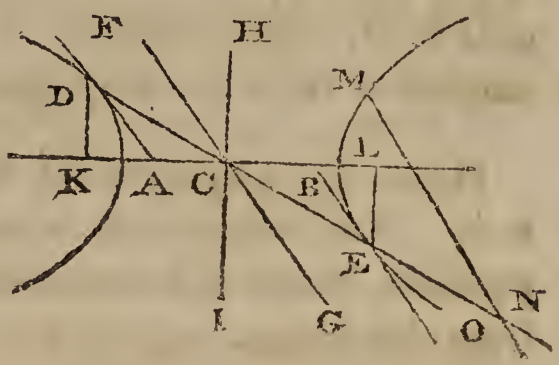
10. The *Axis*, or *Transverse Diameter*, of a conic section is the line or distance AB between the vertices.

Hence the axis of a parabola is infinite in length, A*b* being only a part of it.

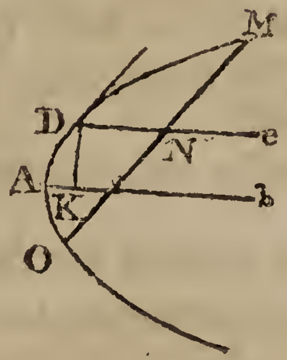
Ellipse.



Opposite Hyperbolas.



Parabola.



11. The *centre* C is the middle of the axis.

Hence the centre of a parabola is infinitely distant from the vertex. And of an ellipse the axes and centre lie within the curve ; but of a hyperbola without.

12. A *Diameter* is any right line, as AB or DE, drawn through the centre, and terminated on each side by the curve; and the extremities of the diameter, or its intersections with the curve, are its *vertices*.

Hence all the diameters of a parabola are parallel to the axis, and infinite in length; and therefore A*b* and D*e* are only parts of two diameters. And hence also every diameter of the ellipse and hyperbola have two vertices; but of the parabola only one; unless we consider the other as at an infinite distance.

13. The *Conjugate* to any diameter is the line drawn through the centre, and parallel to the tangent of the curve at the vertex of the diameter. So FG, parallel to the tangent at D, is the conjugate to DE; and HI, parallel to the tangent at A, is the conjugate to AB.

Hence the conjugate HI of the axis AB is perpendicular to it. And hence there is no conjugate to a diameter of the parabola, unless it be considered as at an infinite distance from the vertex.

14. An *Ordinate* to any diameter is a line parallel to its conjugate, or to the tangent at its vertex, and terminated by the diameter and curve. So DK, EL, are ordinates to the axis AB; and MN, NO, ordinates to the diameter DE.

Hence the ordinates to the axis are pependicular to it.

15. An *Absciss* is a part of any diameter contained between its vertex and an ordinate to it; as AK or BK, or DN or EN.

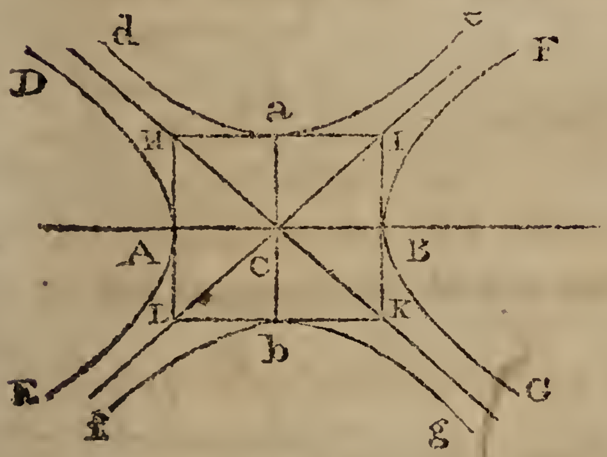
Hence in the ellipse and hyperbola, every ordinate has two abscisses; but in the parabola, only one; the other vertex of the diameter being infinitely distant.

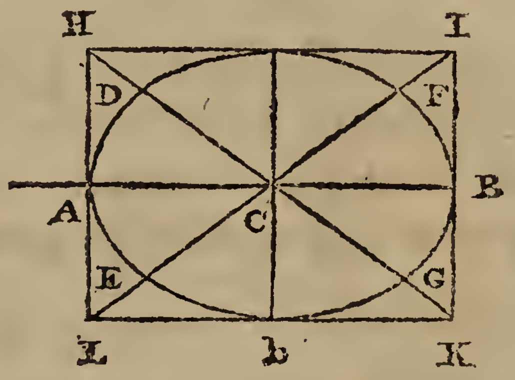
16. The *Parameter* of any diameter is a third proportional to that diameter and its conjugate.

17. The *Focus* is the point in the axis, where the ordinate is equal to half the parameter ; as K and L, where DK or EL is equal to the semiparameter.

Hence, the ellipse and hyperbola have each two foci ; but the parabola only one.

18. If DAE, FBG be two opposite hyperbolas, having AB for their first or transverse axis, and *ab* for their second or conjugate axis ; and if *dae*, *fbg* be two other opposite hyperbolas, having the same axis, but in a contrary order, namely, *ab* their first axis, and AB their second ; then these two latter curves *dae*, *fbg*, are called the *conjugate hyperbolas* to the two former DAE, FBG ; and each pair of opposite curves mutually *conjugate* to the other.





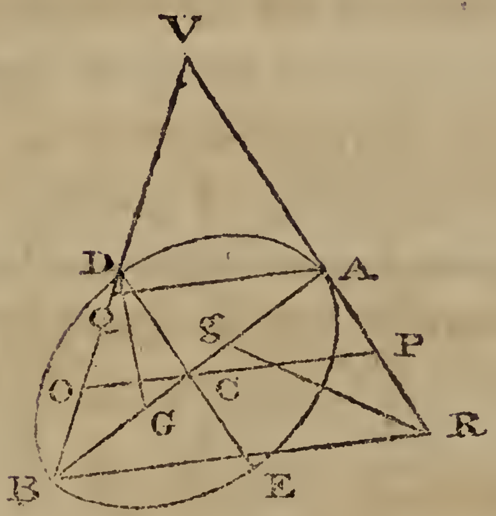
19. And if tangents be drawn to the four vertices of the curves, or extremities of the axis, forming the inscribed rectangle HIKL ; the diagonals HCK, ICL of this rectangle are called the *asymptotes* of the curves. And if these asymptotes intersect at right angles, or the inscribed rectangle be a square, or the two axes AB and *ab* be equal, then the hyperbolas are said to be *right-angled*, or *equilateral*.

Scholium. The rectangle, inscribed between the four conjugate hyperbolas, is similar to a rectangle, circumscribed about an ellipse by drawing tangents, in like manner, to the four extremities of the two axes; and the asymptotes or diagonals, in the hyperbola, are analogous to those in the ellipse, cutting this curve in similar points and making the pair of equal conjugate diameters.

Moreover, the whole figure, formed by the four hyperbolas, is, as it were, an ellipse turned inside out, cut open at the extremities D, E, F, G, of the said equal conjugate diameters, and those four points drawn out to an infinite distance, the curvature being turned the contrary way, but the axes, and the rectangle passing through their extremities, continuing fixed.

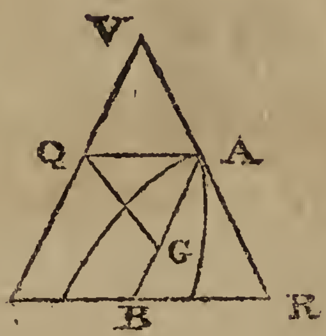
Cor. 1.

Ellipse.



Hyperbola.

Parabola.



In the ellipse, the semiconjugate axis CD, or CE, is a mean proportional between CO and CP, the parts of the diameter OP of a circle, drawn through the centre C of the ellipse, and parallel to the base of the cone.

For DE is a double ordinate, or diameter, in this circle, being perpendicular to OP as well as to AB.

In like manner, in the hyperbola, the length of the semiconjugate axis CD, or CE, is a mean proportional between CO and CP, drawn parallel to the base, and meeting the sides of the cone in O and P.

Or, if AO′ be drawn parallel to the side VB, and meet PC produced in O′, making CO′=CO ; and on this diameter O′P a circle be drawn parallel to the base : then the semiconjugate CD, or CE, will be an ordinate of this circle, being perpendicular to O′P as well as to AB.

Or, in both figures, the whole conjugate axis DE is a mean proportional between QA and BR, parallel to the base of the cone.

For, because AB is double AC, or CB, therefore, by similar triangles, QA is double OC, and BR double CP ; consequently

DE^2 or $2CD \cdot 2CE$, or $2CO \cdot 2CP = QA \cdot BR$, or $QA : DE :: DE : BR$.

In the parabola both the transverse and conjugate are infinite ; for AB and BR are both infinite.

Cor. 2. In all the sections AG will be equal to the parameter of the axis, if QG be drawn making the angle AQG equal to the angle BAR.

For, by the definition, AB : DE :: DE : p the parameter.

But, by Cor. 1, BR : DE :: DE : AQ ;

Therefore, AB : BR :: AQ : p.

But, by similar triangles, AB : BR :: AQ : AG ;

And therefore AG$=p$ the parameter.

In like manner Bg will be equal to the parameter p, if Rg be drawn to make the angle BRg = the angle ABQ. For here also AB : AQ :: BR : Bg=p.

Cor. 3. Hence the upper hyperbolic section, or section of the oppsite cone, is equal and similar to the lower section.

For the two sections have the same transverse or first axis AB, and the same conjugate or second axis DE, which is the mean proportional between AQ and RB ; they have also equal parameters AG, Bg. So that the two opposite sections make, as is were, but the two opposite ends of one entire section or hyperbola, the two being every where mutually equal and similar, like the two halves of an ellipse, with their ends turned the contrary way.

Cor. 4. And hence, although both the transverse and conjugate axes in the parabola be infinite, yet the former is infinitely greater than the latter, or has an infinite ratio to it.

For the transverse has the same ratio to the conjugate, as the conjugate has to the parameter, that is, as an infinite to to a finite quantity, which is an infinite ratio.

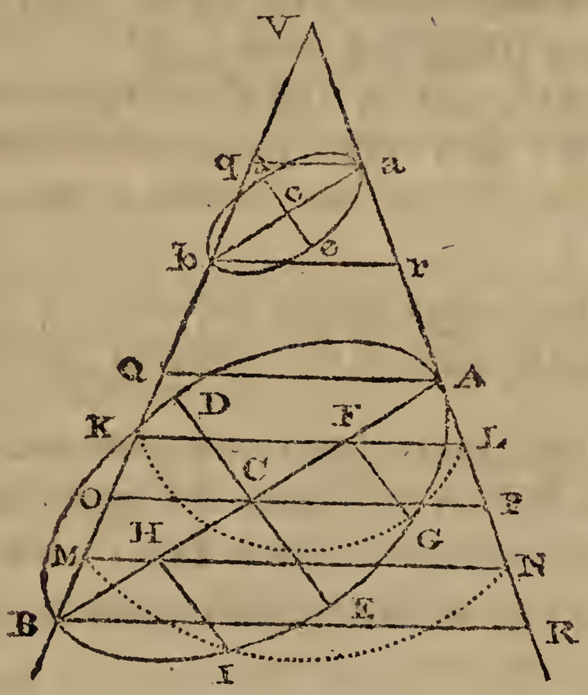
ELLIPSE.

PROPOSITION I.

The squares of the ordinates of the axis are to each other as the rectangles of their abscisses.

Let RVB be a plane passing through the axis of the cone; AEBD another section of the cone perpendicular to the plane RVB, but the oblique to another plane passing through the axis perpendicularly to this; AB the axis of this elliptic section; and FG, HI, ordinates perpendicular to it. Then

$$FG^2 : HI^2 :: AF \cdot FB : AH \cdot HB.$$



For through the ordinates FG, HI, draw the circular sections KGL, MIN, parallel to the base of the cone, having KL, MN, for their diameters, to which FG, HI, are ordinates, as well as to the axes of the ellipse.

Now, by the similar triangles AFL, AHN, and BFK, BHM, we have

$$AF : AH :: FL : HN,$$
$$\text{and } FB : HB :: KF : MH;$$

Hence taking the rectangles of the corresponding terms, we have $AF \cdot FB : AH \cdot HB :: KF \cdot FL : MH \cdot HN$. But, by the nature of the circle, $KF \cdot FL = FG^2$, and $MH \cdot HN = HI^2$;

therefore, $AF \cdot FB : AH \cdot HB :: FG^2 : HI^2$. Q. E. D.

Cor. 1. All the parallel sections are similar figures, or have their two axes in the same proportion; that is, $AB : ab :: DE : de$.

For, by similar triangles, $AB : ab :: AQ : aq$,

And $AB : ab :: RB : rb$;

Therefore, by composition, $AB^2 : ab^2 :: AQ \cdot RB : aq \cdot rb$.

But $AQ \cdot RB = DE^2$, and $aq \cdot rb = de^2$;

Therefore, $AB^2 : ab^2 :: DE^2 : de^2$,

Or, $AB : ab :: DE : de$.

Cor. 2. Hence also, as the property is the same for the ordinates on both sides of the diameter, it follows, that

1. At equal distances from the centre, or from the vertices, the ordinates on both sides are equal, or that the double ordinates are bisected by the axis; and that the whole figure, made up of all the double ordinates, is also bisected by the axis.

2. The two foci are equally distant from the centre, or from either vertex.

Cor. 3. When the angle, which the plane of the section makes with the base of the cone, increases till it become equal to the angle made by the side of the cone and the base, or till the section be parallel to the opposite side of the cone; then the axis becomes infinitely long, and the ellipse degenerates into a parabola; and the abscisses are to each other as the squares of their ordinates.

For, as then the infinites FB and HB are in a ratio of equality, the general property,

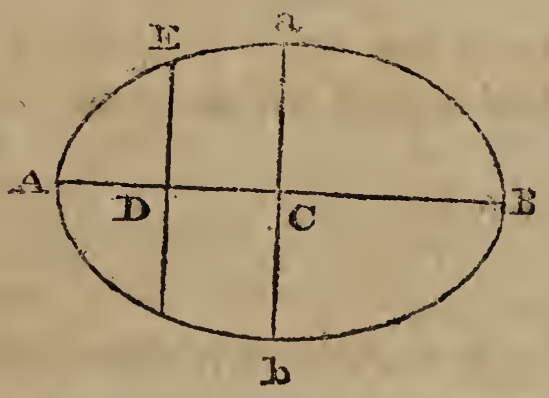
namely, $AF \cdot FB : AH \cdot HB :: FG^2 : HI^2$,

becomes $AF : AH :: FG^2 : HI^2$.

PROPOSITION II.

As the square of the transverse axis :
Is to the square of the conjugate ::
So is the rectangle of the abscisses :
To the square of their ordinate.

That is, $AB^2 : ab^2$, or $AC^2 : aC^2 :: AD \cdot DB : DE^2$.



For, by Prop. I. $AC \cdot CB : AD \cdot DB :: Ca^2 : DE^2$;

But if C be the centre, then $AC \cdot CB = AC^2$, and Ca is the semiconjugate.

Therefore $AC^2 : AD \cdot DB :: aC^2 : DE^2$;

Or, by permutation, $AC^2 : aC^2 :: AD \cdot DB : DE^2$;

Or, by doubling, $AB^2 : ab^2 :: AD \cdot DB : DE^2$.

Q. E. D.

Cor. 1. Or, the same property is $CA^2 : Ca^2 :: CA^2 - CD^2 : DE^2$,

Or $AB^2 : ab^2 :: CA^2 - CD^2 : DE^2$.

Because the rectangle $AD \cdot DB = CA^2 - CD^2$.

Cor. 2. Or, As the tranverse :
Is to its parameter ::
So is the rectangle of the abscisses :
To the square of their ordinate.

For by divis. $AB : \frac{ab^2}{AB} :: CA^2 - CD^2 : DE^2$.

That is, $AB : p :: AD \cdot DB$ or $CA^2 - CD^2 : DE^2$;

where p is the parameter $\frac{ab^2}{AB}$, by the definition of it.

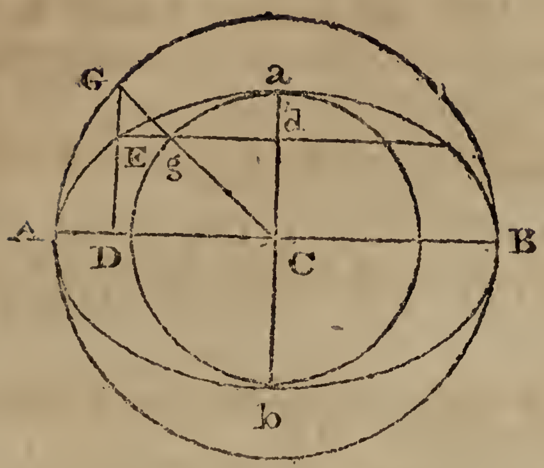
Cor. 3. In the parabola, the parameter is a third proportional to any absciss and its ordinate. For when the axis A is infinitely long, the curve becomes a parabola, and the infinites AB, DB, are then in a ratio of equality; and then the last property,

namely, $AB : p :: AD \cdot DB : DE^2$;
or $AB \cdot DE : AD \cdot DB :: p : DE$,
becomes $DE : AD :: p : DE$,
or $AD : DE :: DE : p$.

PROPOSITION III.

As the square of the conjugate axis :
Is to the square of the transverse axis ::
So is the rectangle of the abscisses of the conjugate, or the difference of the squares of the semiconjugate and distance of the centre from any ordinate of that axis :
To the square of that ordinate.

That is, $Ca^2 : CB^2 :: ad \cdot db$ or $Ca^2 - Cd^2 : dE^2$.



For, draw the ordinate ED to the transverse AB,

Then, by Cor. 1, Prop. II. $CA^2 : Ca^2 :: CA^2 - CD^2 : DE^2$.

But $CD^2 = dE^2$, and $DE^2 = Cd^2$,

Therefore $CA^2 : Ca^2 :: CA^2 - dE^2 : Cd^2$,

Or by alternation, $CA^2 : CA^2 - dE^2 :: Ca^2 : Cd^2$,

And by division, $CA^2 : dE^2 :: Ca^2 : Ca^2 - Cd^2$,

And by altern. $CA^2 : Ca^2 :: dE^2 : Ca^2 - Cd^2$,

And by invers. $Ca^2 : CA^2 :: Ca^2 - Cd^2 : dE^2$.

Q. E. D.

Cor. 1. If two circles be described on the two axes as diameters, one inscribed within the ellipse, and the other circumscribed about it; then an ordinate in the circle will be to the corresponding ordinate in the ellipse, as the axis of this ordinate is to the other axis.

That is, $CA : Ca :: DG : DE$,

and $Ca : CA :: dg : dE$.

For, by the nature of the circle, $AD \cdot DB = DG^2$; therefore, by the nature of the ellipse

$$CA^2 : Ca^2 :: AD \cdot DB \text{ or } DG^2 : DE^2,$$

$$\text{or } CA : Ca :: DG : DE;$$

In like manner, $Ca : CA :: dg : dE$.

Moreover, by equality, $DG : DE$ or $Cd :: dE$ or $DC : dg$.

Therefore CgG is a continued straight line.*

Cor. 2. Hence also, the ellipse is a mean proportional between the two circles.

For, as the ellipse and circle are made up of the same number of corresponding ordinates, which are all in the same proportion of the two axes, it follows, that the areas of the whole circle and ellipse, as also of any like parts of them, are in the same proportion of the two axes, or as the square of the diameter to the rectangle of the two axes; that is, the areas of the two circles, and of the ellipse, are as the square of each axis and the rectangle of the two. †

---

* Or, since $DG : DE :: DC : dg$,
by division $DG : DG - DE = EG :: DC : DC - dg = Eg$; therefore, since DC and Eg are parallel, CgG is one side of the triangle CDG.

† By Cor. 1, $CA : Ca :: DG : DE$;

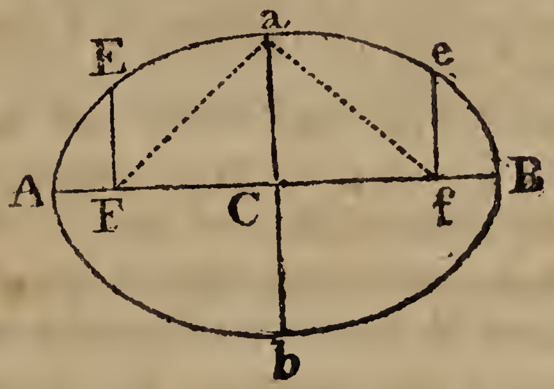
and $Ca : CA :: dg : dE$.

Therefore, since the ellipse and either of the circles consist of the same number of corresponding ordinates, it follows, that $AC : aC ::$ circle $ABG :$ ellipse $AaBb$,
and $aC : AC ::$ circle $abg :$ ellipse $AaBb$;
the rectangles of the corresponding terms are
$AC \cdot aC : aC \cdot AC ::$ circle $ABG \cdot$ circle $abg :$ ellipse $AaBb^2$;

PROPOSITION IV.

The square of the distance of the focus from the centre is equal to the difference of the squares of the semiaxes. Or, the square of the distance between the foci is equal to the difference of the squares of the two axes.

That is, $CF^2 = CA^2 - Ca^2$,
or, $Ff^2 = AB^2 - ab^2$.



For, to the focus F draw the ordinate FE; which, by the definition, will be the semiparameter. Then, by the nature of the curve,

$$CA^2 : Ca^2 :: CA^2 - CF^2 : FE^2;$$

and, by the definition of the parameter,

$$CA^2 : Ca^2 :: Ca^2 : FE^2;$$

therefore $Ca^2 = CA^2 - CF^2$;

and, by addition and subtraction, $CF^2 = CA^2 - Ca^2$;

or, by doubling, $Ff^2 = AB^2 - ab^2$. Q. E. D.

---

but, since the two first terms are equal, the other two are equal also;

that is, circle ABG · circle abg = ellipse $AaBb^2$;

therefore circle ABG : ellipse $AaBb$ :: ellipse $AaBb$: circle abg.

Cor. 1. The two semiaxes, and the focal distance from the centre, are the sides of a right-angled triangle CFa; and the distance Fa from the focus to the extremity of the conjugate axis is $=AC$ the semitransverse.

For, as above, $CA^2 - Ca^2 = CF^2$,
and, by right-angled triangles, $Fa^2 - Ca^2 = CF^2$;
therefore, $CA = Fa$, and $AB = Fa + fa$.

Cor. 2. The conjugate semiaxis Ca is a mean proportional between AF, FB, or between Af, fB, the distances of either focus from the two vertices.

For, $Ca^2 = CA^2 - CF^2 = \overline{CA + CF} \cdot \overline{CA - CF} = AF \cdot FB$.

Cor. 3. The same rectangle $AF \cdot FB$ of the focal distances from either vertex is also equal to the rectangle $AC \cdot FE$ under the semitranverse and its semiparameter.

For this last is equal to the square of the semiconjugate, by the definition of the parameter.

Or, $AF : FE :: AC : FB$.

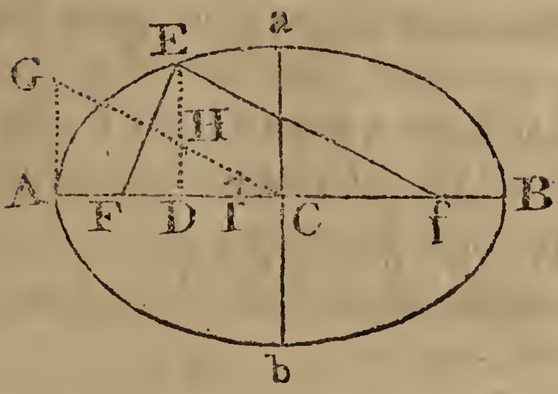
PROPOSITION V.

The difference between the semitranverse and a line, drawn from the focus to any point in the curve, is equal to a fourth proportional to the semitranverse, the distance from the centre to the focus, and the distance from the centre to the ordinate belonging to that point of curve.

That is, $AC - FE = CI$, or $FE = AI$;

and $fE - AC = CI$, or $fE = BI$.

Where $CA : CF :: CD : CI$ the fourth proportional to CA, CF, CD.



For, by right-angled triangles, $FE^2 = FD^2 + DE^2$.

Now, draw AG parallel and equal to Ca the semiconjugate; and join CG, meeting the ordinate DE in H.

Then, by Prop. II. $CA^2 : AG^2 :: CA^2 - CD^2 : DE^2$;
and, by sim. tri. $CA^2 : AG^2 :: CA^2 - CD^2 : AG^2 - DH^2$;
consequently, $DE^2 = AG^2 - DH^2 = Ca^2 - DH^2$.
Also $FD = CF - CD$, and $FD^2 = CF^2 - 2CF \cdot CD + CD^2$;
therefore $FE^2 = CF^2 + Ca^2 - 2CF \cdot CD + CD^2 - DH^2$.

But, by Prop. IV. $Ca^2 + CF^2 = CA^2$;
and by supposition, $2CF \cdot CD = 2CA \cdot CI$;
therefore, $FE^2 = CA^2 - 2CA \cdot CI + CD^2 - DH^2$.

But, by supp. $CA^2 : CD^2 :: CF^2$ or $CA^2 - AG^2 : CI^2$;
and, by sim. tri. $CA^2 : CD^2 :: CA^2 - AG^2 : CD^2 - DH^2$;
therefore $CI^2 = CD^2 - DH^2$;
consequently, $FE^2 = CA^2 - 2CA \cdot CI + CI^2$.

And the root or side of this square is $FE = CA - CI = AI$.

In the same manner is found $fE = CA + CI = BI$.

Q. E. D.

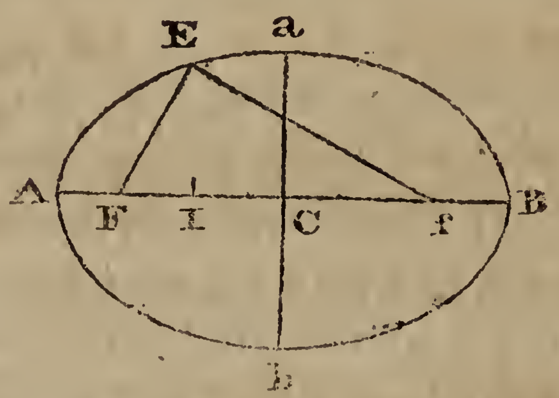
Cor. 1. Hence CI or $CA - FE$ is a fourth proportional to CA, CF, CD.

Cor. 2. And $fE - FE = 2CI$; that is, the difference between two lines, drawn from the foci to any point in the curve, is double the fourth proportional to CA, CF, CD.

PROPOSITION VI.

The sum of two lines, drawn from the foci to meet in any point of the curve, is equal to the transverse axis.

That is, $FE+fE=AB$.

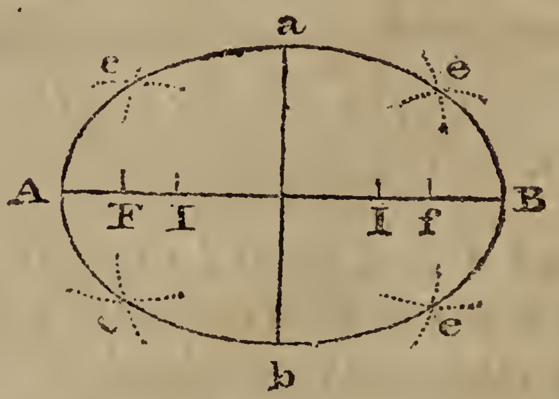


For, by the last Prop. $FE=CA-CI=AI$,

And, by the same, $fE=CA+CI=BI$;

Therefore, by addition, $FE+fE=AB$.

COR. Hence is derived the common method of describing the curve mechanically by points, or with a thread, thus :



In the transverse take the foci F, f, and any point I. Then with the radii AI, BI, and centres F, f, describe arcs intersecting in e, which will be a point in the curve. In like manner, assuming other points I, as many other points e will be found in the curve. Then, with a steady hand, draw the curve line through all the points of intersection e.

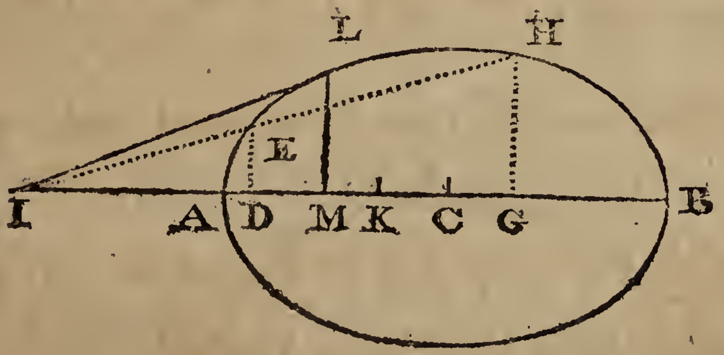
Or, take a thread of the length of AB the transverse axis, and fix its two ends in the foci F, f, by two pins. Then carry a pen or pencil round by the thread, keeping it always stretched, and its point will trace out the curve line.

Oct 22nd 1816

PROPOSITION VII.

If from any point I in the axis produced, a line IL be drawn touching the curve in a point L; and the ordinate LM be drawn; and if C be the centre, or the middle of AB; then shall CM be to CI as the square of AM to the square of AI.

That is, CM : CI :: AM^2 : AI^2.



For, from the point I draw any line IEH to cut the curve in two points E and H, from which let fall the perpendiculars ED and GH; and bisect DG in K.

Then, by Prop. I. $AD\cdot DB : AG\cdot GB :: DE^2 : GH^2$, and, by sim. tri. $ID^2 : IG^2 :: DE^2 : GH^2$; therefore, by equality, $AD\cdot DB : AG\cdot GB :: ID^2 : IG^2$.

But $DB = CB + CD = AC + CD = AG + DC - CG = 2CK + AG$,
and $GB = CB - CG = AC - CG = AD + DC - CG = 2CK + AD$;
therefore, $AD\cdot 2CK + AD\cdot AG : AG\cdot 2CK + AD\cdot AG :: ID^2 : IG^2$,
by division, $AD\cdot 2CK + AD\cdot AG : AG\cdot 2CK - AD\cdot 2CK :: ID^2 : IG^2 - ID^2$,
by inv. $AG\cdot 2CK - AD\cdot 2CK : AD\cdot 2CK + AD\cdot AG :: IG - ID^2 : ID^2$,
by alt. $DG\cdot 2CK : IG^2 - ID^2 :: AD\cdot 2CK + AD\cdot AG : ID^2$.

But $IG^2 - ID^2 = \overline{IG+ID}\cdot\overline{IG-ID} = 2IK\cdot DG$;
therefore, $DG\cdot 2CK : DG\cdot 2IK :: AD\cdot 2CK + AD\cdot AG : ID^2$,
or $2CK : 2IK :: AD\cdot 2CK + AD\cdot AG : ID^2$,
or $AD\cdot 2CK : AD\cdot 2IK :: AD\cdot 2CK + AD\cdot AG : ID^2$;
by alt. $AD\cdot 2CK : AD\cdot 2CK + AD\cdot AG :: AD\cdot 2IK : ID^2$.
Therefore, by division, $AD\cdot 2CK : AD\cdot AG :: AD\cdot 2IK : ID^2 - AD\cdot 2IK$,
by alt. $CK : IK :: AD\cdot AG : ID^2 - AD\cdot 2IK$,
and, by comp. $CK : CI :: AD\cdot AG : ID^2 - AD\cdot\overline{2IK - AG}$.

But $-AD\cdot\overline{2IK - AG} = -AD\cdot\overline{IG + ID - AG} = -AD\cdot\overline{ID + IA + AG - AG} = -AD\cdot\overline{ID + IA}$,

And $ID = IA + AD$, and $ID^2 = IA^2 + AD^2 + 2IA\cdot AD$;
consequently, $-AD\cdot\overline{IA + ID} = -AD\cdot\overline{2IA + AD} = -2IA\cdot AD - AD^2$,

and $ID^2 - AD \cdot \overline{2IK - AG} = IA^2 + AD^2 + 2IA \cdot AD - 2IA \cdot AD - AD^2 = IA^2$,

therefore, $CK : CI :: AD \cdot AG : AI$.

But when the line IH, by revolving about the point I, comes into the position of the tangent IL, then the points E and H meet in the point L, and the points D, K, G, coincide with the point M; and then the last proportion becomes $CM : CI :: AM^2 : AI^2$.

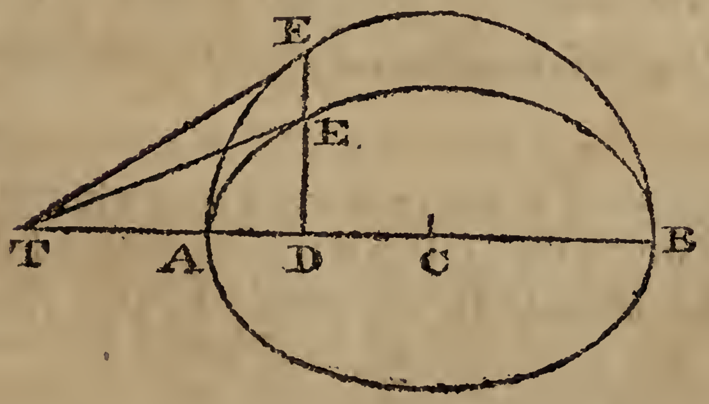
Q. E. D.

PROPOSITION VIII.

If a tangent and ordinate be drawn from any point in the curve, meeting the transverse axis; the semitransverse will be a mean proportional between the distances of the said two intersections from the centre.

That is, CA is a mean proportional between CD and CT;

or CD, CA, CT, are continued proportionals.



For, by Prop. VII. $CD : CT :: AD^2 : AT^2$,
that is, $CD : CT :: \overline{CA - CD}^2 : \overline{CT - CA}^2$,
or $CD : CT :: CD^2 + CA^2 : CA^2 + CT^2$,
and $CD : DT :: CD^2 + CA^2 : CT^2 - CD^2$,
or $CD : DT :: CD^2 + CA^2 : \overline{CT + CD} \cdot DT$.
or $CD^2 : CD \cdot DT :: CD^2 + CA^2 : CD \cdot DT + CT \cdot DT$,
hence $CD^2 : CA^2 :: CD \cdot DT : CT \cdot DT$,
and $CD^2 : CA^2 :: CD : CT$;
therefore $CD : CA :: CA : CT$.

Q. E. D.

Cor. 1. Since CT is always a third proportional to CD, CA; if the points D, A, remain constant, then will the point T be constant also, and therefore all the tangents will meet in this point T, which are drawn from E, of every ellipse described on the same axis AB, where they are cut by the common ordinate DEE, drawn from the point D.

Cor. 2. Hence a tangent is easily drawn to the curve from any point, either in the curve or without it.

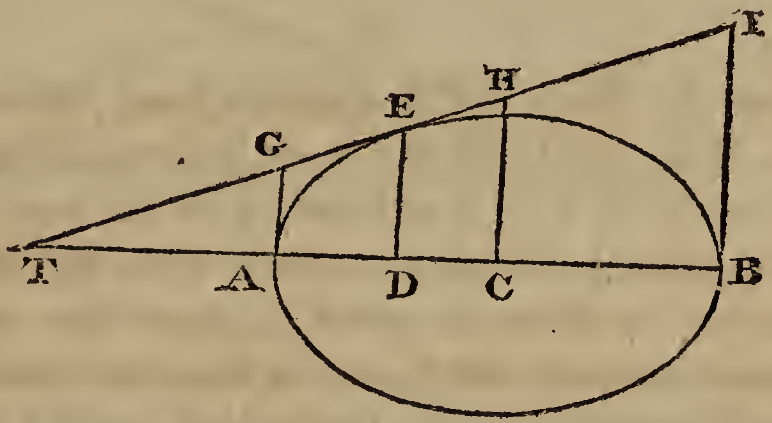
First, if the given point E be in the curve; draw the ordinate DE of the diameter AC; and in the diameter produced take CT a third proportional to CD, CA. Then join TE for the tangent required.

But, if the point T be given any where without the curve; join CT, in which take CD a third proportional to CT, CA; and draw the ordinate DE. Then join TE, as before.

PROPOSITION. IX.

If there be any tangent meeting four perpendiculars to the axis drawn from these four points, namely, the centre, the two extremities of the axis, and the point of contact; those four perpendiculars will be proportionals.

That is, AG : DE :: CH : BI.



For, by Prop. VIII. TC : AC :: AC : DC,

therefore, by div. TA : AD :: TC : AC or CB,

and, by comp. TA : TD :: TC : TB,

and hence, by sim. tri. AG : DE :: CH : BI.

Q. E. D.

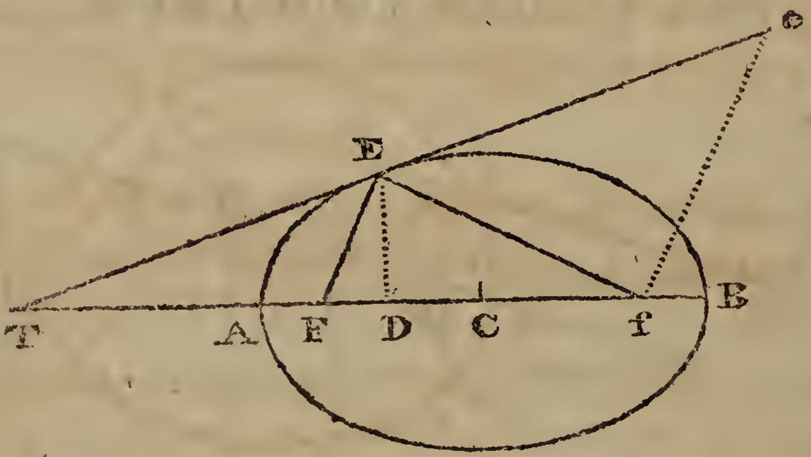
Cor. Hence TA, TD, TC, TB, and TG, TE, TH, TI, } are also proportionals.

For these are as AG, DE, CH, BI, by similar triangles.

PROPOSITION X.

If there be any tangent, and two lines drawn from the foci to the point of contact; these two lines will make equal angles with the tangent.

That is, the $\angle$ TEF $= \angle f$Ee.



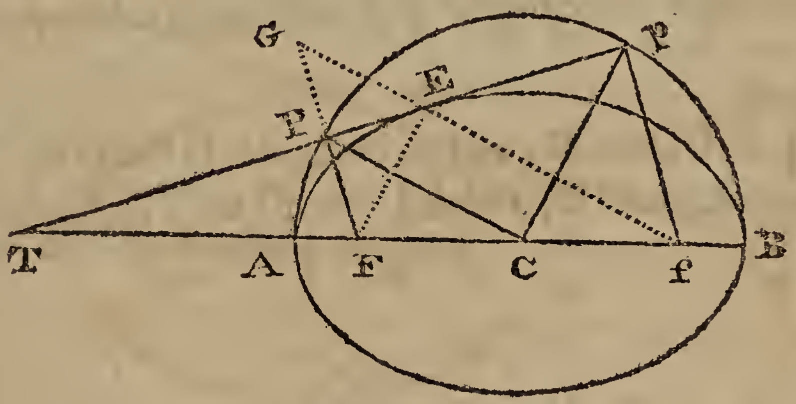
For, draw the ordinate DE, and fe parallel to FE.
By Prop. V. Cor. 1, CA : CD :: CF : CA—FE,
and by Prop. VIII. CA : CD :: CT : CA;
therefore, CT : CF :: CA : CA — FE;
and by add. and sub. TF : Tf :: FE : 2CA—FE, or fE, by Prop. VI.
But, by sim. tri. TF : Tf :: FE : fe;
therefore, fE $= fe$, and consequently $\angle e = \angle f$Ee.
But, because FE is parallel to fe, the $\angle e = \angle$ FET;
therefore, the $\angle$ FET $= \angle f$Ee. Q. E. D.

Cor. As opticians find, that the angle of incidence is equal to the angle of reflection, it appears from our Proposition, that rays of light issuing from one focus, and meeting the curve in every point, will be reflected into lines drawn from the other focus. So the ray fE is reflected into FE. And this is the reason why the points F, f, are called *foci*, or burning points.

PROPOSITION XI.

If a line be drawn from either focus perpendicular to a tangent to any point of the curve, the distance of their intersection from the centre will be equal to the semitransverse axis.

That is, if FP, fp, be perpendicular to the tangent TPp, then shall CP and Cp, be each equal to CA or CB.



For through the point of contact E draw FE, and fE meeting FP produced in G. Then the ∠GEP=∠FEP, being each equal to the ∠fEp, and the angles at P being right, and the side PE being common, the two triangles GEP, FEP, are equal in all respects, and so GE=FE, and GP=FP. Therefore, since FP=$\frac{1}{2}$ FG, and FC=$\frac{1}{2}$ Ff, and the angle at F common, the side CP will be = $\frac{1}{2}$ fG or $\frac{1}{2}$ AB, that is, CP=CA or CB.

And in the same manner Cp=CA or CB. Q. E. D.

Cor. 1. A circle described on the transverse axis, as a diameter, will pass through the points P, p; because all the lines CA, CP, Cp, CB, being equal, will be the radii of the circle.

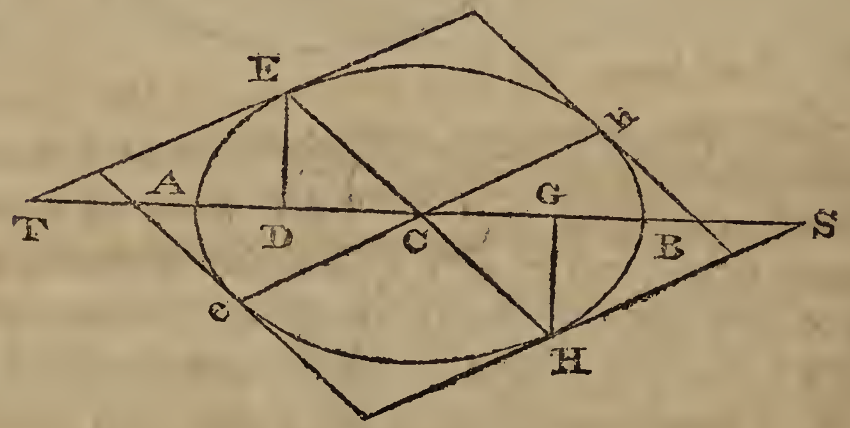
Cor. 2. CP is parellel to fE, and Cp parallel to FE.

COR. 3. If at the intersections of any tangent, with the circumscribed circle, perpendiculars to the tangent be drawn, they will meet the transverse axis in the two foci. That is, the perpendiculars PF, *pf*, give the foci F, *f*.

PROPOSITION XII.

The equal ordinates, or the ordinates at equal distances from the centre, on the opposite sides and ends of an ellipse have their extremities connected by one right line passing through the centre, and that line is bisected by the centre.

That is, if CD=CG, or the ordinate DE=GH, then shall CE=CH, and ECH will be a right line.



For, when CD = CG, then also is DE = GH, by Prop. I. Cor. 2.

But the ∠D=∠G being both right angles; therefore, the third side CE=CH, and the ∠DCE=∠GCH, and consequently, ECH is a right line. [redacted]

COR. 1. And conversely, if ECH be a right line passing through the centre; then shall it be bisected by the centre, or have CE = CH; also DE will be = GH, and CD = CG.

COR. 2. Hence also, if two tangents be drawn to the two ends E, H, of any diameter EH, they will be parallel to each other, and will cut the axis at equal angles, and at equal distances from the centre.

For, the two CA, CD, being equal to the two CG, CB, the third proportionals CT, CS, will be equal also; then the two sides CE, CT, being equal to the two CH, CS, and the included angle ECT equal to the included angle HCS, all the other corresponding parts are equal: and so the ∠ T = ∠ S, and TE is parallel to HS.

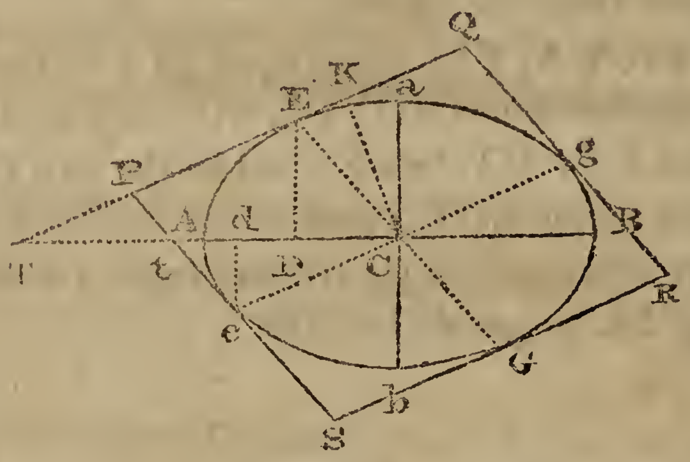
COR. 3. And hence the four tangents, at the four extremities of any two conjugate diameters, form a parallelogram circumscribing the ellipse, and the pairs of opposite sides are each equal to the corresponding parallel conjugate diameters.

For, if the diameter *eh* be drawn parallel to the tangent TE or HS, it will be the conjugate to EH, by the definition; and the tangents to *eh* will be parallel to each other, and to the diameter EH for the same reason.

PROPOSITION XIII.

All the parallelograms circumscribed about an ellipse are equal to one another, and each equal to the rectangle of the two axes.

That is, the parallelogram PQRS = the rectangle AB·*ab*.



Let EG, *eg*, be two conjugate diameters parallel to the sides of the parallelogram, and dividing it into four less and equal parallelograms. Also, draw the ordinates DE, *de*, and CK perpendicular to PQ; and let the axis CA produced meet the sides of the parallelogram, produced if necessary, in T and *t*.

Then, by Prop. VIII. CT : CA :: CA : CD,
and C*t* : CA :: CA : C*d*;
therefore, by equality, CT : C*t* :: C*d* : CD;
but, by sim. tri. CT : C*t* :: TD : C*d*,
therefore, by equality, TD : C*d* :: C*d* : CD,
and ∴ the rectangle TD·DC = the square Cd^2.

Again, by Prop. VIII. CD : CA :: CA : CT,
or, by division, CD : CA :: DA : AT,
and, by composition, CD : DB :: AD : DT;
consequently, the rectangle CD·DT = Cd^2 = AD·DB.

But, by Prop. II. $CA^2 : Ca^2 :: (AD \cdot DB \text{ or}) \; Cd^2 : DE^2$;

therefore, $CA : Ca :: Cd : DE$;

in like manner, $CA : Ca :: CD : de$,

or $Ca : de :: CA : CD$.

But, by Prop. VIII. $CT : CA :: CA : CD$;

therefore, by equality, $CT : CA :: Ca : de$.

But, by sim. tri. $CT : CK :: Ce : de$;

therefore, by equality, $CK : CA :: Ca : Ce$,

and the rectangle $CK \cdot Ce = CA \cdot Ca$.

But the rectangle $CK \cdot Ce =$ the parallelogram CEPe,

therefore the rect. $CA \cdot Ca =$ the parallelogram CEPe,

and consequently the rect. $AB \cdot ab =$ the paral. PQRS.

Q. E. D.

Cor. 1. The rectangles of every pair of conjugate diameters are to one another reciprocally as the sines of their included angles.

For the areas of their parallelograms, which are all equal among themselves, are equal to the rectangles of the sides, or conjugate diameters, multiplied by the sines of their contained angles, the radius being 1. That is, the rectangle of every two conjugate diameters, drawn into the sine of their contained angle, is equal to the same constant quantity. And therefore the rectangle of the diameters is inversely as the sine of their contained angle.

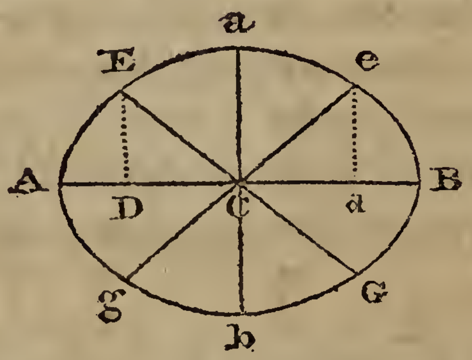
Cor. 2. As it is proved in this proposition, that every circumscribing parallelogram of an ellipse is a constant quantity; so it may hence be shown, that each of the spaces EAeP, EagQ, gBGR, GbeS, between the curve and the tangents, is equal to a constant quantity.

For, since every diameter bisects the ellipse, the conjugate diameters EG, eg, divide the ellipse into four equal sectors CEAe, CEag, CgBG, CGbe; but the same conjugate diameters divide also the whole tangential parallelogram PQRS into four equal parts, or small parallelograms CEPe, CEQg, CgRG, CGSe; and therefore the differences between these small parallelograms and the sectors, which are the said external spaces, must be all equal among themselves. And as the ellipse and circumscribing parallelograms both remain constant, the difference of their fourth parts will also be a constant quantity. That is, the said external parts are each equal to the same constant quantity.

PROPOSITION XIV.

The sum of the squares of every pair of conjugate diameters is equal to the same constant quantity, namely, the sum of the squares of the two axes.

That is, $AB^2 + ab^2 = EG^2 + eg^2$, where EG, eg, are any conjugate diameters.



For, draw the ordinates ED, ed.

Then, by step 10, last dem. $Cd^2 = AD \cdot DB$,

hence $Cd^2 = CA^2 - CD^2$,

therefore, $CA^2 = CD^2 + Cd^2$;

in like manner, $Ca^2 = DE^2 + de^2$;

therefore, the sum $CA^2 + Ca^2 = CD^2 + DE^2 + Cd^2 + de^2$.

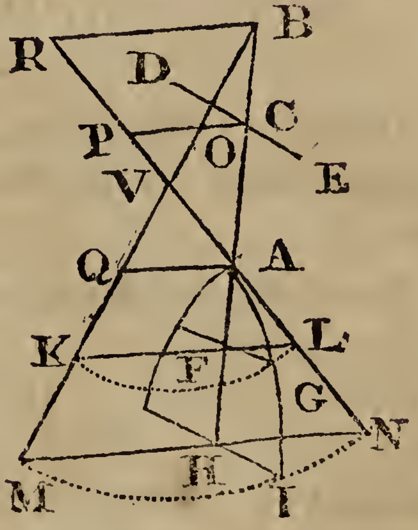
But, by right-angled triangles, $CE^2 = CD^2 + DE^2$,
and $Ce^2 = Cd^2 + de^2$;
therefore, the sum $CE^2 + Ce^2 = CD^2 + DE^2 + Cd^2 + de^2$;
consequently, $CA^2 + Ca^2 = CE^2 + Ce^2$;
or, by doubling, $AB^2 + ab^2 = EG^2 + eg^2$.

Q. E. D.

HYPERBOLA.

PROPOSITION I.

The squares of the ordinates of the axis are to each other as the rectangles of their abscisses.



Let AVB be a plane passing through the vertex and axis of the opposite cones; AGIH another section of them perpendicular to the plane of the former; AB the axis of the hyperbolic sections; and FG, HI, ordinates perpendicular to it. Then

$$FG^2 : HI^2 :: AF \cdot FB : AH \cdot HB$$

For, through the ordinates FG, HI, draw the circular sections KGL, MIN, parallel to the base of the cone, having KL, MN, for their diameters, to which FG, HI, are ordinates, as well as to the axis of the hyperbola.

Now, by the similar triangles AFL, AHN, and BFK, BHM, we have AF : AH :: FL : HN,

and FB : HB :: KF : MH ;

hence, taking the rectangles of the corresponding terms, we have the rect. $AF \cdot FB : AH \cdot HB :: KF \cdot FL : MH \cdot HN$.

But, by the nature of the circle, $KF \cdot FL = FG^2$, and $MH \cdot HN = HI^2$;

therefore, the rect. $AF \cdot FB : AH \cdot HB :: FG^2 : HI^2$.

Q. E. D.

Cor. 1. All the parallel sections are similar figures, or have their two axes in the same proportion; that is,

$$AB : ab :: DE : de.$$

For, by sim. tri. $AB : ab :: AQ : aq$,

and $AB : ab :: RB : rb$;

therefore, by comp. $AB^2 : ab^2 :: AQ \cdot RB : aq \cdot rb$.

But $AQ \cdot RB = DE^2$, and $aq \cdot rb = de^2$;

therefore $AB^2 : ab^2 :: DE^2 : de^2$,

or $AB : ab :: DE : de$.

Cor. 2. Hence also, as the property is the same for the ordinates on both sides of the diameter, it follows, that

1. At equal distances from the centre, or from the vertices, the ordinates on both sides are equal, or that the double ordinates are bisected by the axis ; and that the whole figure, made up of all the double ordinates, is also bisected by the axis.

2. The two foci are equally distant from the centre, or from either vertex.

Cor. 3. When the angle, which the plane of the section makes with the base of the cone, decreases till it become equal to the angle made by the side of the cone and the base, or till the section be parallell to the opposite side of the cone ; then the axis becomes infinitely long, and the hyperbola degenerates into a parabola ; and the abscisses are to each other as the squares of their ordinates.

For then the infinites FB and HB are in a ratio of equality, and the general property,

namely, $AF \cdot FB : AH \cdot HB :: FG^2 : HI^2$,
becomes $AF : AH :: FG^2 : HI^2$.

PROPOSITION II.

As the square of the transverse axis :
Is to the square of the conjugate ::
So is the rectangle of the abscisses :
To the square of their ordinate.

That is, $AB^2 : ab^2$ or $AC^2 : aC^2 :: AD \cdot DB : DE^2$.

For, by Prop. I. $AC \cdot CB : AD \cdot DB :: Ca^2 : DE^2$;
But, if C be the centre, then $AC \cdot CB = AC^2$, and Ca is the semiconjugate ;
therefore, $AC^2 : AD \cdot DB :: aC^2 : DE^2$;
or, by permutation, $AC^2 : aC^2 :: AD \cdot DB : DE^2$;
or, by doubling, $AB^2 : ab^2 :: AD \cdot DB : DE^2$.

Q. E. D.

Cor. 1. Or, $AB^2 : ab^2 :: CD^2 - CA^2 : DE^2$.
For as the rectangle $AD \cdot DB = CD^2 - CA^2$,
the same property is $CA^2 : Ca^2 :: CD^2 - CA^2 : DE^2$,

Cor. 2. Or, as the transverse
Is to its parameter,
So is the rectangle of the abscisses
To the square of their ordinate.

For, by div. $AB : \frac{ab^2}{AB} :: CD^2 - CA^2 : DE^2$,

that is, $AB : p :: AD \cdot DB$ or $CD^2 - CA^2 : DE^2$; where

p is the parameter $\frac{ab^2}{AB}$, by the definition of it.

Cor. 3. When the axis AB is infinitely long, the curve becomes a parabola, and the infinites AB, DB, are then in a ratio of equality; and then the last property,

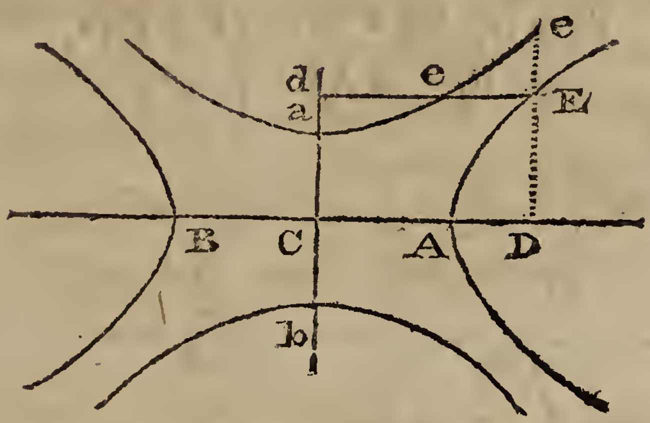
namely, $AB : p :: AD \cdot DB : DE^2$;
or $AB \cdot DE : AD \cdot DB :: p : DE$;
becomes $DE : AD :: p : DE$,
or $AB : DE :: DE : p$.

That is, in the parabola the parameter is a third proportional to any absciss and its ordinate.

PROPOSITION III.

As the square of the conjugate axis :
Is to the square of the transverse axis ::
So is the sum of the squares of the semiconjugate, and the distance of the the centre from any ordinate of this axis :
To the square of that ordinate.

That is, $Ca^2 : CA^2 :: Ca^2 + Cd^2 : dE^2$.



For, draw the ordinate ED to the transverse AB. Then, by Prop. I. $CA^2 : Ca^2 :: CD^2 - CA^2 : DE^2$.

But $CD^2 = dE^2$, and $DE^2 = Cd^2$,
therefore, $CA^2 : Ca^2 :: dE^2 - CA^2 : Cd^2$,
or, by alt. $CA^2 : dE^2 - CA^2 :: Ca^2 : Cd^2$,
and, by comp. $CA^2 : dE^2 :: Ca^2 : Ca^2 + Cd^2$,
and, by alt. and inv. $Ca^2 : CA^2 :: Ca^2 + Cd^2 : dE^2$.
In like manner, $CA^2 : Ca^2 :: CA^2 + CD^2 : De^2$.

Q. E. D.

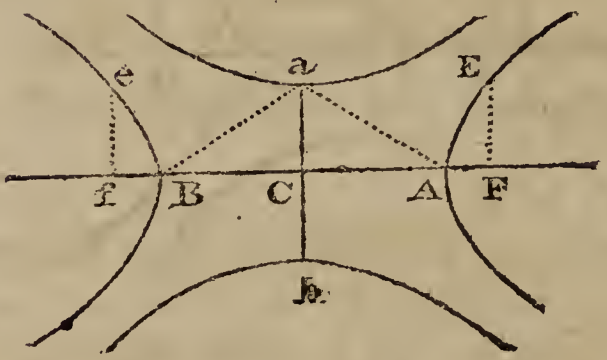
Cor. By the last Prop. $CA^2 : Ca^2 :: CD^2 - CA^2 : DE^2$, and by this Prop. $CA^2 : Ca^2 :: CD^2 + CA^2 : De^2$; therefore, $DE^2 : De^2 :: CD^2 - CA^2 : CD^2 + CA^2$. In like manner, $de^2 : dE^2 :: Cd^2 - Ca^2 : Cd^2 + Ca^2$.

PROPOSITION IV.

The square of the distance of the focus from the centre is equal to the sum of the squares of the semiaxes.

Or, the square of the distance between the foci is equal to the sum of the squares of the two axes.

$$\text{That is, } CF^2 = CA^2 + Ca^2,$$
$$\text{or } Ff^2 = AB^2 + ab^2.$$



For, to the focus F draw the ordinate FE; which, by the definition, will be the semiparameter. Then by the nature of the curve $CA^2 : Ca^2 :: CF^2 - CA^2 : FE^2$; and, by the def. of the param. $CA^2 : Ca^2 :: Ca^2 : FE$;

therefore, $Ca^2 = CF^2 - CA^2$;
and, by add. $CF^2 = CA^2 + Ca$;
or, by doubling, $Ff^2 = AB^2 + ab^2$.

Q. E. D.

Cor. 1. The two semiaxes and the focal distance from the centre are the sides of a right-angled triangle CAa; and the distance $Aa = CF$ the focal distance.

For, as above, $CA^2 + Ca = CF^2$,
and, by right-angled triangles, $Ca^2 + CA^2 = Aa^2$,
therefore, $CF = Aa$, and $Ff = Aa + Ba$.

Cor. 2. The conjugate semiaxis Ca is a mean proportional between AF, FB, or between Af, fB, the distances of either focus from the two vertices.

For $Ca^2 = CF^2 - CA^2 = \overline{CF + CA} \cdot \overline{CF - CA} = AF \cdot FB$.

Cor. 3. The same rectangle $AF \cdot FB$ of the focal distances from either vertex, is also equal to the rectangle $AC \cdot FE$ under the semitransverse and its semiparameter; since this last is equal to the square of the semiconjugate, by the definition of the parameter.

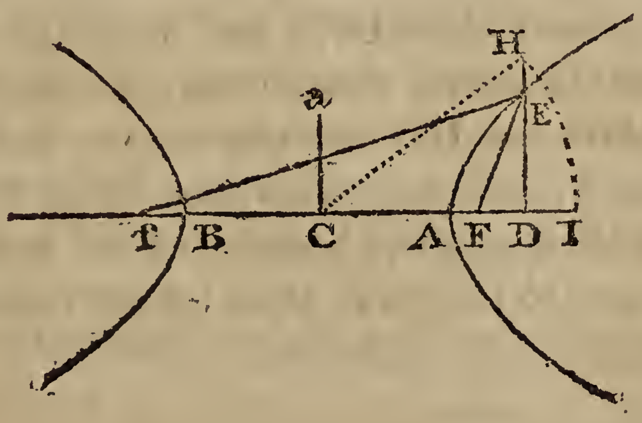
$$\text{Or } AF : FE :: AC : FB.$$

PROPOSITION V.

The sum or difference between the semitransverse and a line, drawn from the focus to any point in the curve, is equal to a fourth proportional to the semitransverse, the distance from the centre to the focus, and the distance from the centre to the ordinate belonging to that point of the curve.

That is, $AC + FE = CI$, or $FE = AI$;
and $fE - AC = CI$, or $fE = BI$.

Where $CA : CF :: CD : CI$ the fourth proportional to CA, CF, CD.



For, by right-angled triangles, $FE^2 = FD^2 + DE^2$.

Now draw AG parallel and equal to Ca the semiconjugate; and join CG, meeting the ordinate DE produced in H. Then, by Prop. II. $CA^2 : AG^2 :: CD^2 - CA^2 : DE^2$; and, by sim. tri. $CA^2 : AG^2 :: CD^2 - CA^2 : DH^2 - AG^2$; consequently, $DE^2 = DH^2 - AG^2 = DH^2 - Ca^2$.

Also, $FD = CF \backsim CD$, and $FD^2 = CF^2 - 2CF \cdot CD + CD^2$; therefore, $FE^2 = CF^2 - Ca^2 - 2CF \cdot CD + CD^2 + DH^2$.

But, by Prop. IV. $CF^2 - Ca^2 = CA^2$,
and, by supposition, $2CF \cdot CD = 2CA \cdot CI$;
therefore, $FE^2 = CA^2 - 2CA \cdot CI + CD^2 + DH^2$.
But, by supp. $CA^2 : CD^2 :: CF^2$ or $CA^2 + AG^2 : CI^2$;
and, by sim. tri. $CA^2 : CD^2 :: CA^2 + AG^2 : CD^2 + DH^2$;
therefore, $CI^2 = CD^2 + DH^2 = CH^2$;
consequently, $FE^2 = CA^2 - 2CA \cdot CI + CI^2$.
And the root or side of this square is $FE = CI - CA = AI$.
In the same manner is found $fE = CI + CA = BI$.

Q. E. D.

Cor. 1. Hence $CH = CI$ is a fourth proportional to CA, CF, CD.

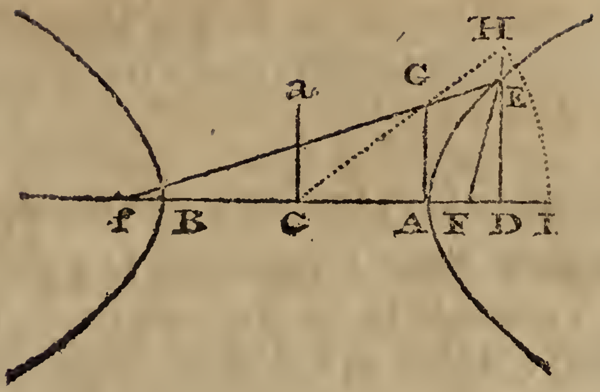
Cor. 2. And $fE + FE = 2CH$ or $2CI$; or FE, CH, fE, are in continued arithmetical progression, the common difference being CA the semitransverse.

Cor. 3. From the demonstration it appears, that $DE^2 = DH^2 - AG^2 = DH^2 - Ca^2$. Consequently DH is every where greater than DE; and so the asymptote CGH never meets the curve, though they be ever so far produced: but DH and DE approach nearer and nearer to a ratio of equality, as they recede farther from the vertex, till at an infinite distance they become equal, and the asymptote is a tangent to the curve at an infinite distance from the vertex.

PROPOSITION VI.

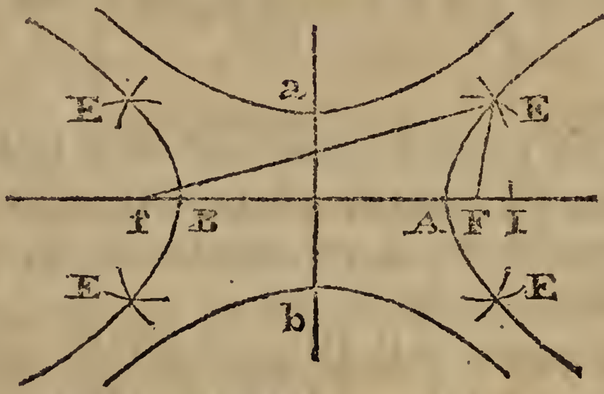
The difference of two lines, drawn from the foci to meet in any point of the curve, is equal to the transverse axis.

That is, $fE - FE = AB$.



For, by the last Prop. $FE = CI - CA = AI$,
and, by the same, $fE = CI + CA = BI$;
therefore, by subtraction, $fE - FE = AB$.

Cor. Hence is derived the common method of describing the curve mechanically by points, thus:



In the transverse AB produced, take the foci F, f, and any point I. Then with the radii AI, BI, and centres F, f, describe arcs intersecting in E, which will be a point in the curve. In like manner assuming other points I, as many other points will be found in the curve.

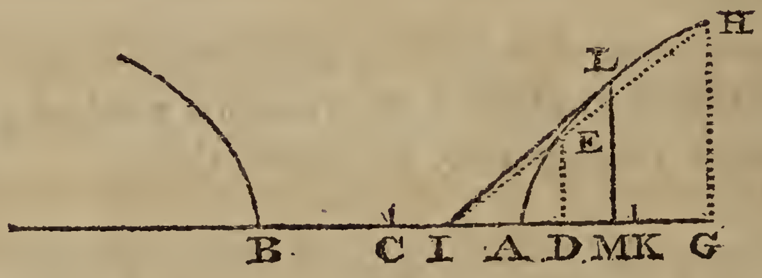
Then, with a steady hand, draw the curve line through all the points of intersection E.

In the same manner are constructed the other two hyperbolas, using the axis ab instead of AB.

PROPOSITION VII.

If from any point I, in the axis produced, a line IL be drawn touching the curve in a point L; and the ordinate LM be drawn; and if C be the centre or the middle of AB; then shall CM be to CI as the square of AM to the square of AI.

That is, $CM : CI :: AM^2 : AI^2$.



For, from the point I draw any line IEH to cut the curve in 2 points E and H; from which let fall the perpendiculars ED, HG; and bisect DG in K.

Then, by Prop. I. $AD \cdot DB : AG \cdot GB :: DE^2 : GH^2$,
and, by sim. tri. $ID : IG^2 :: DE^2 : GH^2$;
therefore, by equality, $AD \cdot DB : AG \cdot GB :: ID^2 : IG^2$.

But $DB = CB + CD = CA + CD = CG + CD - AG = 2CK - AG$,
and $GB = CB + CG = CA + CG = CG + CD - AD = 2CK - AD$;
therefore, $AD \cdot 2CK - AD \cdot AG : AG \cdot 2CK - AD \cdot AG :: ID^2 : IG^2$.
By div. $AD \cdot 2CK - AD \cdot AG : AG \cdot 2CK - AD \cdot 2CK :: ID^2 : IG^2 - ID^2$,
by inv. $AG \cdot 2CK - AD \cdot 2CK : AD \cdot 2CK - AD \cdot AG :: IG^2 - ID^2 : ID^2$,
by alt. $DG \cdot 2CK : IG^2 - ID^2 :: AD \cdot 2CK - AD \cdot AG : ID^2$.

But $IG^2 - ID^2 = \overline{IG + ID} \cdot \overline{IG - ID} = 2IK \cdot DG$;

theref. $DG \cdot 2CK : DG \cdot 2IK :: AD \cdot 2CK - AD \cdot AG : ID^2$,
or $2CK : 2IK :: AD \cdot 2CK - AD \cdot AG : ID^2$,
or $AD \cdot 2CK : AD \cdot 2IK :: AD \cdot 2CK - AD \cdot AG : ID^2$;
by alt. $AD \cdot 2CK : AD \cdot 2CK - AD \cdot AG :: AD \cdot 2IK : ID^2$;
therefore, by division, $AD \cdot 2CK : AD \cdot AG :: AD \cdot 2IK : AD \cdot 2IK - ID^2$;
by alt. $CK : IK :: AD \cdot AG : AD \cdot 2IK - ID^2$;

and by div. $CK : CI :: AD \cdot AG : ID^2 - AD . \overline{2IK - AG}$.

But $- AD . \overline{2IK - AG} = - AD \cdot \overline{IG + ID - AG} =$

$- AD . \overline{ID + IA + AG - AG} = - AD \cdot \overline{ID + IA}$;
$ID = IA + AD$, hence $ID^2 = IA^2 + AD^2 + 2IA \cdot AD$.

Consequently, $- AD \cdot \overline{ID + IA} = - AD \cdot \overline{2IA + AD} =$
$- 2IA \cdot AD - AD^2$,

and $ID^2 - AD \cdot \overline{2IK - AG} = IA^2 + AD^2 + 2IA \cdot AD - 2IA \cdot AD - AD^2 = IA^2$;
therefore, $CK : CI :: AD \cdot AG : IA^2$.

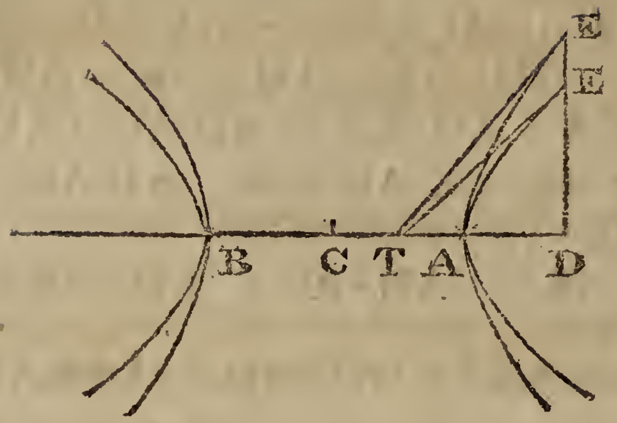
But when the line IH, by revolving about the point I, comes into the position of the tangent IL, and the ordinate LM being drawn, then the points E and H meet in the point L, and the points D, K, G, coincide with the point M; and then the property in the proposition becomes $CM : CI :: AM^2 : AI^2$.

Q. E. D.

PROPOSITION VIII.

If a tangent and ordinate be drawn from any point in the curve, meeting the transverse axis, the semitransverse will be a mean proportional between the distances of the said two intersections from the centre.

That is, CA is a mean proportional between CD and CT; or CD, CA, CT, are continued proportionals.



For, by Prop. VII. $CD : CT :: AD^2 : AT^2$,

that is, $CD : CT :: \overline{CD - CA}^2 : \overline{CA - CT}^2$,
or $:: CD^2 + CA^2 : CA^2 + CT^2$,
and $CD : DT :: CD^2 + CA^2 : CD^2 - CT^2$,

or $CD : DT :: CD^2 + CA^2 : \overline{CD + CT} \cdot DT$,
or $CD^2 : CD \cdot DT :: CD^2 + CA^2 : CD \cdot DT + CT \cdot TD$;
hence $CD^2 : CA^2 :: CD \cdot DT : CT \cdot TD$,
and $CD^2 : CA^2 :: CD : CT$;
therefore, $CD : CA :: CA : CT$.

Q. E. D.

Cor. 1. Since CT is always a third proportional to CD, CA; if the points D, A, remain constant, then will the point T be constant also; and therefore all the tangents will meet in this point T, which are drawn from E of every hyperbola described on the same axis AB, where they are cut by the common ordinate DEE, drawn from the point D.

Cor. 2. Hence a tangent is easily drawn to the curve, from any point, either in the curve or without it.

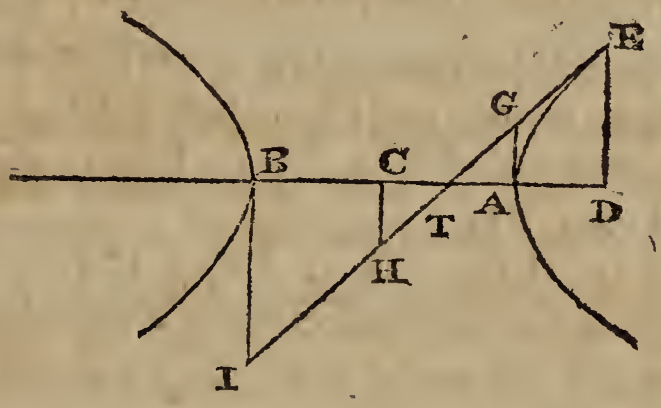
First, if the given point E be in the curve; draw the ordinate DE of the diameter AC; and in the diameter produced take CT a third proportional to CD, CA. Then join TE for the tangent required.

But if the point T be given any where without the curve; join CT, in which take CD a third proportional to CT, CA; and draw the ordinate DE. Then join TE, as before.

PROPOSITION IX.

If there be any tangent meeting four perpendiculars to the axis drawn from these four points, namely, the centre, the two extremities of the axis, and the point of contact; those four perpendiculars will be proportionals.

That is, AG : DE :: CH : BI.



For, by Prop. VIII.	TC : AC :: AC : DC,
therefore, by div.	TA : AD :: TC : AC or CB,
and, by comp.	TA : TD :: TC : TB,
and hence, by sim. tri.	AG : DE :: CH : BI.

Q. E. D.

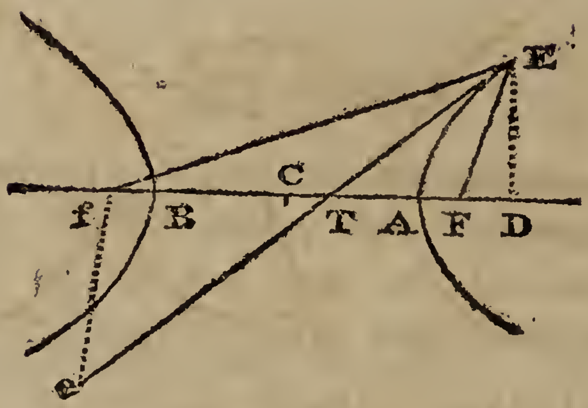
Cor. Hence TA, TD, TC, TB, and TG, TE, TH, TI, } are also proportionals.

For these are as AG, DE, CH, BI, by similar triangles.

PROPOSITION X.

If there be any tangent, and two lines drawn from the foci to the point of contact, these two lines will make equal angles with the tangent.

That is, the $\angle FET = \angle fEe$



For, draw the ordinate DE, and fe parallel to FE.
By Prop. V. Cor. 1, CA : CD :: CF : CA + FE,
and, by Prop. VIII. CA : CD :: CT : CA;
therefore, CT : CF :: CA : CA + FE;
and, by sub. and add. TF : Tf :: FE : 2CA + FE
or fE by Prop. VI.
But, by sim. tri. TF : Tf :: FE : fe;
therefore, fE = fe, and consequently $\angle e = \angle fEe$.
But, because FE is parallel to fe, the $\angle e = \angle FET$;
therefore, the $\angle FET = \angle fEe$.

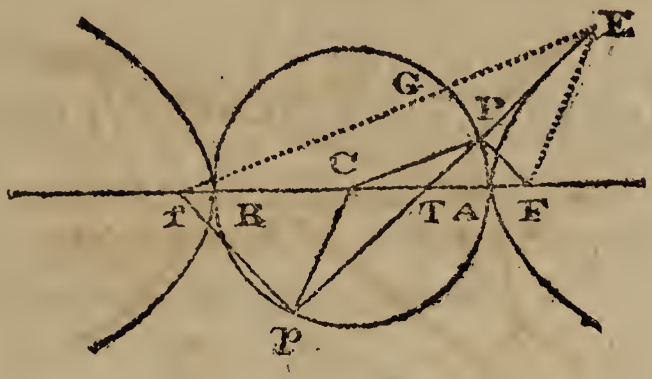
Q. E. D.

Cor. As opticians find, that the angle of incidence is equal to the angle of reflection, it appears from our proposition, that rays of light, issuing from one focus and meeting the curve in every point, will be reflected into lines drawn from the other focus. So the ray fE is reflected into FE. And this is the reason why the points F, f, are called *foci*, or burning points.

PROPOSITION XI.

If a line be drawn from either focus, perpendicular to a tangent to any point of the curve, the distance of their intersection from the centre will be equal to the semitransverse axis.

That is, if FP, fp, be perpendicular to the tangent PTp, then shall CP and Cp be each equal to CA or CB.



For, through the point of contact E draw FE and fE meeting FP produced in G. Then, the ∠GEP=FEP, being each equal to the ∠fEp, and the angles at P being right, and the side PE being common, the two triangles GEP, FEP, are equal in all respects, and so GE=FE, and GP=FP. Therefore, since FP= $\frac{1}{2}$ FG, and FC= $\frac{1}{2}$ Ff, and the angle at F common, the side CP will be = $\frac{1}{2}$ fG or $\frac{1}{2}$ AB, that is, CP=CA or CB.

Q. E. D.

Cor. 1. A circle described on the transverse axis, as a diameter, will pass through the points P, p; because all the lines CA, CP, Cp, CB, being equal, will be radii of the circle.

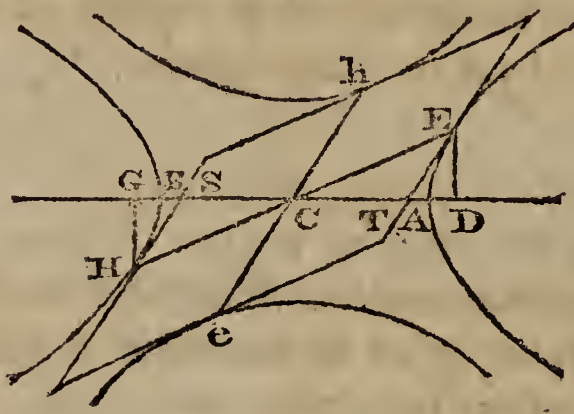
Cor. 2. CP is parallel to fE, and Cp parallel to FE.

Cor. 3. If at the intersections of any tangent with the circle described on the transverse axis, perpendiculars to the tangent be drawn, they will meet the transverse axis in the two foci. That is, the perpendiculars PF, pf, give the foci F, f.

PROPOSITION XII.

The equal ordinates, or the ordinates at equal distances from the centre, on the opposite sides and ends of a hyperbola, have their extremities connected by one right line passing through the centre, and that line is bisected by the centre.

That is, if CD=CG, or the ordinate DE=GH;
then shall CE=CH, and ECH will be a right line.



For, when CD=CG, then also is DE=GH, by Prop. I, Cor. 2.
But the ∠D=∠G, being both right angles;
therefore the third side CE=CH, and the ∠DCE=∠GCH, and consequently, ECH is a right line.

Cor. 1. And conversely, if ECH be a right line passing through the centre, then shall it be bisected by the centre, or have CE=CH; also DE will be =GH, and CD =CG.

Cor. 2. Hence also, if two tangents be drawn to the two ends E, H, of any diameter EH, they will be parallel to each other, and will cut the axis at equal angles, and at equal distances from the centre.

For, the two CD, CA, being equal to the two CG, CB, the third proportionals CT, CS, will be equal also; then the two sides CE, CT, being equal to the two CH, CS, and the included angle ECT equal to the included angle HCS, all the other corresponding parts are equal: and so the ∠T=∠S, and TE is parallel to HS.

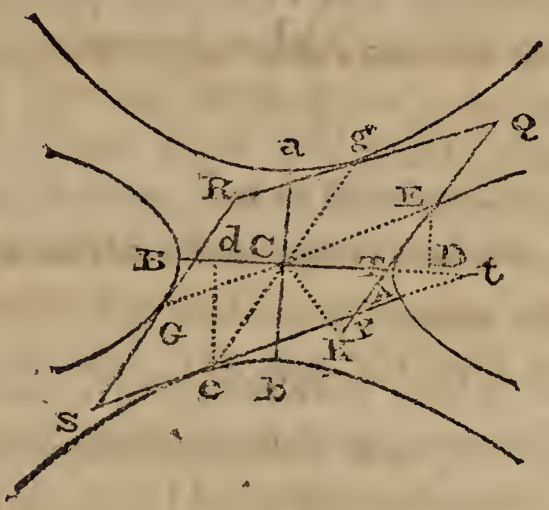
COR. 3. And hence the four tangents, at the four extremities of any two conjugate diameters, form a parallelogram inscribed between the hyperbolas, and the pairs of opposite sides are each equal to the corresponding parallel conjugate diameters.

For, if the diameter eh be drawn parallel to the tangent TE or HS, it will be the conjugate to EH, by the definition, and the tangents to eh will be parallel to each other, and to the diameter EH for the same reason.

PROPOSITION XIII.

All the parallelograms inscribed between the four conjugate hyperbolas are equal to one another, and each equal to the rectangles of the two axes.

That is, the parallelogram PQRS = the rectangle AB·ab.



Let EG, eg, be two conjugate diameters parallel to the sides of the parallelogram, and dividing it into four less and equal parallelograms. Also draw the ordintes DE, de, and CK perpendicular to PQ; and let the axis produced meet the sides of the parallelogram produced, if necessary, in T and t.

Then, by Prop. VIII.	CT	:	CA	::	CA	:	CD,
and	Ct	:	CA	::	CA	:	Cd;
therefore, by equality,	CT	:	Ct	::	Cd	:	CD;

but, by sim. tri. CT : Ct :: TD : Cd,
therefore, by equality, TD : Cd :: Cd : CD,
and the rectangle $TD \cdot DC =$ the square Cd^2.

Again, by Prop. VIII. CD : CA :: CA : CT,
or, by division, CD : CA :: DA : AT,
and, by composition, CD : DB :: DA : DT;
consequently, the rectangle $CD \cdot DT = Cd^2 = AD \cdot DB$.
But, by Prop. II. $CA^2 : Ca^2 :: (AD \cdot DB \text{ or}) \; Cd^2 : DE^2$,
therefore, CA : Ca :: Cd : DE;
in like manner, CA : Ca :: CD : de,
or Ca : de :: CA : CD.

But, by Prop. VIII. CT : CA :: CA : CD;
therefore, by equality, CT : CA :: Ca : de.
But, by sim. tri. CT : CK :: Ce : de;
therefore, by equality, CK : CA :: Ca : Ce,
and the rectangle $CK \cdot Ce = CA \cdot Ca$.
But the rectangle $CK \cdot Ce =$ the parallelogram CEPe,
therefore, the rectangle $CA \cdot Ca =$ the parallelogram CEPe,
and consequently the rect. $AB \cdot ab =$ the paral. PQRS.

Q. E. D.

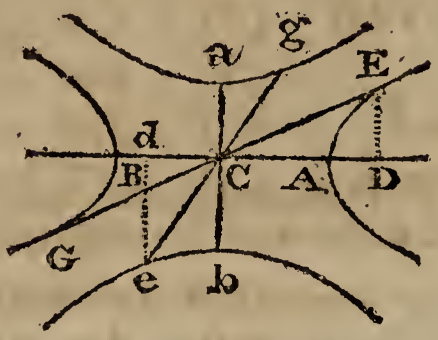
Cor. The rectangles of every pair of conjugate diameters are to one another reciprocally as the sines of their included angles.

For the areas of their parallelograms, which are all equal among themselves, are equal to the rectangles of the sides, or conjugate diameters, multiplied by the sines of their contained angles, the radius being 1. That is, the rectangle of every two conjugate diameters, drawn into the sine of their contained angle, is equal to the same constant quantity. And therefore the rectangle of the diameters is inversely as the sine of their contained angle.

PROPOSITION XIV.

The difference of the squares of every pair of conjugate diameters is equal to the same constant quantity, namely, the difference of the squares of the two axes.

That is, $AB^2 - ab^2 = EG^2 - eg^2$, where EG, eg, are any conjugate diameters.



For, draw the ordinates ED, ed.

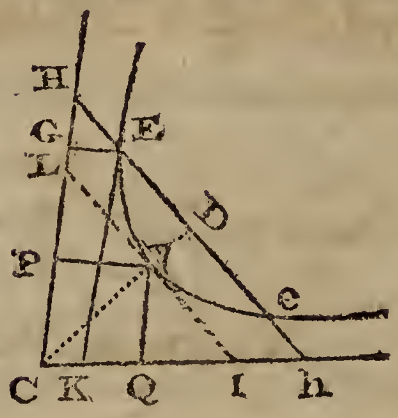
Then, by step 10, last dem. $Cd^2 = AD \cdot DB$,
hence $Cd^2 = CD^2 - CA^2$,
therefore, $CA^2 = CD^2 - Cd^2$;
in like manner, $Ca^2 = de^2 - DE^2$;
therefore, the diff. $CA^2 - Ca^2 = CD^2 + DE^2 - Cd^2 - de^2$.
But, by right-angled triangles, $CE^2 = CD^2 + DE^2$,
and $Ce^2 = Cd^2 + de$;
therefore, $CE^2 - Ce^2 = CD^2 + DE^2 - Cd^2 - de^2$.
Consequently, $CA^2 - Ca^2 = CE^2 - Ce^2$;
or, by doubling, $AB^2 - ab^2 = EG^2 - eg^2$.

Q. E. D.

PROPOSITION XV.

All the parallelograms are equal, which are formed between the asymptotes and curve, by lines drawn parallel to the asymptotes.

That is, the lines GE, EK, AP, AQ, being parallel to the asymptotes CH, CI; then the parallelogram CGEK = parallelogram CPAQ.



For, let A be the vertex of the curve, or extremity of the semitransverse axis AC, perpendicular to which draw AL, or Al, which will be equal to the semiconjugate. Also, draw HEDeh parallel to Ll.

Then, by Prop. II. $CA^2 : AL^2 :: CD^2 - CA^2 : DE^2$,
and, by parallels, $CA^2 : AL^2 :: CD^2 : DH^2$;
therefore, by sub. $CA^2 : AL^2 :: CA^2 : DH^2 - DE^2$,
or rect. $HE \cdot Eh$;
consequently, the square $AL^2 =$ the rect. $HE \cdot Eh$.
But, by sim. tri. $PA : AL :: GE : EH$,
and, by the same, $QA : Al :: EK : Eh$;
therefore, by comp. $PA \cdot AQ : AL^2 :: GE \cdot EK : HE \cdot Eh$,
and, because $AL^2 = HE \cdot Eh$, therefore, $PA \cdot AQ = GE \cdot EK$.

But, the parallelograms CGEK, CPAQ, being equiangular, are as the rectangles $GE \cdot EK$ and $PA \cdot AQ$.

And therefore the parallelogram GK = the parallelogram PQ. That is, all the inscribed parallelograms are equal to one another.

Cor. 1. Because the rectangle GEK or CGE is constant, therefore GE is reciprocally as CG, or $CG : CP :: PA : GE$. And hence the asymptote continually approaches toward the curve, but never meets it: for GE decreases continually as CG increases; and it is always of *some* magnitude, except when CG is supposed to be infinitely great, for then GE is infinitely small, or nothing. So that the asymptote CG may be considered as a tangent to the curve at a point infinitely distant from C.

10. Given $3x^n - 2x^{\frac{n}{2}} - \frac{p}{9} = \frac{r}{9}$; to find x.

Ans. $x = \overline{\pm\sqrt{\frac{\overline{r+p}\ 1.3}{27}} + \frac{1}{3}}\Big|^{\frac{2}{n}}$.

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

1. To find two numbers, whose difference is 8 and product 240.

Let x equal the less number,
Then will $x+8$ equal the greater,
And $x \times \overline{x+8} = x^2 + 8x = 240$ by the question;
Whence $x^2 + 8x + 16 = 240 + 16 = 256$ by completing the square;
Also $x + 4 = \sqrt{256} = 16$ by evolution;
And therefore $x = 16 - 4 = 12 =$ less number, and $12 + 8 = 20 =$ greater.

2. To divide the number 60 into two such parts, that their product may be 864.

Let $x =$ greater part,
Then will $60 - x =$ less,
And $x \times \overline{60 - x} = 60x - x^2 = 864$ by the question,
That is, $x^2 - 60x = -864$;
Whence $x^2 - 60x + 900 = -864 + 900 = 36$ by completing the square;
Also $x - 30 = \sqrt{36} = 6$ by extracting the root;
And therefore $x = 6 + 30 = 36 =$ greater part,
And $60 - x = 6 - 36 = 24 =$ less.

3. Given the sum of two numbers $= 10$ (a), and the sum of their squares $= 58$ (b); to find those numbers.

Let $x =$ greater of those numbers,
Then will $a - x =$ less;

And $x^2+\overline{a-x}|^2=2x^2+a^2-2ax=b$ by the question,

Or $x^2+\frac{a^2}{2}-ax=\frac{b}{2}$ by division,

Or $x^2-ax=\frac{b}{2}-\frac{a^2}{2}=\frac{b-a^2}{2}$ by transposition;

Whence $x^2-ax+\frac{a^2}{4}=\frac{b-a^2}{2}+\frac{a^2}{4}=\frac{2b-a^2}{4}$ by completing the square;

Also $x-\frac{a}{2}=\sqrt{\frac{2b-a^2}{4}}$ by extracting the root;

And therefore $x=\pm\sqrt{\frac{2b-a^2}{4}}+\frac{a}{2}=$ greater number,

And $a-\frac{a}{2}\pm\sqrt{\frac{2b-a^2}{4}}=\mp\sqrt{\frac{2b-a^2}{4}}+\frac{a}{2}=$ less.

Hence these two theorems, being put into numbers, give 7 and 3 for the numbers required.

4. Sold a piece of cloth for 24l. and gained as much per cent. as the cloth cost me; what was the price of the cloth?

Let $x=$ pounds the cloth cost,

Then $24-x=$ the whole gain;

But $100 : x :: x : 24-x$ by the question,

Or $x^2=100\times\overline{24-x}=2400-100x$;

That is, $x^2+100x=2400$;

Whence $x^2+100x+2500=2400+2500=4900$ by completing the square,

And $x+50=\sqrt{4900}=70$ by extracting the roots,

Consequently, $x=70-5=20=$ price of the cloth.

5. A person bought a number of oxen for 80l. and if he had bought 4 more for the same money, he would have paid 1l. less for each; how many did he buy?

Suppose he bought x oxen,

Then $\frac{80}{x}=$ price of each,

And $\frac{80}{x+4}$ = price of each, if $x+4$ had cost 80l.

But $\frac{80}{x}=\frac{80}{x+4}+1$ by the question,

Or $80=\frac{80x}{x+4}+x$,

Or $80x+320=80x+x^2+4x$,

That is, $x^2+4x=320$;

Whence $x^2+4x+4=320+4=324$ by completing the square;

And $x+2=\sqrt{324}=18$ by evolution;

Consequently $x=18-2=16$=number of oxen required.

6. What two numbers are those, whose sum, product, and difference of their squares, are all equal to each other?

Let x= greater number,

And y= less;

Then $\begin{Bmatrix} x+y=xy \\ x+y=x^2-y^2 \end{Bmatrix}$ by the question,

And $1=\frac{x^2-y^2}{x+y}=x-y$, or $x=y+1$ from the second equation;

Also $\overline{y+1}+y=\overline{y+1}\times y$ from the first equation,

Or $2y+1=y^2+y$,

That is, $y^2-y=1$,

Whence $y^2-y+\frac{1}{4}=1\frac{1}{4}$ by completing the square;

Also $y-\frac{1}{2}=\sqrt{1\frac{1}{4}}=\sqrt{\frac{5}{4}}=\frac{\sqrt{5}}{2}$ by evolution;

Consequently $y=\frac{\sqrt{5}}{2}+\frac{1}{2}=\frac{\sqrt{5}+1}{2}$;

And $x=y+1=\frac{\sqrt{5}+3}{2}$.

7. There are 4 numbers in arithmetical progression, whereof the product of the two extremes is 45, and that of the means 77; what are the numbers?

Let x= less extreme,

And $y=$ common difference;

Then x, $x+y$, $x+2y$, $x+3y$ will be the four numbers;

And $\left\{\begin{array}{l} x\times\overline{x+3y}=x^2+3xy=45 \\ \overline{x+y}\times\overline{x+2y}=x^2+3xy+2y^2=77 \end{array}\right\}$ by the question.

Whence $2y^2=77-45=32$, and $y^2=\frac{32}{2}=16$ by subtraction and division,

Or $y=\sqrt{16}=4$ by evolution,

Therefore $x^2+3xy=x^2+12x=45$ by the first equation,

Also $x^2+12x+36=45+36=81$ by completing the square,

And $x+6=\sqrt{81}=9$ by the extracting of roots,

Consequently $x=9-6=3$,

And the numbers are 3, 7, 11, and 15.

8. To find three numbers in geometrical progression, whose sum shall be 14, and the sum of their squares 84.

Let x, y, and z be the numbers sought;

Then $xz=y^2$ by the nature of proportion,

And $\left\{\begin{array}{l} x+y+z=14 \\ x^2+y^2+z^2=84 \end{array}\right\}$ by the question,

But $x+z=14-y$ by the second equation,

And $x^2+2xz+z^2=196-28y+y^2$ by squaring both sides,

Or $x^2+z^2+2y^2=196-28y+y^2$ by putting $2y^2$ for its equal $2xz$;

That is, $x^2+z^2+y^2=196-28y$ by subtraction,

Or $196-28y=84$ by equality;

Hence $y=\dfrac{196-84}{28}=4$ by transposition and division.

Again, $xz=y^2=16$, or $x=\dfrac{16}{z}$ by the first equation,

And $x+y+z=\dfrac{16}{z}+4+z=14$ by the 2d equation,

Or $16+4z+z^2=14z$, or $z^2-10z=-16$,

Whence $z^2-10z+25=25-16=9$ by completing the square;

And $z-5=\sqrt{9}=3$, or $z=3+5=8$;

Consequently $x=14-y-z=14-4-8=2$, and the numbers are 2, 4, 8.

9. The sum (s) and the product (p) of any two numbers being given; to find the sum of the squares, cubes, biquadrates, &c. of those numbers.

Let the two numbers be denoted by x and y;

Then will $\left\{\begin{array}{l} x+y=s \\ xy=p \end{array}\right\}$ by the question,

But $\overline{x+y}|^2=x^2+2xy+y^2=s^2$ by involution,

And $x^2+2xy+y^2-2xy=s^2-2p$ by subtraction,

That is, $x^2+y^2=s^2-2p=$ sum of the squares.

Again, $\overline{x^2+y^2}\times\overline{x+y}=\overline{s^2-2p}\times s$ by multiplication,

Or $x^3+xy\times\overline{x+y}+y^3=s^3-2sp$,

Or $x^3+sp+y^3=s^3-2sp$ by substituting sp for its equal $xy\times\overline{x+y}$;

And therefore $x^3+y^3=s^3-3sp=$ sum of the cubes.

In like manner, $\overline{x^3+y^3}\times\overline{x+y}=\overline{s^3-3sp}\times s$ by multiplication,

Or $x^4+xy\times\overline{x^2+y^2}+y^4=s^4-3s^2p$,

Or $x^4+p\times\overline{s^2-2p}+y^4=s^4-3s^2p$ by substituting $p\times\overline{s^2-2p}$ for its equal $xy\times\overline{x^2+y^2}$;

And consequently, $x^4+y^4=s^4-3s^2p-p\times\overline{s^2-2p}=s^4-4s^2p+2p^2=$ sum of the biquadrates, or fourth powers; and so on, for any power whatever.

10. The sum (a) and the sum of the squares (b) of four numbers in geometrical progression being given; to find those numbers.

Let x and y denote the two means,

Then will $\frac{x^2}{y}$ and $\frac{y^2}{x}$ be the two extremes, by the nature of proportion.

Also, let the sum of the two means $=s$, and their product $=p$.

And then will the sum of the two extremes $=a-s$ by the question,

And their product $=p$ by the nature of proportion.

$$\text{Hence}\left\{\begin{array}{l} x^2+y^2=s^2-2p \\ \frac{x^4}{y^2}+\frac{y^4}{x^2}=\overline{a-s}|^2-2p \end{array}\right\}\text{ by the last problem,}$$

And $x^2+y^2+\frac{x^4}{y^2}+\frac{y^4}{x^2}=s^2+\overline{a-s}|^2-4p=b$ by the question,

Again, $\frac{x^2}{y}+\frac{y^2}{x}=a-s$ by the question,

Or $x^3+y^3=xy\times\overline{a-s}=p\times\overline{a-s}$.

But $x^3+y^3=s^3-3sp$ by the last problem,

And therefore $p\times\overline{a-s}=s^3-3sp$ by equality,

Or $pa-ps+3ps=pa+2ps=s^3$,

$$\text{Or } p=\frac{s^3}{a+2s};$$

Whence $s^2+\overline{a-s}|^2-4p=s^2+\overline{a-s}|^2-\frac{4s^3}{a+2s}=b$ by substitution,

$$\text{Or } s^2+\frac{b}{a}s=\frac{a^2-b}{2}\text{ by reduction.}$$

And $s=\sqrt{\frac{a^2-}{2}+\frac{{}^2}{4a^2}}-\frac{b}{2a}$ by completing the square, and extracting the root.

And from this value of s all the rest of the quantities p, x, and y may be readily determined.

QUESTIONS FOR PRACTICE.

1. What two numbers are those, whose sum is 20, and their product 36? Ans 2 and 18.

2. To divide the number 60 into two such parts, that their product may be to the sum of their squares in the ratio of 2 to 5. Ans. 20 and 40.

3. The difference of two numbers is 3, and the difference of their cubes is 117; what are those numbers?

Ans. 2 and 5.

4. A company at a tavern had 8l. 15s. to pay for their reckoning; but, before the bill was settled, two of them sneaked off, and then those, who remained, had 10s. a piece more to pay than before; how many were there in the company? Ans. 7.

5. A grazier bought as many sheep as cost him 60l. and after reserving 15 out of the number, he sold the remainder for 54l. and gained 2s. a head by them; how many sheep did he buy? Ans. 75.

6. There are two numbers, whose difference is 15, and half their product is equal to the cube of the less number; what are those numbers? Ans. 3 and 18.

7. A person bought cloth for 33l. 15s. which he sold again at 2l. 8s. per piece, and gained by the bargain as much as one piece cost him; required the number of pieces.

Ans. 15.

8. What number is that, which being divided by the product of its two digits, the quotient is 3; and if 18 be added to it, the digits will be inverted? Ans. 24.

9. What two numbers are those, whose sum multiplied by the greater is equal to 77; and whose difference multiplied by the less is equal to 12? Ans. 4 and 7.

10. The sum of two numbers is 8, and the sum of their cubes is 152; what are the numbers?

Ans. 3 and 5.

11. The sum of two numbers is 7, and the sum of their fourth powers is 641; what are the numbers?

Ans. 2 and 5.

12. The sum of two numbers is 6, and the sum of their fifth powers is 1056; what are the numbers?

Ans. 2 and 4.

13. The sum of four numbers in arithmetical progression is 56, and the sum of their squares is 864; what are the numbers?

Ans. 8, 12, 16, and 20.

14. To find four numbers in geometrical progression, whose sum is 15, and the sum of their squares 85.

Ans. 1, 2, 4, and 8.

15. Given $\overline{x^2-\frac{a^4}{x^2}}\Big|^{\frac{1}{2}}+\overline{x^2-\frac{a^4}{x^2}}\Big|^{\frac{1}{2}}=2a$; to find the value of x.

Ans. $x=\overline{\tfrac{1}{2}a^2+\sqrt{\frac{5a^4}{4}}}\Big|^{\frac{1}{2}}$.

CUBIC AND HIGHER EQUATIONS.

A CUBIC EQUATION, or equation of the third degree or power, is one, that contains the third power of the unknown quantity: as $x^3-ax^2+bx=c$.

A *biquadratic*, or double quadratic, is an equation, that contains the fourth power of the unknown quantity: as $x^4-ax^3+bx^2-cx=d$.

An *equation of the fifth power*, or degree, is one, that contains the fifth power of the unknown quantity: as $x^5-ax^4+bx^3-cx^2+dx=e$.

An *equation of the sixth power*, or degree, is one, that contains the sixth power of the unknown quantity: as $x^6-ax^5+bx^4-cx^3+dx^2-ex=f$.

And so on, for all other higher powers. Where it is to be noted, however, that all the powers, or terms in the equation, are supposed to be freed from surds, or fractional exponents.

There are various particular rules for the resolution of cubic and higher equations; but they may be all easily resolved by the following rule of Double Position.

For, draw the tangent AH parallel to DC, making the triangle FHA similar to GCD.

Then $DC = 2AH$, because $DT = 2DA$; consequently $DG = 2FA = \frac{1}{2}P$.

Cor. 3. The tangent at the vertex AH is a mean proportional between AF and AD.

For, because FHT is a right angle,
therefore, AH is a mean between AF, AT,
or between AF, AD,
because $AD = AT$.
Likewise FH is a mean between FA, FT,
or between FA, FC.

Cor. 4. The tangent TC makes equal angles with FC and the axis FT, or with the line ICK drawn parallel to the axis.

For, because $FT = FC$,
therefore, the $\angle FCT = \angle FTC =$ its altern. $\angle ICT$.
Also, the angle $GCF =$ the angle GCK.

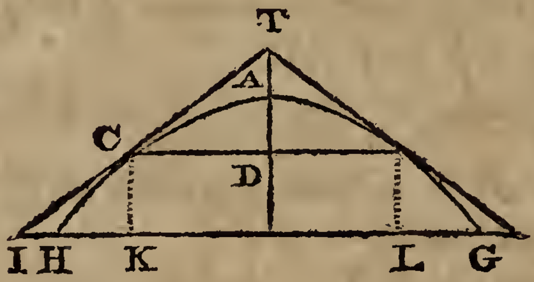
Cor. 5. And because the angle of incidence GCK is = the angle of reflection GCF; therefore a ray of light, falling on the curve in the direction KC, will be reflected to the focus F. That is, all rays, parallel to the axis, are reflected to the focus, or burning point.

PROPOSITION VII.

If an ordinate be drawn to the point of contact of any tangent, and another ordinate produced to cut the tangent; it will be

As the difference of the ordinates :
Is to the difference added to the external part ::
So is double the first ordinate :
To the sum of the ordinates.

That is, KH : KI :: KL : KG.



For, by Prop. I. Cor. 1, P : DC :: DC : DA,
and P : 2DC :: DC : DT or 2DA.
But, by sim. tri. KI : KC :: DC : DT ;
therefore, by equality, P : 2DC :: KI : KC,
or P : KI :: KL : KC.

Again, by Prop. II. P : KH :: KG : KC ;
therefore, by equality, KH : KI :: KL : KG.

Q. E. D.

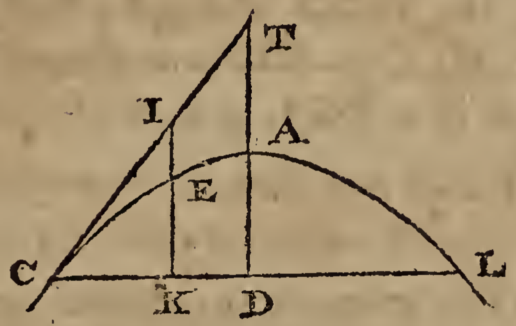
Cor. 1. Hence, by composition and division,
we have KH : KI :: GK : GI,
and HI : HK :: HK : KL,
also IH : IK :: IK : IG ; that is,
IK is a mean proportional between IG and IH.

Cor. 2. And from this last property we can easily draw a tangent to the curve from any given point I. For, draw IHG perpendicular to the axis, and take IK a mean proportional between IH, IG ; then draw KC parallel to the axis, and C will be the point of contact, through which and the given point I the tangent IC is to be drawn.

PROPOSITION VIII.

If there be any tangent, and a double ordinate drawn from the point of contact, and also any line parallel to the axis, limited by the tangent and double ordinate, then shall the curve divide that line in the same ratio, as the line divides the double ordinate.

That is, IE : EK :: CK : KL.



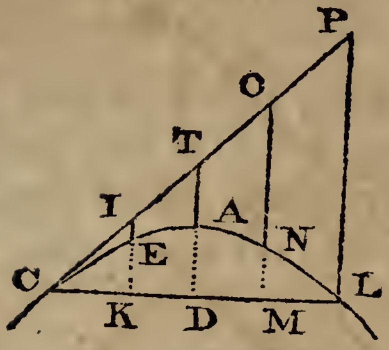
For, by sim. tri. CK : KI :: CD : DT or 2DA ;
but, by the def. of the param. P : CL :: CD : 2DA ;
therefore, by equality, P : CK :: CL : KI.
But, by Prop. II. P : CK :: KL : KE ;
therefore, by equality, CL : KL :: KI : KE ;
and, by division, CK : KL :: IE : EK.

Q. E. D.

PROPOSITION IX.

The same being supposed as in Prop. VIII. then shall the external part of the line between the curve and tangent be proportional to the square of the intercepted part of the tangent, or to the square of the intercepted part of the double ordinate.

That is, IE is as CI^2 or as CK^2 ;
and IE, TA, ON, PL, &c.
are as CI^2, CT^2, CO^2, CP^2, &c.
or as CK^2, CD^2, CM^2, CL^2, &c.



For, by Prop. VIII. IE : EK :: CK : KL,
or, by mult. IE : EK :: CK^2 : CK · KL.

But, by Prop. II. Cor. EK is as the rectangle CK · KL, and therefore, IE is as CK^2, or as CI^2.

Q. E. D.

Cor. 1. As this property is common to every position of the tangent, if the lines IE, TA, ON, &c. be appended to the points I, T, O, &c. and moveable about them, and of such lengths, that their extremities E, A, N, &c. be in the curve of a parabola, in some one position of the tangent; then making the tangent revolve about the point C, the extremities E, A, N, &c. will always form the curve of some parabola, in every position of the tangent.

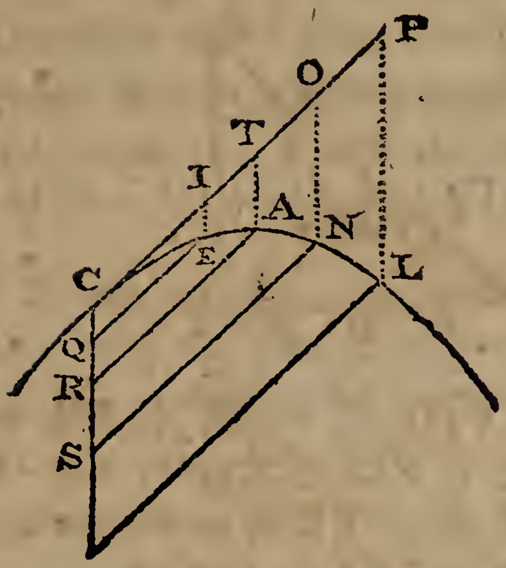
Cor. 2. The parameter of the axis is also a third proportional to IE and CK.

For, by this Prop. EK : KL :: IE : CK;
and, by Prop. II. EK : KL :: CK : P;
therefore, by equality, IE : CK :: CK : P.

PROPOSITION X.

The abscisses of any diameter are as the squares of their ordinates.

That is, CQ, CR, CS, &c.
are as QE^2, RA^2, SN^2, &c.
Or, CQ : CR :: QE^2 : RA^2, &c.



For, draw the tangent CT, and the externals EI, AT, NO, &c. parallel to the axis, or to the diameter CS.

Then, because the ordinates QE, RA, SN, &c. are parallel to the tangent CT, by the definition of them, therefore all the figures IQ, TR, OS, &c. are parallelograms, whose opposite sides are equal,

namely, IE, TA, ON, &c.

are equal to CQ, CR, CS, &c.

Therefore, by Prop. IX. CO, CR, CS, &c.

are as CI^2, CT^2, CO^2, &c.

or as their equals, QE^2, RA^2, SN^2. &c.

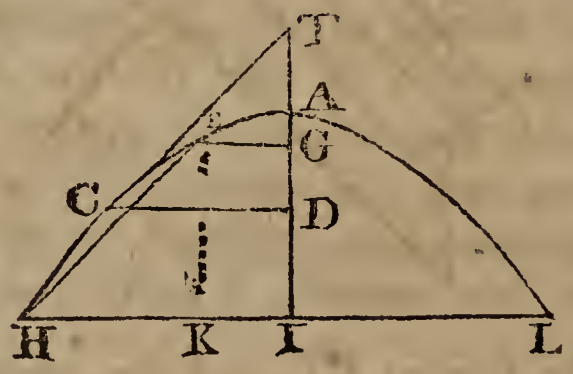
Q. E. D.

Cor. Here, as in Prop. II. the difference of the abscisses is as the difference of the squares of their ordinates, or as the rectangles under the sum and difference of the ordinates, the rectangles under the sum and difference of the ordinates being equal to the rectangle under the difference of the abscisses and the parameter of that diameter, or a third proportional to any absciss and its ordinate.

PROPOSITION XI.

If a line be drawn parallel to any tangent, and cut the curve in two points, then if two ordinates be drawn to the intersections, and a third to the point of contact, these three ordinates will be in arithmetical progression, or the sum of the extremes will be equal to double the mean.

That is, $EG + HI = 2CD$.



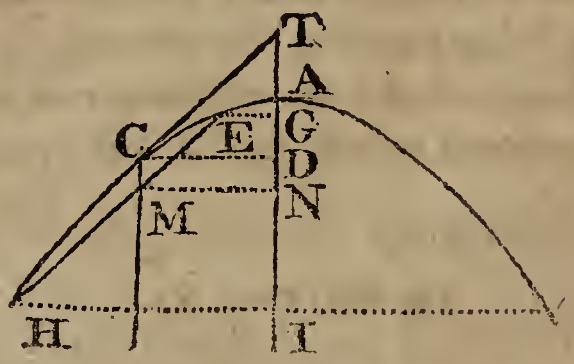
For, draw EK parallel to the axis, and produce HI to L. Then, by sim. tri. $EK : HK :: TD$ or $2AD : CD$; but, by Prop. II. $EK : HK :: KL : P$ the param. therefore, by equality, $2AD : KL :: CD : P$. But, by Prop. I. Cor. 1, $2AD : 2CD :: CD : P$; therefore, the second terms are equal, $KL = 2CD$, that is, $EG + HI = 2CD$.

Q. E. D.

PROPOSITION XII.

Any diameter bisects all its double ordinates, or lines parallel to the tangent at its vertex.

That is, $ME = MH$.



For, to the axis AI draw the ordinates EG, CD, HI, and MN parallel to them, which is equal to CD.

Then, by Prop. XI. $2MN$, or $2CD = EG + HI$,

therefore M is the middle of EH.

And for the same reason all its parallels are bisected.

Q. E. D

Scholium. Hence, as the abscisses of any diameter and their ordinates have the same relations as those of the axis, namely, that the ordinates are bisected by the diameter, and their squares proportional to the abscisses ; so all the other properties of the axis and its ordinates and abscisses, before demonstrated, will likewise hold good for any diameter and its ordinates and abscisses. And also those of the parameters, understanding the parameter of any diameter as a third proportional to any absciss and its ordinate.

DIALING.

DEFINITIONS.

1. A DIAL is an instrument, that shows time by means of the rays of light, proceeding from some celestial body. When it is constructed to show time by means of the sun's rays, it is called a *sun dial;* and when by means of the moon's rays, a *moon dial.* The former is principally used; and is therefore to be considered as intended by the term *dial*, when used for a particular instrument without a distinguishing epithet.

2. DIALING, or GNOMONICS, is that branch of Mathematics, which teaches the construction and use of dials.

3. *Dial surface* is a surface, on which the hour lines of a dial are drawn. This surface may be plane, and is then denominated a *dial plane*; or it may be convex, concave, cylindrical, &c. The most common and useful kinds of dials are those, which have the hour lines described on a plane surface.

4. Dial planes are parallel to, or coincident with, great circles of the sphere; and some dials receive their distinguishing denominations from those circles respectively.

5. When the plane of the dial is parallel to the horizon, it is called a *horizontal dial.*

6. An *equinoctial dial* is that, which has its plane parallel to the equator.

7. A *polar dial* has its plane parallel to that of the meridian, which is perpendicular to the meridian of the place.

8. *Erect*, or *vertical*, dials have their planes perpendicular to the horizon, or coincident with those of vertical circles.

9. A *direct dial* has its plane vertical, and facing one of the four cardinal points, south, north, east, or west. And hence it is common to denominate them respectively *south*, *north*, *east*, *west*, *dials*. The planes of the two former are coincident with, or parallel to, those of the prime vertical; and the planes of the two latter are coincident with, or parallel to, those of the meridian.

Note. The polar, east, and west, dials may be called *meridian dials*.

10. A *declining dial* has its plane vertical, and making an oblique angle with the meridian. They are *south-east*, *south-west*, *north-east*, and *north-west*, *declining dials*.

11. The plane of an *inclining dial* makes an acute angle with the horizon on the side of its face.

12. The plane of a *reclining dial* makes an obtuse angle with the horizon on the side of its face.

13. *Declination* of a plane is the arc of the horizon, contained between the plane and the prime vertical; or, which is the same, the arc of the horizon, contained between the meridian and a plane perpendicular to that plane. The latter is used, and is always reckoned from the south or north toward the east or west.

14. *Reclination*, or *proclination*, of a plane is the angle it makes with a vertical plane, or its angular distance from the zenith; the reclination being on the side of the obtuse angle, and the proclination on the opposite.

15. But the *inclination* of a plane is properly the angle it makes with the horizon.

16. The *centre* of a dial is the point, in which all the hour lines meet, or toward which they tend.

17. The *stile*, or *gnomon*, of a dial is a pin, rod, or plate,

raised perpendicularly on the plane of the dial; by the shadow of which, as an index, the time is shown among the hour lines.

18. The *substile* is the line, on which the stile is erected; it always passes through the centre of the dial.

19. When the stile is in the form of a triangle, the *height of the stile* is the angle, which the upper edge of the stile makes with the substile, the angular point being at the centre of the dial. But if the stile be a perpendicular pin, or rod, the *height* of it is the length; and if a parallelogram, the *height* is that of the upper edge.

20. The intersection of the plane of the dial with that of the meridian of the place, passing through the stile, is called the *meridian of the dial*, or the XII *o'clock hour line.*

21. The meridians, whose planes pass through the stile and make angles at 15°, 30°, 45°, 60°, 75°, and 90°, with the meridian of the place, are called *hour circles;* and their intersections with the plane of the dial, *hour lines.*

22. The *substile's distance from the meridian* is the angle, which the subtile makes with the XII o'clock line.

23. *Hour angle* is the angle, which any hour line on the plane of the dial makes with the substile.

24. The *horizontal line* is a line, drawn parallel to the horizon on the plane of any dial; and is the intersection of the dial plane with the horizontal plane, passing through that point of the stile, whose shadow shows the time.

25. The *equinoctial line* is the intersection of the plane of the equinoctial and the dial plane.

26. The *contingent line* is a line, drawn through the foot of the stile perpendicular to the substile; and serves instead of the equinoctial for finding the hour points upon it, through which the hour lines are to be drawn. And when the contingent does not pass through the foot of the stile, it represents the equinoctial.

27. *Plane's difference of longitude* is the angle, contained between the meridian of the place and the meridian of the plane, that is, the meridian perpendicular to the plane, or the substile. This is also called *inclination of meridians*.

The following *Observations* will appear evident on a little reflection.

1. As the earth performs a rotation on its axis from west to east in 24 hours, the sun appears in that time to perform a revolution round the earth from east to west. And hence the shadow of an opaque body, exposed to the sun, being constantly on the opposite side, moves with a corresponding velocity in the opposite direction.

2. If it be imagined, that the plane of a great circle of the earth parallel to, or coincident with, the plane of any dial is visible and distinguishable ; that the axis is a small opaque rod ; and that the rest of the earth is perfectly transparent : then, were the place of the shadow of the axis observed and marked on that plane every hour of a day, the hour lines would be obtained on it, and the time would be afterward shown by the situation of the shadow.

3. If a dial plane, placed as a tangent on the surface of the earth, be parallel to the plane of a great circle, and have for a stile a small rod parallel to the axis, and hour lines corresponding to those of the great circle ; then the time, supposed to be shown, would be nearly the same on both. For a semidiameter of the earth is so small in respect to the distance of the sun, as not to occasion any observable difference in this case.

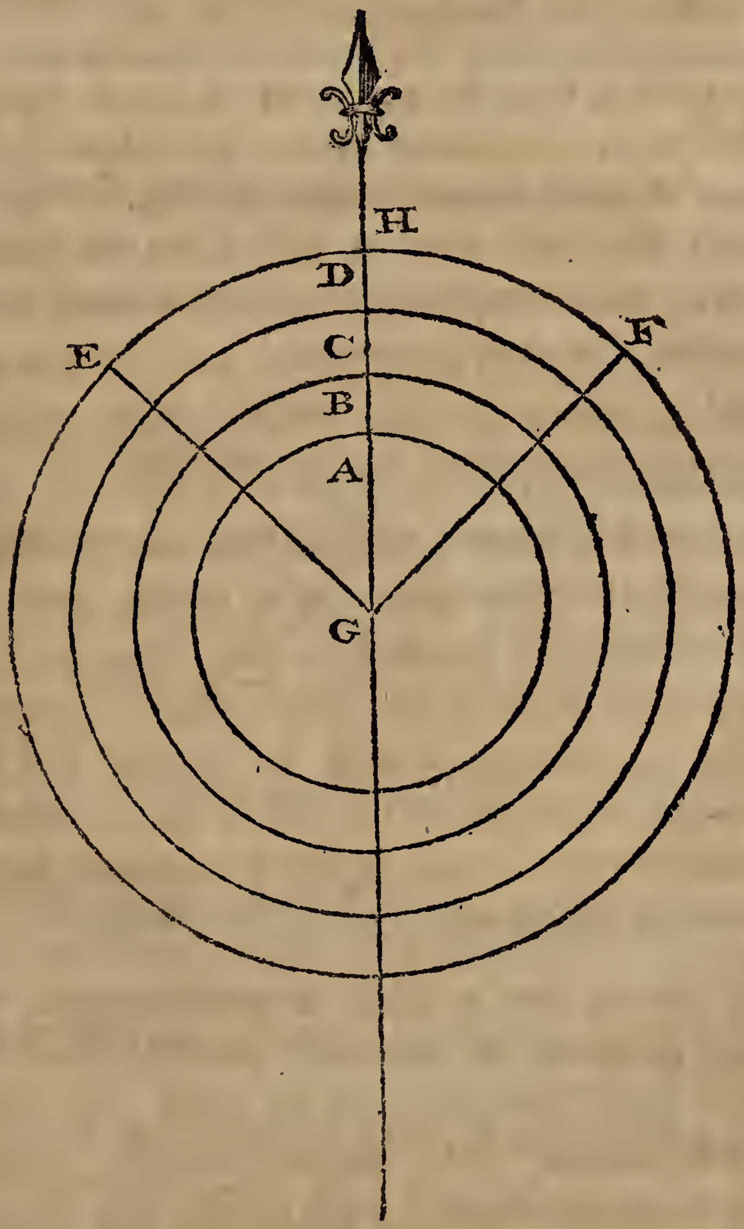
4. The time, shown by a dial, is what elapses while the sun's rays are incident on its face ; and the hour lines, &c. on every dial must be such, as to mark the corresponding parts of this time. And the order of the numbers of the hour lines proceeds from the substile in the direction of the motion of the stile's shadow.

PROBLEMS.

PROBLEM I.

To draw a meridian line.

Let a time piece be well regulated, so as to show the time, when the sun is on the meridian. Then at the instant, when the sun is on the meridian, let a line be drawn on a plane horizontal surface, in the direction of the shadow of a suspended plumb line ; and it will be a meridian line.



Or,

1. Find an even horizontal surface, or make the smooth surface of a board, about a foot square, accurately level; and describe on the surface four or five concentric circles A, B, C, D, at the distance of about $\frac{1}{4}$ inch from one another, and make the diameter of the outmost nearly a foot.

2. In the centre G fix perpendicularly a pin or rod of about $\frac{1}{8}$ inch in diameter, and of such length, that its whole shadow may fall on the inmost circle four hours or longer; its termination being made round and smooth.

3. Mark the points on the circumference of a circle, where the end of the shadow crosses it both before and after noon; as E and F.

4. Bisect the arc, included between the points E, F; and the bisecting line GH, drawn from the centre, is a meridian line.

NOTE 1. As many meridian lines being found, in a similar manner, as there are circles, if they do not coincide, the meridian line, drawn in a mean direction among them, may be considered as the most accurate.

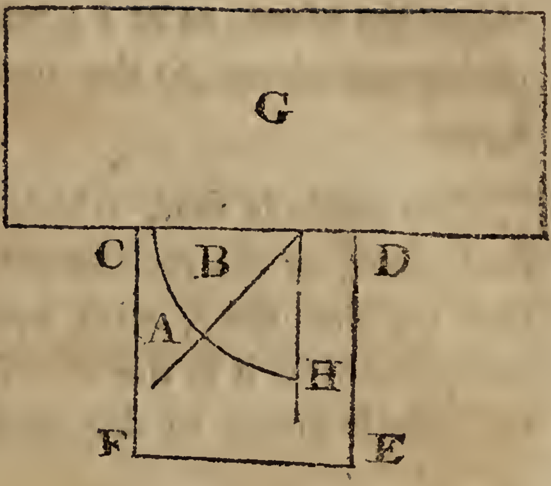
NOTE. 2. If the latter method be applied, the result will be most satisfactory about the time of a solstice, when the sun's change of declination, during the interval between the observations of the forenoon and afternoon, may not produce a sensible error in the result.

PROBLEM II.

To find the declination of a plane.

1. Draw a meridian line AB on the horizontal surface of a board CDEF, having its edge applied to a horizontal line CD on the given plane G, and continue it to the plane.

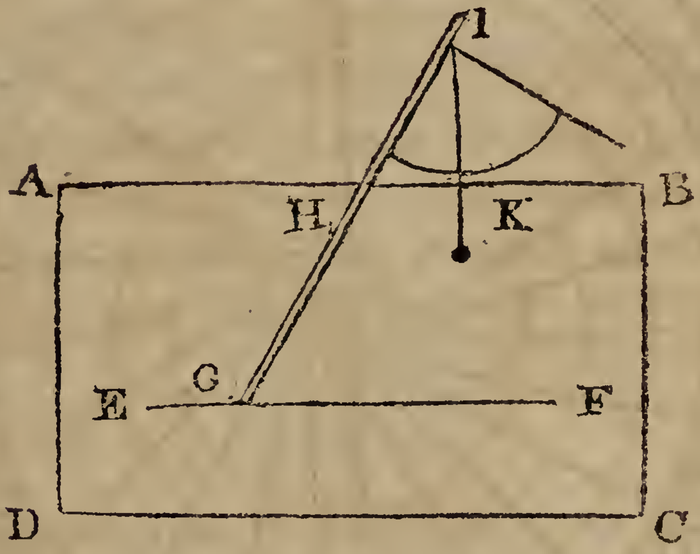
2. Measure the angle ABH, formed by the meridian line and a line BH, drawn on the board perpendicular to the plane from the point of it, where the meridian line meets it; and the declination, being this angle, is obtained.



Note. The angle may be measured with compasses and a chord line; or with a quadrant, applied to the board with its centre at the angular point, and one radius close to that part of the plane DC, which makes an acute angle with the meridian line.

PROBLEM III.

To find the inclination or reclination of a plane.



1. On the given plane ABCD find a horizontal line EF with a quadrant, square and plumb line, or level.

2. Draw GH on the plane perpendicular to EF.

3. Measure the angle GIK, formed by GH and a plumb line IK, suspended from any convenient point of GH, or of a straight rule applied to it, if necessary, and continued

above the plane, as at I ; which may be easily done by applying aquadrant to the under side, as represented in the figure.

This angle is that, which the plane makes with a vertical plane, or the reclination on one side and the proclination on the other, with respect to the vertical ; and the compliment of it is the acute angle, which the plane makes with the horizon, or the proper *inclination.*

PROBLEM IV.

To draw geometrically an equinoctial dial, or a horizontal dial at a pole.

FIGURE 1.

1. Describe the small circle *a* of the same diameter as the stile, for the foot of the stile ; and the large circle A on the same centre, and with any convenient radius.

2. Draw CO for the meridian ; and XII b parallel to CO, and touching the circle a in the point b, for the XII o'clock line.

3. Divide the circumference of the large circle A into 24 equal parts, beginning at XII ; and number them I, II, III, &c. toward the east, (that is, the left in figure 1) omitting the hours not shown on the dial at the place, for which it is intended ; and draw the hour lines from those points to the small circle a, so that they may touch it in points following each other in the same direction from b.*

---

* If it be imagined, that the eye is at the centre of the earth, and can thence view the hour circles, as they appear on a dial plane, which touches its surface ; then these representations, or projections, of the hour circles will be the hour lines of a dial, described on that plane. The drawing is therefore a kind of perspective, the centre of the earth being the point of sight, and the dial plane the picture. A line being drawn from the top of the stile perpendicularly to the dial plane, its length represents the radius of the earth, its top the centre, its foot the point of the surface, where the dial plane touches it ; the centre of the dial the projected pole ; and a line drawn from the top of the perpendicular to the centre of the dial the earth's axis, some part or the whole of which represents what constitutes the stile, or the essensial part of it. From these principles the reason of the rules for the geometrical construction of dials may be apprehended, and their truth demonstrated.

In this dial, which may be considered as the most simple, it is evident, that as the dial plane touches the earth at a pole, the projected pole is at the foot of the perpendicular ; and as the hour circles all meet in the pole at equal angles, the hour lines must be drawn from the centre at equal angles, if the stile be supposed to be only a line ; and as the axis is perpendicular to the dial plane, the stile must also be perpendicular, and erected at the centre.

NOTE 1. The half hours, quarters, &c. on any dial are found by dividing the hour arcs into 2, 4, &c. equal parts, and drawing lines in like manner as for the hours.

NOTE 2. In any dial, two other concentric circles B, D, may be described with convenient radii, or, when the form is not circular, two other lines may be drawn on each side at convenient distances, for containing the subdivisions and numbers.

NOTE 3. The stile of this dial is a cylindrical pin of any convenient length, fixed on the small circle a, perpendicular to the dial plane.

NOTE 4. When this dial is used, it must have its plane set parallel to the equator, with CO in the plane of the meridian, and of course making an angle with the horizon equal to the elevation of the equator, or complement of the latitude; and a line, drawn through C perpendicular to CO, or the VI o'clock line, horizontal.

FIGURE 2.

NOTE 5. If the stile be considered as a mere line, its foot will be the centre C, from which the hour lines are to be drawn.

NOTE 6. At a pole one dial of this kind will show all the hours of the day ; but in any other place the two, that are necessary for the two poles, must be used in connexion to show time through the year ; the northern answering this purpose when the sun is north of the equator, and the southern when it is south of it. In northern latitudes the former is called the *superior equinoctial dial*, and the latter the *inferior equinoctial dial ;* and the contrary in southern latitudes.

These two dials may be described on opposite faces of the same plane, and have for stiles the same pin passing through it. Figure 1 is the superior, and figure 2 the inferior equinoctial for our latitude.

PROBLEM V.

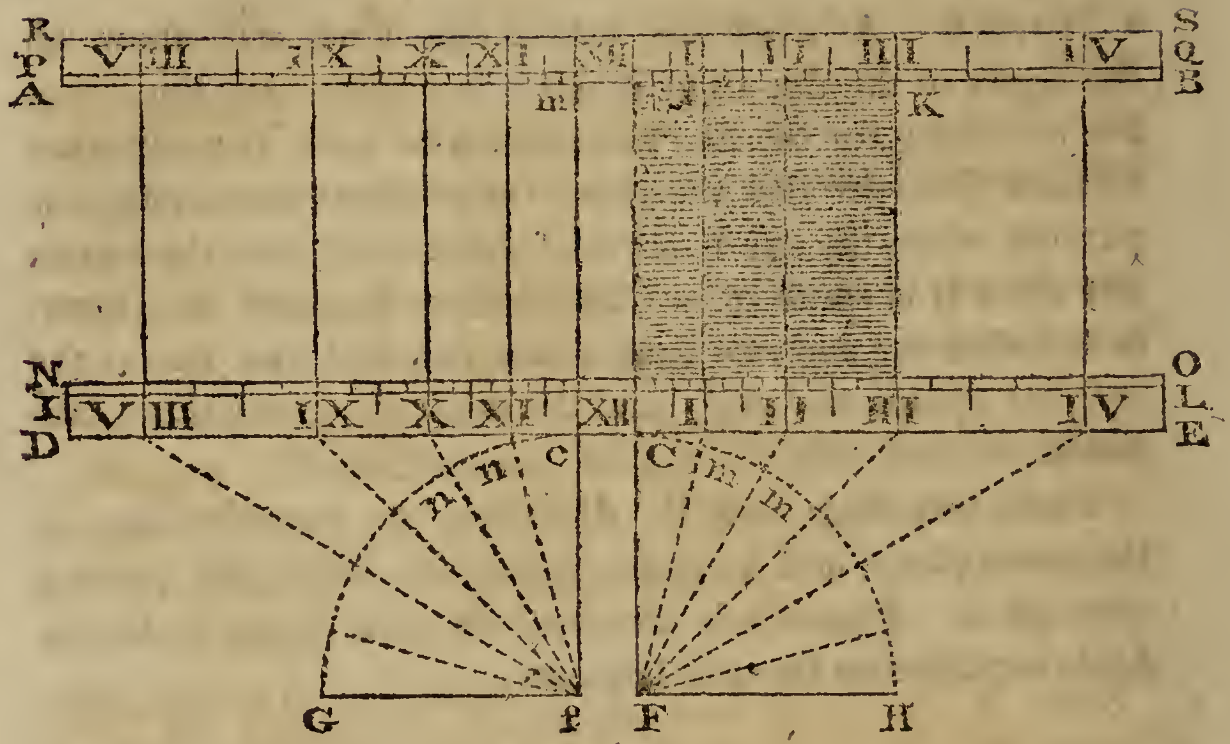
To draw geometrically a polar dial, or a horizontal dial at the equator.

1. Draw two parallel lines AB, DE, at a convenient distance for inscribing the hour lines.

2. Draw MCF, mcf, perpendicularly from AB across DE, making CF, cf, each equal to the height of the stile, and Cc equal to its thickness, for the XII o'clock line.

3. On the centres F, f, and with any convenient radius, as FC, describe the quadrants CH, cG. Divide each quadrant into 6 equal parts, at m, m ; n, n, &c. and draw lines from the respective centres F, f, through all the points of division till they intersect the line DE in the points I, II, &c. numbered toward the east from the XII o'clock line. Then through the points I, II, &c. draw lines from DE to AB parallel to MC ; and they will be the hour lines.

FIGURE 1.

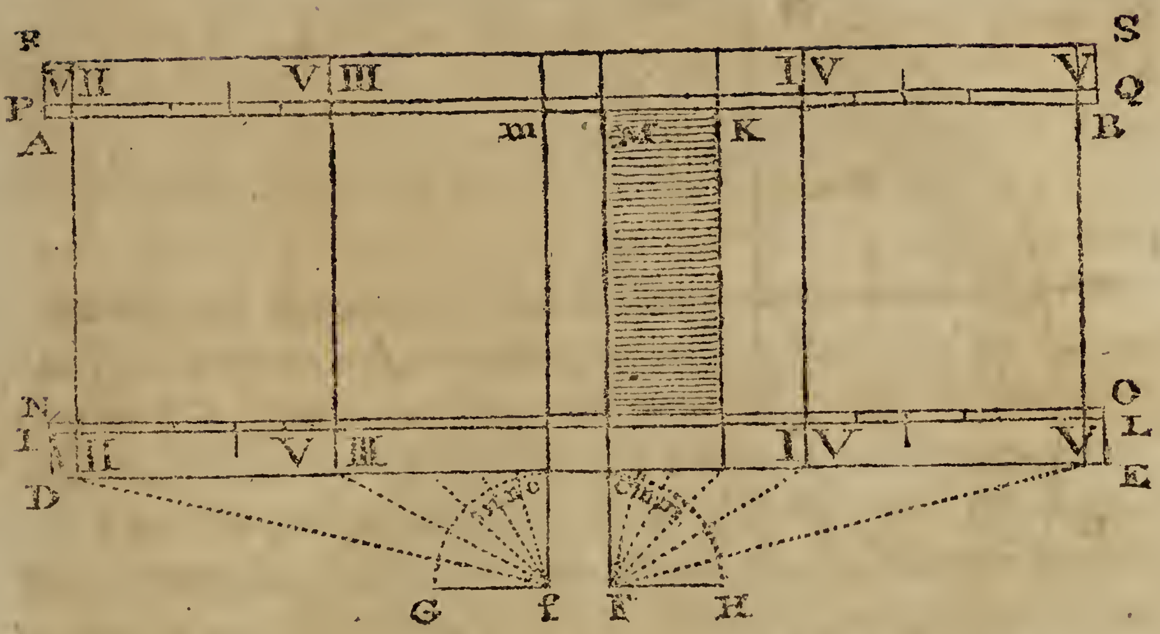


Note 1. The stile of this dial may be a plate in form of a parallelogram, of the thickness Cc, length MC, and height MK = CF; and it must be set on MCcm, perpendicular to the plane of the dial; and the shadow of the top points to the time.

Note 2. When this dial is used, its plane must be set parallel to that meridian, which is perpendicular to the meridian of the place; that is, with the XII o'clock line parallel to the axis of the earth, and a line perpendicular to it, as AB, horizontal.

Note 3. If the stile be considered as a mere geometrical plane, the line MC may be taken for its base, and a semicircle must be described on F, and divided into 12 parts; and the rest of the process will be the same as before. A similar observation is applicable to other dials.

FIGURE 2.



NOTE 4. At the equator one dial of this kind is sufficient to show the time ; but at any other place two, described on the opposite faces of the plane, are necessary ; one, called the *superior polar dial*, for the time between 6 A. M. and 6 P. M. and the other, called the *inferior polar dial*, for the rest of the day. Figure 1 is a superior, and figure 2 an inferior polar dial for our latitude, only some divisions on the dial plane are omitted for want of room.

PROBLEM VI.

To draw geometrically a horizontal dial for any latitude.

1. Describe a circle M, with a convenient radius, and assume a convenient point A out of the centre, for the centre of the dial, and through A draw the meridian, or XII o'clock line AB, and at the distance Aa, equal to the thickness of the stile, draw ab parallel to AB.

2. Make the angle BAC equal to the latitude, suppose 42° 23′ 28″. Assume a point D in the line CA, and draw DE perpendicular to AB, and it will represent a rod used for a perpendicular stile on the point E. Draw DF perpendicular to AC; and through F draw the contingent line GH perpendicular to AB, which here represents the equinoctial.

3. Make FI and fi each equal to DF, and with centres I, i, and any radius, as FI, describe the quadrants FK, fk, and divide each into 6 equal parts. Through the points of division draw lines from the centres I, i, to intersect GH in the hour points m, m; n, n, &c. Then, from the centres A, a, draw lines through the points m, m; n, n, &c

as Am, Am ; and an, an, &c. and they will be the hour lines, to be numbered I, II, III, &c. from XII toward the east. The VI o'clock hour line must be drawn parallel to GH, or perpendicular to the meridian ; and the hours before and after VI are found by producing the opposite hour lines through the centre ; which may be done also on other dials, that show more than 12 hours of the day.

Note. 1. In order to find an hour line, when the line from the centre of the dividing circle does not cut the contingent line within the plane of the dial ; draw a line gd near the boundary parallel to the substile, intersecting the line from the centre of the dividing circle in d. Then, by Prob. XVI. of Geometry, find a fourth proportional to if, fa, and dc ; and set it from c to h. A line, drawn from the centre of the dial through the point h, as aV in this example, will be the hour line required.

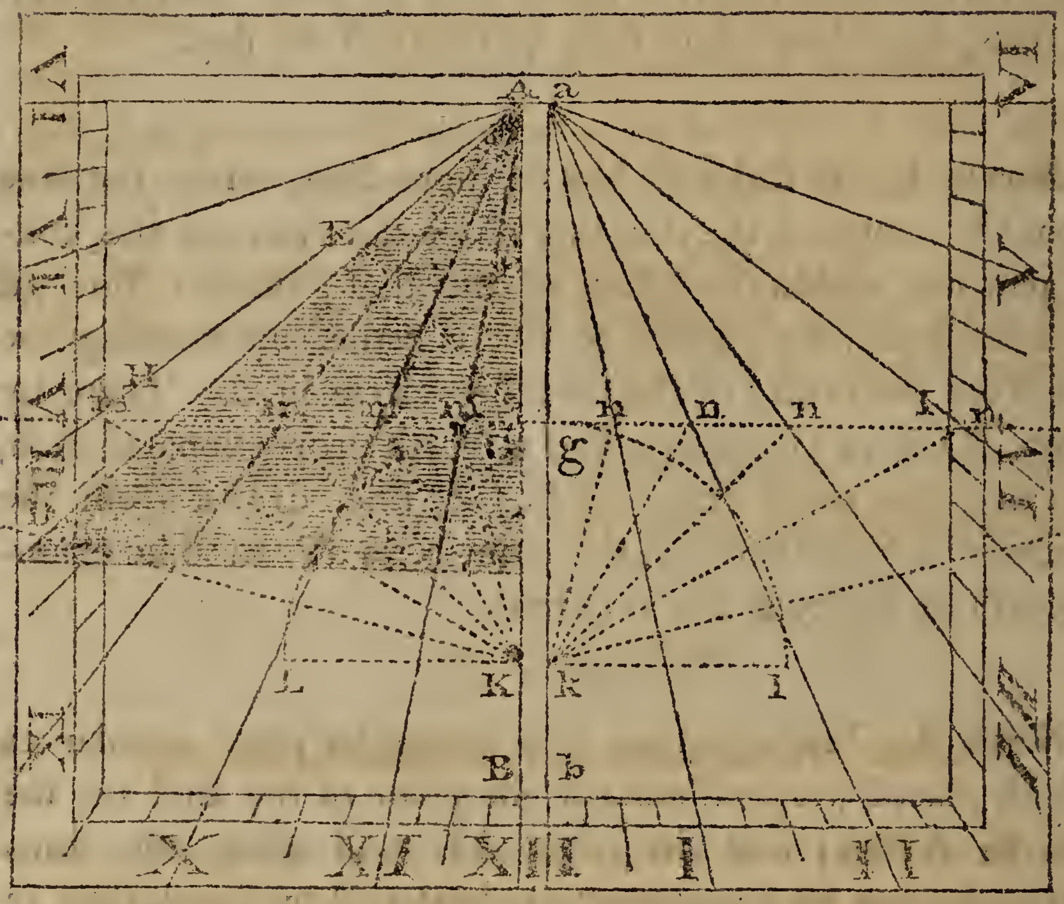
Note 2. The stile may be a triangular plate similar to AED, placed perpendicular to the plane of the dial on the substile ABba ; and the edge AD will show the time. This stile may be lengthened or shortened by producing or curtailing the side AD. Or the gnomon may be a perpendicular pin on E, and in height equal to ED ; and then the shadow of the top will point out the time.

Note 3. When this dial is used, its plane must be horizontal, and the XII o'clock line on a meridian line, with the end B toward the north.

Note 4. This dial shows all the hours of the artificial day at any season of the year.

PROBLEM VII.

To draw geometrically an erect direct south dial.



1. Take a convenient point A, and through A draw the meridian, or XII o'clock line, AB; and at the distance Aa, equal to the thickness of the stile, draw ab parallel to AB.

2. Make the angle BAD equal to the compliment of the latitude of the place. Assume a point E in the line AD, and draw EF perpendicular to AB. Draw EG perpendicular to AD, and through G draw the contingent line HI perpendicular to AB, representing the equinoctial.

3. Make GK, gk, each equal to GE, and with any radius, as GK, and K, k, as centres, describe the quadrants GL, gl, and divide each into six equal parts. Through the

points of division draw lines from the centres K, k, cutting HI in the hour points m, m ; n, n, &c. Then lines, drawn from the centres A, a, through the points m, m ; n, n, &c. will be the hour lines ; and are to be numbered I, II, III, &c. from XII toward the east.

Note 1. The stile may be a triangular plate similar to AEF, placed perpendicular to the plane of the dial on AGga. Or it may be a perpendicular pin erected at F, and equal in height to FE.

Note 2. This dial is set up with the face toward the south ; IH being horizontal from east to west, and BA vertical, with the end A upward.

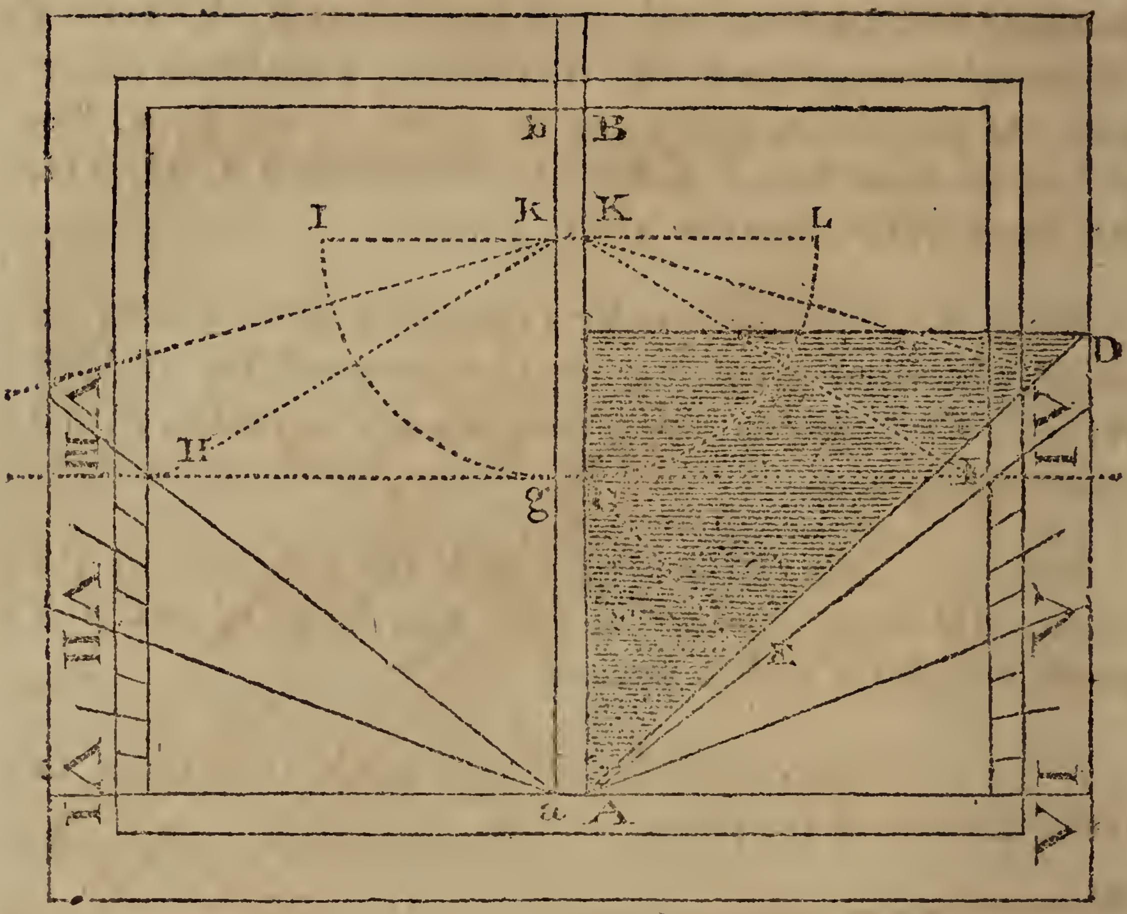
Note 3. This dial shows the time while the sun is on the south side of the prime vertical.

PROBLEM VIII.

To draw geometrically an erect direct north dial.

1. Proceed as in drawing a south dial, except with respect to the hour lines, which must be those, that may be shown while the sun is on the north side of the prime vertical, and the numbering of them from the substile must be in the contrary direction, or toward the west.

2. Then turn it upside down, setting the face directly northward, or making the stile point to the north pole ; and it will be the dial required.



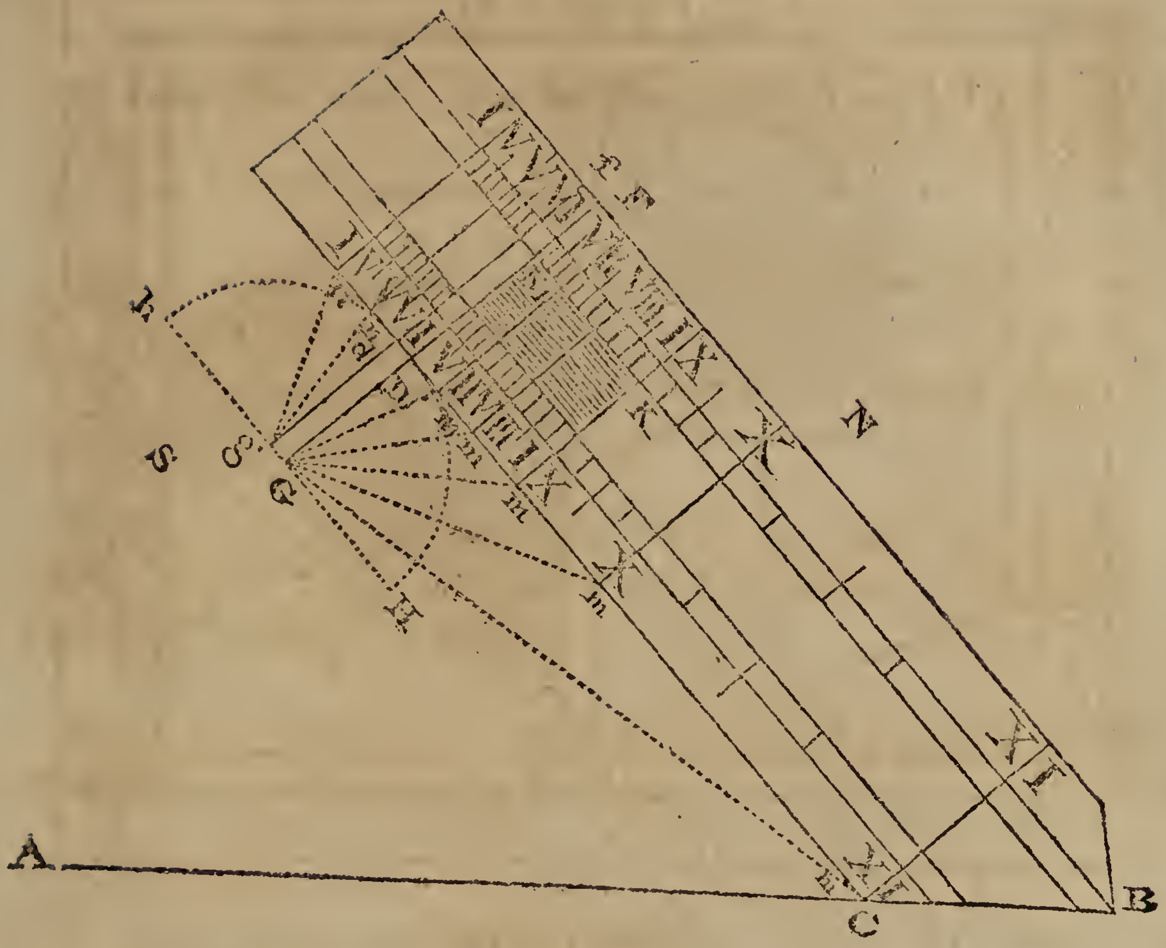
PROBLEM IX.

To draw geometrically an erect direct east dial.

1. Draw the horizontal line AB ; and assume a point C on the right, through which draw the contingent line CD, making the angle ACD equal to the elevation of the equator, or complement of the latitude.

2. Through any point D of the line DC draw FDG perpendicular to DC, and through d at the distance Dd, equal to the thickness of the stile, draw fdg parallel to FDG, for the VI o'clock hour line and the substile.

3. On FG, fg, take DG, dg, each equal to the height of the stile ; and with the centres G, g, and any radius, as DG, describe the quadrants DH, dh, and divide each into six equal parts. Through the points of division draw lines from the centres G, g, cutting CD in the hour points m, m ; n, n, &c. Then through the points m, m ; n, n, &c. draw lines parallel to DF ; numbering them VII, VIII, &c.



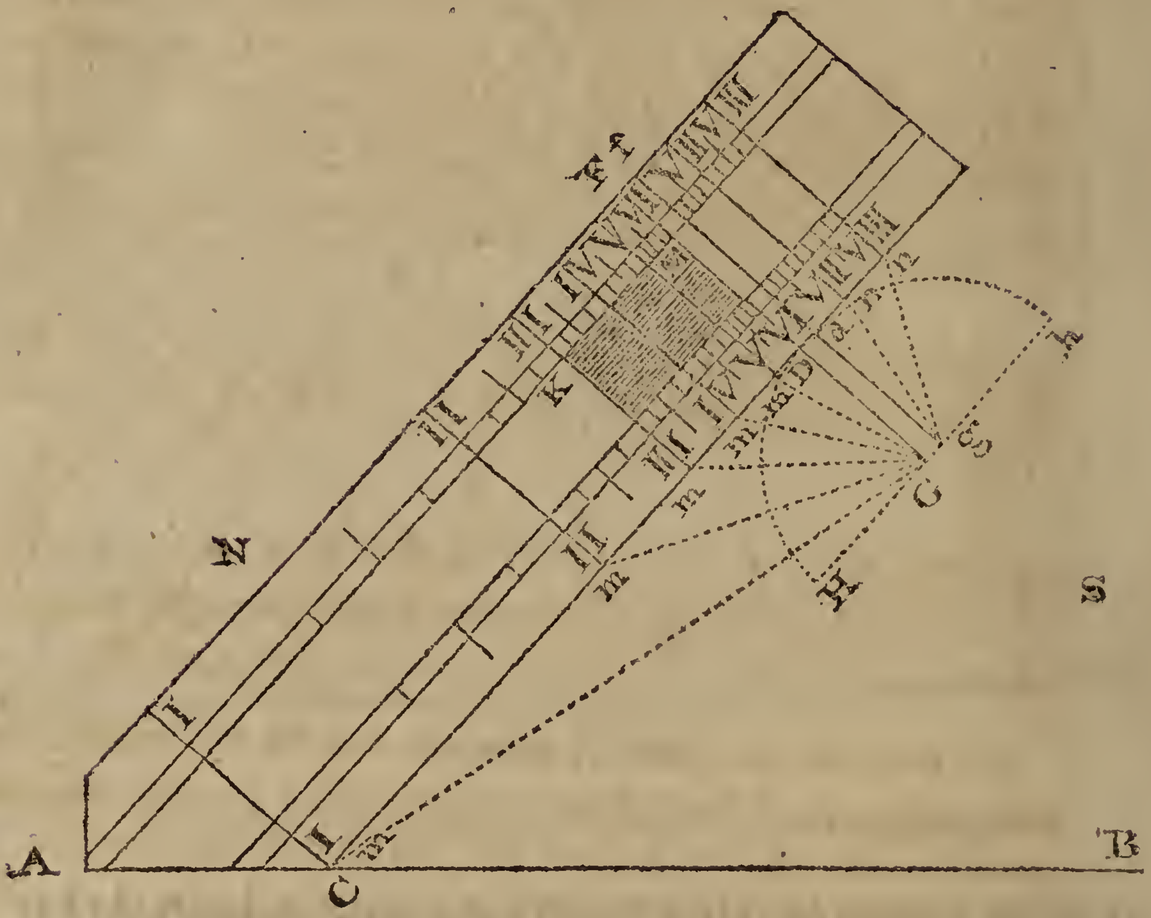
from the substile toward the north or right, the hours before VI being toward the south or left; and they will be the hour lines required.

NOTE 1. The stile may be a plate in form of a parallelogram, placed perpendicular to the dial plane on the substile, its thickness being equal to Dd, and its height equal to MK = DG.

NOTE 2. This dial shows the time of the forenoon.

PROBLEM X.

To draw geometrically an erect direct west dial.



Proceed as in drawing an east dial, except in assuming the point C and making the angle ACD, which must be on the left, and in numbering the hour lines, which are from the substile VII, VIII, &c. toward the south, the hours before VI being toward the north.

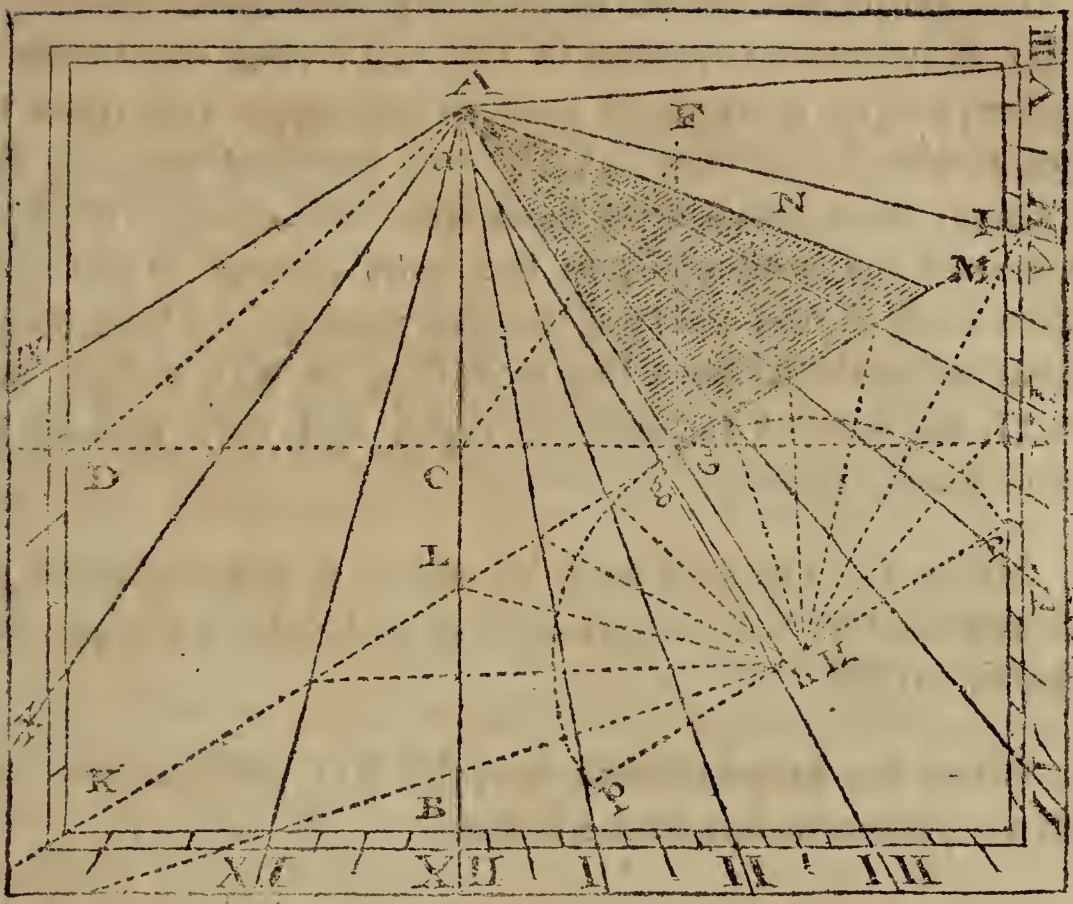
NOTE. This dial shows the time of the afternoon.

PROBLEM XI.

To draw geometrically a south erect declining dial.

EXAMPLE.

The declination being 36° west, and the latitude of the place 42° 23′ 28″ north ; required the construction of the dial.



1. Assume the point A near the top for the centre of the dial ; through A draw the meridian AB ; in AB take any point C, through which draw the horizontal line DCE.

2. Make the angle CAD equal to the complement of the latitude, cutting the horizontal line in D.

3. Make the angle ECF equal to the complement of the declination toward the right, because the declination is west ; it must be toward the left, when the declination is east.

4. Make CF equal to CD ; from F draw FG parallel to AC, to cut the horizontal line in G ; then FG is the height of the stile, and G the foot of it.

5. Through A and G draw AGH, and agh parallel to AGH at the distance Gg, equal to the thickness of the stile, for the substile ; and through G draw ILK perpendicular to AGH, cutting the meridian in L, for the contingent line.

6. In GI make GM equal to GF ; join A, M, for the stile ; from G draw GN perpendicular to AM, and make GH, gh, each equal to GN, and draw hL.

7. With centres H, h, and any convenient radius, as HG, describe the quadrants GP, gp ; take one sixth of either, and set it on them successively each way from the point, where hL cuts gp, for the dividing points ; then through these points draw lines from the centres H, h, to intersect the contingent in the hour points. Then draw lines from A, the centre of the dial, through the hour points, numbering them from the meridian, or XII o'clock line, I, II, &c. toward the east, or right, and they will be the hour lines required.

Note 1. The stile may be either a perpendicular pin represented by GF, or a triangular perpendicular plate similar to AGM.

Note 2. In a declining dial, the XII o'clock line must be set perpendicular to the horizon.

Note 3. The construction, when the declination is the same number of degrees from the south toward the east, is the same, except in making the angle ECF toward the left.

Note 4. For a north erect declining dial draw a south erect dial, whose declination is the same, and on the same side of the meridian, marking only the hours, that will be useful. Then, as north dials are only south dials inverted, turn the upper side downward, number the hours in the opposite direction, and it becomes the dial required. Thus for a north-east make a south-east, and for a north-west make a south-west declining dial.

PROBLEM XII.

To draw geometrically a direct south reclining, or a direct north inclining dial.

1. If a south plane recline, and the reclination be less than the complement of the latitude, add the reclination to

the latitude ; the sum is the latitude, for which an erect south dial being drawn, it will be the dial required.

2. If the reclination be equal to the complement of the latitude, then a horizontal dial at the equator will be the dial required.

3. If the reclination be greater than the complement of the latitude, subtract the complement of the latitude from the reclination, and the remainder is the latitude, for which a horizontal dial being drawn, it will be the dial required.

4. These are all the varieties of direct south reclining dials. And for a direct north inclining dial, make a direct south one for the reclining side of the plane, number the hours the contrary way from the XII o'clock line, turn the upper side downward on the given side of the plane, and it will be the dial required.

PROBLEM XIII.

To draw geometrically a direct north reclining, or a direct south inclining dial.

1. If the reclination of the north plane be less than the latitude, add the reclination to the complement of the latitude ; and the sum is the latitude, for which a horizontal dial being drawn, and placed with the north side properly elevated, it will be the reclining dial required.

2. If the reclination be equal to the latitude, construct a horizontal dial for the elevated pole, or the superior equinoctial dial.

3. If the reclination be greater than the latitude, add the complement of the reclination to the latitude ; and the sum is the latitude, for which a horizontal dial being constructed, it will be adapted to the reclining plane.

4. A direct north reclining dial becomes a direct south inclining dial, adapted to the other side of the same plane, when the upper side is turned downward.

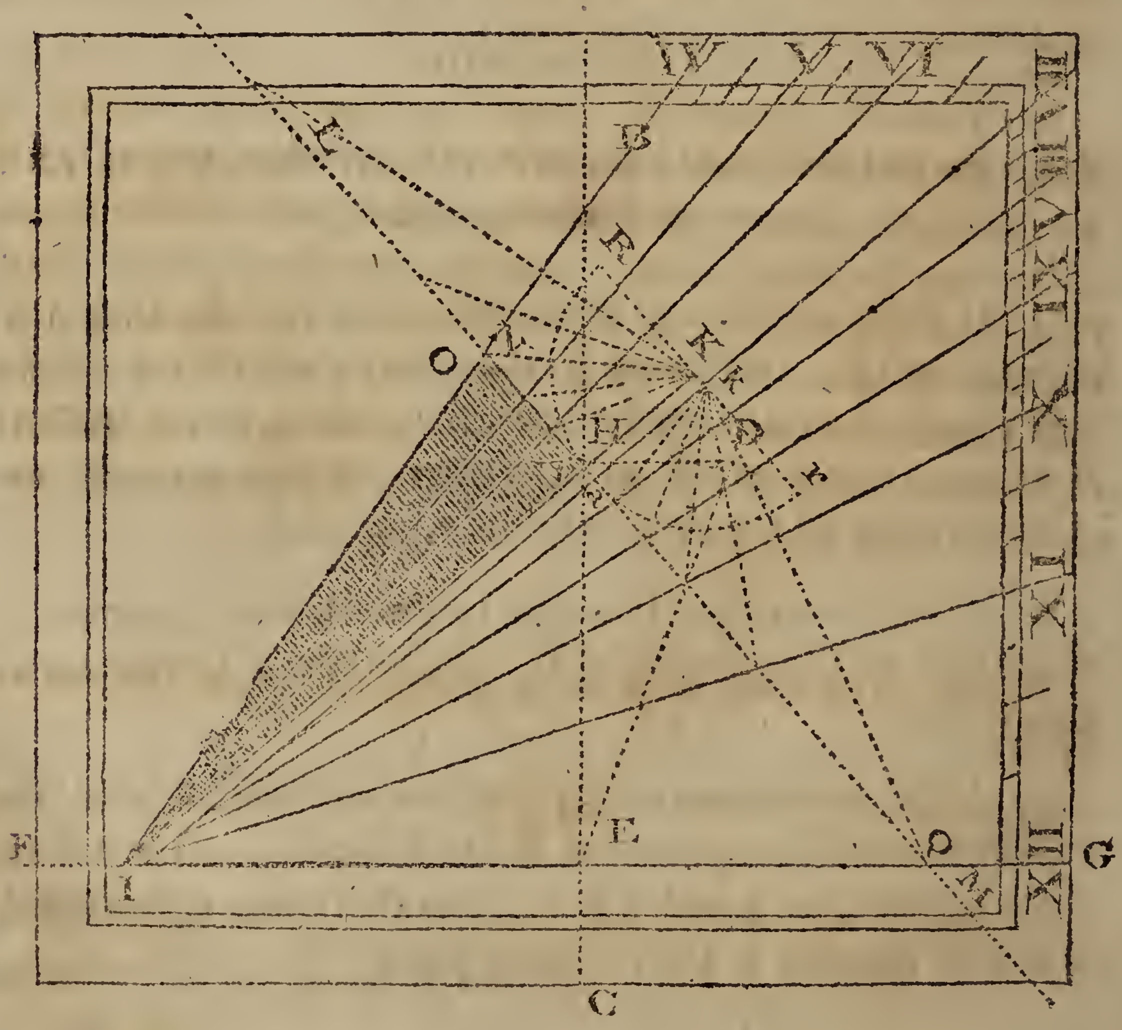
PROBLEM XIV.

To draw geometrically an inclining or reclining east dial.

EXAMPLE.

In latitude 42° 23′ 28″, required an east dial reclining 21°.

1. Assume a point A for the foot of the stile, through which draw the perpendicular line BC, and AD perpendicular to BC, or parallel to the horizon, and equal to the height of the stile.



2. Make the angle ADE equal to the complement of the reclination or proclination downward, if it recline; but upward, if it incline, to cut the vertical line BC in E. And through E draw the meridian, or XII o'clock line, FEG perpendicular to BC.

3. Make EH equal to ED, and the angle EHI equal to the complement of the latitude on the left, when the plane reclines; but on the right, when it inclines; and I is the centre of the dial, which point and the meridian IC will be at the top, when the plane inclines. Through I and A draw IAK, and iak parallel to IAK at the distance Aa, equal to the thickness of the stile, for the substile.

4. Through A draw LM perpendicular to the substile for the contingent line, on which make AN=AD, and draw IN for the stile.

5. Draw AO perpendicular to IN, and make AK, ak, each equal to AO. From k draw kQ to the intersection Q of the contingent and meridian.

6. With the centres K, k, and any convenient radius, as KA, describe the quadrants AR, ar; set one sixth of either successively on them each way from the point where kQ cuts ar, for the dividing points; and through these points draw lines from the centres K, k, to intersect the contingent in the hour points. Then lines, drawn from the centre of the dial I through these points, and numbered from IQ, the XII o'clock line, with XI, X, &c. upward on the north side, are the hour lines required.

NOTE. The stile may be a triangular plate similar to IAN.

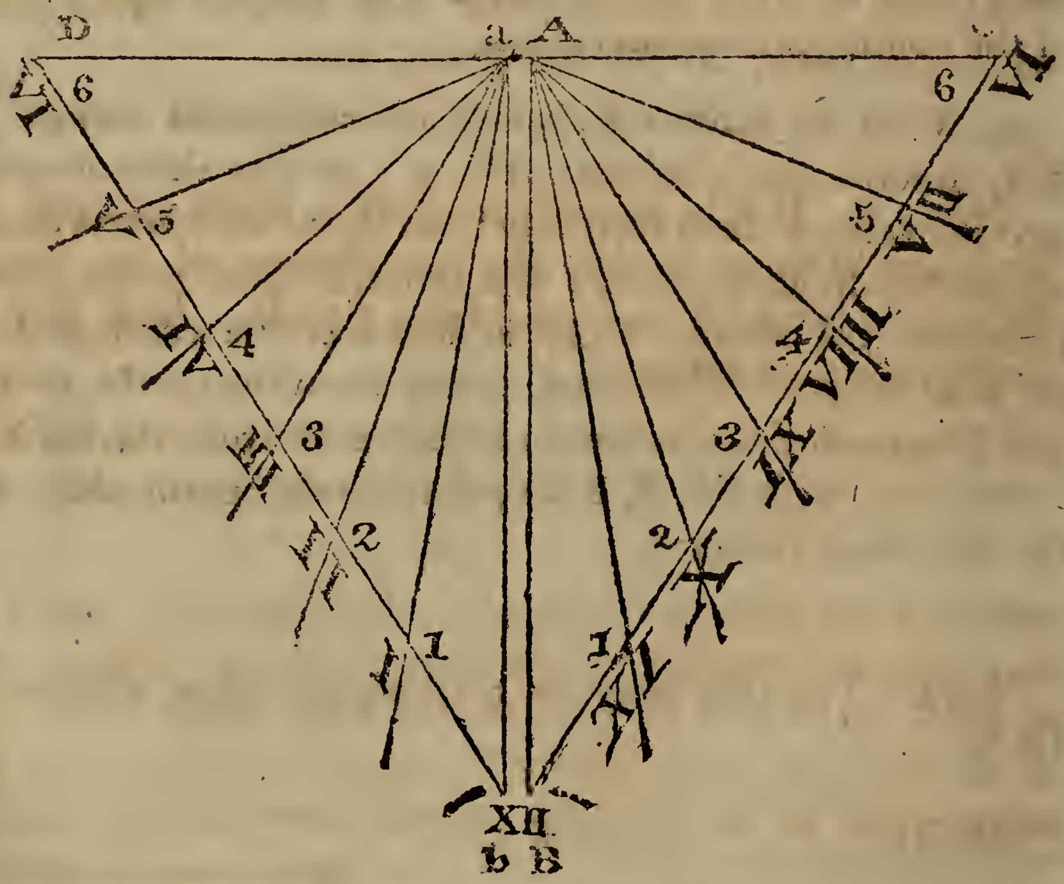
PROBLEM XV.

To draw geometrically an inclining or reclining west dial.

The construction of this dial differs from that of the last only in the following particulars. The angle EHI, which is made equal to the complement of the latitude, must be taken on the right, when the dial reclines ; but on the left, when it inclines. The meridian IG is at the bottom, if the plane recline ; but at the top, if it incline. And the hours must be numbered the contrary way.

PROBLEM XVI.

To draw a horizontal dial by means of the dialing scales ; or the hour line and line of latitudes on the Plane Scale.



1. Draw AB, and ab parallel to AB at the distance Aa, equal to the thickness of the stile, for the substile and XII o'clock line.

2. Through A draw CD perpendicular to AB, for the VI o'clock line.

3. Take the extent of the latitude of the place from the line of latitudes, and set it from A to C, and from a to D; with centres C, D, and the length of the hour line as radius, describe arcs cutting AB in B, and ab in b; and join C, B, and D, b.

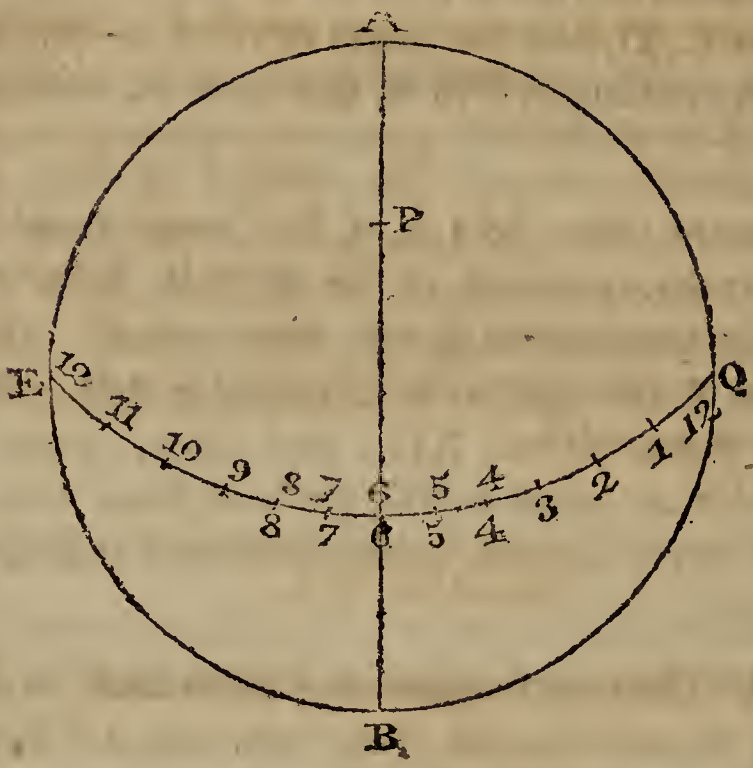
4. Take the extent of 1, 2, 3, &c. hours from the hour line, and set them successively on BC, bD, from B toward C, and from b toward D, for the hour points. Then lines, drawn from A through the hour points in CB, will be the hour lines from VI to XII; and those, drawn from *a* through the hour points in bD, the hour lines from XII to VI. The rest is obvious, being performed as before directed.

NOTE 1. The line *Inclination of Meridians*, on the Plane Scale, may be used instead of the hour line, 15° being allowed for one hour, 30° for two hours, &c.

NOTE 2. If it be required to draw a south vertical dial by means of these Scales, the complement of the latitude must be set from A to C, and from a to D. And then we may proceed as before.

PROBLEM XVII.

To draw a dial on the surface of a sphere.



Assume two diametrically opposite points on the sphere, for the two poles ; and let P be one of them. With centre P, and radius equal to the distance of a point on the surface in the middle between P and the other pole, describe a circle EQ, for the equator, and divide it into 24 equal parts for hours. These are to be numbered from east to west ; with six on the top, the hours of the forenoon above the equator, and those of the afternoon below it.

NOTE. The sphere is to be placed so, that the elevation of the pole P may be equal to the latitude, or the axis parallel to the earth's axis, and the line, or meridian, APB in the meridian of the place. Then the circle bounding the illuminated hemisphere shows the time at its intersection with the equator EQ.

PROBLEM XVIII.

To find the hour of the night by a sun dial, when the moon shines on it.

Find the moon's age, or the number of days elapsed since a change or full. Add $\frac{4}{5}$ of this number, being the time in hours corresponding to the moon's distance from the sun, to the time shown by the moon on the dial, and the sum, or its excess above 12, when it exceeds that number, is the time required.

PROBLEM XIX.

To draw an equinoctial moon dial.

With radii of convenient length, describe two concentric circles A, B; divide the circumference of each into 29 equal parts, and join the corresponding points. On another plane, which is circular, and its radius equal to CB, describe an equinoctial solar dial, as, for example, the superior one D.

NOTE. This dial has the same stile in the centre C, and its plane is set in the same position, as the equinoctial solar dial. The circle D is to be put on B concentric with it, and moveable about the centre. Then the XII o'clock line on D is to be set to the moon's age on the plane AB.

SPHERIC GEOMETRY.

DEFINITIONS.

1. SPHERIC GEOMETRY is the doctrine of the Sphere, particularly of the circles described on it; with the method of projecting the same on a plane, and measuring their arcs and angles when projected.

2. Circles of the sphere, whose planes pass through the centre, are called *great circles*, and all others *small circles*.

3. A straight line, drawn through the centre of any circle of the sphere perpendicular to its plane, and limited on both sides by the surface of the sphere, is called the *axis of that circle*.

4. The *poles of a circle* of the sphere are the extremities of its axis.

5. By *the distance of two points* on the surface of the sphere is meant the arc of a great circle, intercepted between them.

6. *To project an object* is to represent every point of it on the same plane, as it appears to the eye in a certain position.

7. That plane, on which the object is projected, is called *the plane of Projection*; and the point, where the eye is supposed to be situated, *the projecting point*.

8. *The orthographic projection of the sphere* is that, in which a great circle is assumed as the plane of projection, and a point at an infinite distance in the axis of it produced as the projecting point.

9. *The stereographic projection of the sphere* is that, in which a great circle is assumed as the plane of projection, and one of its poles as the projecting point.

10. The great circle, on the plane of which the projection is made, is called *the primitive.*

11. *A direct circle* is that, whose plane is directly opposite to the eye, or perpendicular to the axis of the eye, when directed to the centre of the primitive.

12. *A right circle* is that, whose plane is coincident with the axis of the eye.

13. *An oblique circle* is that, whose plane is oblique to the axis of the eye.

14. *The line of measures* of a circle of the sphere is that diameter of the primitive, produced indefinitely, which is perpendicular to the line of common section of the circle and the primitive.

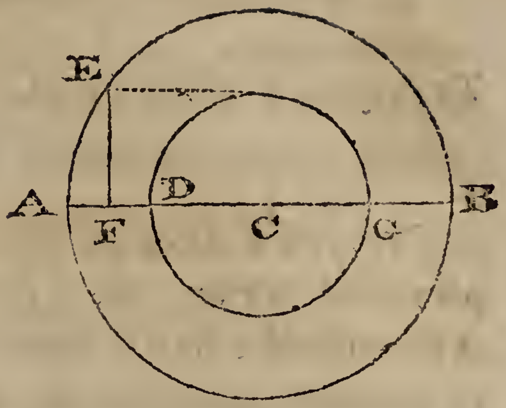
Note. The projection or representation of any point is where the straight line, drawn from the projecting point through it, intersects the plane of projection.

ORTHOGRAPHIC PROJECTION.

PROBLEM I.

To project a circle parallel to the primitive.

Take the complement of its distance from the primitive, and set it from A to E; and with centre C, and radius CD = perpendicular EF, describe the circle DG*.



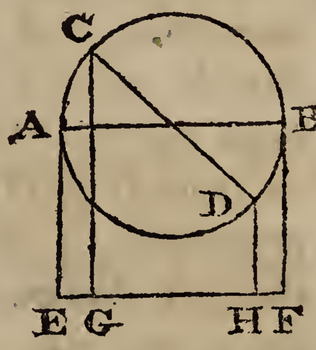
---

* The following are laws of the orthographic projection.

1. The rays coming from the eye, being at an infinite distance, and making the projection, are parallel to each other, and perpendicular to the plane of projection.

2. A right line, perpendicular to the plane of projection, is projected into a point, where that line meets the said plane.

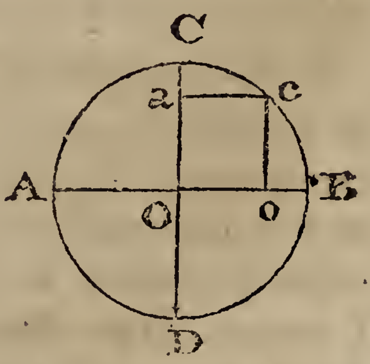
3. A right line, as AB, or CD, not perpendicular, but either parallel or oblique to the plane of the projection, is projected into a right line, as EF, or GH, and is always comprehended between the extreme perpendiculars AE and BF, or CG and DH.



4. The projection of the right line AB is the greatest, when AB is parallel to the plane of the projection.

5. Hence it is evident, that a line parallel to the plane of the projection is projected into a right line equal to itself; but a line, that is oblique to the plane of projection, is projected into one, that is less than itself.

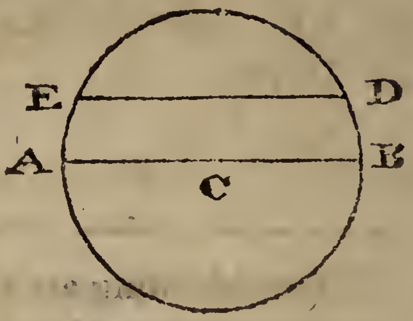
6. A plane surface, as ACBD, perpendicular to the plane of the projection is projected into the right line, as AB, in which it cuts that plane.—Hence it is evident, that the circle ACBD perpendicular to the plane of projection, passing through its centre, is projected into that diameter AB, in which it cuts the plane of



PROBLEM II.

To project a right circle, or one, that is perpendicular to the plane of projection.

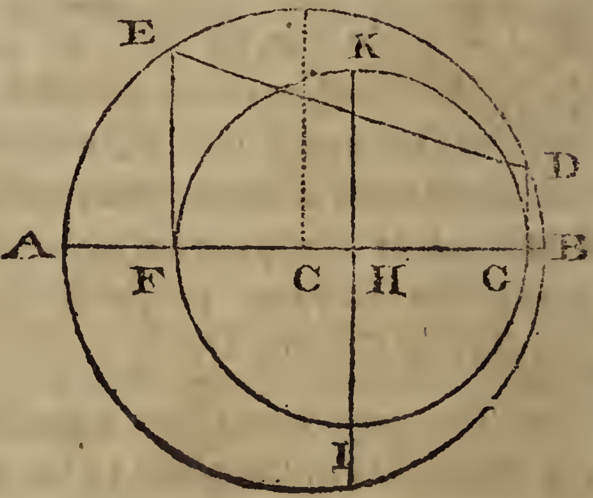
Through the centre C of the primitive draw the diameter AB, and take the distance from its parallel great circle, and set it from A to E, and from B to D, and draw ED, the right circle required.



PROBLEM III.

To project a given oblique circle.

Draw the line of measures AB, and take the circle's nearest distance from the primitive, and set from B to D upward, if it be above the primitive; or downward, if below; likewise take its greatest distance, and set from A to E, draw ED,



---

the projection. Also any arc, as Cc, is projected into Oo, equal to ca, the right sine of that arc; and the complemental arc cB is projected into oB, the versed sine of the same arc cB.

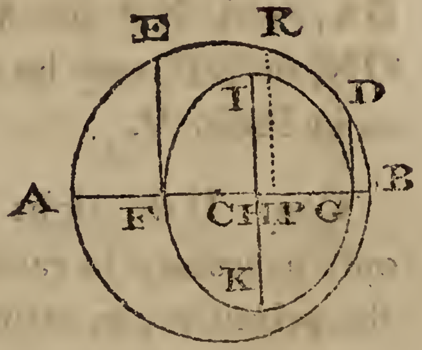
7. A circle parallel to the plane of the projection is projected into a circle equal to itself, having its centre the same with the centre of the projection, and its radius equal to the cosine of its distance from the plane. And a circle, oblique to the plane of the projection, is projected into an ellipse, whose greater axis is equal to the diameter of the circle, and its less axis equal to double the cosine of the obliquity of the circle, to a radius equal to half the greater axis.

and let fall the perpendiculars EF, DG; bisect FG in H, and erect the perpendicular KHI, making KH = HI = half ED; then describe an ellipse, whose transverse is IK, and conjugate FG; and that will represent the given circle.

PROBLEM IV.

To find the pole of a given ellipse.

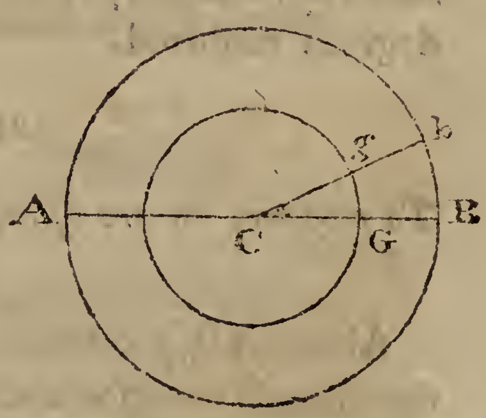
Through the centre C of the primitive draw the conjugate of the ellipse; on the extreme points F, G, erect the perpendiculars FE, GD, or set the transverse IK from E to D, bisect ED in R, and let fall RP perpendicular to AB; then is P the pole.



PROBLEM V.

To measure an arc of a parallel circle; or to set any number of degrees on it.

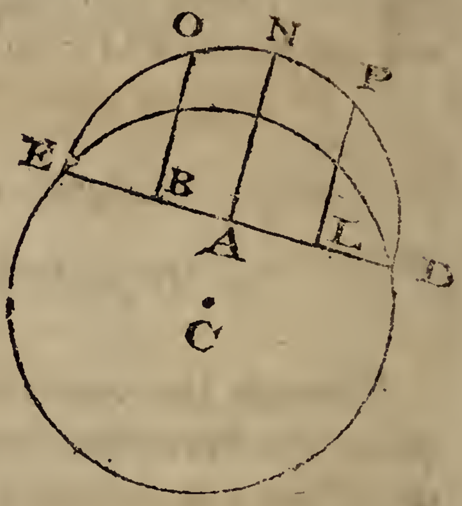
With the radius of the parallel, and centre C, describe a circle Gg, draw CGB and Cgb; then Bb will measure the given arc Gg; or Gg will contain the given number of degrees, set from B to b.



PROBLEM VI.

To measure any part of a right circle

In the right circle ED, let EA = AD; and let AB be the part to be measured. On the diameter ED, describe the semicircle END, draw AN, BO, LP, perpendicular to ED. Then ON is the measure of BA, and NP of AL; and ON, or NP, may be measured as in Prob. V.



Cor. If the right circle pass through the centre, it is only necessary to raise perpendiculars on it, which will cut the primitive, as required.

PROBLEM VII.

To set any number of degrees on a right circle.

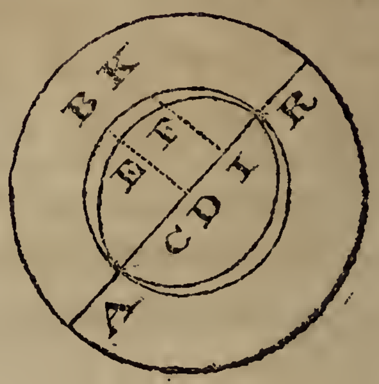
[See Figure under last Problem.]

On ED, the given right circle, describe the semicircle END; then, by Prob. V. set off NP = the given degrees, and draw PL perpendicular to ED; then AL contains the degrees required.

PROBLEM VIII.

To measure an arc of an ellipse; or to set any number of degrees on it.

About AR, the transverse axis of the ellipse, describe a circle ABR; erect the perpendiculars BED, KFI, on AR; then BK is the measure of EF, or EF is the representation of the ark BK. And BK is to be measured, or any degrees set on it, as in Prob. V.



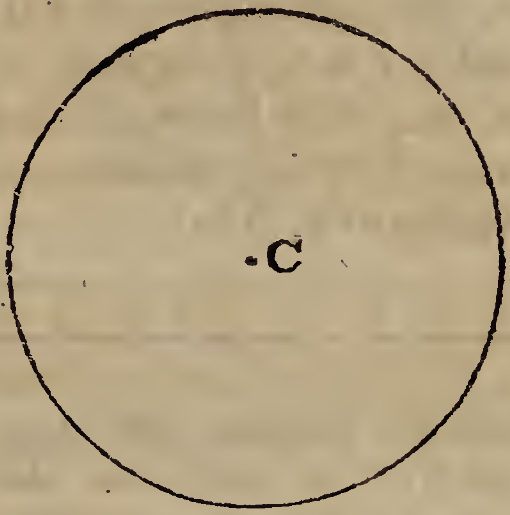
STEREOGRAPHIC PROJECTION.

PROBLEM I.

To find the poles of any projected great circle.

1. *The poles of the primitive circle.*

They are in the centre C.*

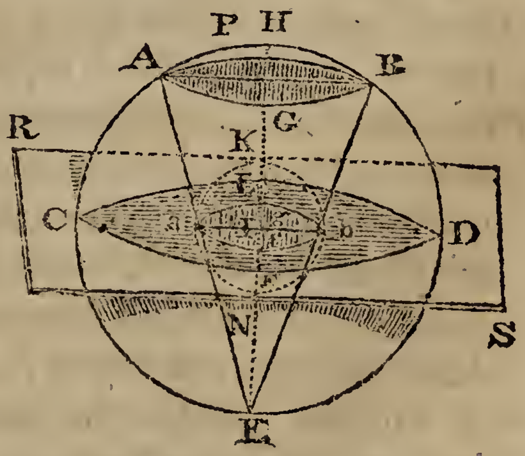


* The following are laws of the stereographic projection.

1. In this projection a right circle, or one perpendicular to the plane of projection, and passing through the eye, is projected into a line of half tangents.

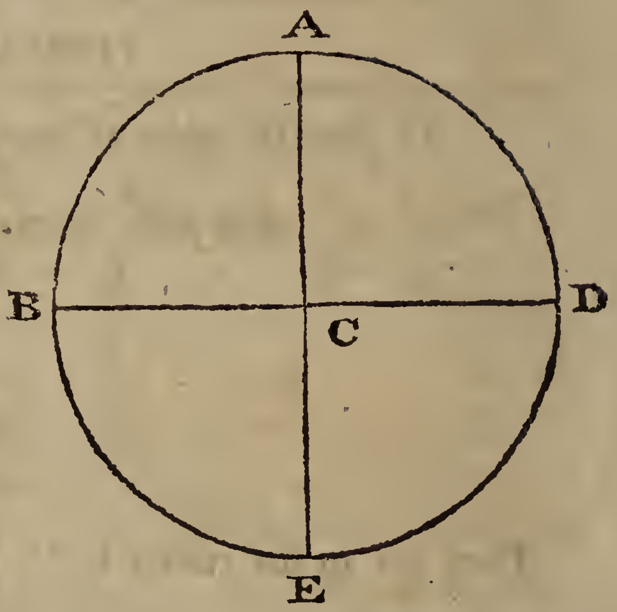
2. The projection of all other circles, not passing through the projecting point, whether parallel or oblique, is into circles.

Thus, let ACEDB represent a sphere, cut by a plane RS, passing through the centre I, perpendicular to the diameter EH, drawn from E, the place of the eye; and let the section of the sphere by the plane RS be the circle CFDL, whose poles are H

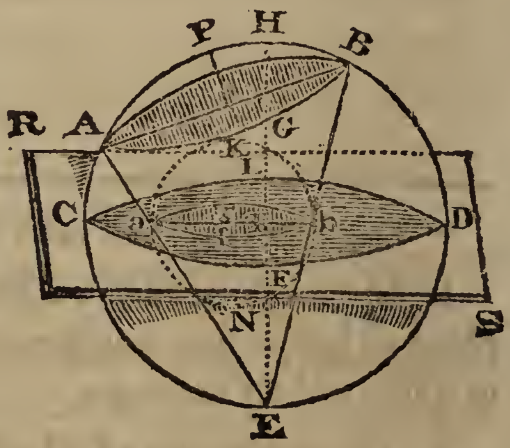


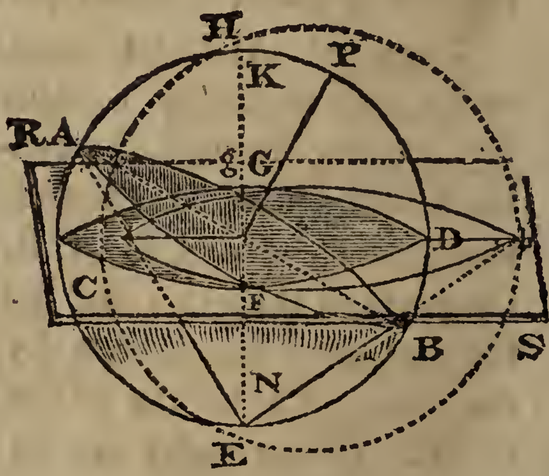
2. *The poles of a right circle.*

Let the right circle be BD.



Draw the diameter AE at right angles to the given right circle, and the extremities A and E will be the poles of it. Also B and D are the poles of the right circle AE.

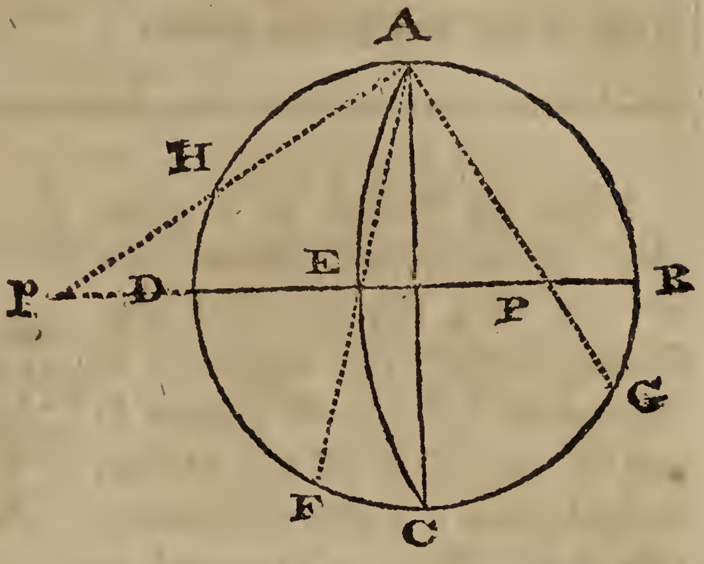




and E. Suppose now, that AGB is a circle on the sphere to be projected, whose pole most remote from the eye is P ; and the visual rays from the circle AGB, meeting in E, form the cone AGBE, of which the triangle AEB is a section through the vertex E, and the diameter of the base is AB ; then will the figure agbf, which is the projection of the circle AGB, be itself a circle. Hence, the middle of the projected diameter is the centre of the projected circle, whether it be a great circle or a small one. Also the poles and centres of all circles, parallel to the plane of projection, fall in the centre of the projection ; and all oblique great circles cut the primitive circle in two points diametrically opposite.

3. *The poles of an oblique circle.*

Let the given oblique circle be AEC.

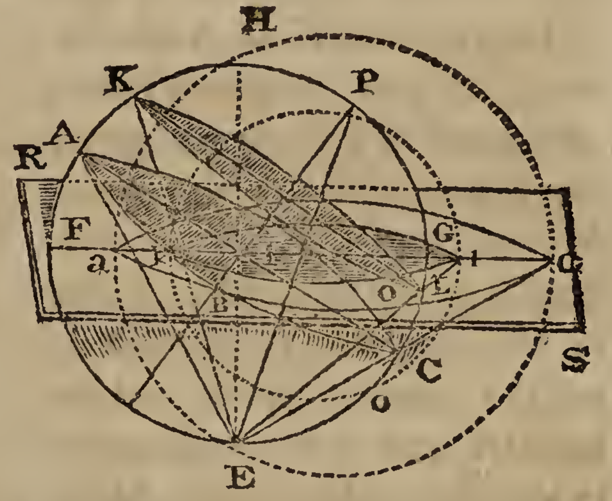


Draw the perpendicular diameters AC, BD; reduce the point E to F in the primitive circle; make FG and FH each equal to a

---

3. The projected diameter of any circle subtends an angle at the eye, equal to the distance of that circle from its nearest pole, taken on the sphere; and that angle is bisected by a right line joining the eye and that pole.

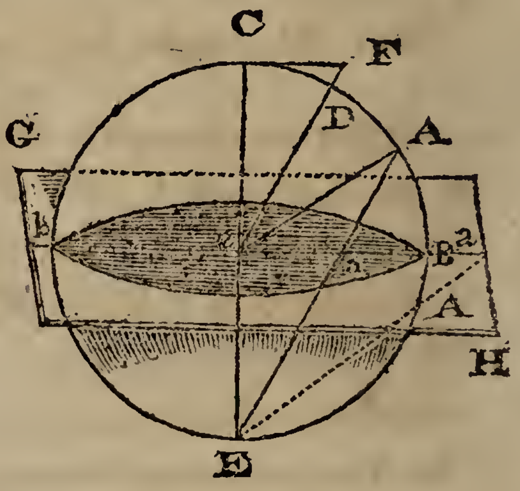
Thus, let the plane RS cut the sphere HFEG through its centre I; and let ABC be any oblique great circle, whose diameter AC is projected into ac; and KOL any small circle parallel to ABC, whose diameter KL is projected into kl. The distances of those circles from their pole P, are the arcs AHP, KHP; and the angles aEc, kEl, are the angles at the eye, subtended by their projected diameters, ac and kl. Then is the angle aEc measured by the arc AHP, and the angle kEl by the arc KHP; and those angles are bisected by EP.



4. Any point of a sphere is projected at such a distance from the centre of projection, as is equal to the tangent of half the arc, intercepted between that point and the pole opposite to the eye, the semidiameter of the sphere being radius.

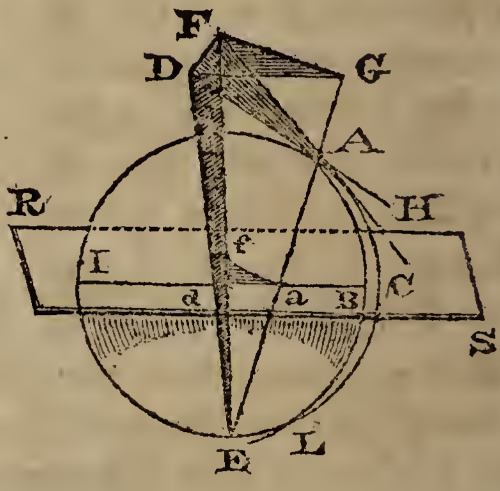
quadrant; reduce G to the point P on the diameter BD and H to the point p on the same diameter continued. Then P, p, will be the poles.

Thus, let CbEB be a great circle of the sphere, whose centre is c, GH the plane of projection, cutting the diameter of the sphere in b and B; also E and C the poles of the section by that plane; and a the projection of A. Then ca is equal to the tangent of half the arc AC, as is evident by drawing CF = the tangent of half that arc, and joining cF.



5. The angle, made by two projected circles, is equal to the angle, which these circles make on the sphere.

For, let ACE and ABL be two circles on a sphere intersecting in A; E the projecting point; and RS the plane of projection, in which the point A is projected in *a*, in the line IC, the diameter of the circle ACE. Also let DH and FA be tangents to the circles ACE and ABL. Then will the projected angle daf be equal to the spheric angle BAC.



6. The distance between the poles of the primitive circle and an oblique circle is equal to the tangent of half the inclination of those circles; and the distance of their centres is equal to the tangent of their inclination, the semidiameter of the primitive being radius.

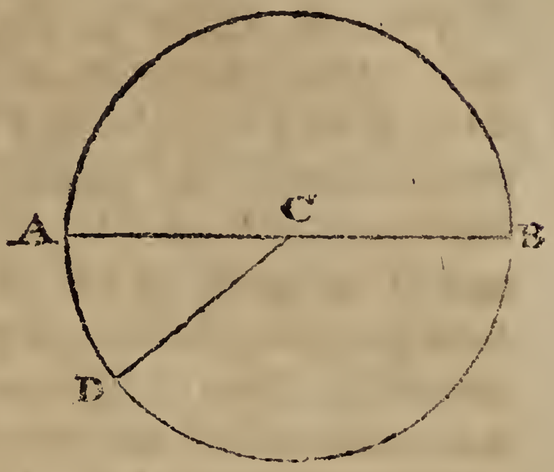
PROBLEM II.

At any given point to project a given angle.

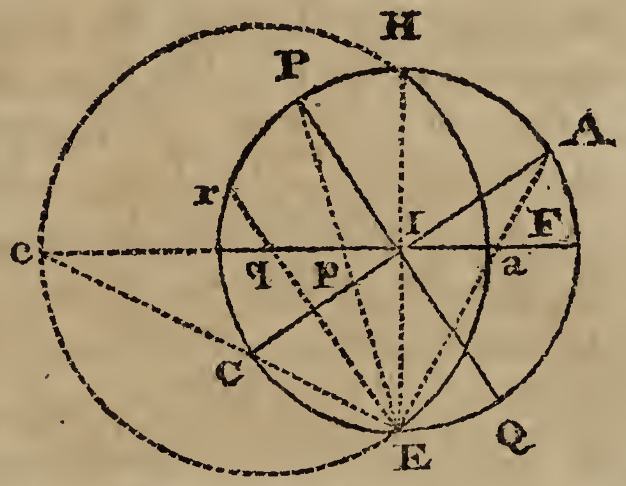
1. *When the angle is at the centre of the primitive.*

Let the angle be 36°.

Draw the diameter AB; from A to D set 36°, the given number of degrees. Then ACD will be the angle required.

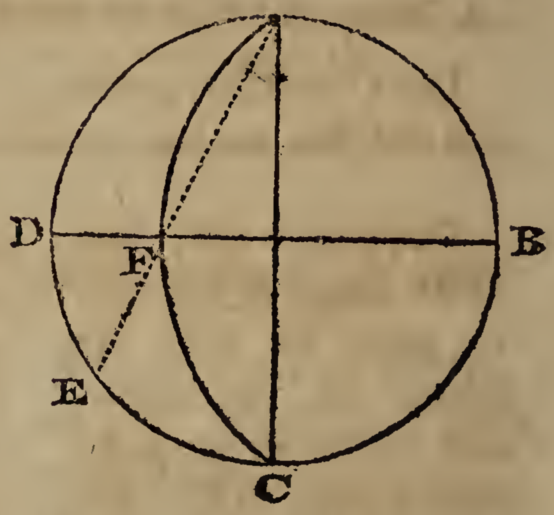


For, let AC be the diameter of a circle, whose poles are P and Q, and inclined to the plane of projection in the angle AIF; and let a, c, p, be the projections of the points A, C, P; also let HaE be the projected oblique circle, whose centre is q. Now when the plane of projection becomes the primitive circle, whose pole is I; then is Ip = tangent of half the angle AIF, or of half the arc AF; and Iq = tangent of AF, or of the angle FHa = AIF.



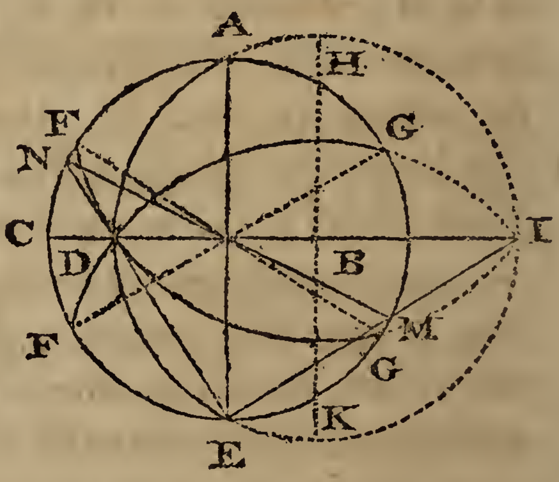
7. If through any given point in the primitive circle an oblique circle be described, then the centres of all other oblique circles, passing through that point, will be in a right line; drawn through the centre of the first oblique circle, and perpendicular to a line passing through that centre, the given point, and the centre of the primitive circle.

2. *When the angular point is in the periphery of the primitive.*



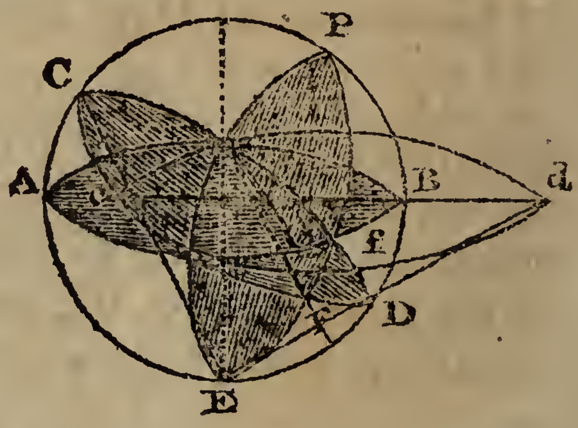
Let A be the point, and the angle 36°.

Through the given point A draw the diameter AC; cross it perpendicularly with BD; from D to E in the periphery set 36°, the given angle; reduce E to the point F; through A, F, C, draw the circle AFC. Then DAF will be the angle required.



Thus, let GACE be the primitive circle, ADEI a great circle described through D, its centre being B. HK is a right line drawn through B perpendicular to a right line CI, passing through D, B, and the centre of the primitive circle. Then the centres of all other great circles, as FDG, passing through D, will fall in the line HK.

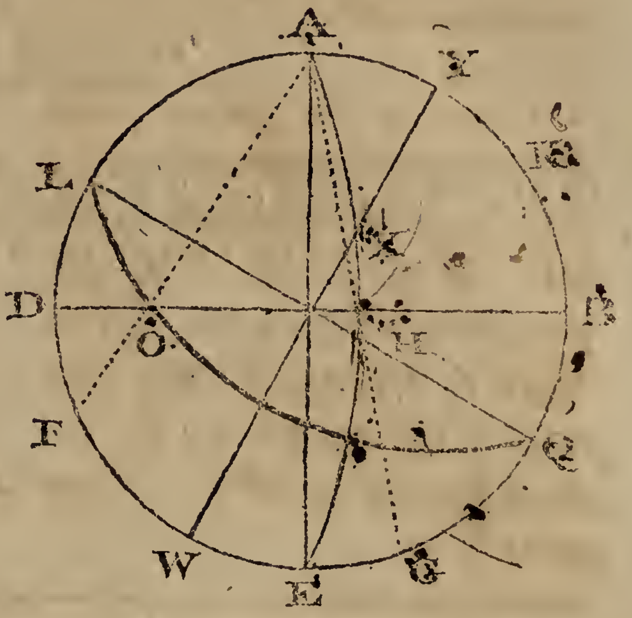
8. Equal arcs of any two great circles of the sphere will be intercepted between two other circles, drawn on the sphere through the remotest poles of those great circles.



For, let PBEA be a sphere, on which AGB and CFD are two great circles, whose remotest poles are E and P; and through these poles let the great circle PBEC and the small circle PGE be drawn, cutting the great circles AGB and CFD in the points B, G, D, F. Then are the intercepted arcs BG and DF equal to one another.

3. *When the angular point is in the plane of the primitive.*

Let O be the point, and 58° the angle.



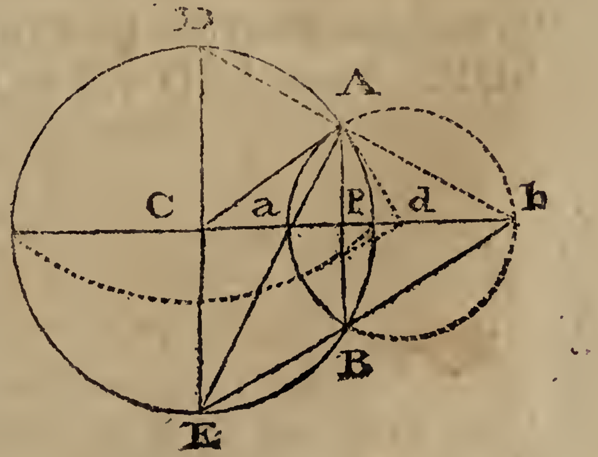
Draw a right line DOB through the given point O and the centre of the circle, and cross it at right angles with AE. Reduce O to F in the primitive; make EG equal to DF; reduce G to H; through A, H, E, project a circle. Set the given angle from A to K, and by laying a rule on O, K, cut the oblique circle at X. Draw through X the perpendicular diameters WXY, QL; and project the circle LOQ. Then LOD is the angle required.

---

9. If lines be drawn from the projected pole of any great circle, cutting the peripheries of the projected circle and plane of projection; the intercepted arcs of those peripheries are equal.

10. The radius of any less circle, whose plane is perpendicular to that of the primitive circle, is equal to the tangent of that less circle's distance from its pole; and the secant of that distance is equal to the distance of the centres of the primitive and less circle.

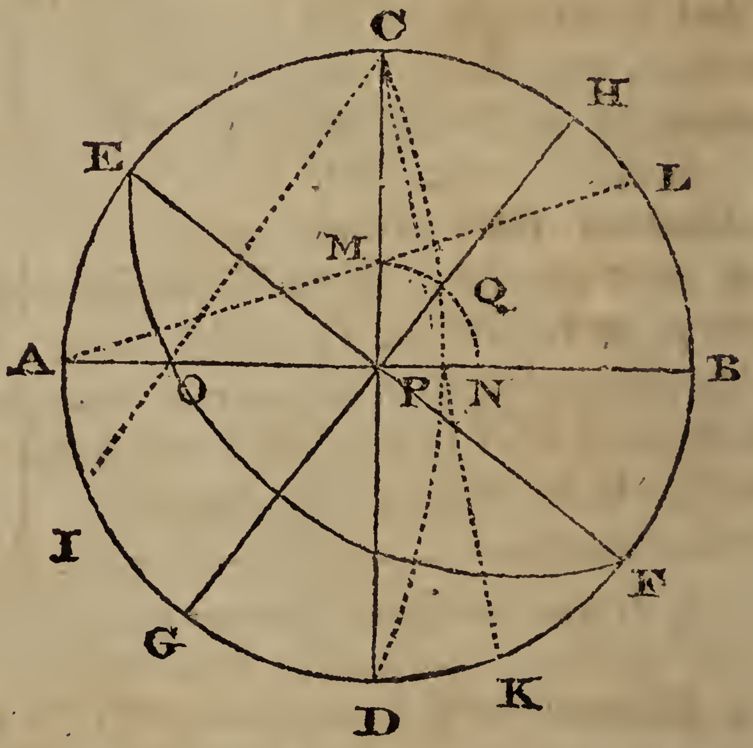
For, let P be the pole, and AB the diameter of a less circle, its plane being perpendicular to that of the primitive circle, whose centre is C; then d being the centre of the projected less circle, da is equal to the tangent of the arc PA, and dC = the secant of PA.



PROBLEM III.

To project a great circle through a given point, which shall make a given angle with the primitive circle.

Let O be the given point, and 36° the given angle.



Through O & the centre draw AB, with the perpendicular diameter CD.—Reduce O to I in the primitive circle; make DK = AI; and find P, the pole of O. Through C, P, D, draw the circle CPD; make BL = the given angle; reduce L to M in the right circle DC, and draw the quadrant MN. Through the point Q, where MN cuts the oblique circle CPD, draw the diameter GQH, with the perpendicular diameter FE; through the extremities of which and O draw the circle EOF. Then AEO will be the required angle.

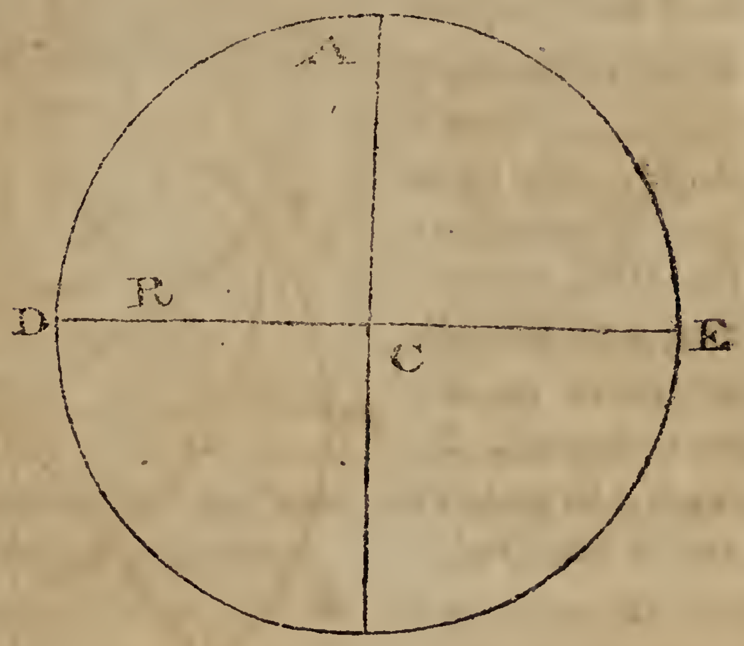
PROBLEM IV.

To draw a great circle through two given points in the plane of the primitive.

1. *When one of the points is in the centre, and the other in the periphery, or elsewhere in the primitive.*

Let C, and A or R, be the points.

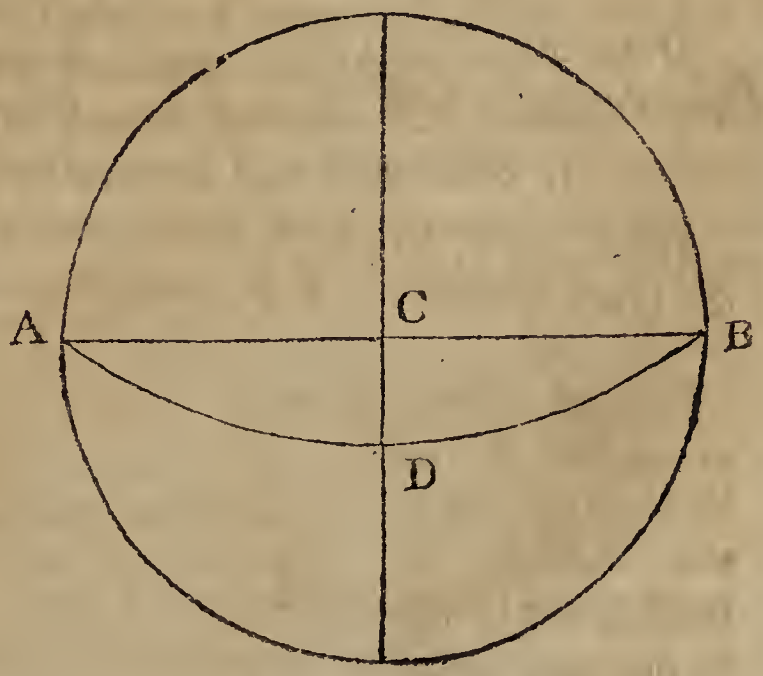
Draw a right line through the centre and the other given point, as ACB, or DCE; and it will be the circle required.



2. *When one point is in the circumference of the primitive, and the other out of the centre.*

Let A, D, be the given points.

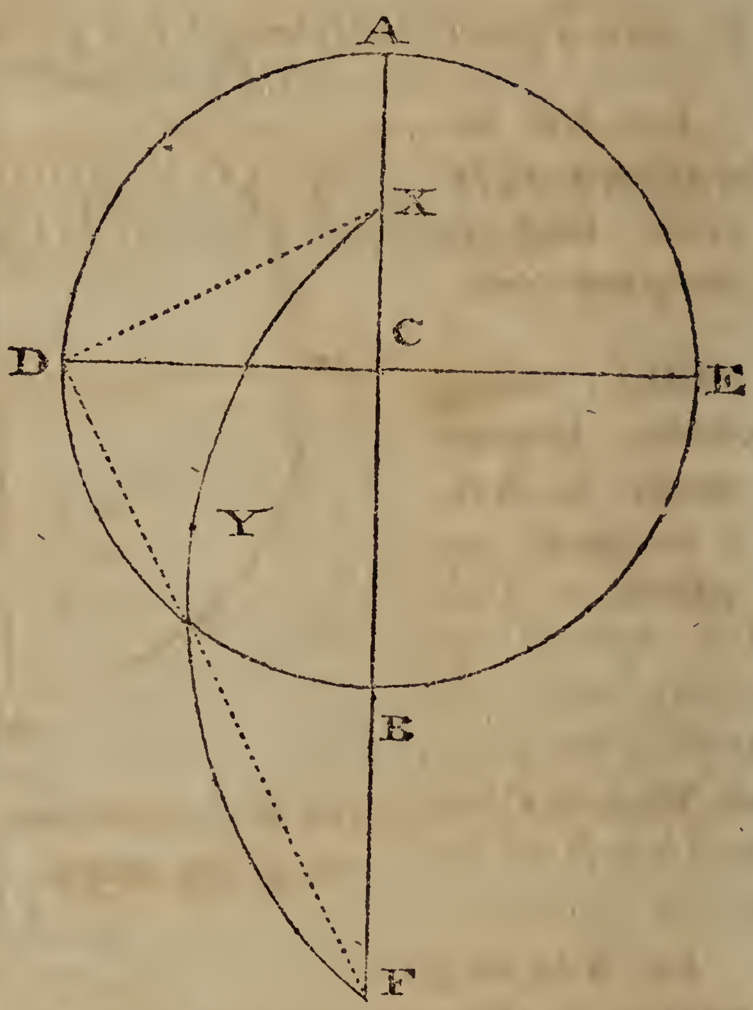
Through A, the given point in the circumference, draw the diameter AB; and an oblique circle, described through A, D, B, will be the projection of the great circle required.



3. *When neither of the points is in the centre or circumference.*

Let X and Y be the points.

Draw through one point the diameter AXCB and the perpendicular diameter DE. Connect X, D, and raise the perpendicular DF, concurring with AXCB produced in F; and through X, Y, F, draw the circle required.



PROBLEM V.

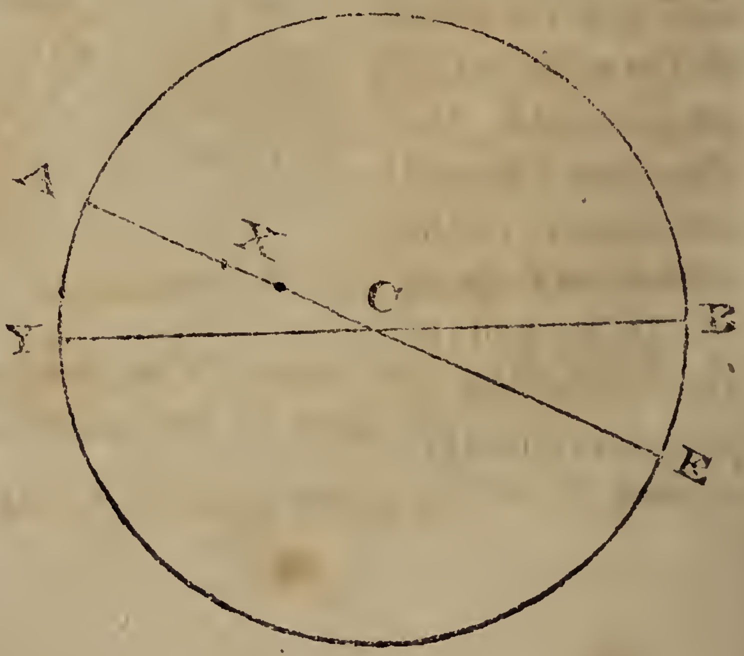
Through a given point, in a projectea great circle, to draw another great circle perpendicular to it.

Draw the arc of a great circle through the pole of the given circle and the given point. Hence

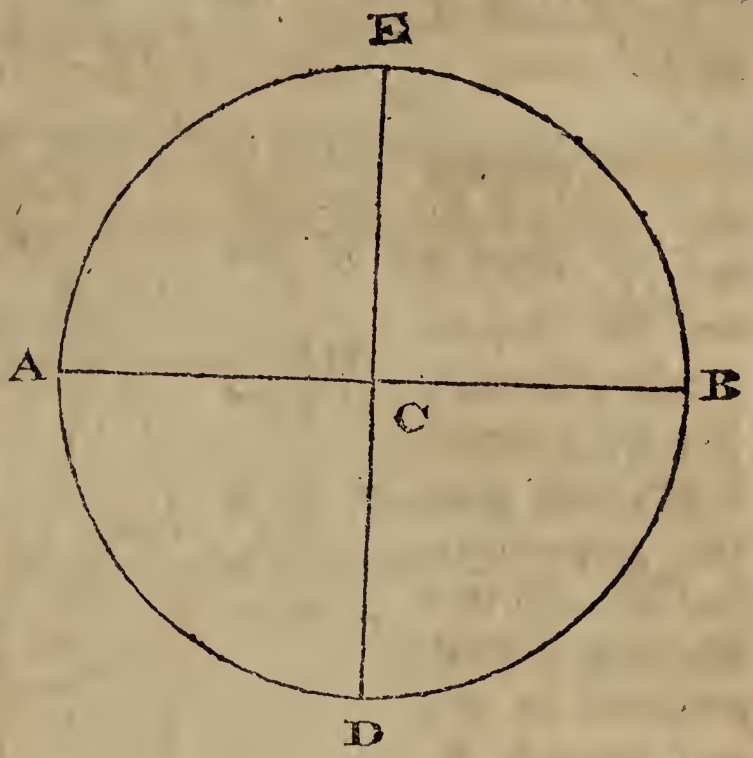
1. *When the given circle is the primitive.*

Let X, or Y, be the given point.

Through the given point X, or Y, draw the diameter AXCE, or BCY; and it will be the perpendicular circle required.



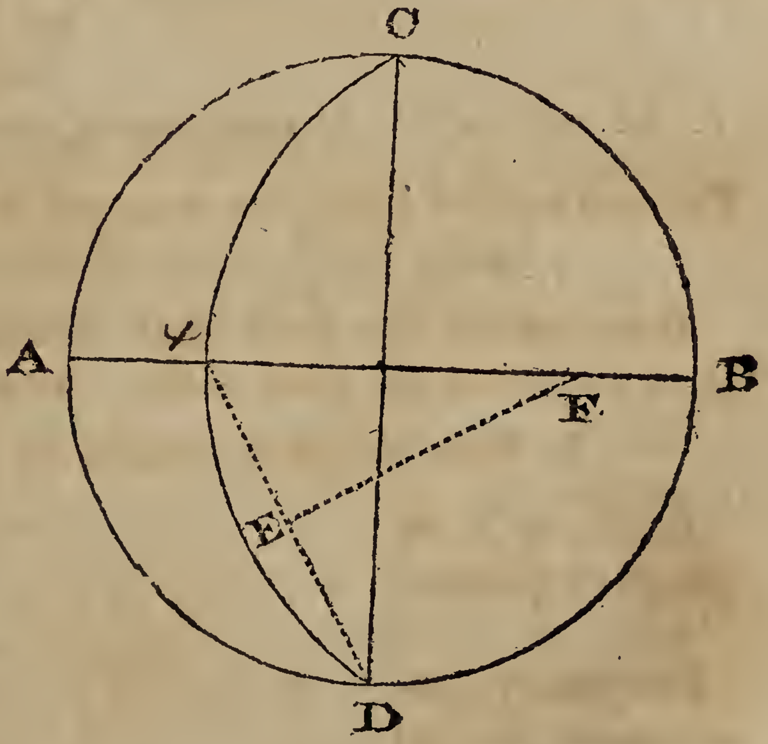
2. *When the given circle is a right circle, and the given point the centre of the primitive.*



Let AB be the given right circle, and C the given point.

DE, being drawn perpendicular to AB, is the circle required.

3. *When the given circle is a right one, and the given point out of the centre.*



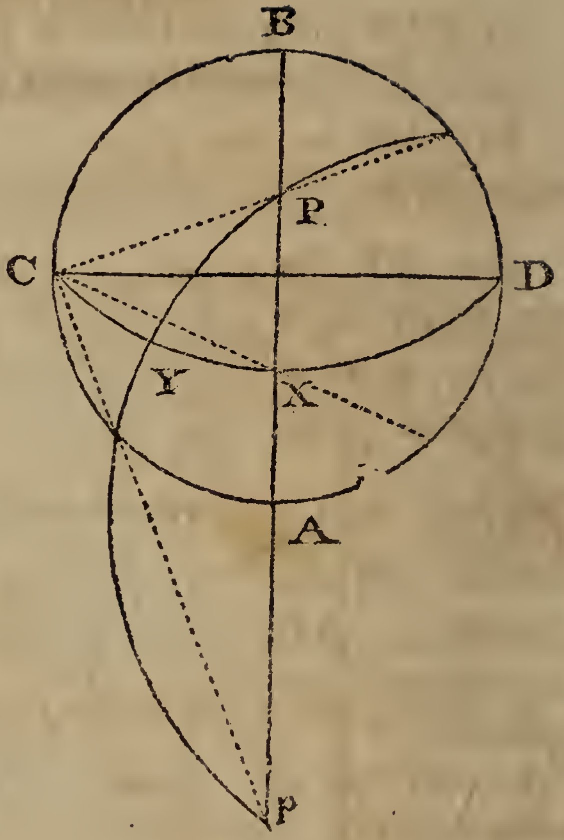
Let X be the given point in the given right circle AB.

Then draw AXB, with the perpendicular diameter CD. Connect the points D and X. Bisect DX in E; and from E draw EF at right angles to DX. With the centre F project the circle CXD, which will be the circle required.

4. *When the given circle is an oblique circle, and the given point in the middle of it.*

Let X be the given point in CXD.

Through X and the centre draw the diameter AXB, and it will be the circle required.



5. *When the given circle is an oblique one, and the given point out of the middle of it.*

Let Y be the given point in CXD.

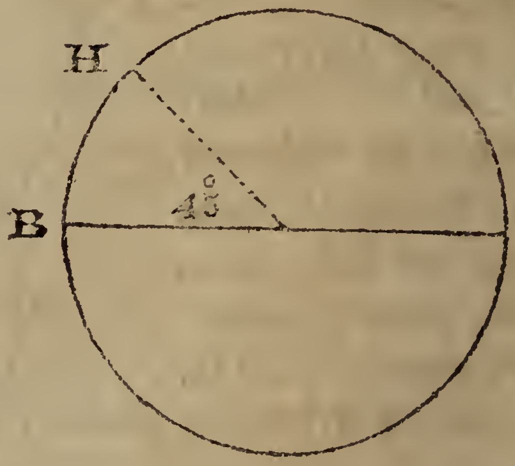
Find the poles P, p of the given oblique circle, and through P, Y, p, draw the circle required.

PROBLEM VI.

To set any number of degrees on a projected great circle.

1. *When the given circle is the primitive.*

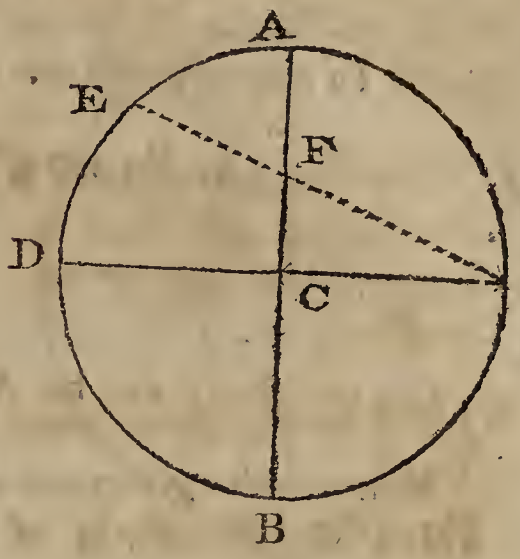
Let the number of degrees be 45.



With a line of chords set the given degrees on the circumference, as from B to H.

2. *On a given right circle from the centre.*

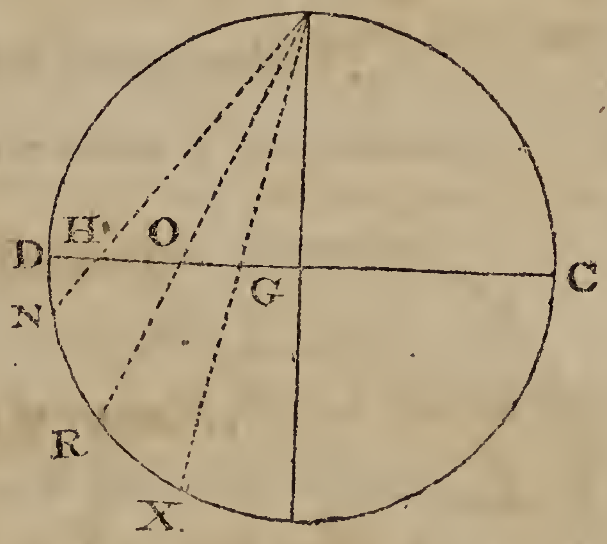
Let AB be the given right circle.



Set the number of degrees on the primitive, as from D to E; reduce E to F. Then FC will be equal to the number of degrees required.

3. *On a given right circle from a point out of the centre.*

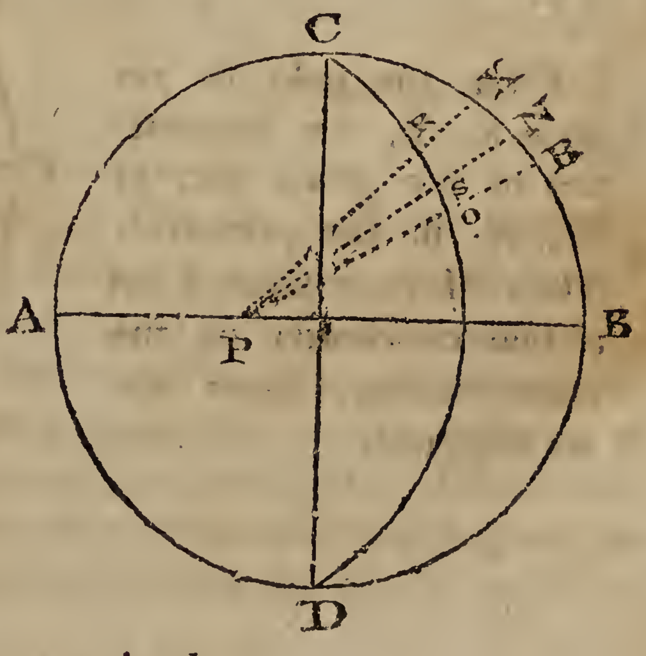
Let O be the given point on DC.



Reduce the given point O to R in the primitive. Set the given number of degrees from R to N, or to X; reduce N, or X, to H, or G. Then OH will be equal to RN, and OG to RX.

4. *From a given point on an oblique circle.*

Let S be the given point on CSD.



Find the pole P of the oblique circle; from P reduce the given point S to N in the primitive; set the number of degrees on the side required from N to X, or to B; draw the lines PX, or PB. Then SR, or SO, is the arc required.

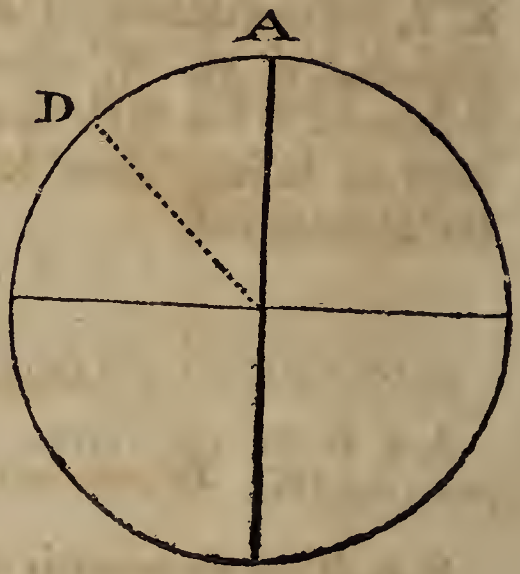
PROBLEM VII.

To measure any arc of a projected great circle.

1. *An arc of the primitive circle.*

Let the given arc be AD.

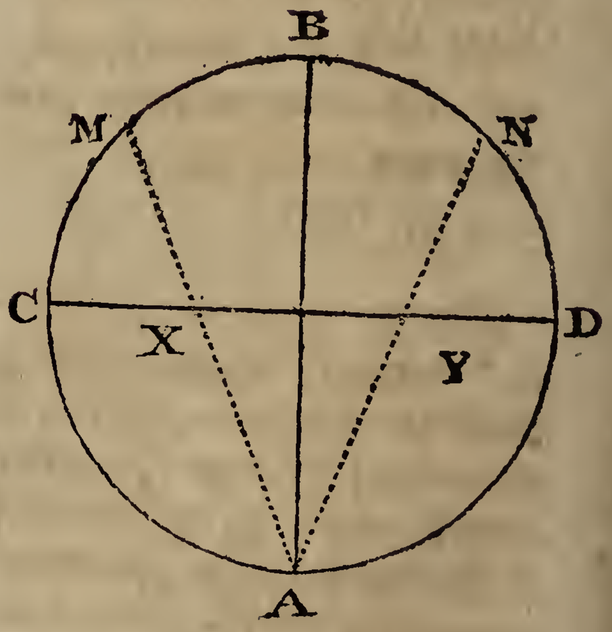
Measure the given arc AD with a line of chords of the same radius.



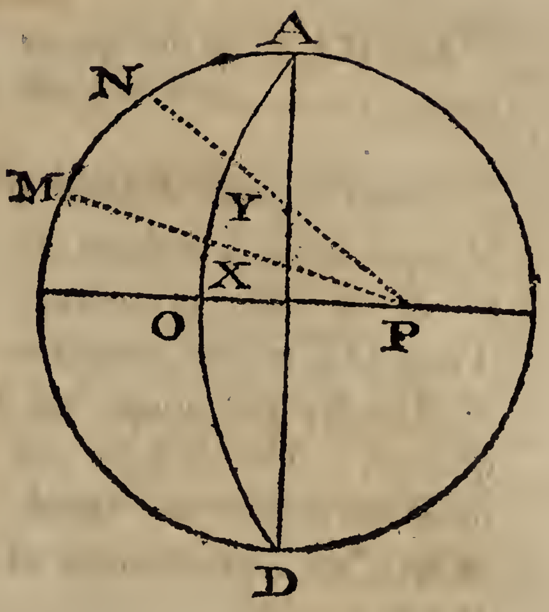
2. *An arc of a right circle.*

Let the given arc be XY.

From the pole A reduce X, Y, the extremities of the given arc, to M, N, in the primitive. Then MN, measured on a line of chords of the same radius, gives the arc required.



3. *An arc of an oblique circle.*



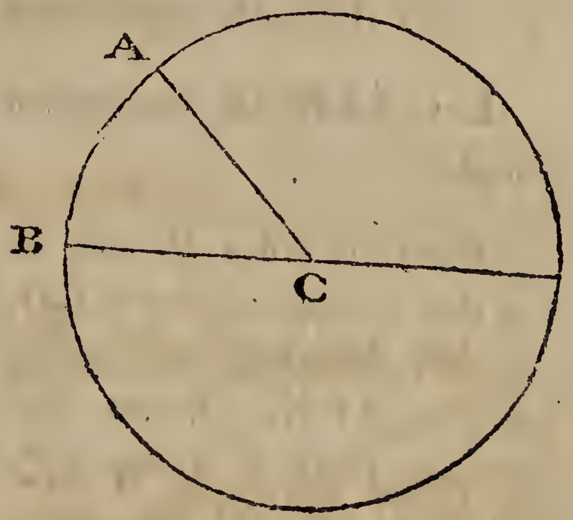
Let the given arc be XY.

Find the pole P; reduce X, Y, the extremities of the given arc, to M, N, in the primitive. Then MN, measured as before, gives XY, the arc required.

PROBLEM VIII.

*To measure any spheric angle.**

1. *When the angle is at the centre of the primitive.*



Let ACB be the given angle.

Continue the arcs, if necessary, to the circumference, and the intercepted arc AB is the measure of the angle required.

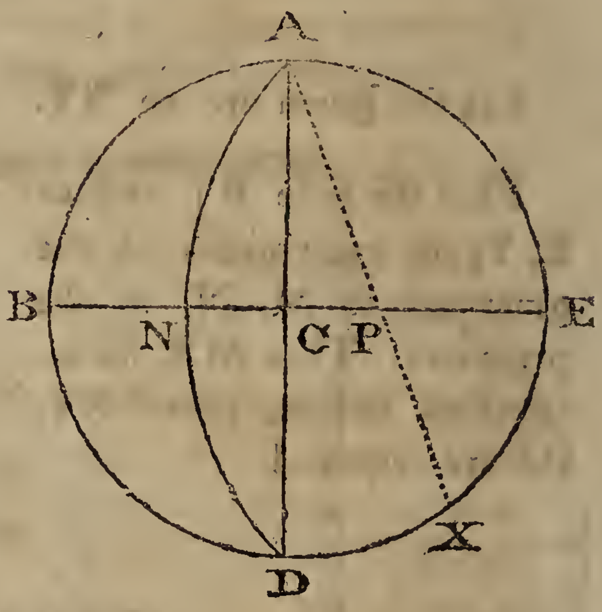
---

* A spheric angle is measured by the arc of a great circle, intercepted between the two sides, that form the angle, at the distance of 90° from the angular point. Or the measure of a spheric angle is equal to the arc of a great circle, whose pole is the angular point, and which passes through the two poles of those great circles, which form the given angle. Hence a rule on the angular point and the poles of the arcs, which form the angle, cuts off, on the primitive, the measure of the angle required.

2. *When the angle is at the circumference of the primitive.*

Let BAN be the given angle.

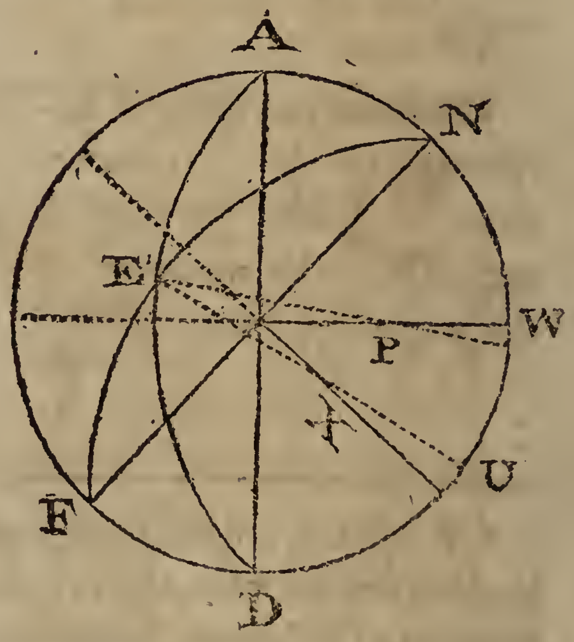
Find the pole P of the oblique circle; reduce P to X in the primitive. Then DX is the measure of BAN, the angle required. Also XE is the measure of the angle NAC, or complement of BAN.



3. *When the angle is in the plane of the primitive.*

Let AEN be the given angle.

Find the poles P and X of the oblique circles AED, NEF, forming the given angle AEN. From the angular point E reduce P, X, to U, W, on the primitive. Then UW is the measure of the angle AEN, or DEF.— And the supplement of UW is the measure of the angle NED, or AEF.



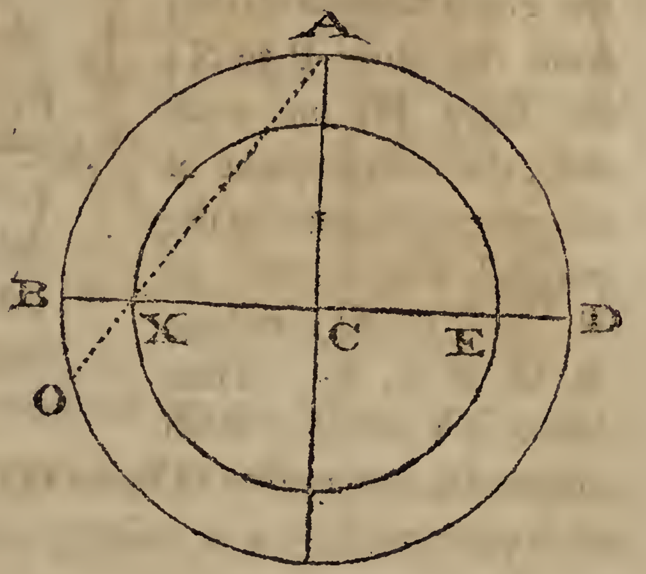
PROBLEM IX.

To draw a small circle parallel to a given great circle, at any given distance.

1. *When the small circle required is parallel to the primitive.*

Let the distance be 13°.

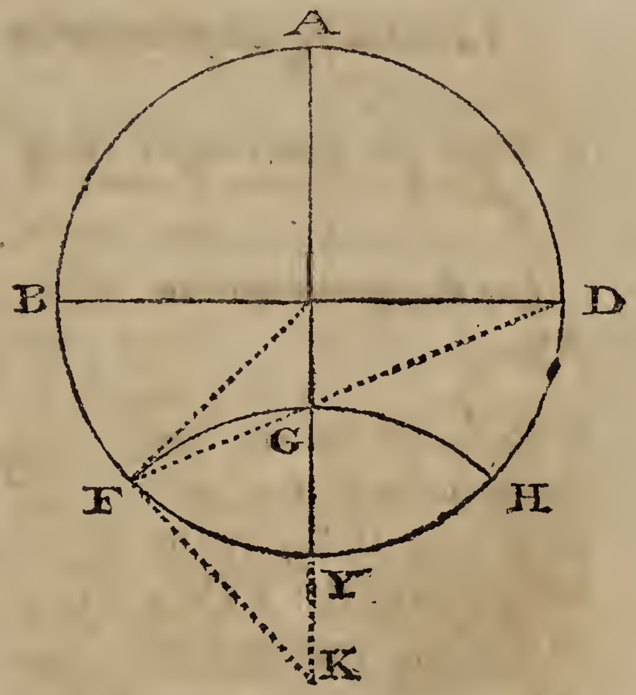
Set the given distance from B to O; reduce O to X on the right circle BD. Then the circle XE, described with the centre C, and radius CX, will be the parallel circle required.



2. *When the small circle required is parallel to a right circle.*

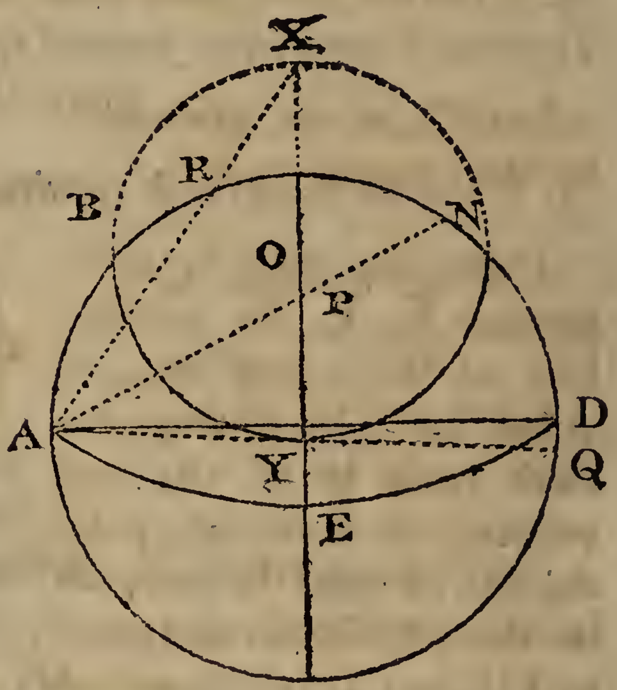
Let the distance be 45° from the right circle BD.

Set the given distance from B to F; reduce F to G; and through F, G, draw the circle FGH, which will be parallel to the given right circle BD, at the distance of 45°.



NOTE. The point K, which is the centre of the circle FGH, may be found by drawing the tangent FK from the point F. For, the centre is where it intersects the diameter AY produced.

3. *When the small circle required is parallel to an oblique circle.*



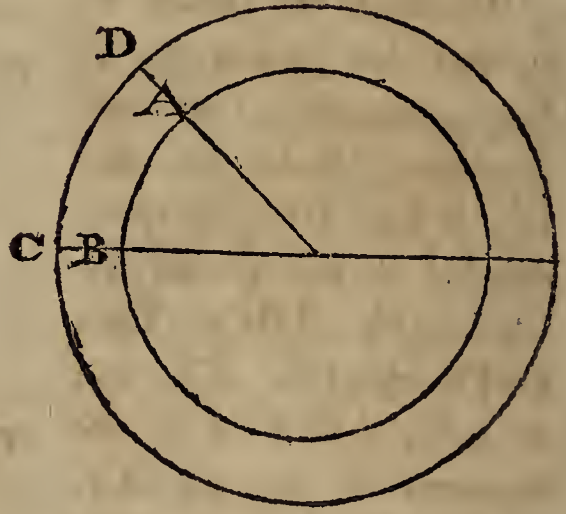
Let the distance be 30° from the oblique circle AED.

Find the pole P of the given oblique circle; from A reduce P to N; set from N, on each side, the complement of the given distance of the parallel, that is 60°, to R, and to Q; reduce R, Q, to X, Y. Then bisect XY in O; with centre O, and radius OX, or OY, draw XBY for the circle required.

PROBLEM X.

To measure any arc of a projected small circle.

1. *When the given small circle is parallel to the primitive.*

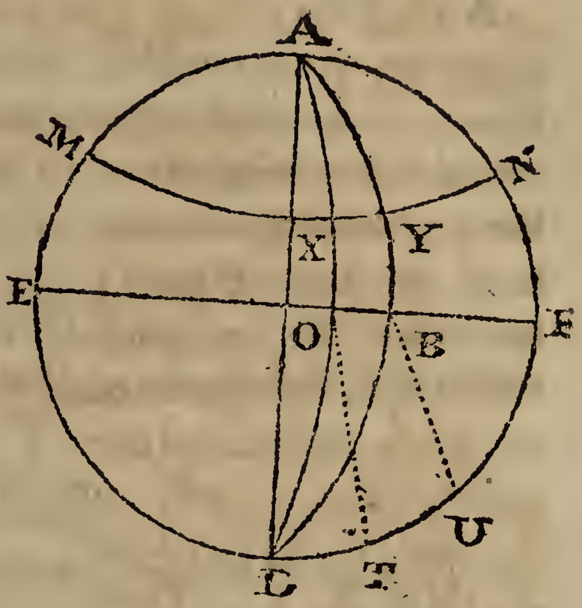


Let the given arc be AB.

Reduce A, B, the extremities of the given arc, to D, C, in the primitive circle. Then DC is the measure of AB.

Cor. Any arc of the primitive circle, as DC, is reducible, in this manner, to the similar arc AB of a given parallel circle.

2. *When the given small circle is parallel to a right circle.*

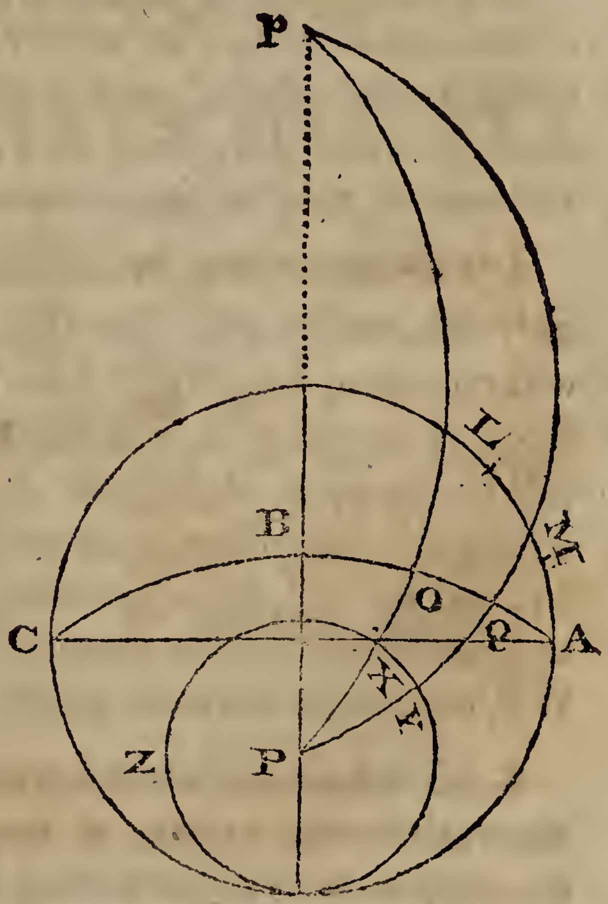


Let XY be the given arc of MN, parallel to EF.

Draw two oblique circles through X, Y, the extremities of the given arc, as AXD, AYD, cutting the right circle in O, B, and passing through the poles A, D. Then OB, reduced to the primitive, is equal to TU, and TU is the measure of the arc XY.

Cor. By inverting the process any number of degrees may be set on a small circle, parallel to a right circle, from any given point.

3. *When the given small circle is parallel to an oblique circle.*



Let XY be the given arc of XYZ, parallel to ABC.

Through the poles P, p, and the points X, Y, the extremities of the given arc, draw the circles PXp, PYp, perpendicular to the given oblique circle ABC. Then OQ is equal to XY, and to LM. And LM is the measure of the arc XY.

Cor. By inverting the process any number of degrees may be set on a small circle, parallel to an oblique circle.

PROJECTIONS OF THE SPHERE.

A PROJECTION OF THE SPHERE is a perspective representation of the circles of the sphere. Each of the following figures, or general projections on the planes of great circles, contains a hemisphere. In the stereographic projection, the hemisphere, opposite to the eye, falls within the primitive, to which hemisphere this kind of projection is generally limited. If it be applied to any part of the hemisphere next to the eye, the representation extends beyond the primitive.

PROBLEM I.

To project the sphere stereographically on the plane of the equator.

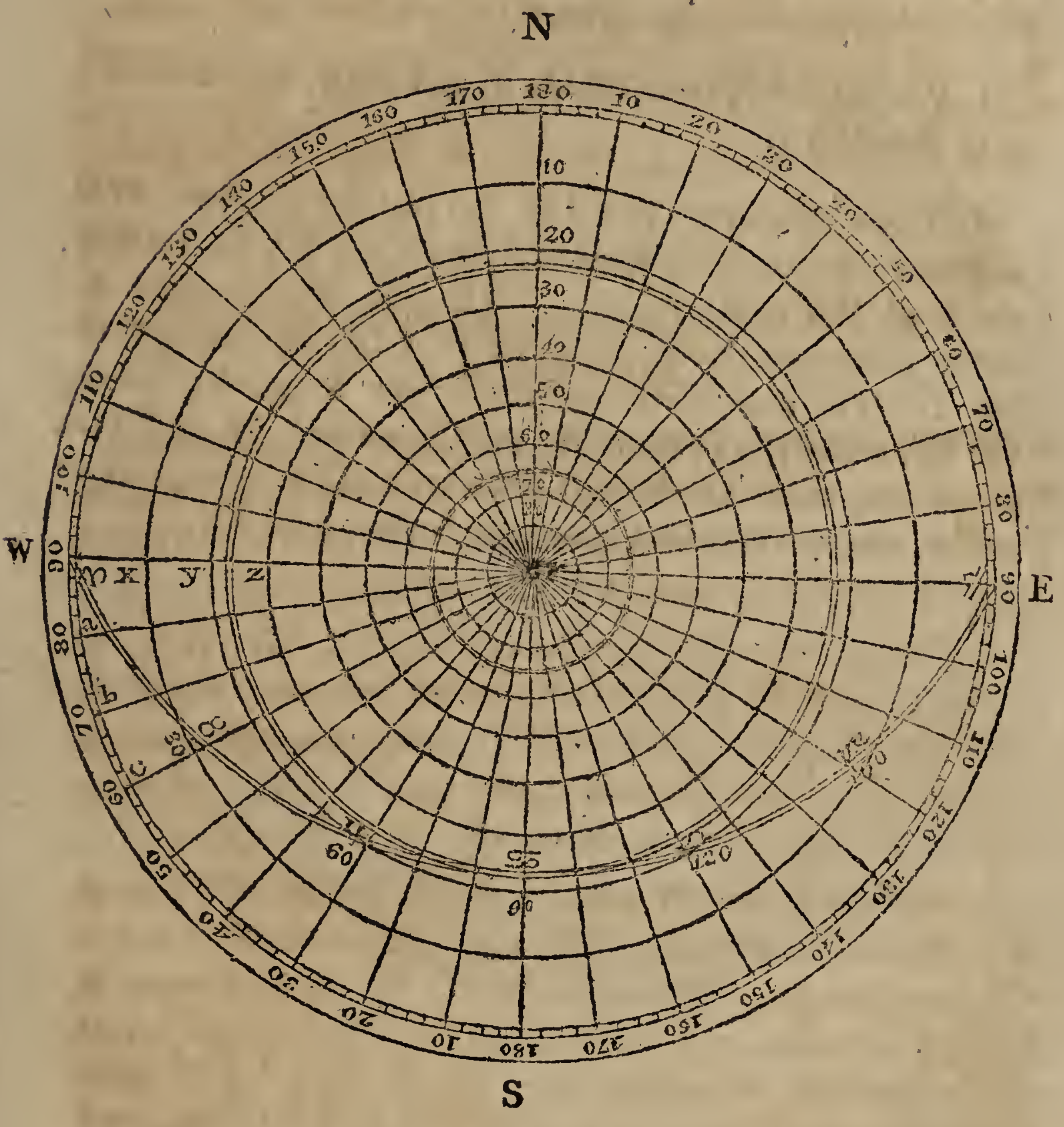
1. Draw the primitive circle WNES, of a convenient bigness, to represent the plane of the equator; and draw the diameters WE, NS, at right angles to each other, which here represent the equinoctial and solstitial colures.

2. Divide each of the quadrants NE, ES, SW, WN, into nine equal parts; by which means the equator will be divided into 36 parts, each $= 10^\circ$. Then through each of these divisions draw lines to P, the pole. Each of these 10° may be divided more minutely.

3. Lay a rule over N and a, and it will cross PW at x; over Nb, and it will cross PW at y; and over Nc, and it will cross at z, &c. With radii Px, Py, Pz, &c. and centre P, through the points x, y, z, &c. draw concentric circles, which will be parallels of latitude 10° distant from each other. The tropic and polar circle may be projected by setting 23° $28'$, and 66° $32'$, from W toward S, reducing them to WP, and describing circles through the points of intersection in WP, concentric with the parallels of latitude.

4. If the projection be intended for the northern hemisphere, the tropic is that of cancer, and the northern half of the ecliptic is to be projected through W♋E. Inthe projection of the southern hemisphere is the tropic of capricorn, and the southern half of the ecliptic touching it.

Stereographic Projection on the Equator.



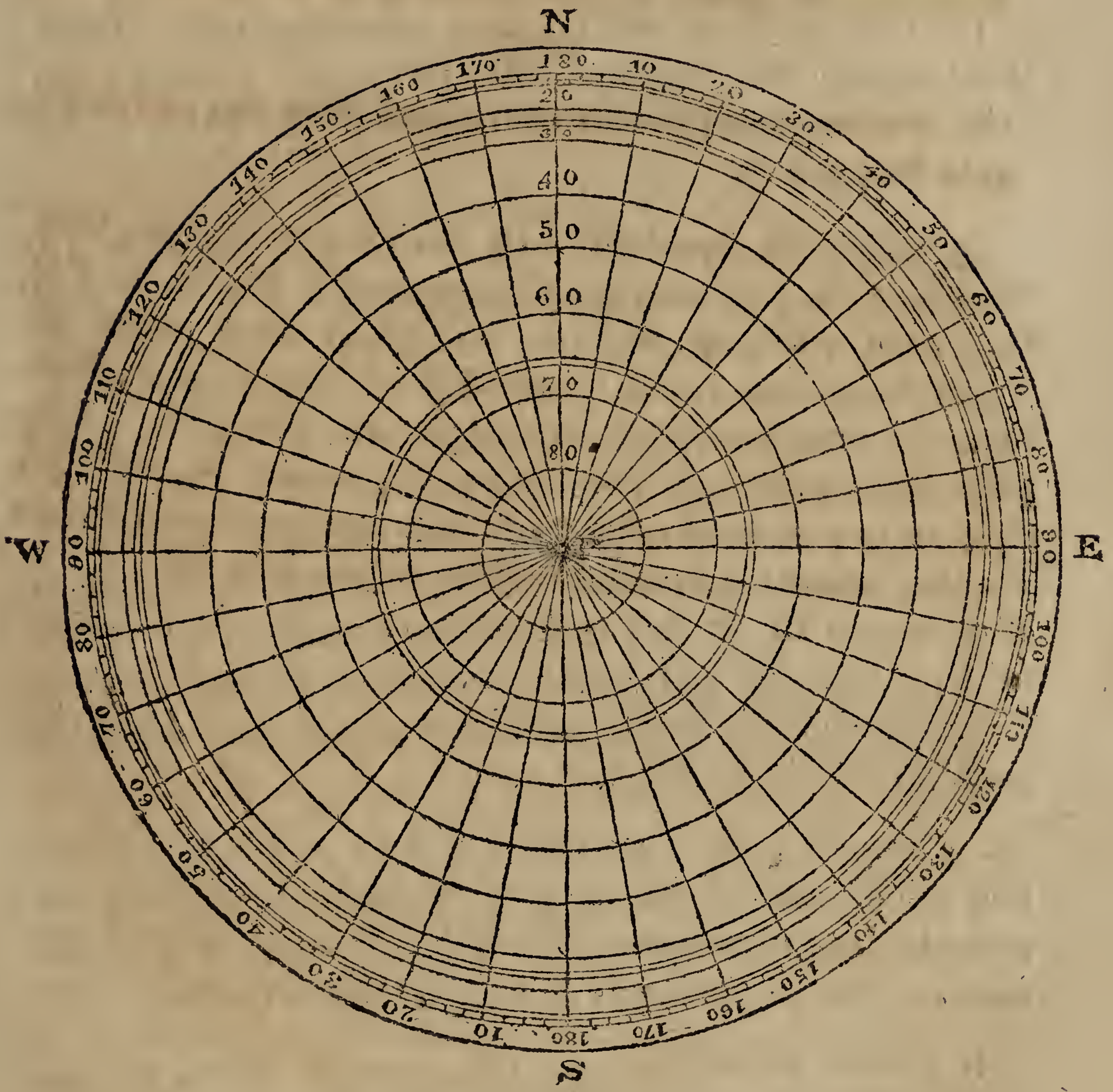
PROBLEM II.

To project the sphere orthographically on the plane of the equator.

1. Project the equator, divide it, and draw the meridians, as in Problem I.

2. Lay a rule over 100, 110, &c. on the quadrant WN, and the respective correspondent points 80, 70, &c. on EN, and mark PN in 80, 70, &c. Through these points describe concentric circles for parallels of latitude, as in Problem I. The polar circle and tropic are found by setting 23° 28′, and 66° 32′, from W and E toward N, and then proceeding as for parallels of latitude, passing through those points, where a rule laid across, as before, cuts PN.

Orthographic pojection on the equator.



PROBLEM III.

To project the sphere stereographically on the plane of the meridian.

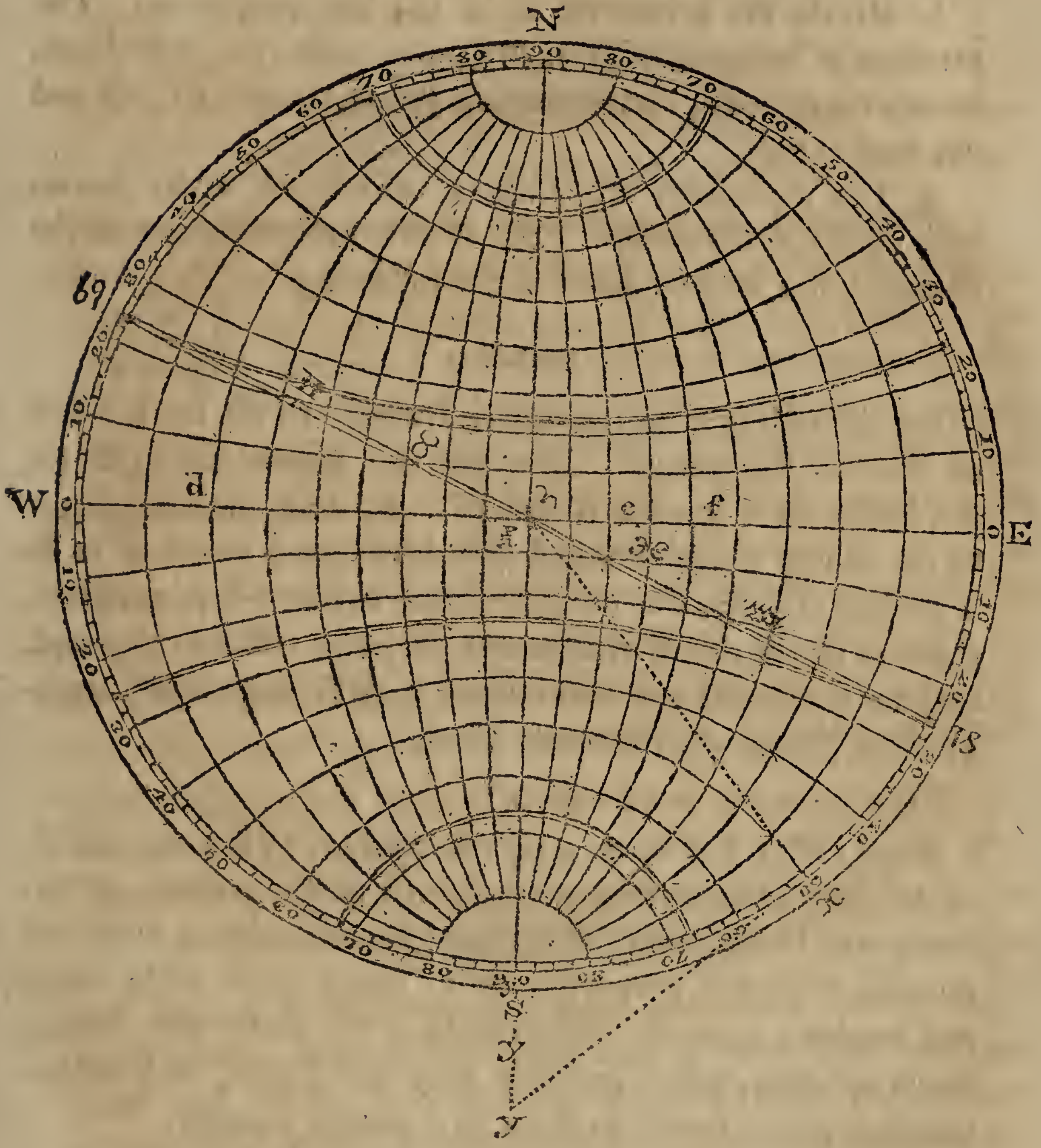
1. Draw the circle NESW, of a convenient size. Draw the diameter WE, which, in this case, is the equator; and cross it at right angles with the meridian NS, which is the equinoctial colure.

2. Divide the quadrants, each into 9 equal parts, at 10, 20, &c. which reduce to the meridian NS at a, b, &c. And through 10a, 20b, &c. draw the parallels of latitude, the centres of which may be easily found by a square. For, if any radius be drawn, as Px, the perpendicular xy will intersect the extended diameter NS at the centre. Hence also is derived an easy method of projecting the parallels of latitude, viz. by continuing the diameter NS, without any reduction of the points 10, 20, &c. and applying one side of a square to P10, P20, &c. and the other will show the centres y, y, &c. from which, with the radius y10, y20, &c. draw the parallels of latitude required.

3. The tropics and polar circles may be projected by setting 23° 28′, and 66° 32′, on each side of the equator on the primitive, and then drawing parallels of latitude at those distances. The projection of the ecliptic is also obvious.

4. For the meridians, lay a rule from N to 10, 20, &c. reckoned both from W and E toward S, and reduce them to c, d, &c. on WE; and through SaN, SbN, &c. draw the meridians required; the centres of which will always be in the diameter WE continued, and distant from P on the opposite side twice as many degrees as the respective meridians are from the primitive. Thus, e is the centre of NeS; f, of NdS, &c.

Stereographic projection on the meridian.



PROBLEM IV.

To project the sphere orthographically on the plane of the meridian.

1. Divide the primitive as in the last projection. The parallels of latitude, tropics, and polar circles are right lines, drawn through the corresponding points 10 and 10, 20 and 20, &c.

2. The meridians are ellipses, which are easily drawn with elliptical compasses. But if this instrument cannot be obtained, use may be made of the following

METHOD.

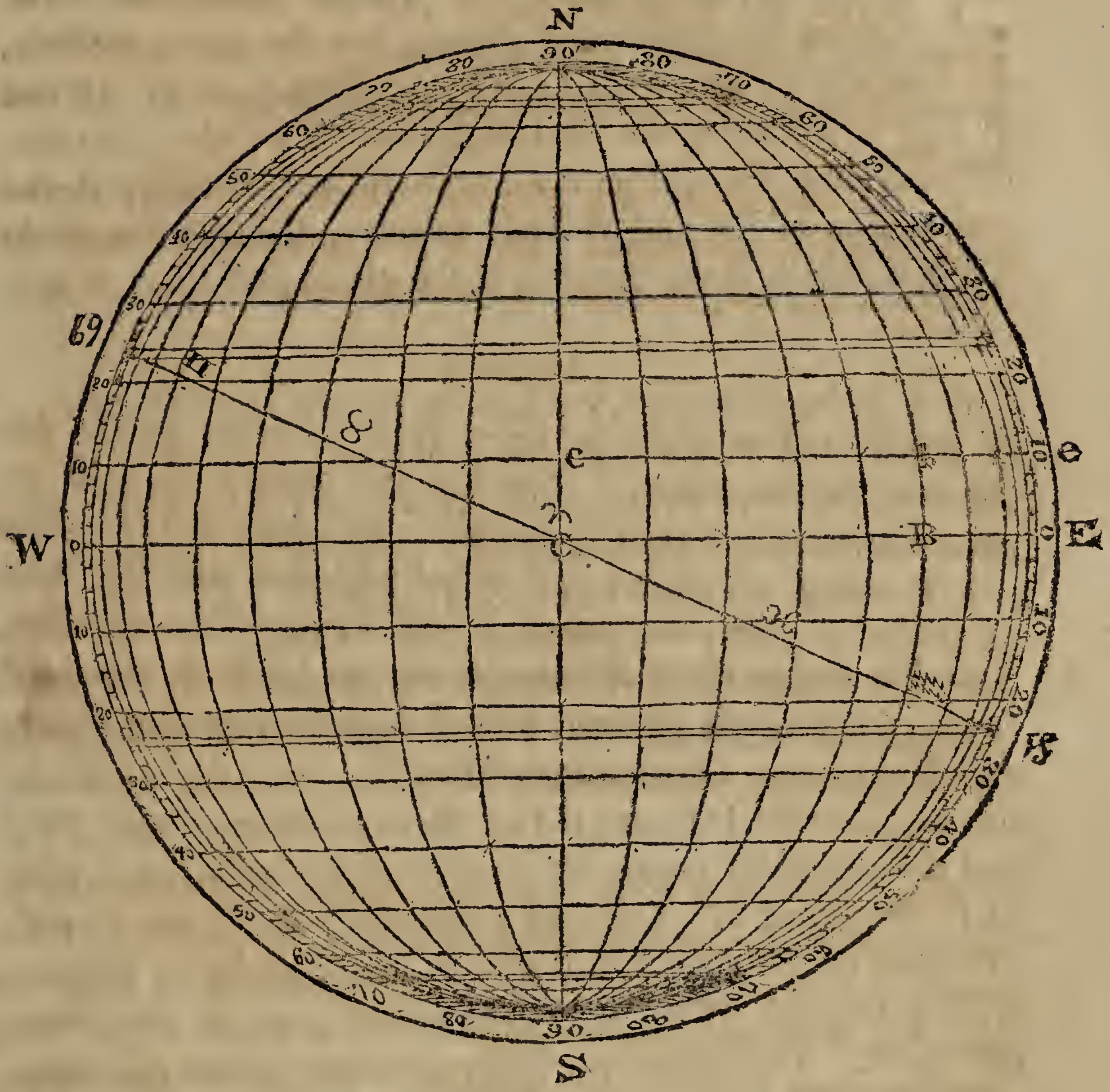
Set the radius of the primitive from 90 to 90 on the line of sines on the sector. Then set the parallel sines of 10, 20, &c. from C toward E and W. By adapting the sector to the radius of each parallel of latitude, and marking it in a similar manner, the points of the equator and parallels, through which the meridians are to pass, will be obtained, and the meridians are then drawn, each through the corresponding points, with a steady hand.

Or,

Since CE : CB :: ce : cb universally, WE being divided as before, the corresponding points of the parallels of latitude may be obtained, by finding geometrically a sufficient number of fourth proportionals to three given right lines, and marking them on the parallels. To make the ellipses more accurate, other parallels may be drawn, and corresponding points found on them in a similar manner.

NOTE. This projection is commonly called the *Analemma.*

Orthographic projection on the meridian.



PROBLEM V.

To project the sphere stereographically on the plane of the horizon.

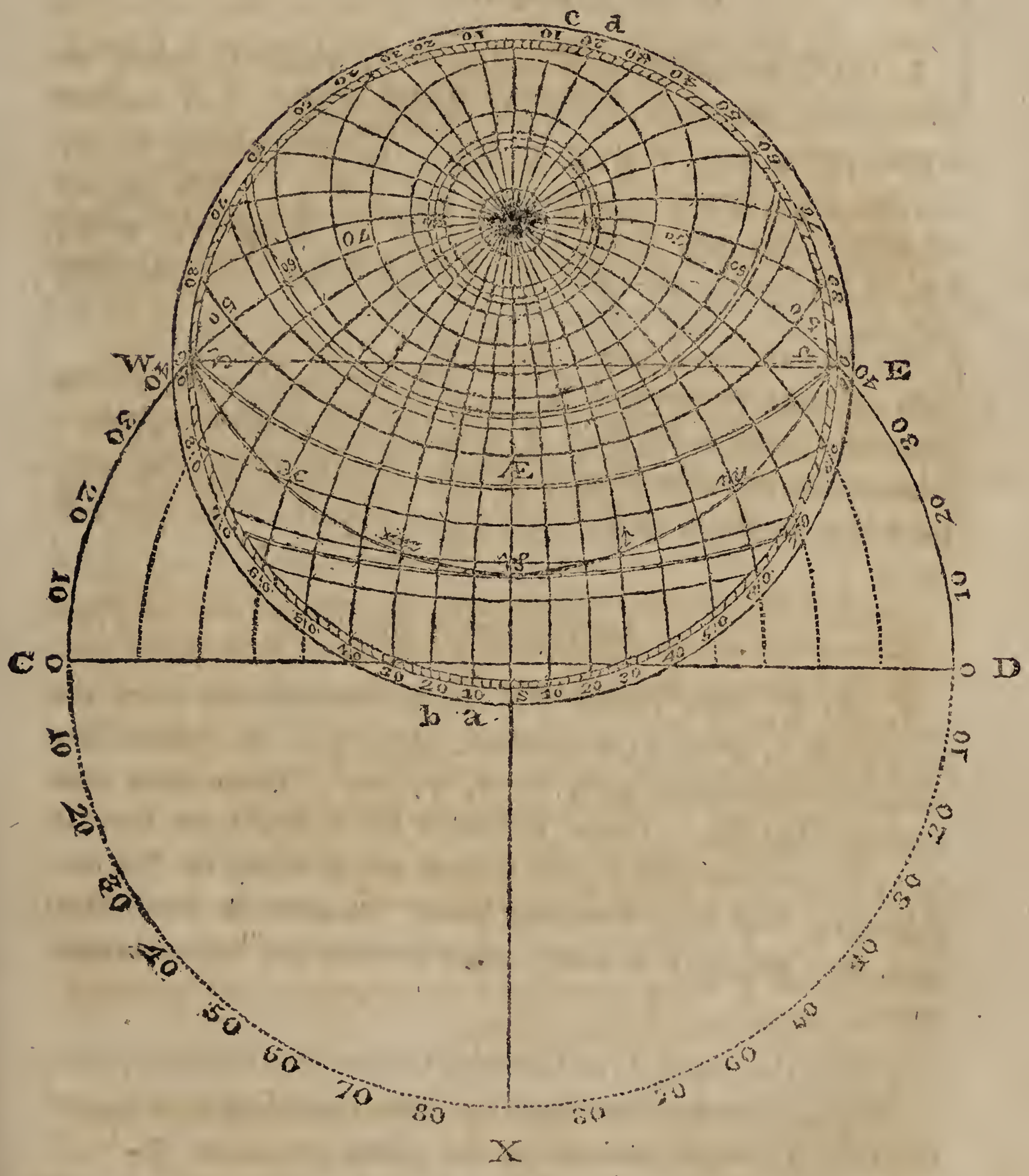
1. Describe a circle, with any convenient radius, as XCWPED ; divide it into four quarters, and subdivide each of them into nine equal parts. Set the given latitude, suppose $42^\circ\ 23'\ 28''$, from D to E, and from C to W. Draw WE, which will be the east and west line ; also continue SP to N, and SN will be the first meridian. Lastly, at the intersection of SN and WE, as a centre, describe the circle SWNE, which is the horizon of the place, and plane of projection.

2. For the meridians, project the circles aP, bP, &c. on the plane SCWPED, in the same manner as in the stereographic projection on the plane of the meridian ; and continue them beyond P, through the plane of the horizon, to c, d, &c. with the same radius and centre.

3. For the parallels of latitude, and consequently the polar circles, tropics, and equator, lay a rule over W and the several divisions on the quadrant PD ; also over the divisions on the quadrant Dx, and reduce them to the meridian PS ; and through these points the parallels will pass on that side of P, or toward S. Then lay a rule from W to the several divisions on PC, which will give as many points on PN extended, through which the parallels will pass on the other side of P. The distances of corresponding points on opposite sides of P being considered as diameters, the parallels, &c. are to be described on them.

NOTE. To know how large the meridian projection must be to form a horizontal one of any given diameter, say, as cosine of the latitude : radius :: the given semidiameter : the semidiameter of the meridian projection.

Stereographic projection on the horizon.



PROBLEM VI.

To project the sphere stereographically on the plane of the ecliptic.

1. On C, as centre, with radius C♋, which is taken at discretion, draw the ecliptic, which divide into twelve equal parts, viz. ♈ ♉, ♉ ♊, &c. To the points ♈, ♉, ♊, &c. right lines, drawn from the centre C, are circles of longitude; the most remarkable of which are ♈ ♎, and ♋ ♑, one the equinoctial, and the other the solstitial, colure.

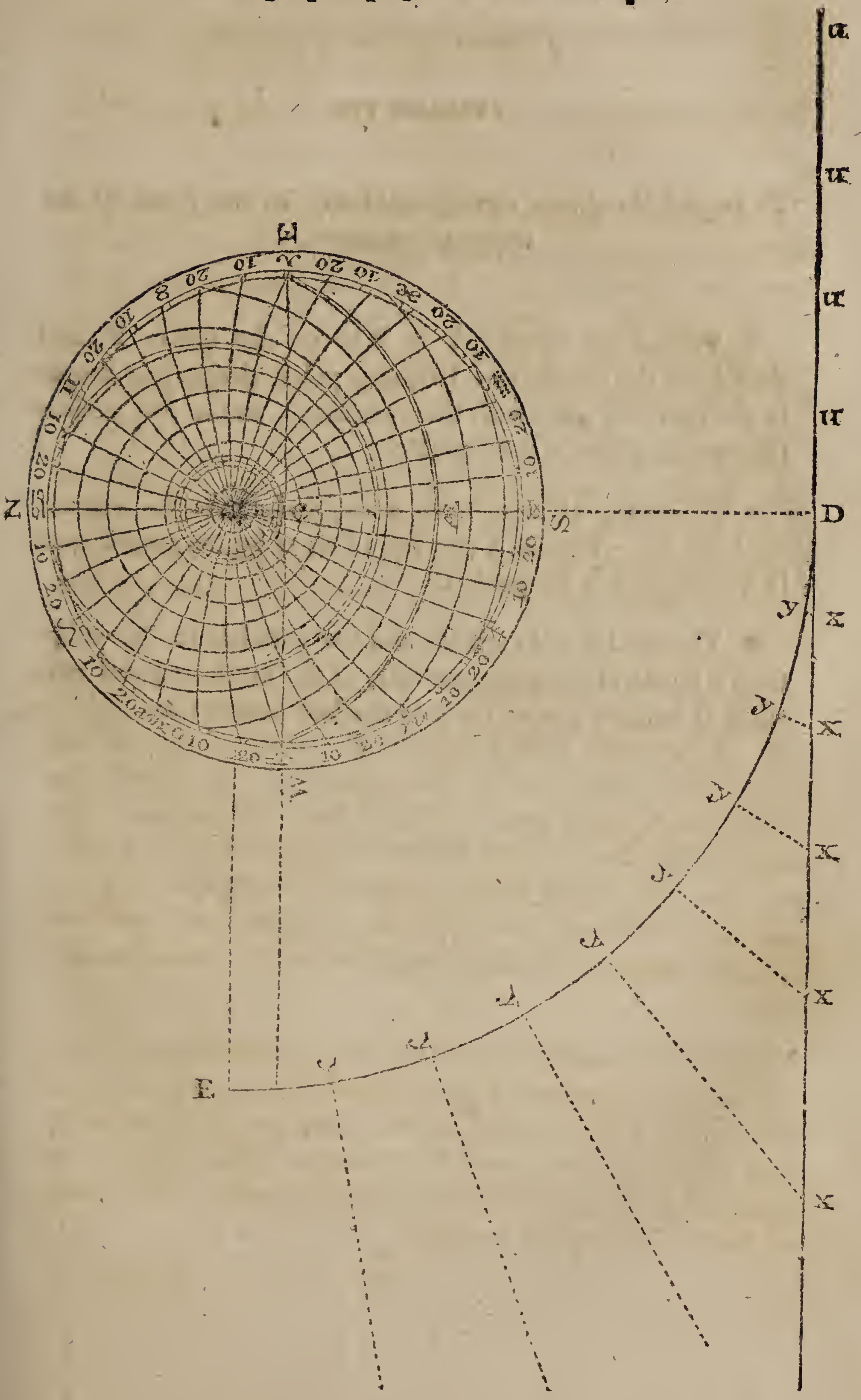
2. From ♈ toward ♉ set 23° 28′, and reduce the bounding point to P; then P will be the pole of the world; whence the meridians and parallels of latitude may be projected, as in the last Problem; but they are here projected by

ANOTHER METHOD.

1. At P, with radius PD = the semidiameter of the circle ♈P♎, project the quadrant Dyy, &c. E, which divide into nine equal parts at y, y, &c. Then draw the tangent Dxx, &c. From P reduce the points y, y, &c, to x, x, &c. which will be the centres of as many of the meridians. And these divisions being transferred from D to u, u, &c. will give as many other centres for other meridians.

2. For the parallels of latitude, tropics, equator and polar circles, reduce the pole P to O; from O, on each side, set 10° to a, a, which reduce to the diameter ♋ ♑ at b, b. And from b on one side of P to b on the other, will be the diameter of the parallel of 10° from the pole. The others are found in a similar manner.

Stereographic projection on the ecliptic.



PROBLEM VII.

To project the sphere stereographically on the plane of the tropic of capricorn.

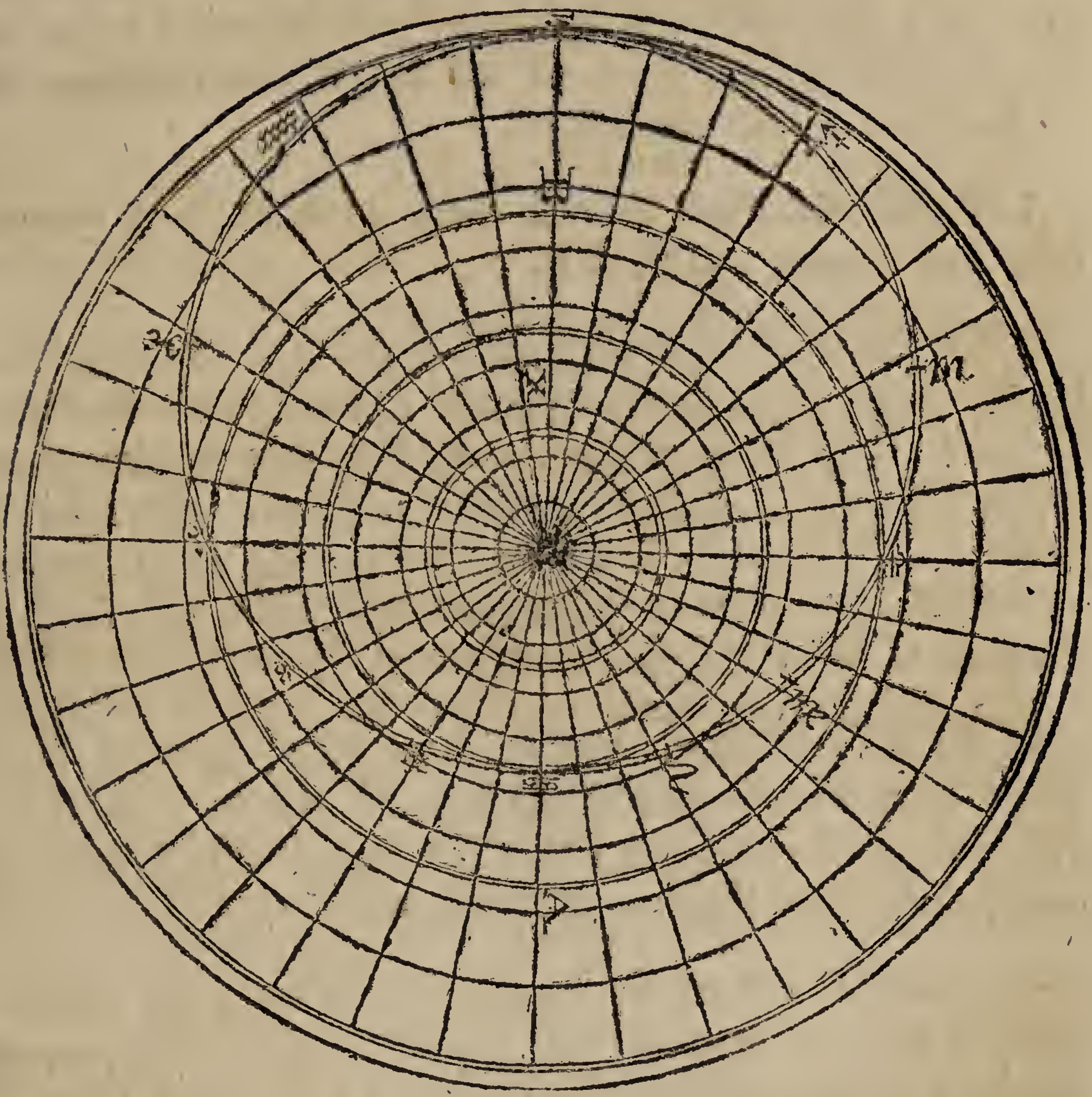
1. Project the equator A ♎ B ♈ on the centre P, and then draw the parallels of latitude, polar circle, and tropic in the northern part, with the meridians, as in the projection on the plane of the equator.

2. Bisect the distance ♑ ♋ at x, and draw the ecliptic ♈ ♉ ♊, &c. Divide it into the 12 signs by the proper rule in Stereographic Projection.

3. The circles of longitude and the parallels to the ecliptic are found in the same manner, as the meridians and parallels of latitude respectively in the last projection.

Stereographic Projection on Capricorn.

N

W  E

S

PROBLEM VIII.

To project the sphere stereographically on the plane of a circle oblique to the horizon.

EXAMPLE.

Required a stereographic projection of the sphere on the plane of a great circle, declining 24° 30′ westward from the south at Cambridge, in lat. 42° 23′ 28″ N, and reclining 36° 30′ from the zenith northward.

1. Project the great oblique circle HDOL, and quarter it by the diameters HO, DL.

2. Set the reclination northward from O to A, considering the upper as the northern part; reduce A to Z; then Z is the zenith of Cambrige, and Q is that of the place, where the oblique circle is a horizontal plane. Also reduce a, 90° distant from A, to z; aud through H, z, O, project HWzSO, the horizon of Cambridge, and continue it through O to E.

3. Set the western declination from H to y, from D to c and from O to d. From the pole Z reduce y, c, d, to W, S, E, on the horizon, which will be the west, south, and east, points of the horizon of Cambridge.

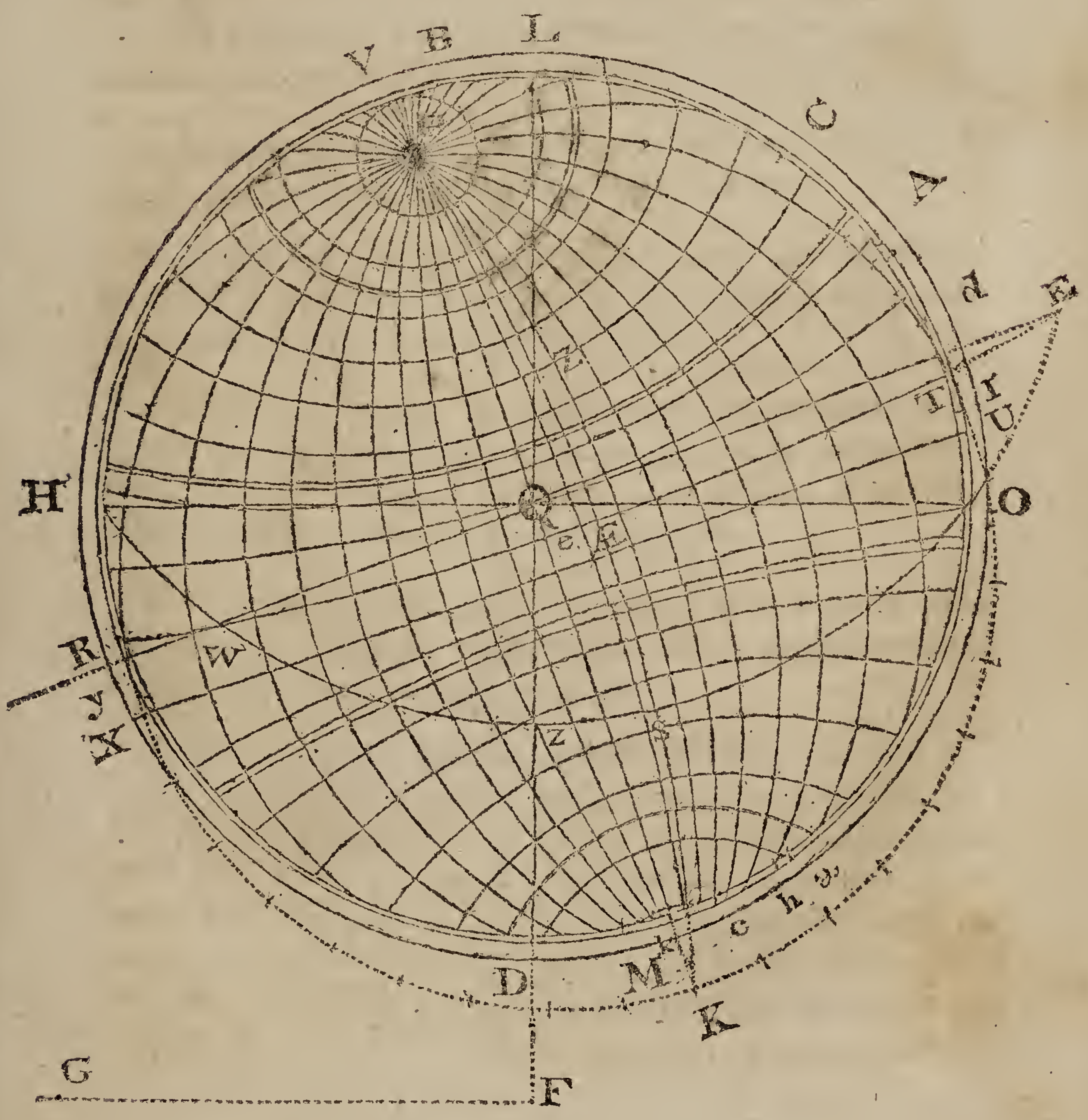
4. From Q to F set the tangent of 53° 30′, the complement of the reclination 36° 30′; from F raise the perpendicular FG, and continue the diameter EW till it intersect FG. Then with centre G, and radius GS, describe PZSK, the meridian of Cambridge.

5. A rule on W, Z, cuts the primitive at C. Set the complement of the latitude from C to B; and a rule over W, B, cuts the meridian at P, which is the north pole. Set a quadrant faom B to I, and also from I to M. Then a rule on W, I, cuts the meridian at Æ, the point where the equinoctial intersects it; and on W, M, it cuts the meridian produced in K, the south pole.

6. Draw the equinoctial RWÆTE through the three points W, Æ, E ; and VQK, the axis, and meridian of the place Q.

7. Through the three points R, P, T, describe the circle PRKT, and draw the diameter UX perpendicular to PQK. Then describe the meridians, and parallels of latitude, as in the stereographic projection on the plane of the horizon ; finding the intersections of the parallels with the axis by reducing from the point R.

Stereographic Projection on an oblique Circle.



NOTE. Every great circle of the sphere is the horizon of a certain place. And when the sphere is projected on an oblique circle, it is easy to determine the latitude of the place, where the primitive would be horizontal, and also its difference in longitude from the place, for which the projection is made.

Thus, in this projection, VP = Qe = the latitude ; and eÆ = kh = the difference, in longitude from Cambridge, of the place, where this oblique circle would be the horizon.

SPHERIC TRIGONOMETRY.

DEFINITIONS.

1. SPHERIC TRIGONOMETRY teaches the relations and calculation of the side and angles of spheric triangles.

2. A *spheric triangle* is a figure on the surface of a sphere, bounded by three arcs of great circles.

3. A spheric triangle, like a plane one, is *equilateral*, *isoscelar*, or *scalenous*, according as the three sides, two of them only, or no two of them, are equal.

4. A spheric triangle is *right-angled*, or *rectangular*, if it have one right angle, or more; *quadrantal*, or *rectilateral*, if one of the sides be a quadrant; and *oblique*, if it have neither a right angle, nor a quadrantal side.

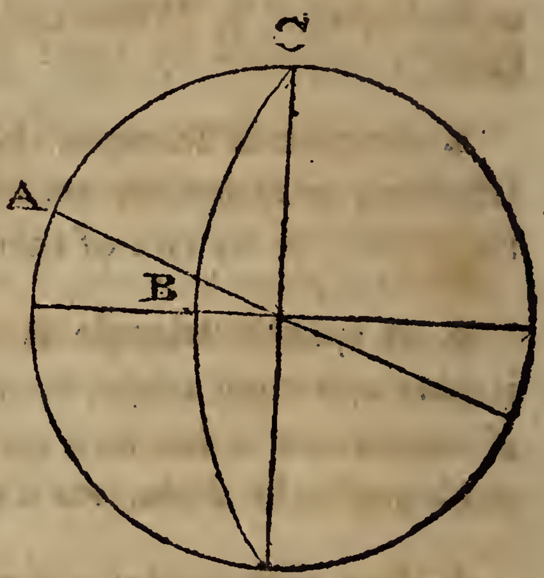
5. In a right-angled spheric triangle, as in a plane one, the side, opposite to the right angle, is called the *hypotenuse;* and the other two the *legs*, or *sides*.

6. Two sides of a spheric triangle are said to be *alike*, or of the *same affection* or *kind*, when they are each greater or less than a quadrant; and *unlike*, or of *different affection* or *kind*, when one is greater and the other less than a quadrant. Also two angles are *alike*, or of the *same affection* or *kind*, when they are both acute or obtuse; and *unlike*, or of *different affection* or *kind*, when one is acute and the other obtuse.

7. The *circular parts* of a right-angled spheric triangle are *five*, namely, the two legs, and the complements of the hypotenuse and the two oblique angles.

8. The right angle is not considered as separating the legs; and if no other part be situated between two of three circular parts, and thus separate them, that, which is in the middle, is called the *middle part;* and the other two *adjacent extremes*. But if one of three circular parts be separated from the other two, it is called the *middle part*, and the other two *opposite extremes*.

NOTE. To illustrate the meaning and application of these trigonometrical terms, first introduced into this branch of science by LORD NAPIER, a spheric triangle is annexed, together with the adjacent and opposite extremes, corresponding to any given mean or middle part.



In the figure, AC may be the base, AB the perpendicular, BC is the hypotenuse, C the angle at the base, and then B the angle at the perpendicular.—Then

If the middle part be	The adjacent extremes are	The opposite extremes are
AB	AC and Co. B	Co. BC and Co. C
AC	AB and Co. C	Co. BC and Co. B
Co. BC	Co. B and Co. C	AC and AB
Co. B	AB and Co. BC	AC and Co. C
Co. C	AC and Co. BC	AB and Co. B

GENERAL PROPERTIES OF SPHERIC TRIANGLES.

The following are among the properties, common to plane and spheric triangles, and the demonstrations in both cases are similar.

1. The greater, equal, or less side subtends the greater, equal, or less angle; and the contrary.

2. Two triangles are equal, 1. When the three sides of one are respectively equal to the three sides of the other. 2. When two sides and the included angle of one are respectively equal to two sides and the included angle of the other. 3. When two angles and the included side of one are respectively equal to two angles and the included side of the other. 4. When they have equal angles above equal bases.

Among the differences of plane and spheric triangles the following are very remarkable.

1. In a plane triangle, if only the three angles be given, only the *relative* lengths of the sides can be found. But in a spheric triangle, the sides being circular arcs, whose values are expressed in degrees as well as those of the angles, if the three angles be given, the sides may be thence determined.

2. Two angles always determine the third in a plane triangle; but never in a spheric triangle.

THEOREM. I.

The sum of any two sides of a spheric triangle is greater than the third.*

---

* Demonstration. The shortest distance between any two points on the surface of a sphere is the arc of a great circle passing through them. The sides of a spheric triangle then

THEOREM II.

The sum of the three sides of any spheric triangle is less than 360°.*

THEOREM III.

The sum of the three angles of any spheric triangle is greater than two right angles, and less than six.†

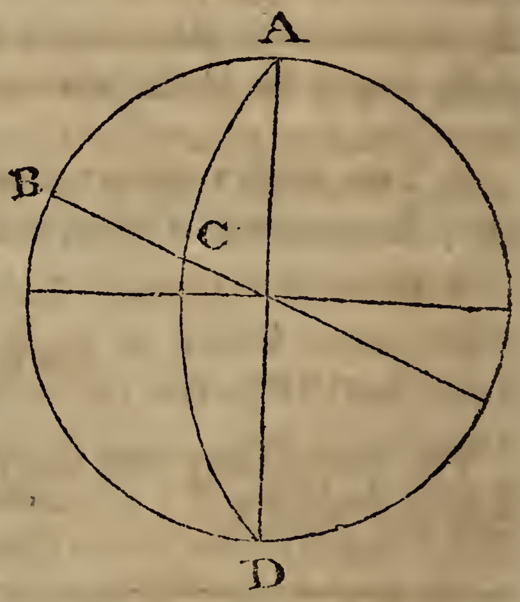
---

are the shortest distances between the angular points on the sphere, as the sides of a plane triangle are the shortest distances between the angular points on a plane. Therefore the sum of any two sides of a spheric triangle is greater than the third. Q. E. D.

* Demonstration. Let the sides AB, AC, containing any angle A, be produced till they meet again in D; and the arcs ABD, ACD, will be each 180°, since all great circles bisect each other; therefore ABD+ACD = 360°. But by the last Theo. DB+DC>BC; consequently, the sum of the three sides is less than 360°. Q. E. D.

[See the figure in the next note.]

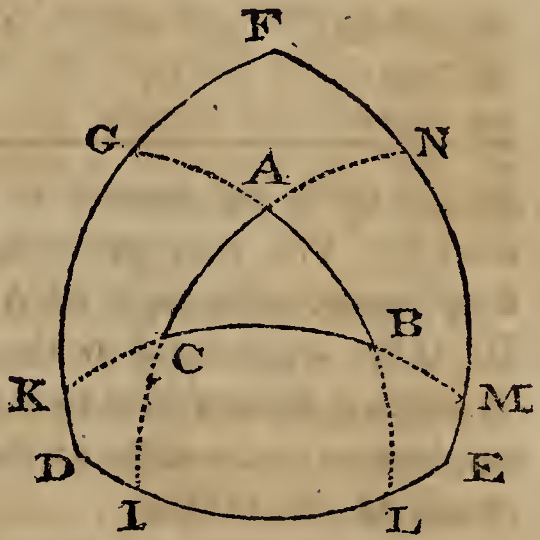
† Demonstration. If the sides AB, AC, BC, of the spheric triangle ABC, be supposed indefinitely short, they will approach indefinitely near to right lines, and the spheric surface indefinitely near to a plane surface; therefore the triangle may, in such case, be considered as a plane triangle. But the sum of the angles of a plane triangle is only equal to two right angles; therefore, while the sides of a spheric triangle are of a finite magnitude, the sum of its angles is greater than two right angles. Q. E. 1°D.



Cor. 1. Hence a spheric triangle may have its three angles either right or obtuse ; and therefore, if two be given, the third cannot be thence determined.

Cor. 2. If the three angles of a spheric triangle be each right or obtuse, the three sides are also each equal to, or greater than, 90° ; and if each of the angles be acute, each of the sides is less than 90° ; and the contrary.

THEOREM IV.



If from the three angles B, A, and C, of a spheric triangle BAC, as poles, there be described on the surface of the sphere three arcs of great circles FD, DE, and EF, which by their intersections form another spheric triangle DEF ; each side of the new triangle will be the supplement to the angle, which is at its pole. and each of its angles the supplement to that side in the triangle BAC, to which it is opposite.*

---

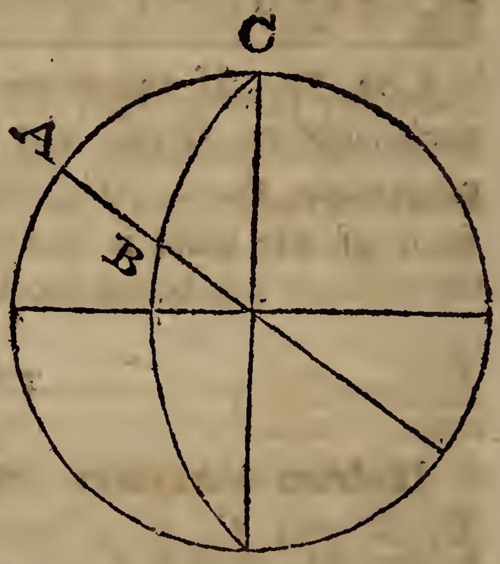
2. Every angle is less than two right angles ; therefore the sum of the three angles of every spheric triangle is less than six right angles. Q. E. 2° D.

* Demonstration. Let the sides AB, AC, and BC, of the triangle BAC, be produced till they meet those of the triangle DEF in the points G, L ; I, N ; K, M ; then, since the point A is the pole of the arc DILE, the distance of the points A, E, is 90° ; and, since C is the pole of the arc EF, the distance of the points C and E is also 90° ; therefore the point E is the pole of the arc AC. We may prove in a similar manner, that F is the pole of BC, and D the pole of AB. Hence evidently DL = 90°, and IE = 90° ; and therefore DL +IE = DL+EL+IL = DE + IL = 180°. Consequently the

RECTANGULAR SRHERIC TRIGONOMETRY.

THEOREM I.

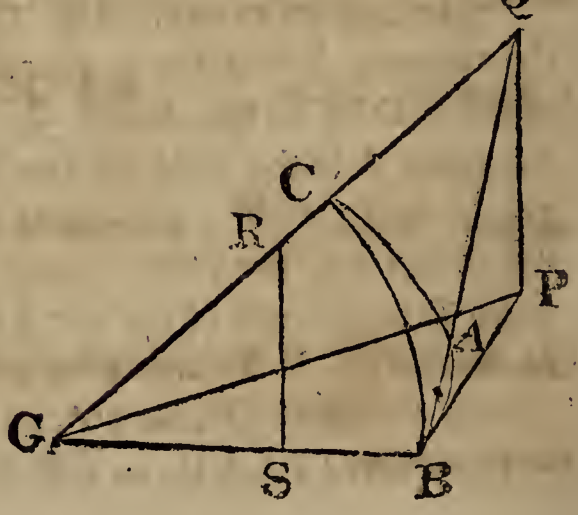
In any right-angled spheric triangle BAC, radius is to the sine of the hypotenuse, as the sine of either angle to that of its opposite leg ; and the contrary.*



arc DE is the supplement to the angle BAC, measured by the arc IL. We may prove in the same manner, that EF is the supplement to the angle BCA, measured by the arc MN, and that DF is the supplement to the angle ABC, measured by GK. Whence it follows, that each side of the triangle DEF is the supplement to the angle in the triangle BAC, which is at its pole. Q. E. 1°D.

2. Since the arcs AL and BG are each 90°, AL+BG = GL +AB =180° ; but GL is the measure of the angle EDF, and consequently AB its supplement. We may prove in the same manner, that AC and BC are the supplements to the angles at E and F. Therefore the angles of the triangle DEF are supplemental to the sides of the triangle BAC, which are opposite to them. Q. E. 2°D.

* Demonstration. Let GBPQ be a pyramid, composed of four right-angled triangles GBQ, GBP, GPQ, and BPQ ; and let AB, AC, and BC, be three circular arcs, described with the centre G of the sphere, and its radius GB, and evidently forming a spheric triangle BAC, right-angled at A.



Cor. If two right-angled spheric triangles have one common angle, the sines of their hypotenuses are as the sines of their legs, opposite to this angle.

Since the planes GQP, GBP, are perpendicular to each other, if the radius be made equal to unity, the values, specified in the following table, may be easily obtained for all the parts of the triangle; it being recollected, that tang. $=\frac{\text{sin.}}{\text{cos.}}$, and cot. $=\frac{RR}{\text{tang.}}=\frac{\text{cos.}}{\text{sin.}}$.

The Arc or Angle.	The Sine.	The Cos.	The Tang.
1. Arc BC, or ∠QGB	$\frac{BQ}{GQ}$	$\frac{BG}{GQ}$	$\frac{BQ}{BG}$
2. Arc BA, or ∠BGP	$\frac{BP}{GP}$	$\frac{GB}{GP}$	$\frac{BP}{BG}$
3. Arc AC, or ∠BGQ	$\frac{QP}{GQ}$	$\frac{GP}{GQ}$	$\frac{QP}{GP}$
4. ∠ABC, or QBP	$\frac{PQ}{BQ}$	$\frac{BP}{BQ}$	$\frac{QP}{BP}$
5. ∠BCA	$\frac{BP\times GQ}{BQ\times GP}$	$\frac{QP\times BG}{GP\times BQ}$	$\frac{BP\times GQ}{QP\times BG}$.

Now, to show how these several expressions are demonstrated, it is sufficient to give merely the demonstration of the first line. For this purpose assume any line GR, and regard it as the radius of the Tables; let fall the perpendicular RS on GB. Then it is evident, that RS is the sine of the arc BC, and GS its cosine. In the similar triangles GRS, GQB, as QG : QB :: GR $=1$: RS $=$ sin. BC $=\frac{QB}{QG}$; and QG : GB :: GR $=1$: GS $=$ cos. BC $=\frac{GB}{QG}$; whence is deduced tang. BC $=\frac{BQ}{BG}$, and cot. BC $=\frac{BG}{BQ}$. The other expressions are demonstrated in the same manner.

THEOREM II.

As radius is to the cosine of either angle, so is the tangent of the hypotenuse to the tangent of the leg adjacent to this angle.

That is, R : cos. B :: tang. BC : tang. AB;

Or, R : cos. C :: tang. BC : tang. AC.

COR. If two right-angled spheric triangles have one common leg, the tangents of their hypotenuses are in the inverse ratio of the cosines of the angles adjacent to this leg.

THEOREM III.

As radius is to the cosine of one of the legs, so is the cosine of the other leg to that of the hypotenuse.

That is, R : cos. AB :: cos. AC : cos. BC;

Or, R : cos. AC :: cos. AB : cos. BC.

The expressions for the cotangents, being obtainable by merely inverting those for the tangents, are not inserted in the table.

The truth of the first six Theorems is proved by substituting the particular values of the terms of the proportions to be demonstrated, and then comparing the product of the extremes with that of the means. For the two products will always be found to be exactly equal.

Thus, Theorem I. R : sin. BC :: sin. B : sin. AC.

That is, $1 : \frac{BQ}{GQ} :: \frac{PQ}{BQ} : \frac{QP}{GQ}$;

$\because \frac{QP}{GQ} = \frac{BQ}{GQ} \times \frac{PQ}{BQ} = \frac{PQ}{GQ}$. Q. E. D.

NOTE. By this Theorem, the expressions for the sine, cosine, and tangent of the angle BCA were obtained.

Cor. If two right-angled spheric triangles have one common leg, the cosines of their hypotenuses are as the cosines of their other legs.

THEOREM IV.

As radius is to the sine of either angle, so is the cosine of the adjacent leg to the cosine of the other angle.

That is, R : sin. B :: cos. AB : cos. C;

Or, R : sin. C :: cos. AC : cos. B.

Cor. If two right-angled spheric triangles have one common leg, the cosines of the angles, opposite to this leg, are to each other as the sines of the adjacent angles.

THEOREM V.

As radius is to the sine of one of the legs, so is the tangent of its adjacent angle to the tangent of the other leg.

That is, R : sin. AB :: tang. B : tang. AC;

Or, R : sin. AC :: tang. C : tang. AB.

Cor. If two right-angled spheric triangles have one common leg, the sines of their other legs are reciprocally as the tangents of the angles at these legs.

Cor. 2. If they have one common angle, the tangents of their legs, opposite to this angle, are as the sines of the legs adjacent to it.

THEOREM VI.

As radius is to the cotangent of one of the angles, so is the cotangent of the other angle to the cosine of the hypotenuse; or, which is the same, radius is to the cosine of the hypotenuse, as the tangent of one angle is to the cotangent of the other.

That is, R : cot. B :: cot. C : cos. BC ;

Or, R : cos. BC :: tang. B : cot. C :: tang. C : cot. B.

NOTE. These six Theorems are sufficient for the solutions of all the cases of right-angled spheric triangles.

THEOREM VII.

The product of radius and the sine of the middle part is equal to the product of the t*a*ngents* of the *a*djacent extremes, or to that of the c*o*sines of the *o*pposite extremes.†

NOTE. This Theorem is *general*, and equally applicable to every case of Rectangular Spheric Trigonometry. It is NAPIER'S *Method of the five circular parts*, and is sometimes called the *Catholic Proposition.*

---

* It will assist the memory in recollecting this proposition to observe, that the second letters in t*a*ngents and c*o*sines are respectively the same with the first letters of *a*djacent and *o*pposite.

† DEMONSTRATION. This Theorem may be demonstrated by substituting for each particular term the value, specified in the small table belonging to the demonstration of the first Theorem. Thus, if we assume AB, in the figure belonging to the definitions, for the middle part, AC and B are the adjacent extremes, and BC and C the opposite extremes ; then, according to this Theor. R × sin. AB = tang. AC × cot. B = sin. BC × sin. C. The expressions of the values of these quantities being taken from the aforesaid table, we have $R \times \frac{BP}{GP} = \frac{QP \times BP}{GP \times QP} = \frac{BQ \times BP \times GQ}{GQ \times BQ \times GP}$; each of which expressions is evidently the

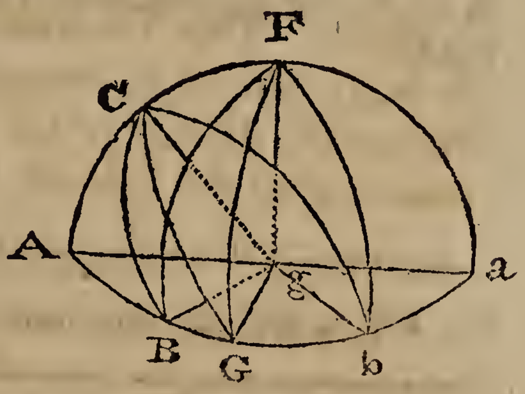
THEOREM VIII.

The angles at the hypotenuse are always of the same affection with their opposite sides ; and the hypotenuse is less or greater than a quadrant, according as the legs are of the same or different affection.*

Note. The converse of this Theorem is true in all its parts. And when the hypotenuse is exactly a quadrant, one or each of the legs is 90°.

---

same, when reduced to its lowest terms. In the same manner may any other case be proved. Therefore the Theorem is true in all cases. Q. E. D.



* Demonstration. If the sides AB and AC be each 90° ; then, since the angle A is right by supposition, the points B and C will be the poles of the arcs AC and AB ; and consequently the angles B and C will be right ones, being measured by arcs, which are supposed to be 90° ; that is, they will be of the same affection with their opposite sides.

To prove, that, when the legs AB and AC are both acute, their opposite angles will be so also ; let the side AC be produced to F, or till it be 90°. Then, as the point F will be the pole of the arc AB, the angle B will be also 90° ; and consequently the angle ABC, which is less than the angle ABF, will be acute. We may prove in the same manner, that the angle at C is acute, when the opposite side AB is acute.

It is equally evident, that the angles B and C, in the triangle BaC, right-angled at a, are obtuse, when the opposite sides aB, aC, are obtuse.

If one of the sides Ab be obtuse, and the other side AC acute, as in the right-angled triangle bAC, the angle at C will also be obtuse, and that at b acute. For, having taken the arc AG=90°

PROBLEM.

In a right-angled spheric triangle, any two of the six parts being given, beside the right angle, to find the other three.

This problem has six cases.

1. When the hypotenuse and a leg are given.
2. When the hypotenuse and an angle are given.
3. When a leg and its opposite angle are given.
4. When a leg and its adjacent angle are given.
5. When the two legs are given.
6. When the two angles are given.

These six cases have sixteen analogies; and the solutions of them, by the general Theorem of NAPIER, immediately follow.

CASE I.

EXAMPLE. Given the hypotenuse $AC = 55^\circ\ 8'$, and the leg $BC = 32^\circ\ 12'$; to find the rest.

---

on ABb, and drawn the arc GC from the point G to the point C, the angle ACG will be right, since G is the pole of the arc AC; whence it follows, that the angle ACb will be obtuse. For the same reason the angle abC, in the right-angled triangle baC, will be obtuse, it being opposite to the obtuse side aC; and consequently the supplement of it AbC acute, that is, of the same affection with its opposite side. Therefore the angles at the hypotenuse are of the same affection with their opposite sides.

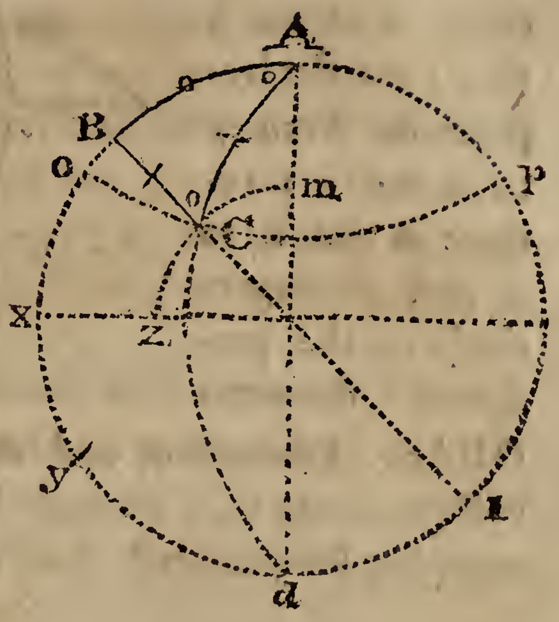
Q. E. 1°. D.

2. It is evident, that BC, considered either as the hypotenuse of the triangle BAC, or that of the triangle BaC, is less than BF; and that the hypotenuse bC, in the triangle baC, is greater than bF; whence it follows, that the hypotenuse of any right-angled spheric triangle is always less than 90°, when the two legs are of the same affection; and greater, when they are of different affection. Q. E. 2°. D.

PROJECTION OF THE TRIANGLE.*

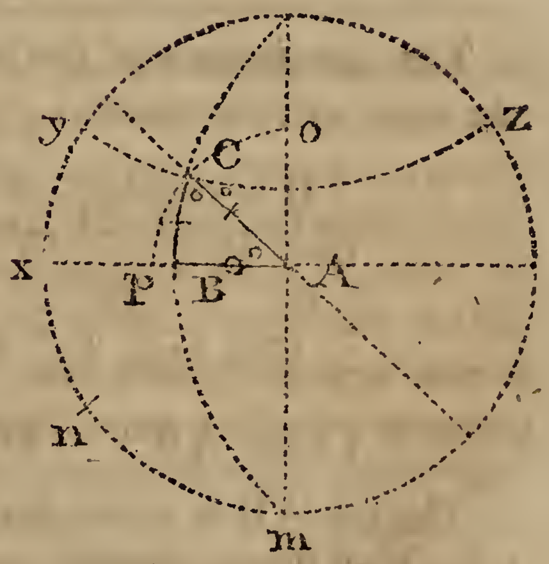
1. At the circumference of the primitive.

Set BC from x to y ; reduce y to z, and draw the quadrant zm. Set the hypotenuse from A to o and p, and draw the parallel oCp ; through C, the point of its intersection with zm, draw the right circle BCl, and the oblique circle ACd.



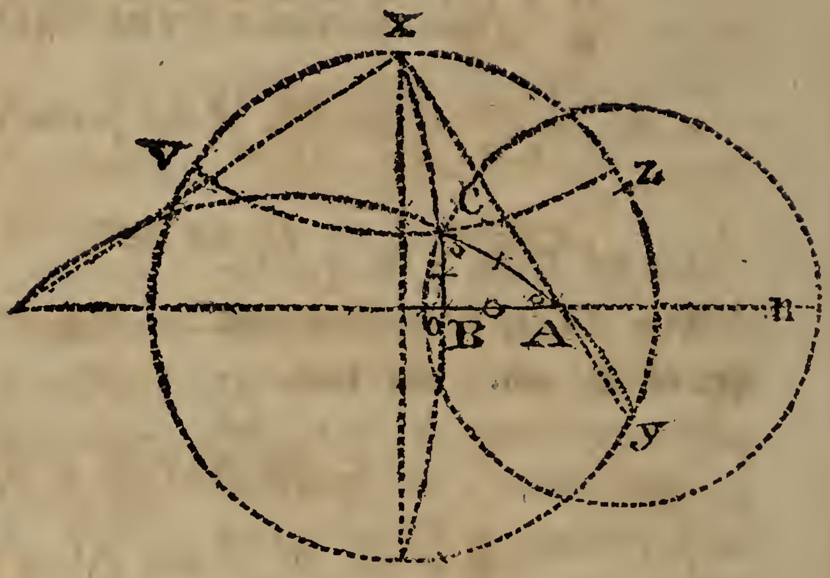
2. At the centre.

Set BC from x to y, and project the parallel circle yCz. Set the hypotenuse from m to n ; reduce n to p ; and draw the quadrant pCo ; through C, the intersection of the parallel and quadrant, draw the right circle CA.



* In each of the following examples of the six Cases, the triangle is projected at the circumference, at the centre, and in the plane, or between the centre and circumference, of the primitive.

3. In the plane.



Draw the parallel circle vCz, as in the last case. Reduce A to y; set the hypotenuse from y to m and z; reduce m and z to o and n respectively on the produced diameter OBAn. Bisect n o, and draw the circle oCn; draw the oblique circle xCd through C; then, by Prob. IV, 2, Stereographic Projection,* draw the hypotenuse AC.

CALCULATION.

1. *To find the other leg.*

The complement of the hypotenuse and the two legs are the three circular parts.

R × cos. hyp. = cos. leg × cos. leg.

Cos. BC†	32° 12′	ar. co.	0·0725305
: R	90°		10·
: : Cos. hyp. AC	55° 8′		9·7571444
: Cos. AB	47° 30′ 4″		9·8296749

The leg AB is acute, because the hypotenuse and given leg are of the same affection.

* In the rest of this work, whenever there is reference to a Problem, without specification of the branch, to which it belongs, it is to be understood as a Problem in Stereographic Projection.

† If the middle part be required, radius must be the first term; but if one of the extremes be required, the other extreme

2. *To find the angle* C.*

R × cos. ∠ C = tang. BC × cot. AC.

R	90°	10·
: Tang. BC	32° 12′	9·7991569
: : Cot. AC	55 8	9·8430739
: Cos. C	63 58 30″	9·6422308

The ∠ C is acute, because the hypotenuse and given leg are of the same affection.

3. *To find the* ∠ A.

R × sin. BC = sin. AC × sin. A.

Sin. AC	55° 8′	ar. co.	0·0859296
: R	90		10·
: : Sin. BC	32 12		9·7266264
: Sin. A	40 30 5″		9·8125560

The ∠ A is acute, because the hypotenuse and given leg are of the same affection.

CASE 2.

EXAMPLE. Given the hypotenuse AC = 55° 8′, and the angle A = 40° 30′ 5″ ; to find the rest.

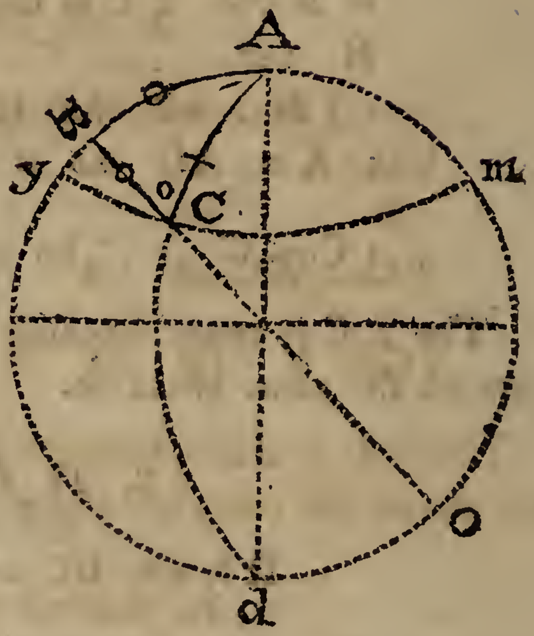
---

must be the first term. And it must be recollected, that the cosine of the complement is the sine, and the tangent of the complement the cotangent, and the contrary.

* The three sides and right angle being known, the other angles may be determined by this proposition.—The sines of the sides are proportional to the sines of the opposite angles. But if an angle be first required, the process must be as above.

PROJECTION OF THE TRIANGLE.

1. At the circumference.

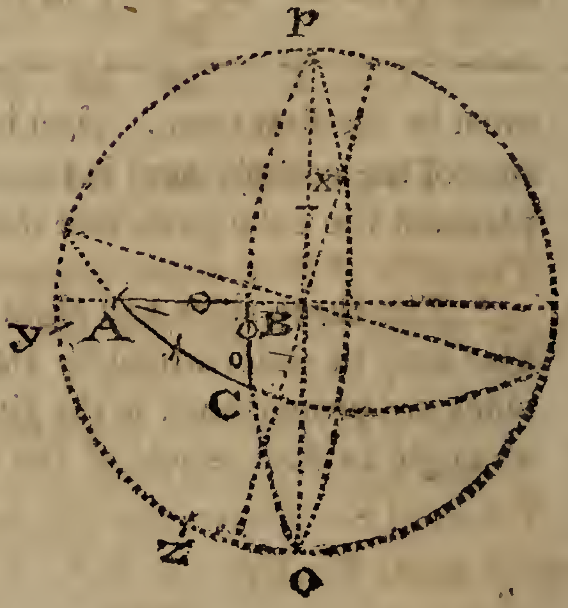


Project the angle A, by Prob. II, 2. Set the hypotenuse from A to y and m; draw the parallel circle yCm, by Prob. IX, 2; and through C, the point of intersection with ACd, draw the right circle BCo.

2. At the centre.

Project the angle, by Prob. II, 1. Set the hypotenuse from x to y; reduce y to C; and project the oblique circle pCBO.

3. In the plane.



Project the angle, by Prob. II, 3. From x, the pole of AC, reduce A to y; set the hypotenuse from y to z; reduce z to C; and through C draw the oblique circle pCo.

CALCULATION.

1. *To find the angle* C.

R × cos. hyp. = cot. A × cot. C.

Cot. A	40°	30′	5″	ar. co.	9·9315202
: R	90				10·
:: Cos. AC	55	8			9·7571444
: Cot. C	63	58	30		9·6886646

The ∠ C is acute, because the hypotenuse and given angle are of the same affection.

2. *To find the opposite leg* BC.

R × sin. BC = sin. A × sin. AC.

R	90°			10·
: Sin. AC	55	8′		9·9140704
: : Sin. A	40	30	5″	9·8125567
: Sin. BC	32	12		9·7266271

The side BC is acute, because the hypotenuse and given angle are of the same affection.

3. *To find the adjacent leg* AB.

R × cos. A = cot. AC × tang. AB.

Cot. AC	55°	8′		ar. co.	0·1569261
: R	90				10·
: : Cos. A	40	30	5″		9·8810365
: Tang. AB	47	30	4		10·0379626

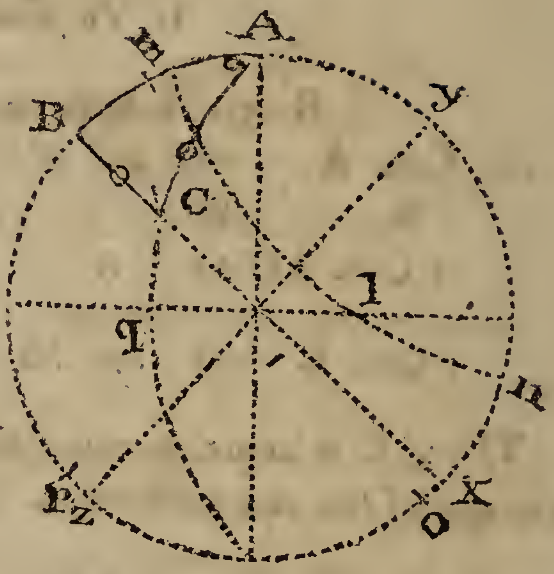
The side AB is acute, because the hypotenuse and given angle are of the same affection.

CASE 3.

Example. Given the leg AB=47° 30′ 4″, and its opposite angle C=63° 58′ 30″; to find the rest.

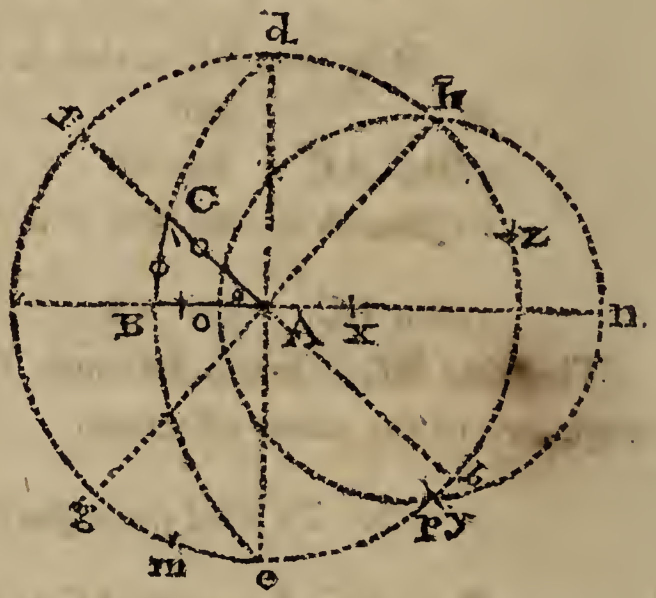
PROJECTION OF THE TRIANGLE.

1. At the circumference.



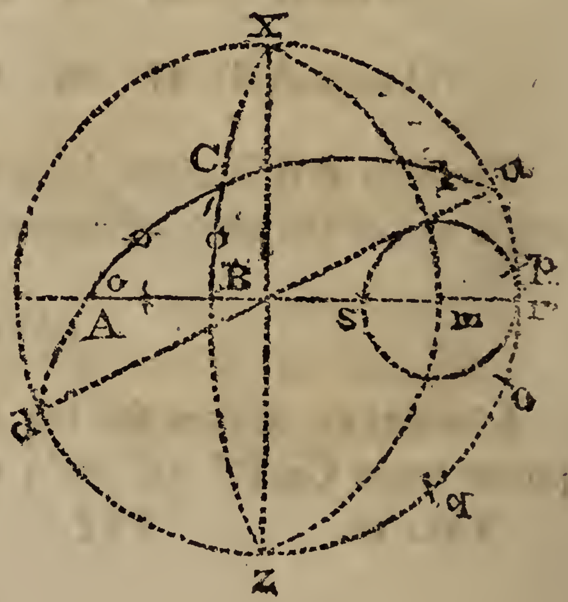
Make AB equal to the given leg; draw BCx, and yz perpendicular to it. Set the given angle C from y to m and n; and draw the parallel circle mln. Lay a rule over A, l, and it will cut the primitive at o; make op a quadrant; reduce p to q; through q draw the oblique circle ACq.

2. At the centre.



Draw AB, by Prob. VI, 2; and through B draw the oblique circle dBe. Find its pole x; reduce x to y; set the given angle from y to m and z; reduce m, z, to o, n. Bisect on; and draw the parallel circle poh; through h, the point where it intersects the primitive, draw the diameter hg, and bf perpendicular to it.

3. In the plane.



Draw the given leg AB, by Prob. VI, 3; and project the oblique circle xBz; through its pole m draw xmz; and reduce m to o. Set the complement of the angle C from o to p and q; and draw the parallel circle rls; through the point of intersection l draw the right circle ud; and through u, A, d, draw uCAd

CALCULATION.

1. *To find the other angle* A.

R × cos. C = cos. AB × sin. A.

Cos. AB	47° 30′ 4″	ar. co.	0·1703258
: R	90		10·
: : Cos. C	63 58 30		9·6422303
: Sin. A	40 30 5		9·8125561

The angle A is here ambiguous, or doubtful, as it does not appear from the data alone, whether BC, A, and AC, be each greater or less than 90°; but the projection shows, that they are each less than 90°.

2. *To find the other leg* BC.

R × sin. BC = tang. AB × cot. C.

R	90°	10·
: Tang. AB	47 30′ 4″	10·0379644
: : Cot. C	63 58 30	9·6886626
: Sin. BC	32 12	9·7266270

The side BC is ambiguous, as before observed.

3. *To find the hypotenuse* AC.

R × sin. AB = sin. C × sin. AC.

Sin. C	63° 58′ 30″	ar. co.	0·0464323
: R	90		10·
:: Sin. AB	47 30 4		9·8676386
: Sin. AC	55 8		9·9140709

The hypotenuse is ambiguous, as before observed.

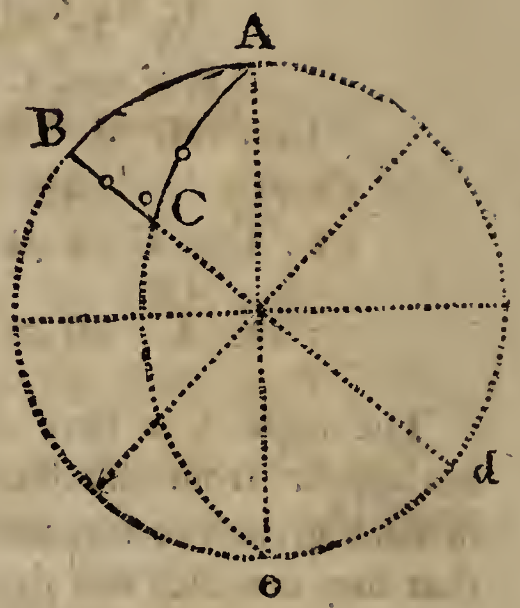
CASE 4.

EXAMPLE. Given the leg AB = 47° 30′ 4″, and the adjacent angle A = 40° 30′ 5″; to find the rest.

PROJECTION OF THE TRIANGLE.

1. At the circumference.

Set the given leg AB from A to B; draw ACe, to make the angle A, by Prob. II, 2; and draw BCd.



2. At the centre.

Make the angle A, by Prob. II, ' ; and set AB from A to B, by Prob. VI. 2 ; and draw the oblique circle dBe.

3. In the plane.

Draw the angle A, by Prob. II, 3 ; reduce A to x ; set AB from x to y ; reduce y to B ; and project the circle nBm.

CALCULATION.

1. *To find the other leg* BC.

R × sin. AB = cot. A. × tang. BC.

Cot. A	40°	30′	5″	ar. co.	9·9315202
: R	90				10·
: : Sin. AB	47	30	4		9·8676386
: Tang. BC	32	12			9·7991588

The side BC is of the same affection as the given angle A.

2. *To find the other angle* C.

R × cos. C = cos. AB × sin. A.

R	90°			10·
: Sin. A	40	30′	5″	9·8125567
: : Cos. AB	47	30	4	9·8296742
: Cos. C	63	58	30	9·6422309

The angle C is like the given leg AB.

3. *To find the hypotenuse* AC.

R × cos. A = tang. AB × cot. AC.

Tang. AB	47°	30′	4″	ar. co.	9·9620356
: R	90				10·
:: Cos. A	40	30	5		9·8810365
: Cot. AC	55	8			9·8430721

The hypotenuse AC is acute, because the given leg and angle are alike.

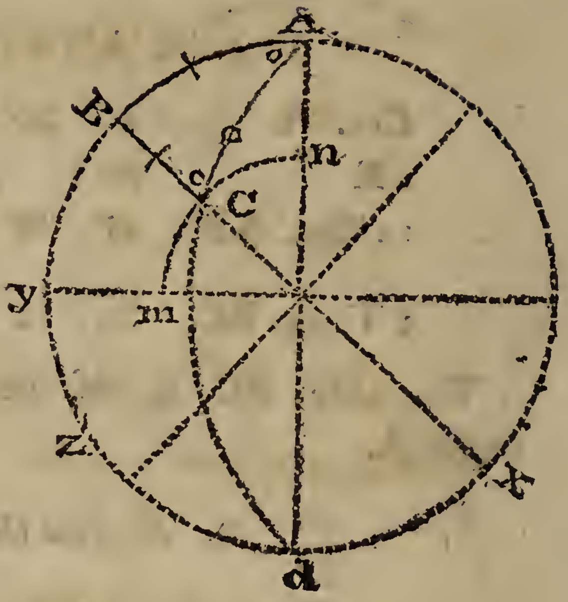
CASE 5.

EXAMPLE. Given the two legs, namely, AB = 47° 30′ 4″, and BC = 32° 12′; to find the rest.

PROJECTION OF THE TRIANGLE.

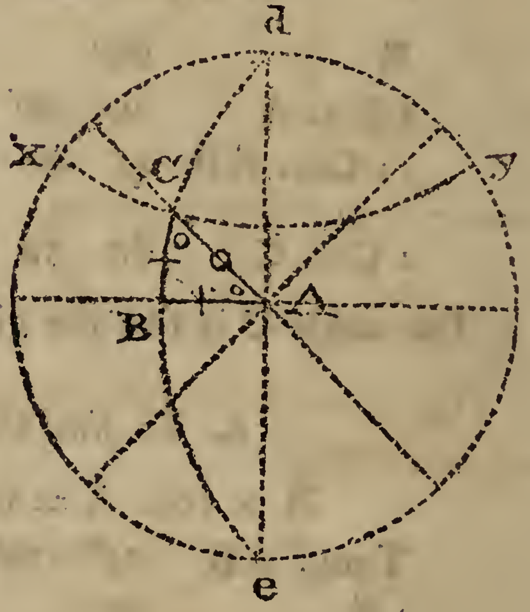
1. At the circumference.

Set AB on the circumference, and draw the right circle BCx. Set BC from y to z; reduce z to m; draw the quadrant mCn; and through C, where it intersects BCx, draw the oblique circle ACd.



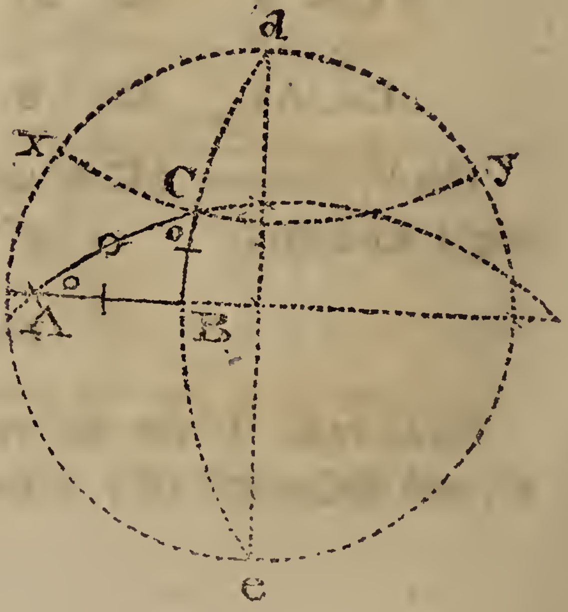
2. At the centre.

Set AB from A to B, by Prob. VI, 2; through d, B, e, project the oblique circle dBe; on which set the other leg BC, by means of the parallel circle xCy.



3. In the plane.

Project AB, by Prob. VI, 3; and draw BC, by means of the oblique circle dCBe, and the parallel circle xCy. Through the points A, C, by Prob. IV, 2, project the hypotenuse AC.



CALCULATION.

1. *To find the hypotenuse* AC.

R × cos. AC = cos. AB × cos. BC.

R	90°	10·
: Cos. AB	47 30′ 4″	9·8296742
: : Cos. BC	32 12	9·9274695
: Cos. AC	55 8	9·7571437

The hypotenuse is acute, because the legs are alike.

2. *To find the angle* A.

R × sin. AB = tang. BC × cot. A.

Tang. BC	32° 12′	ar. co.	0·2008431
: R	90		10·
: : Sin. AB	47 30 4″		9·8676386
: Cot. A	40 30 5		10·0684817

The angle A is acute, being of the same affection with the opposite leg.

3. *To find the other angle* C.

R × sin. BC = tang. AB × cot. C.

Tang. AB	47° 30′ 4″	ar. co.	9·9620356
: R	90		10·
: : Sin. BC	32 12		9·7266264
: Cot. C	63 58 30		9·6886620

The angle C is acute, being like the opposite leg.

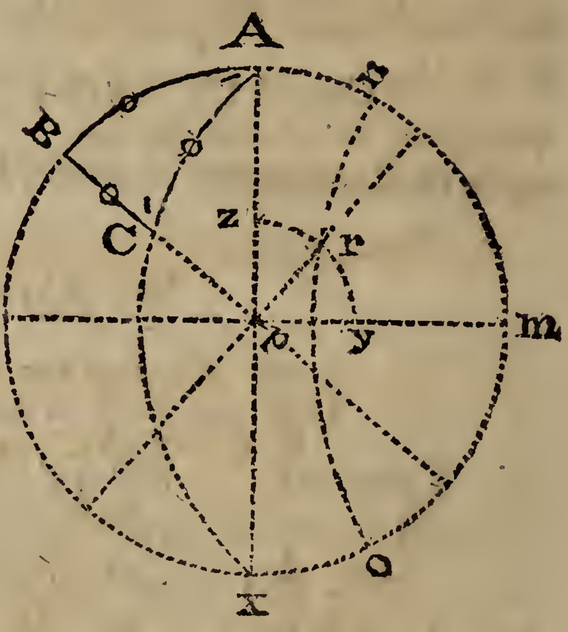
This analogy is like the second.

CASE 6.

EXAMPLE. Given the two angles, namely, A = 40° 30′ 5″, and C = 63° 58′ 30″ ; to find the rest.

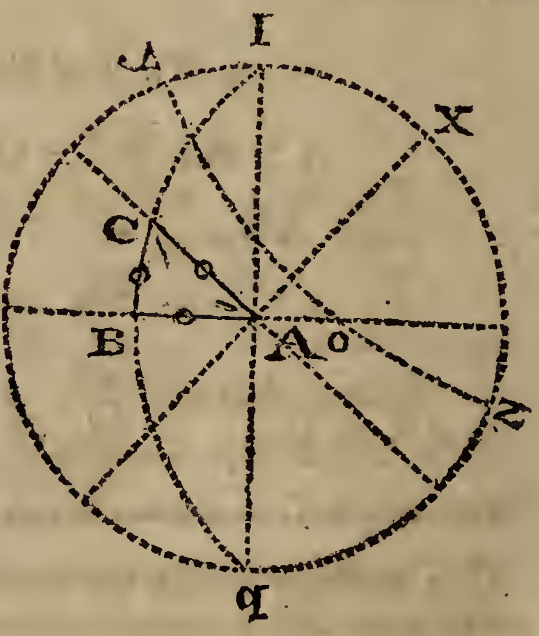
PROJECTION OF THE TRIANGLE.

1. At the circumference.



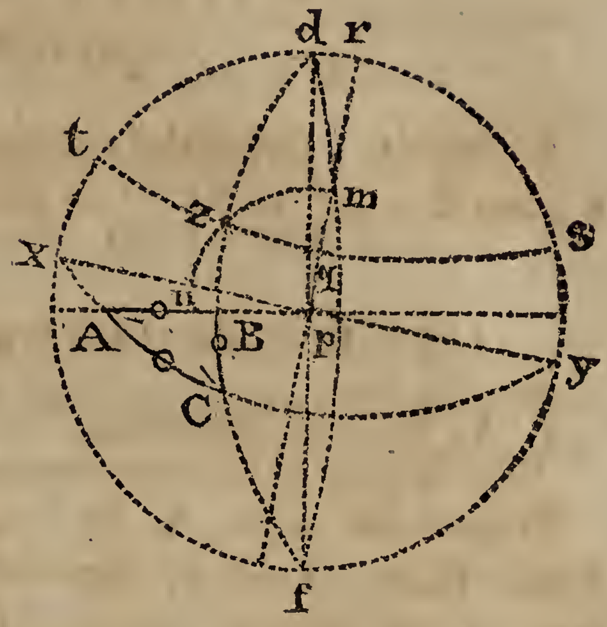
Project the angle A, by Prob. II, 2. Find y, the pole of the oblique circle ACx ; and draw the quadrant yz. Set the angle C from m to n and o ; draw the parallel circle nro ; through r, its intersection with the quadrant, draw the right circle rp, and BCp perpendicular to it.

2. At the centre.



Project the angle A, by Prob. II, 1. Set the angle C from x to y and z ; and draw the parallel circle yoz ; the point of intersection o will be the pole of the oblique circle ICBq.

3. In the plane.



Project the angle A, by Prob. II, 3. Find m, the pole of xACy ; draw the quadrant mn ; set the angle C from r to s and t ; draw the parallel circle tqs ; and through the point z, its intersection with the quadrant mn, draw the oblique circle dzBCf.

CALCULATION.

1. *To find the hypotenuse* AC.

R × cos. AC = cot. A × cot. C.

R	90°			10·
: Cot. A	40	30′	5″	10·0684798
:: Cot. C	63	58	30	9·6886626
: Cos. AC	55	8	1*	9·7571424 +

The hypotenuse is acute, the angles being of the same kind.

2. *To find the leg* AB.

R × cos. C = sin. A × cos. AB.

Sin. A	40°	30′	5″	ar. co.	0·1874433
: R	90				10·
:: Cos. C	63	58	30		9·6422303
: Cos. AB	47	30	4		9·8296736

The leg AB is acute, like the opposite angle.

* The log. of the 4th term, in this and the third analogy, gives more than half a second beside 55° 8′, on account of the fraction of a second, neglected in other parts of the triangle.

3. *To find the other leg* BC.

R × cos. A = sin. C × cos. BC.

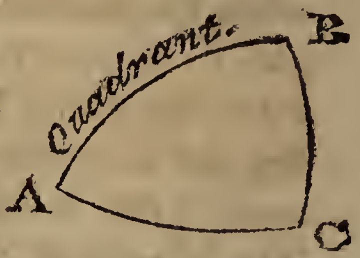
Sin. C	63°	58′	30″	ar. co.	0·0464323	
: R	90				10·	
:: Cos. A	40	30	5		9·8810365	
: Cos. BC	32	12	1		9·9274688	+

The leg BC is acute, like the opposite angle.

This analogy is like the second.

RECTILATERAL SPHERIC TRIGONOMETRY.

If the supplemental triangle to a rectangular spheric triangle be drawn, by Theor. IV. General Properties of Spheric Triangles, one side will be a quadrant, being the supplement of a right angle; and therefore it will be a quadrantal or rectilateral spheric triangle. And as the sine, cosine, and tangent of an arc are the same as the sine, cosine, and tangent of its supplement respectively, the equations for the rectangular are all equally applicable to its supplemental rectilateral triangle. Hence a rectilateral spheric triangle ABC may be solved like a rectangular one by the method of five circular parts. The quadrantal side, called the *quadrant*, or *radius*, like the right angle, is not considered as separating the adjacent parts. The two angles adjacent to the quadrant, the complement of the other angle, opposite to the quadrant and called the *hypotenusal angle*, and the complements of the other two sides are to be regarded as the five circular parts.



Thus, in the triangle ABC, AB is the quadrant, or radius, the angle A, the complement of AC, the complement of the hypotenusal angle C, the complement of CB and the angle B are the circular parts.

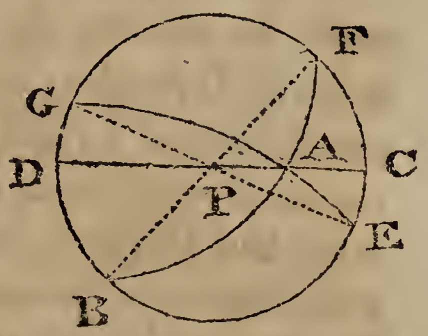
The ambiguous cases are the same as in rectangular triangles, that is, when one side and its opposite angle are given.

OBLIQUE SPHERIC TRIGONOMETRY.

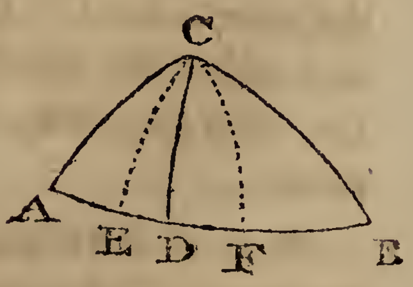
THEOREM I.

In a spheric triangle, if the angles at the base be of the same affection, the perpendicular, falling on the base from the opposite angle, is within the triangle ; if they be of different affection, it is without.*

Cor. If the two least sides of a spheric triangle be of the same affection, the perpendicular on the other will be within the triangle.†



* Demonstration. In the triangle GAB, the base being GB and its pole P, the perpendicular must pass through the pole P ; let it fall on the base at D, and on the base of FAE at C. Then, as the angles PGB and PBG are both right, AGB and ABG are both obtuse, when the perpendicular CD falls between them. Also the angles PFE and PEF are both right ; therefore, in the triangle FAE, the angles F and E are both acute, when AC falls within. But the perpendicular AC falls without the triangle BAE, whose angle ABE at the base is acute, and the other AEB obtuse. The same will be found true in any other triangle. Hence the proposition is inferred. Q. E. D.



† Demonstration. Let AB be the greatest side ; make BE = BC, and AF = AC. Then, as either of the equal sides of an isoscelar triangle is of the same affection as its opposite angle, the angle BEC is of the same affection as CB, and AFC of the same affection as AC. But AC, BC, are of the same affection, and consequently the angles EFC, FEC.

THEOREM II.

A perpendicular being drawn, the product of the sine of one segment of the base and the tangent of its adjacent angle is equal to the product of the sine of the other segment and the tangent of its adjacent angle.*

THEOREM III.

In a spheric triangle, the sines of the sides are as the sines of their opposite angles.†

PROBLEM.

In an oblique spheric triangle three parts of the six being given, to find the rest.

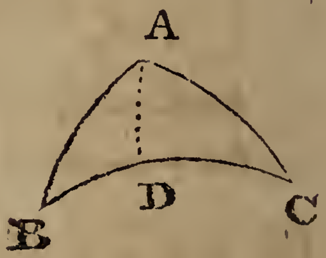
This Problem has six cases.

Therefore, by this Theorem, the perpendicular CD falls within the triangle ECF or ACB. Q. E. D.

NOTE. If AB be less than AC, or CB, but greater than their supplements, they being of the same affection, the perpendicular will be within. For this and the adjoining triangle, whose common base is AB, have the same perpendicular.

* DEMONSTRATION. Let AB, last figure, be the base, and CD the perpendicular. Then, by Theor. V, Rect. Spher. Tri.
sin. BD : R : : tang. DC : tang. B,
and sin. AD : R : : tang. DC : tang. A ;
∵ sin. BD × tang. B = sin AD × tang. A. Q. E. D.

† DEMONSTRATION. Let BAC be a spheric triangle, and from any angle, as A, draw a perpendicular AD to the opposite side BC. Then, by the Catholic Proposition,



R × sin. AD = sin AB × sin. B,
R × sin. AC = sin. AC × sin. C ;
∵ sin. AB × sin. B = sin. AC × sin C,
and sin. AB : sin. AC : : sin. C : sin. B. Q. E. D.

1. When two sides and an opposite angle are given.
2. When two sides and the included angle are given.
3. When two angles and an opposite side are given.
4. When two angles and the included side are given.
5. When the three sides are given.
6. When the three angles are given.

In order to solve the four first cases, a perpendicular may be drawn from an angle to its opposite side, continued if necessary; by which mean, if the perpendicular be within the given triangle, it is divided into two rectangular triangles; but if without, a rectangular triangle is joined to the given oblique triangle. And thus two rectangular spheric triangles are formed, by solving which the required solution may be obtained.

For the purpose of having sufficient data in one of the rectangular triangles, it is directed to draw the perpendicular so, that two of the data may be in one of the rectangular triangles.

In each of the four first cases, a side and its adjacent angle being given; let this side be set on the primitive, and the perpendicular drawn from its unknown adjacent angle to the side, with which it forms the given adjacent angle. The perpendicular being drawn, the segments of the base and their opposite angles are determined, according to the Catholic Proposition, from the following proportions.

As Cot. AC : R :: Cos. C : tang. PC.
As Cot. AB : R :: Cos. B : tang. PB.

And,

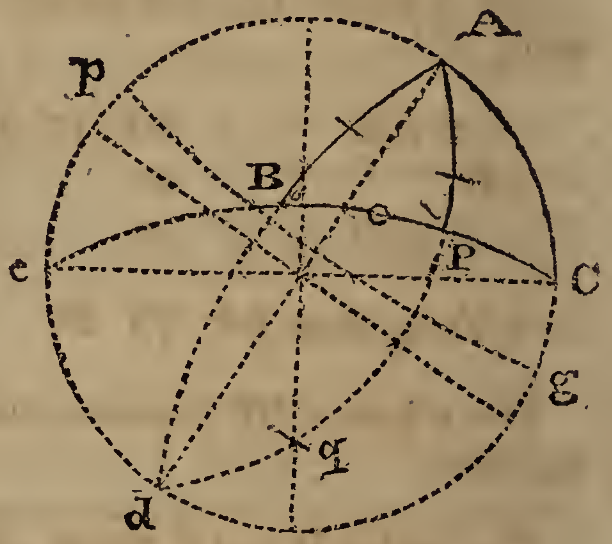
As Cot. C : R :: Cos. AC : Cot. PAC.
As Cot. B : R :: Cos. AB : Cot. BAP.

The sum or difference of the segments, or angles, must be taken for the base, or its opposite angle respectively, according as the perpendicular falls within, or without the given triangle.

CASE 1.

EXAMPLE. Given two sides, namely, AC = 58°, and AB = 79° 17′ 14″, and the opposite angle C = 62° 34′ 6″; to find the rest.

PROJECTION OF THE TRIANGLE.



The primitive and perpendicular diameters being drawn, as in every other case; set AC = 58° on the primitive from C to A; draw the diameter Ad; draw the oblique circle CBe, making the angle BCA = 62° 34′ 6″; about A describe the parallel circle pg, at the distance of 79° 17′ 14″ = AB; through the point B, where the parallel circle pg cuts the oblique circle CBe, and the points A and d, describe a circle. Then ABC is the triangle required.

Draw the perpendicular AP, which is the arc of a great circle, projected through q, the pole of eBC, A and d.

CALCULATION.

1 *To find the angle* B.

Sin. AB	79° 17′ 14″	ar. co.	0·0076359
: Sin. C	62 34 6		9·9481982
:: Sin. AC	58		9·9284205
: Sin. B	50		9·8842546

According to the data, the angle B may be either acute or obtuse, but the projection shows it to be acute.

2. *To find the side or base* BC.

Cot. AC	58°	ar. co.	0·2042108
: R	90		10·
:: Cos. C	62 34′ 6″		9·6634092
: Tang. Seg. PC	36 24		9·8676200

The segment PC is acute, the hyp. AC and the angle C being alike.

Cot. AB	79° 17′ 14″	ar. co.	0·7231180
: R	90		10·
:: Cos. B	50		9·8080675
: Tang. Seg. BP	73 36		10·5311855

The segment BP is acute, the hyp. AB and angle B being alike.

The angles B and C being alike, the perpendicular falls within the triangle ; therefore BP + PC = BC = 110°.

3. *To find the angle* A.

Sin. AC	58°	ar. co.	0·0715795
: Sin. B	50		9·8842540
:: Sin. BC	110		9·9729358
: Sin. A*	121 54′ 56″		9·9288193

According to the data, the angle A may be either acute or obtuse, but the projection shows it to be obtuse.

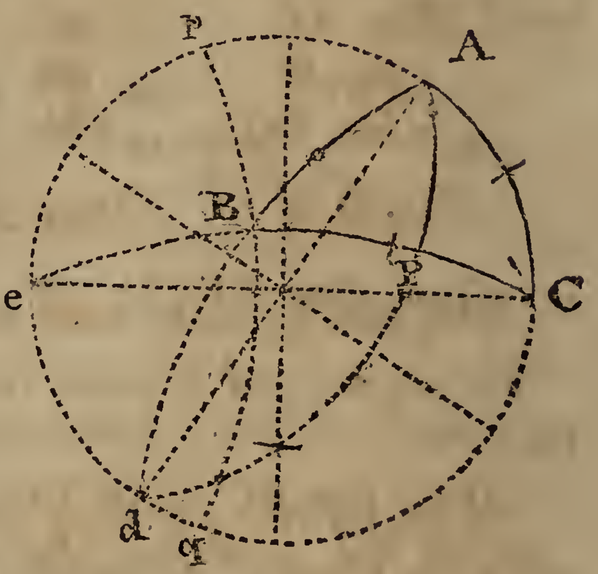
---

* Or the angle A might have been found before BC, and then BC determined as A is here.

CASE 2.

EXAMPLE. Given two sides, namely, AC = 58°, and BC = 110°, and the included angle C = 62° 34′ 6″; to find the rest.

PROJECTION OF THE TRIANGLE.



Draw the oblique circle eBC, to make the angle at C = 62° 34′ 6″; set 58° from C to A; about e, at the distance of 70°, the supplement of 110°, draw the parallel circle pq, which will cut the oblique circle eBC in B; draw the diameter Ad; and through d, B, A, describe a circle. Then ABC is the triangle required.

Draw the perpendicular AP.

CALCULATION.

1. *To find the segment* PC.

Cot. AC	58°	ar. co.	0·2042108
: R	90		10·
:: Cos. C	62 34′ 6″		9·6634092
: Tang. PC	36 24		9·8676200

This analogy is the same as the second of Case 1. 110° — 36° 24′ = 73° 36′ = BP.

2. *To find the angle* B.

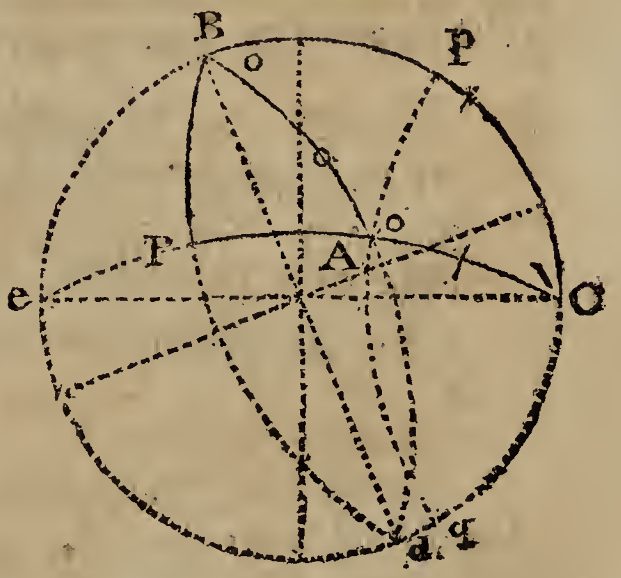
Sin. Seg. BP	73° 36′	ar. co.	0·0180392
: Sin. PC	36 24		9·7733614
:: Tang. C	62 34 6″		10·2847890
: Tang. B	50 0 1		10·0761896 +

3. *To find the angle* A.

Sin. AC	58°	ar. co.	0·0715795
: Sin. B	50		9·8842540
:: Sin. BC	110		9·9729858
: Sin. A	121 54′ 56″		9·9288193

This analogy is the same as the last of Case 1.

If the side BC = 110° be set on the primitive, the projection will be as in the annexed figure; and the solution as follows.



1. *To find the leg* PA.

Cot. BC	110°	ar. co.	0·4389341
: R	90		10·
:: Cos. C	62 34′ 6″		9·6634092
: Tang. PC	128 18 38		10·1023433

The side PC is obtuse, the hyp. and the angle C being unlike.
PC — AC = the segment AP = 70° 18′ 38″.

2. *To find the perpendicular* PB.

R	90°	10·
: Sin. BC	110	9·9729858
:: Sin. C	62 34′ 6″	9·9481982
: Sin. PB	56 30 56	9·9211840

PB is acute, like its opposite angle C.

3. *To find* BA.

R	90°	10·
: Cos. PA	70 18′ 38″	9·5275291
:: Cos. PB	56 30 56	9·7417113
: Cos. BA	79 17 15	9·2692404 +

The hypotenuse is acute, because the legs are alike.

4. *To find the angle* A.

Sin. AB	79° 17′ 14″	ar. co.	0·0076359
: Sin. C	62 34 6		9·9481982
:: Sin. BC	110		9·9729858
: Sin. A	121 54 56		9·9288199

According to the data, the angle A may be acute or obtuse, but the projection shows it to be obtuse.

5. *To find the angle* B.

Sin. AB	79° 17′ 14″	ar. co.	0·0076359
: Sin. C	62 34 6		9·9481982
:: Sin. AC	58		9·9284205
: Sin. B	50		9·8842546

According to the data, the angle B may be acute or obtuse, but the projection shows it to be acate.

The solution of this case may be performed more concisely in the following manner; the segments PC and PA being found by the first analogy.

6. *To find the angle* BAP, *the supplement of the angle* BAC, *by Theorem* II.

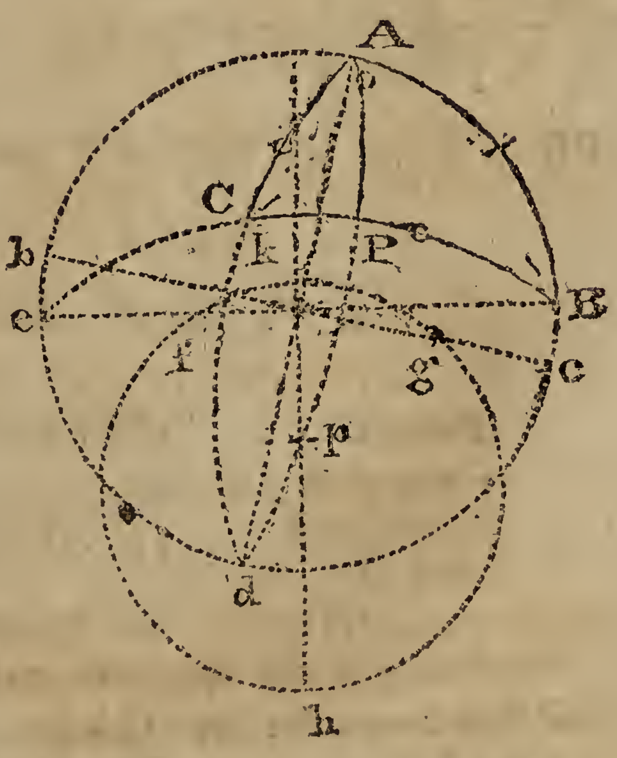
Sin. PA	70°	18′	38″	ar. co.	0·0261646
: Sin. PC	128	18	38		9·8946827
:: Tang. C	62	34	6		10·2847890
: Tang. BAP	58	5	4		10·2056363

180°—58° 5′ 4″=121° 54′ 56″= the angle BAC; which being known, the rest may be found by Theor. III.

CASE 3.

Example. Given two angles, namely, B = 50°, and C = 62° 34′ 6″, and the opposite side AB = 79° 17′ 14″; to find the rest.

PROJECTION OF THE TRIANGLE.



Set 79° 17′ 14″ on the primitive from B to A; draw the oblique circle eCB, making the angle ABC = 50°; draw the diameter Ad, and bc perpendicular to it; about p, the pole of eCB, describe the small circle fgh, parallel to eCB, at the distance of 62° 34′ 6″ = the angle C from the pole p, intersecting bc in the point g, the pole of ACd. Hence find k ; and through A, k, d, draw the oblique circle Akd.

Then ABC will be the triangle required.

Draw the perpendicular AP.

CALCULATION.

1. *To find the side* CA.

Sin. C	62° 34′ 6″ ar. co.	0·0518018
: Sin. AB	79 17 14	9·9923641
: : Sin. B	50	9·8842540
: Sin. CA	58	9·9284199

According to the data, the side CA may be acute or obtuse, but the projection shows it to be acute.

2. *To find the segment* PB.

Cot. AB	79° 17′ 14″ ar. co.	0·7231180
: R	90	10·
: : Cos. B	50	9·8080675
: Tang. PB	73 36	10.5311855

PB is acute, the hyp. AB and the angle B being alike.

3. *To find the other segment* PC.

Tang. C	62° 34′ 6″ ar. co.	9·7152110
: Tang. B	50	10·0761865
: : Sin. PB	73 36	9·9819608
: Sin. PC	36 23 59	9·7733583

PC is acute, the hyp. CA being acute.
PB+PC = the side CB=110°.

4. *To find the angle* A.

Sin. CA	58°	ar. co.	0·0715795
: Sin. B	50		9·8842540
: : Sin. CB	110		9·9729858
: Sin. A	121 54′ 56″		9·9288193

According to the data, the angle A may be acute or obtuse, but the projection shows it to be obtuse.

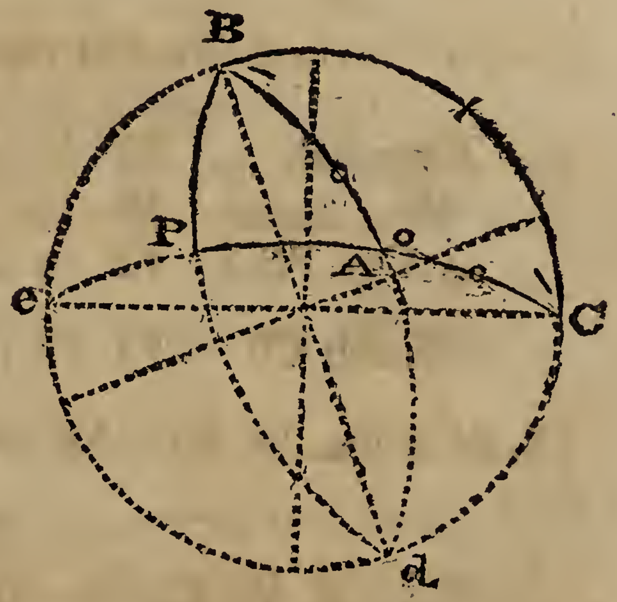
This analogy is the same as the last of Case 1.

CASE 4.

EXAMPLE. Given two angles, namely, B=50°, and C=62° 34′ 6″, and the included side BC=110°; to find the rest.

PROJECTION OF THE TRIANGLE.

Set 110° on the primitive from C to B; draw the oblique circle ePAC, making the angle BCA =62° 34′ 6″; and draw the oblique circle BAd, making the angle CBA =50°. Then ABC is the triangle required.



Draw the perpendicular BP.

CALCULATION.

1. *To find the angle* PBC.

Cot. C	62° 34′ 6″	ar. co.	0·2847890
: R	90		10·
: : Cos. BC	110		9·5340517
: Cot. PBC	123 22 56		9·8188407

The angle PBC is obtuse, the hyp. BC and the angle C being unlike.

PBC — ABC = 123° 22′ 56″ — 50° = 73° 22′ 56″ = the angle PBA.

2. *To find the perpendicular* PB.

R	90°	10·
: Sin. BC	110	9·9729858
: : Sin. C	62 34′ 6″	9·9481982
: Sin. PB	56 30 56	9·9211840

PB is acute, like its opposite angle.

This analogy is the same as the second under the second projection of Case 2.

3. *To find* BA.

Tang. PB	56° 30′ 56″ ar. co.	9·8205267
: R	90	10·
: : Cos. PBA	73 22 56	9·4563443
: Cot. BA	79 17 15	9·2768710 +

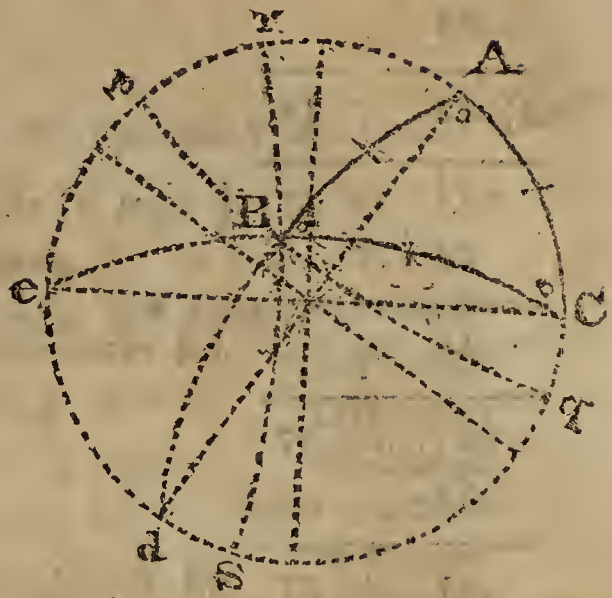
BA is acute, because the side BP and angle PBA are alike.

These being determined, the rest may be found as before, by Theor. III.

CASE 5.

Example. Given the three sides, namely, AB = 79° 17′ 14″, BC = 110°, and AC = 58°; to find the rest

PROJECTION OF THE TRIANGLE.



Set 58° = CA on the primitive from C to A; about A, at the distance of 79° 17′ 14″ = AB, describe the parallel circle pq; draw also the parallel circle rs about e, at the distance of 70°, the supplement of BC; through B, the intersection of pq and rs, draw the oblique circles ABd, and CBe. Then ABC is the triangle required.

CALCULATION.

To find the angle A.

1. From half the sum of the three sides subtract each of the two sides, which contain the required angle.

2. Add together the sines of these two remainders and the co-arcs of the sines of the sides, which contain the angle.

3. Half the sum of these four logarithms will give the sine of half the required angle.*

---

* Let s be put for the sum of the three sides, r for radius, a for the side opposite to the angle sought, b and c for the sides including the said angle. Then a Theorem, demonstrated by Writers on Spherics, may be thus expressed:

$$\text{Sin.}\ \tfrac{1}{2}\ \text{angle} = \sqrt{\frac{r^2 \times \sin.\ \overline{\tfrac{1}{2}s - b} \times \sin.\ \overline{\tfrac{1}{2}s - c}}{\sin.\ b \times \sin.\ c}}.$$

Whence, if logarithms be used for the quantities,

$$\text{Sin.}\ \tfrac{1}{2}\ \text{angle} = \frac{\sin.\ \overline{\tfrac{1}{2}s - b} + \sin.\ \overline{\tfrac{1}{2}s - c} + \text{co-ar.} \sin.\ b + \text{co-ar.} \sin.\ c}{2};$$

and this, expressed by words, is the rule.

79°	17′	14″		
110				
58				
2)247	17	14		
123	38	37	= ½ sum of the three sides.	
79	17	14		
44	21	23	1st remainder, the sine	9·8445513
123	38	37		
58				
65	38	37	2d remainder, the sine	9·9595173
			Co-arc sine 58°	0·0715795
			Co-arc sine 79 17′ 14″	0·0076359
				2)19·8832840
			Sine 60° 57′ 28″	9·9416420
			2	
			121 54 56 = the required ∠ A.	

By a similar operation the angles B and C may be found; but, when one angle is known, and the sides, the other two are most easily determined by Theor. III.

ANOTHER METHOD.

1. Find the difference of the half sum of the three sides, and the side opposite to the required angle.

2. Add together the sine of the said half sum, the sine of the said difference, and the co-arcs of the sines of the two containing sides.

3. Half the sum of these four logarithms will be the co-sine of half the required angle.*

* The same letters being used, as in the note under the last rule, the Theorem may be thus expressed:

Note. This rule is preferable, when the required angle is between 90° and 180°; and the other, when it is less than 90°.

Half the sum of the three sides	123° 38′ 37″,	the sine		9·9203841
Side opposite ∠ A	110			
Difference	13 36 37,	the sine		9·3726945
Co-arc sine	58			0·0715795
Co-arc sine	79 17 14			0·0076359
			2)	19·3722940
Cos.	60 57 28			9·6861470
	2			
Angle A =	121 54 56			

CASE 6.

Example. Given the three angles, namely, A = 121° 54′ 56″, B = 50°, and C = 62° 34′ 6″; to find the rest.

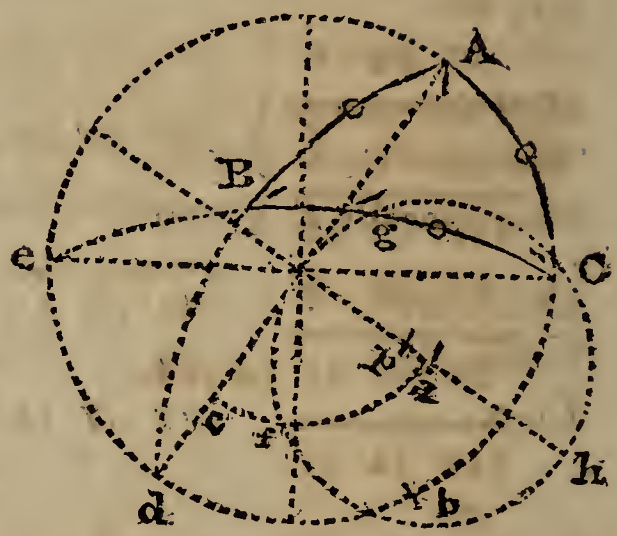
$$\text{Cos.}\ \tfrac{1}{2}\text{angle} = \sqrt{\frac{r^2 \times \text{sin.}\ \overline{\tfrac{1}{2}s - a} \times \text{sin.}\ \tfrac{1}{2}s}{\text{sin.}\ b \times \text{sin.}\ c}}.$$

Whence, logarithms being used,

$$\text{Cos.}\ \tfrac{1}{2}\ \text{angle} = \frac{\text{sin.}\ \overline{\tfrac{1}{2}s - a} + \text{sin.}\ \tfrac{1}{2}s + \text{co-ar. sin.}\ b + \text{co-ar.sin.}\ c}{2};$$

which, expressed by words, is the rule.

PROJECTION OF THE TRIANGLE.



Draw the oblique circle ABd, making the angle BAe = 58° 5′ 4″, the supplement of the given ∠A; set the angle C = 62° 34′ 6″ from d to b; reduce b to a, and describe the quadrant ac; about the pole p of ABd describe the small parallel circle fgh, and the intersection f of fgh and the quadrant ac will be the pole of eBC; describe eBC about f, as its pole, and through e, C. Then ABC is the triangle required.

CALCULATION.

To find the side BC.

1. From half the sum of the three angles subtract each of the angles next to the required side.

2. Add together the cosines of these two remainders and the co-arcs of the sines of each of the adjacent angles.

3. Half the sum of these four logarithms will give the cosine of half the required side.*

---

* Let s be put for the sum of the three angles, a for the angle opposite to the required side, b and c for the adjacent angles; then the Theorem may be expressed thus:

$$\text{Cos. } \tfrac{1}{2} \text{ side} = \sqrt{\frac{r^2 \times \cos. \overline{\tfrac{1}{2}s - b} \times \cos. \overline{\tfrac{1}{2}s - c}}{\sin. b \times \sin. c}};$$

Whence, logarithms being used,

$$\text{Cos. } \tfrac{1}{2} \text{ side} = \frac{\cos. \overline{\tfrac{1}{2}s - b} + \cos. \overline{\tfrac{1}{2}s - c} + \text{co-ar. sin. } b + \text{co-ar. sin. } c}{2};$$

which, in words, is the rule.

121° 54′ 56″					
50					
62 34 6					
2)234 29 2					
117 14 31					
62 34 6					
54 40 25	1st rem.		the cos.	9·7621032	
117 14 31					
50					
67 14 31	2d rem.		the cos.	9·5875321	
∠B 50			co-arc sine	0·1157460	
∠C 62 34 6			co-arc sine	0·0518018	
				2)19·5171831	
		Cos. 55°		9·7585915	
		2			
		110 = the required side BC.			

In a similar manner the other sides may be found, but one side being known, and the angles, the other sides are more easily determined by Theorem III.

ANOTHER METHOD.

1. Find the difference of the half sum of the three angles and the angle, opposite to the required side.

2. Add together the cosine of the said half sum, the cosine of the said difference, and the co-arcs of the sines of the other two angles.

3. Half the sum of these four logarithms will be the sine of half the required side*.

Note, This rule is not accurate, when the required side is near 180° ; but is proper, when the said side is less than 90°. The preceding rule is not accurate, when the said side is near 0°; but is proper, when it is more than 90°.

Half the sum of the three angles		117°	14′	31,	the cos.	9·6606273
∠A opposite BC		121	54	56		
Difference		4	40	25,	the cos.	9·9985536
	∠B	50			co-arc sin.	0·1157460
	∠C	62	34	6	co-arc sin.	0·0518018
						2)19·8267287
	Sine	55°				9·9133643
		2				
		110 = the required side BC.				

SOLUTIONS OF THE FOUR FIRST CASES WITHOUT A PERPENDICULAR.

The solutions of the four first cases of Oblique Spheric Triangles by means of a perpendicular is generally preferred. Another method, in which no perpendicular is required, has also, in some cases, peculiar advantages to recommend it. And it is contained in the following Theorems.

---

* The letters being used to represent the same things, as in the last note, the Theorem may be thus expressed :

$$\text{Sin. } \tfrac{1}{2} \text{ side } = \sqrt{\frac{r^2 \times \cos. \tfrac{1}{2}s \times \cos. \overline{\tfrac{1}{2}s - a}}{\sin. b \times \sin. c}};$$

Whence, logarithms being used,

$$\text{Sin. } \tfrac{1}{2} \text{ side } = \frac{\cos. \tfrac{1}{2}s + \cos. \overline{\tfrac{1}{2}s - a} + \text{co-ar. sin. } b + \text{co-ar. sin. } c}{2},$$

which, in words, is the rule.

THEOREM IV.

As the sine of half the sum of two sides
Is to the sine of half their difference,
So is the cotangent of half their contained angle
To the tangent of half the difference of the other angles.

NOTE. If the sum of the two sides exceed a semicircle, subtract each from 180°, and proceed with the remainders as the sides ; the result will be the supplement of the required angle.

THEOREM V.

As the cosine of half the sum of two sides
Is to the cosine of half their difference,
So is the cotangent of half the contained angle
To the tangent of half the sum of the other angles.

THEOREM VI.

As the sine of half the sum of two angles
Is to the sine of half their difference,
So is the tangent of half the contained side
To the tangent of half the difference of the other sides.

THEOREM VII.

As the cosine of half the sum of two angles
Is to the cosine of half the difference,
So is the tangent of half the contained side
To the tangent of half the sum of the other sides.

These four Theorems may be applied not only to the second and fourth cases, but also to the first and third by only varying the order of the terms. And, as an excercise in their application, examples, given under those cases, may be performed by them, and the results compared with those before obtained.

A SHORT METHOD

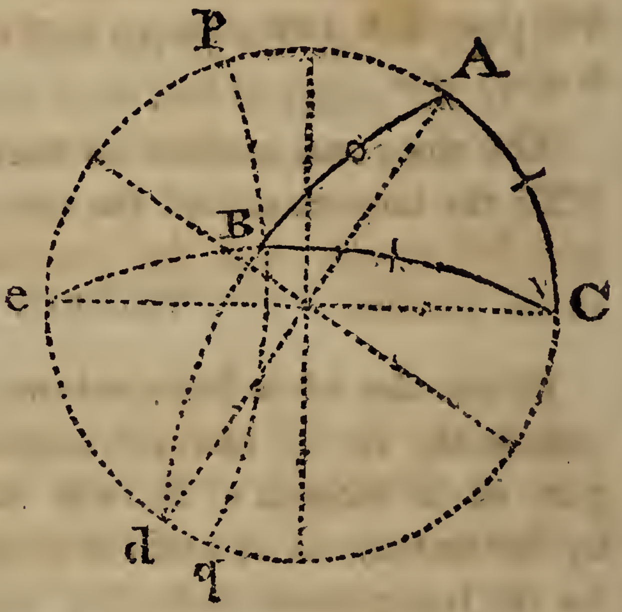
To find the third side, when two sides and the included angle are given.

1. To the sines of the two containing sides add double the sine of half the contained angle.

2. From half the sum of these three logarithms subtract the sine of half the difference of the sides, the remainder is the tangent of an arc.

3. Subtract the sine of this arc from the said half sum of the three logarithms, and the remainder is the sine of half the required side.

EXAMPLE. Given the side AC = 58°, the side BC = 110°, and the contained angle C = 62° 34′ 6″; to find the third side AB.



Side AC = 58°,	sine	9·9284205
Side BC = 110°	sine	9·9729858
½ contained ∠C = 31 17′ 3″	doub. sine	19·4308082
		2)39·3322145
		19·6661072
½ diff. sides 26°	sine	9·6418420
Tangent 46 35′ 59″		10·0242652
		19·6661072
Sine 46 35 59		9·8612783
Subtracted from ½ sum of logs. rem. =		9·8048289

Rem. sine of 39° 38′ 37″
2

79 17 14 = the required side AB.

Note. This case frequently occurs in astronomical calculations.

IMPROVED SOLUTIONS OF CERTAIN CASES OF SPHERIC TRIANGLES.

If an arc be nearer 0 or 180° than 90°, that is, $<45°$ or $> 135°$, it is called an *extreme arc;* but if it be nearer 90° than 0 or 180°, that is, between 45° and 135°, it is called a *mean arc.*

The sines and cosines increase very unequally from 0 to 90°; the increments of the sines near 0 being great, and near 90° small; and the contrary being true with respect to the cosines.

When the term required is obtained by a tangent, or cotangent, it is always accurate, or true to a small part of a second. It will also be exact, when found by the sine of an extreme arc, or the cosine of a mean arc; for the logarithmic difference corresponding to one second of the arc is then large. But it will be the less accurate, the nearer the arc is to 90°, if given by its sine; or to 0 or 180°, if given by its cosine; for the logarithmic difference decreases, and at length vanishes. When the arc is distant 4° from 90°, if given by its sine; or from 0 or 180°, if given by its cosine, the error may amount to one second. And it increases in the same ratio as this distance decreases.

In the solution of spheric triangles error may arise in the use of the common rules from two sources. 1. Inaccuracy in taking out the logarithmic sine or cosine of the first arc, when the tabular difference is large. 2. Uncertainty of the required arc, found by its sine or cosine, when the tabular difference is small.

To prevent error in such cases:—find from the data the tangent or cotangent of some other part in the first operation, and its arc to the fractional part of a second. From that result and some of the original data find in the second operation the required arc. And putting A for the arc or segment, found in the first operation by its tangent to be an extreme arc, use for its logarithmic sine in the second operation l. tang. A+l. cos. A—10; or for the l. cos. A, if it be a mean arc, l. sin. A+10—l. tang. A. But if A be found in the first by its cotangent to be an extreme arc, then in the second operation for its l. sine use l. cos. A+10—l. cot. A; or for l. cos. A, if it be a mean arc, use l. sin. A+l. cot. A—10.* And find the required arc by the sine, if it be extreme and the common rule give it by the cosine; but by the cosine, if it be a mean arc and the common rule give it by the sine; or in either case by a tangent or cotangent.

NOTE 1. The logarithmic sine of a mean arc, or cosine of an extreme arc, is still to be used in the second operation, according to the common rule.†

NOTE 2. In Spheric Trigonometry there are various theorems and proportions, which become propositions in Plane Trigonometry, by the mere substitution of the sides themselves instead of their sines or tangents.

---

* For $\text{sin. A} = \frac{\text{tang. A} \times \text{cos. A}}{\text{R}} = \frac{\text{cos.A} \times \text{R}}{\text{cot. A}}$;

and $\text{cos. A} = \frac{\text{sin. A} \times \text{R}}{\text{tang. A}} = \frac{\text{sin. A} \times \text{cot. A}}{\text{R}}$. And this substitution is made on the consideration, that the logarithmic difference of the sines is large, where that of the cosines is small; and the contrary.

† See Dr. MASKELYNE's excellent *Preface* to TAYLOR's *Tables of Logarithms.*

QUESTIONS FOR PRACTICE IN RIGHT-ANGLED AND OBLIQUE-ANGLED SPHERIC TRIGONOMETRY.

1. *Right-angled.*

1. Given the hypoth. 50°, and a leg 44° 18′ 39″; to find the rest.

Ans.	The other leg	26°	3′	53′
	Opp. angle	35	0	0
	adj. angle	65	46	5.

2. Given the hypoth. 64° 40′ 0″
and an angle 64 38 11; to find the rest.

Ans.	The other ∠	48°	0′	14″
	its adjacent leg	54	41	28
	its opp. leg	42	12	0.

3. Given a leg 26° 3′ 53
its opp. angle 35 0 0; to find the rest.

Ans.	Hypoth.	50°	0	0
	other leg	44	18	39
	other angle	65	46	5.

4. Given a leg 42° 12′ 0
and its adj. ∠ 64 38 31; to find the rest.

Ans.	Other leg	54°	41′	28″
	other angle	48	0	14
	hypoth.	64	40	0.

5. Given a leg 64° 10′ 0″
the other leg 54 28 0; to find the rest.

Ans.	Hypoth.	75°	19′	49″
	an angle	68	29	48
	the other ∠	57	16	1.

6. Given an angle 68° 29′ 48
the other ∠ 57 16 1; to find the rest.

Ans.	A leg	64°	10′	0″
	the other leg	54	28	10
	hypoth.	75	19	49.

2. *Oblique-angled.*

1. Given a side 114° 30′
its opp. angle 125 20
and a side 56 40 ; to find the rest.

Ans. The side 83° 12′ 10″
its opp. angle 62 54 4
its adj. ∠ 48 30.

2. Given a side 83° 12′ 10″
a side 114 30
contained ∠ 48 30; to find the rest.

Ans. The side 56° 40′
the angles { 125 20
62 54 4.

3. Given an angle 130° 0′ 0″
its opp. side 122 0 0
and an angle 62 34 6 ; to find the rest.

Ans. The angle 58° 5′ 4″
its opp. side 70 0 0
its adj. side 79 17 14.

4. Given an angle 58° 5′ 4″
an angle 62 34 6
and included side 122 0 0 ; to find the rest.

Ans. The angle 130° 0′ 0″
The sides { 79 17 14
70 0 0.

5. Given the three sides { 65° 30′ 0″
96 47 50
56 40 0 ; to find the angles.

Ans. The angles { 54° 40′ 0″
48 30 0
117 5 56

6. Given the three angles $\begin{cases} 62^\circ\ 34'\ 6'' \\ 58\quad 5\quad 4 \\ 130\quad 0\quad 0 \end{cases}$; to find the sides.

Ans. The sides $\begin{cases} 70^\circ\ 0'\ 0'' \\ 79\ 17\ 14 \\ 122\quad 0\quad 0. \end{cases}$

SPHERIC ASTRONOMY,

OR,

APPLICATION OF SPHERICS TO ASTRONOMY.

DEFINITIONS.

1. ASTRONOMY is the science, which treats of the stars. It is a branch of mixed Mathematics, or Natural Philosophy.

2. The *stars* are the heavenly bodies, that shine by emitting or reflecting light, namely, the *sun*, *planets*, *comets*, and *fixed stars*.

3. *Plane* or *pure Astronomy* is the part, which determines the magnitudes, distances, motions, and orbits of the said bodies, and deduces them from observations on the appearances.

4. *Physical Astronomy* is the part, which investigates the causes of the celestial motions, and reasons analogically from the principles and laws of motion, which are found to govern terrestrial bodies.

5. By *spheric Astronomy* is here meant that part, which consists in the application of the principles of Spherics to Astronomy.

An observer on the earth conceives himself to be in the centre of a vast concave sphere, and all the celestial bodies to be situated on its surface. And it is evident, that, were his observatory in any other part of the Universe, the same spheric appearance of situation would be exhibited to his view. This sphere is represented by the celestial globe, and on its surface the various motions are apparently performed. It is evident then, that Spherics are extensively applicable to the explanation and determination of celestial phenomena.

6. The apparent celestial sphere, being supposed concentric with the earth, has corresponding circles. Thus, the planes of the terrestrial equator, meridians, horizon, polar circles, and tropics, being extended to the heavens, mark there the corresponding *circles of the celestial sphere.*

7. The *Ecliptic* is the circle, which the sun apparently describes in a year. It cuts the celestial equator or equinoctial in two opposite points, called the *equinoctial points;* one of which is the first point of Aries, or *vernal equinoctial;* and the other the first of Libra, or *autumnal equinoctial point.* And the acute angle at this intersection, which is about 23°28′, is called the *obliquity of the ecliptic.* The ecliptic is divided into twelve equal parts, called *signs*, which are ♈ *Aries*, ♉ *Taurus*, ♊ *Gemini*, ♋ *Cancer*, ♌ *Leo*, ♍ *Virgo*, ♎ *Libra*, ♏ *Scorpio*, ♐ *Sagittarius*, ♑ *Capricornus*, ♒ *Aquarius*, ♓ *Pisces.*

8. The tropic of Cancer touches the ecliptic in the first point of Cancer, and that of Capricorn touches it in the first point of that sign. These points of contact are called the *solstitial points;* the former being the *summer*, and the latter the *winter solstitial point.*

9. The signs on the north side of the equinoctial are called *northern*, and those on the south side the *southern signs.* Also six of the signs in order, beginning with Capricorn, are called *ascending*, and the other six *descending signs.*

10. Motion according to the order of the signs is said to be *forward*, *direct*, or *in consequentia;* but, when contrary to this order, *backward*, *retrograde*, or *in antecedentia.*

11. The highest point of the ecliptic is called the *Nonagesimal degree*, being 90° from each of the intersections of this circle with the horizon.

12. The point of the ecliptic, which is on the meridian, is called the *culminating point*, or *medium cœli* ; and is the distance of the meridian from the vernal equinoctial point.

13. The *zodiac* is a zone extending about 8° on each side of the ecliptic, and bounded by two small parallel circles. Within this zone the motions of all the planets are performed, with the exception of one or two, lately discovered.

14. A *secondary* to a great circle is a great circle passing through its poles, and consequently perpendicular to it.

15. *Meridians*, which with respect to the earth are *circles of terrestrial longitude*, called also *circles of right ascension* and *hour circles*, are secondaries to the equator or equinoctial.

16. *Circles of celestial longitude* are secondaries to the ecliptic.

17. The *zenith* and *nadir* are the poles of the rational horizon; the former being in the direction of a perpendicular from its centre above it, and the latter in the opposite direction below it.

18. The *vertical* or *azimuth circles* are secondaries to the horizon.

19. The *altitude* of a star is the arc of a vertical circle, contained between the star and the horizon. And the *zenith distance* is the complement of the altitude.

20. *Almacanter* is a small circle, parallel to the horizon. The *sensible horizon* is an almacanter, whose plane touches the earth at the station of the spectator, and terminates his view.

21. *Longitude of a star* is the arc of the ecliptic,, contained between the vernal equinoctial point and the star or its circle of longitude, reckoned in the order of the signs. If the body be in our system, the longitude, viewed from the earth,

is called the *geocentric* longitude; viewed from the sun, the *heliocentric* longitude.

22. *Latitude of a star* is the arc of its circle of longitude, contained between the star and the ecliptic. If the body be in our system, the latitude, viewed from the earth, is called the *geocentric* latitude; viewed from the sun, the *heliocentric* latitude.

23. *Parallels of celestial latitude* are small circles, parallel to the ecliptic.

24. *Declination of a star* is the arc of its meridian, contained between the star and the equinoctial. Declination and parallels of declination correspond to terrestrial latitude and parallels of latitude.

25. *Right ascension of a star* is the arc of the equinoctial, contained between the vernal equinoctial point and the star or its meridian, reckoned in the order of the signs.

26. *Ascensional difference* is the arc of the equinoctial, contained between the star's meridian and the point, that rises with the star. Or, with respect to the sun, it is the angle at the pole, formed by the sun's meridian and the 6 o'clock meridian.

27. *Oblique ascension* is the sum or difference of the right ascension and ascentional difference. Or, it is the arc of the equinoctial, contained between the vernal equinoctial point and the point, that rises with the star. And the *oblique descension* is the arc of the equinoctial, contained between the vernal equinoctial point and the point, that sets with the star.

28. *Azimuth of a star* is the angle at the zenith, formed by the star's vertical or azimuth circle and the meridian of the place. Or, it is the arc of the horizon, contained between the meridian and the star's vertical circle.

29. *Amplitude of a star* is the angle at the zenith, formed by the prime vertical and the star's verticle circle. Or, it is the arc of the horizon, contained between the east or west point and that, where the star's vertical circle intersects it.

30. *Angle of position* is the angle at a star, formed by two great circles, one passing through the pole of the equinoctial and the other through the pole of the ecliptic.

31. *Hour of the day*, or *hour angle*, is the arc of the equinoctial, contained between the meridian of the place and the star's meridian.

32. *Nodes* are the points, where the orbits of the primary planets cut the ecliptic, and where the orbits of the secondaries cut the orbits of their primaries. The *ascending* node is the point, where the planet crosses, moving northward; and the *descending* the point, where it crosses, moving southward.

33. Two heavenly bodies are in *conjunction*, when they have the same longitude; in *opposition*, when the difference of their longitudes is 180°; in *quadrature*, when the difference is 90°. And *syzygy* denotes either conjunction or opposition.

NOTE. The character ☉ is used to denote the Sun, ☽ the Moon, ☿ Mercury, ♀ Venus, ⊕ the Earth, ♂ Mars, ♃ Jupiter, ♄ Saturn, ♅ Herschel, ✳ a fixed star;—☌ Conjunction, ☍ Opposition, □ Quadrature;—☊ the ascending Node, ☋ the decending Node.

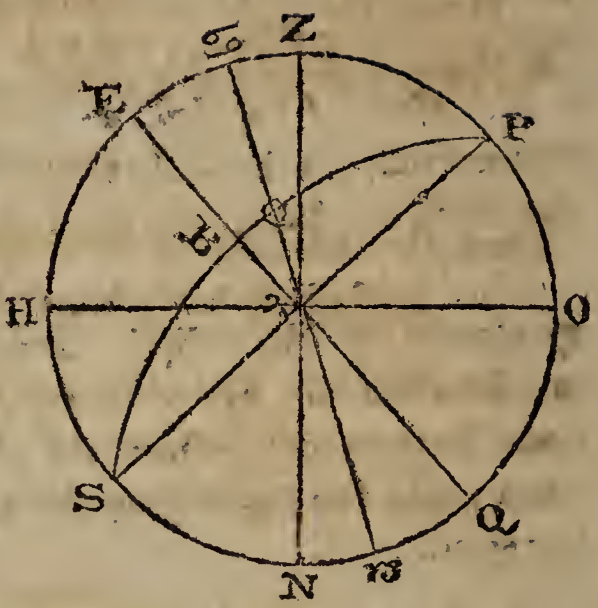
PROBLEMS.

PROBLEM I.

Given the sun's longitude, and the obliquity of the ecliptic; to find the sun's right ascension, and declination.

EXAMPLE. Given the sun's longitude = $1^s\ 11° \ 1' \ 44''$, and the obliquity of the ecliptic = 23° 28′; required the sun's right ascension and declination.

PROJECTION.



Draw the primitive PESQ for the solstitial colure, with the perpendicular diameters EQ for the equinoctial and PS for the equinoctial colure. Set 23° 28′, the obliquity of the ecliptic, from E to ♋; draw the diameter ♋♑ for the ecliptic; set the sun's longitude from ♈ to ⊙ on the ecliptic; and through the points P, ⊙, S, draw a circle, which will be a circle of right ascension.

CALCULATION.

In the right-angled spheric triangle ♈⊙B, ♈⊙ = 41° 1′ 44″, ∠⊙♈B = 23° 28′.

1. *To find the declination* ⊙B.

R × sin. ⊙B = sin. B♈⊙ × sin. ♈⊙.

R	90°	10·
: Sin. B♈⊙	23 28′	9·6001181
:: Sin. ♈⊙	41 1 44″	9·8171947
: Sin. ⊙B	15 9 12 N	9·4173128

2. *To find the right ascension* ♈B.

R × cos. B♈⊙ = cot. ♈⊙ × tang. ♈B.

Cot. ♈⊙	41° 1′ 44″ ar. co.	9·9396053
: R	90	10·
:: Cos. B♈⊙	23 28	9·9625076
: Tang. ♈B	38 35 49	9·9021129

Hence the right ascension = 38° 35′ 49″, which, reduced to time, = 2h. 34′ 23″ 16‴.*

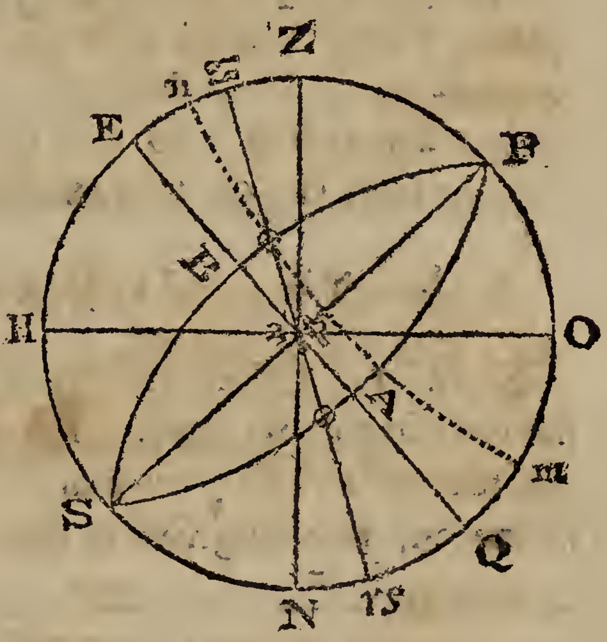
PROBLEM II.

Given the obliquity of the ecliptic, and the sun's declination; to find the sun's longitude and right ascension.

Example. The obliquity of the ecliptic being 23° 28′, and the sun's declination 15° 9′ 12″ N; required the sun's longitude and right ascension.

PROJECTION.

The solstitial and equinoctial colures, equinoctial and ecliptic being described, as in Prob. I, make En and Qm each = the declination; draw the parallel of declination nm, which intersects the ecliptic in ⊙, the sun's place; and through P, ⊙, S, describe a circle.



---

* Degrees may be converted into time by the following rule:

1. Divide the given degrees by 15 for hours; and multiply the remainder by 4 for minutes.

2. Divide the minutes, seconds, &c. in the same manner by 15 for minutes, seconds, &c. and multiply each remainder by 4 for the next lower denomination.

Thus,	38°÷15=	2h. 32′
	35′÷15=	2 20″
	49″÷15=	3 16‴
	38° 35′ 49″=	2h. 34′ 23″ 16‴

The reason of the rule is, that 15° = 1h. and consequently 1° = 4′, &c.

CALCULATION.

In the rectangular spheric triangle ♈☉B, B♈☉ = 23° 28′, and B☉ = 15° 9′ 12″.

1. *To find the longitude* ♈☉.

Sin. B♈☉	23° 28′	ar. co.	0·3998819
: Sin. B☉	15 9 12″		9·4173107
:: R	90		10·
: Sin. ♈☉	41 1 43		9·8171926

2. *To find the right ascension* ♈B.

R	90°	10·
: Tang. B☉	15 9′ 12″	9·4326800
:: Cot. B♈☉	23 28	10·3623894
: Sin. ♈B	38 35 48	9·7950694

NOTE 1. Other Problems may be formed by varying the data in this triangle. Thus, the obliquity of the ecliptic and the sun's right ascension, or the sun's longitude and right ascension, may be given to find the rest.

By a reverse operation, time may be reduced to degrees; that is, multiply the hours by 15, to the product add $\frac{1}{4}$ of the minutes, seconds, &c. each being removed one place higher, or toward the left.

Thus, 2h. 34′ 23″ 16‴
15

30	=2× 15
8 30	=34÷4
5 45	=23÷4
4	=16÷4
38° 35′ 49″	

NOTE 2. While the sun is ascending from Aries to Cancer, its longitude is ♈☉, or the hypotenuse of the triangle ♈☉B. But when the sun has passed the solstitial point of ♋, and is descending toward ♎, its longitude being subtracted from 180°, the remainder is the hypotenuse ☉♎; and B♎ is then the supplement of the right ascension. When the sun's longitude is less than 180°, the declination is north. While the sun is decending from the beginning of ♎ toward ♑, the excess of its longitude above 180° will be the hypotenuse ♎☉; and ♎A, the corresponding arc of right ascension, must be added to 180° for the whole right ascension. When the sun has passed the solstitial point of ♑, and is ascending toward ♈, the longitude being subtracted from 360°, the remainder is the hypotenuse ☉♈; and the corresponding arc of right ascension A♈ must be subtracted from 360°, to give the right ascension. While the longitude is between 180° and 360° the declination is south.

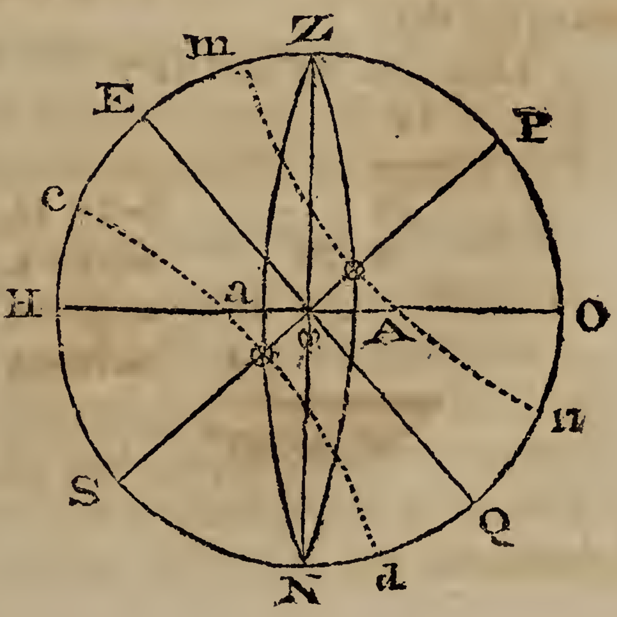
PROBLEM III.

Given the latitude of the place, and the sun's declination; to find the sun's altitude and azimuth at 6 o'clock.

EXAMPLE. Suppose the latitude of the place to be 42° 23′ 28″ N, and the sun's declination 23° 27′ 57″ N; required the sun's altitude and azimuth at 6 o'clock.

PROJECTION.

Describe the primitive for the meridian, HO for the horizon, ZN for the prime vertical. Make OP =42° 23′ 28″ N the latitude; draw the six o'clock hour circle PS, and the equinoctial EQ perpendicular to it. Project the parallel of declination mn, at the distance of 23° 27′ 57″



from the equinoctial on the north side, and it will cut PS in ☉, the sun's place. Through Z, ☉, N, describe the azimuth circle Z☉N, cutting the horizon in A.

CALCULATION.

The things, given and required, are contained in either of the rectangular spheric triangles ♈☉A, or Z☉P, which are supplemental to each other. ♈☉ = the declination, A☉ = the altitude, A♈☉ = the latitude, ♈A = the co-azimuth, ZP = co-latitude, ☉Z = co-altitude, ☉P = co-declination, ∠☉ZP, measured by the arc A☉ = the azimuth.

1. *To find the altitude* A☉.

R	90°	10·
: Sin. ♈☉	23 27′ 57″	9·6001036
:: Sin. A♈☉	42 23 28	9·8287809
: Sin. A☉	15 34 22	9·4288845

2. *To find the azimuth* ☉ZP, *or* AZ☉.

Tang. ☉P co-dec.	66° 32′ 3″ ar. co.	9·6375933
: R	90	10·
:: Sin. ZP co-lat.	47 36 32	9·8683857
: Cot. ☉ZP azim. reckoned from the north	72 13 25	9·5059790

Cor. 1. It is evident, that, as the declination increases, the altitude at 6 increases, but the amizuth decreases; and the contrary, when the declination decreases.

Cor. 2. It is also evident, that, when the sun has no declination, the altitude at 6 o'clock is nothing; for the sun is then at ♈ or ♎ in the horizon; and the azimuth being 90°, the sun will be east in the forenoon, and west in the afternoon. Thus, on the days of the equinoxes, the sun rises and sets at six o'clock, in the east and west points of the horizon.

NOTE 1. The given being the sun's greatest northern declination, the altitude and azimuth at 6 o'clock are found for the longest day. The parallel of southern declination cd, for the shortest day, would cut the 6 o'clock hour circle below the horizon at ☉. Now as the triangles ♈A☉ and ♈a☉ are similar and equal, the depression a☉ below the horizon on the shortest day at 6 o'clock is equal to the altitude A☉ at the same hour on the longest day. The azimuth is also equal, but must be reckoned from the south. And in both cases toward the east A. M. and toward the west P. M.

NOTE 2. The four things concerned in this Problem are the *latitude of the place*, and the *sun's declination*, *altitude* and *azimuth* at 6 o'clock; any two of which being given, the rest may be found.

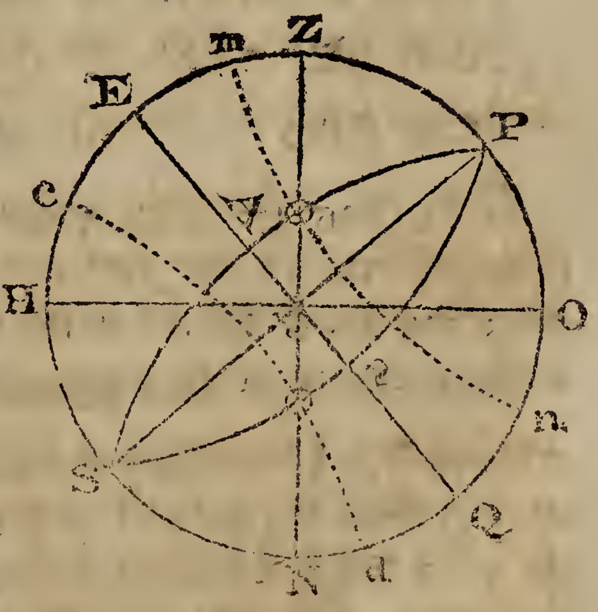
PROBLEM IV.

Given the latitude of the place, and the sun's declination; to find the altitude, and hour of the day, when the sun is east or west.

EXAMPLE. Suppose the latitude of the place to be 42° 23′ 28″ N and the suns declination 23° 27′ 57″ N; required the altitude, and hour, when the sun is east or west.

PROJECTION.

Draw the primitive HZON for the meridian, HO for the horizon, and ZN for the prime vertical; make OP = 42° 23′ 28″, the latitude; draw the 6 o'clock hour circle PS, and the equinoctial EQ perpendicular to it; draw the parallel of declination mn at the distance of 23° 27′ 57″ from EQ,



which will cut the prime vertical in ⊙; and through P, ⊙, S, describe the hour circle P⊙S.

CALCULATION.

The things, given and required, are contained in each of the right-angled spheric triangles PZ⊙, ♈A⊙. For ♈⊙ = the altitude, A⊙ = the declination, A♈⊙ = the latitude, ♈A = the co-hour angle from noon, or hour angle from 6; ⊙Z = the co-altitude, ZP = the co-latitude, P⊙ = the co-declination, and ZP⊙ = the hour angle from noon.

1. *To find the co-altitude ⊙Z, and consequently the altitude.*

Cos. co-lat. PZ	47° 36′ 32″	ar. co.	0·1712191
: R	90		10·
:: Cos. co-dec. P⊙	66 32 3		9·6001036
: Cos. co-alt. ⊙Z	53 47 51		9·7713227

Hence 36° 12′ 9″ is the sun's altitude, when east or west on the longest day.

2. *To find the hour angle from noon* ZP⊙.

R	90°	10·
: Tang. co-lat. PZ	47 36′ 32″	10·0396049
:: Cot. co-dec. P⊙	66 32 3	9·6375933
: Cos. ZP⊙	61 36 17	9·6771982

And 61° 36′ 17″ = 4h. 6′ 25″ 8‴, which, subtracted from 12h. gives 7h. 53′ 34″ 52‴ for the time, when the sun is east; and 4h. 6′ 25″ 8‴ is the time, when the sun is west in the afternoon of the longest day.

NOTE 1. If the sun's declination be given for the shortest day, draw the parallel of declination cd; then, the triangle ♈a⊙ being similar and equal to ♈A⊙, the sun's depression, when east or west, is found by the same operation, as well as the hour, which will be before 6 in the forenoon, and after 6 in the afternoon; and they will be respectively equal to the altitude and hour before found.

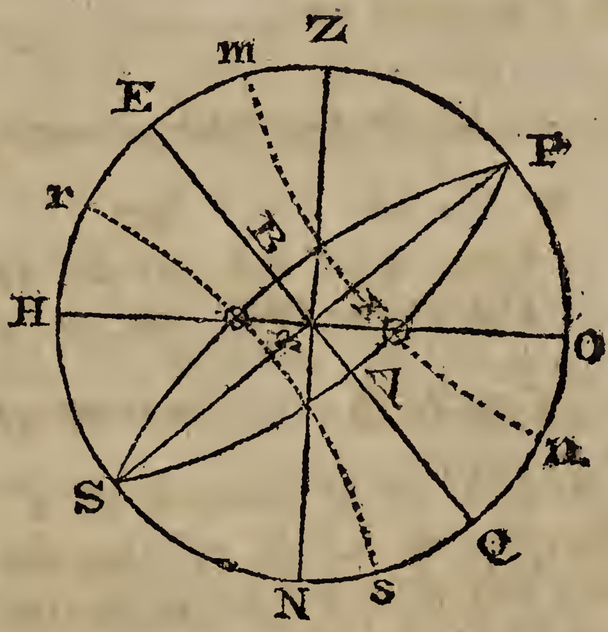
NOTE 2. Of the four things in this Problem, namely, the *latitude of the place*, and the *sun's declination*, *altitude*, and *hour when east or west*, any two being given, the rest may be found.

PROBLEM V.

Given the latitude of the place, and the sun's declination; to find the sun's amplitude, and ascensional difference.

EXAMPLE. Suppose the latitude of the place to be 42° 23′ 28″ N, and the sun's declination 23° 27′ 57″ N; required the sun's amplitude and ascensional difference.

PROJECTION.



Describe the primitive for the meridian, HO for the horizon, ZN for the prime vertical, PS for the six o'clock hour circle or axis, and EQ perpendicular to PS for the equinoctial. Draw mn at the distance of 23° 27′ 57″ from EQ, cutting the horizon in ⊙, the place of the sun's rising or setting; and through ⊙ describe the hour circle P⊙S.

CALCULATION.

On the 21st of June, or the longest day, the sun, having the declination 23° 27′ 57″ N, appears at Cambridge, in lat. 42° 23′ 28″ N, to describe the parallel nm by the diurnal motion. It is in m at noon, and describes the quadrant mx in 6 hours. ⊙x is the ascensional difference, which, being found, converted into time, and added to 6, or subtracted from it, gives the time of sunrise, or sunset. But ⊙x contains the same number of degrees as A♈, the corresponding arc of the equinoctial. Hence in the right-angled spheric triangle ♈⊙A, ♈A = the ascensional difference, ♈⊙ = the amplitude N, ⊙♈A = the co-latitude, and A⊙ = the declination.

1. *To find the amplitude* ♈⊙.

Sin. co-lat. ⊙♈A	47° 36′ 32″	ar. co.	0·1316143
: Sin. dec. ♈⊙	23 27 57		9·6001036
:: R	90		10·
: Sin. am. ♈⊙	32 37 35		9·7317179

2. *To find the ascensional difference* ♈A.

R	90°	10·
: Tang. dec. A⊙	23 27′ 57″	9·6375933
:: Cot. co-lat. A♈⊙	47 36 32	9·9603951
: Sin. asc. diff. ♈A	23 20 42	9·5979884

The ascensional difference 23° 20′ 42″, reduced to time, =1h. 33′ 22″ 48‴ = the time of sunrise before 6, and of sunset after 6 ; or the sun rises at 4h. 26′ 37″, and sets at 7h. 33′ 23″.

Cor. 1. The sun apparently describes the parallel nm in 12h. being in n at midnight and in m at noon ; hence n⊙, expressed in time, is half the night ; and ⊙m, thus expressed, is half the day, making the times of sunrise and sunset the dividing points between the day and night. Therefore double the time of sunrise is the length of the night, and double the time of sunset the length of the day. Thus,

$\overline{4h. 26' 37''} \times 2 = 8h. 53' 14''$ = the shortest night; and $\overline{7h. 33' 23''} \times 2 = 15h. 6' 46''$ = the longest day.

Cor. 2. If the parallel of latitude rs be drawn 23° 27′ 57″ southward from the equinoctial, the hour circle PBS, passing through ⊙, the place, where the sun rises or sets, will form the triangle ♈⊙B, equal and similar to the triangle ♈⊙A. In ♈⊙B, ♈⊙= the amplitude S, ♈B= the assensional difference, which, reduced to time, shows the time of sunrise after 6, and of sunset before 6, on the shortest day.

Cor. 3. Hence it appears, that when the latitude and declination are both of the same denomination, that is, both either north or south, the sun rises before and sets after 6; but when they are of different denominations, the sun rises after and sets before 6.

Cor. 4. It also appears, that when the latitude and declination are of the same denomination, the difference of the right ascension and ascensional difference is the oblique ascension, and their sum is the oblique descension. But when they are of different denominations, their sum is the oblique ascension, and their difference the oblique descension.

Cor. 5. When the declination is equal to, or greater than, the co-latitude of any place, and of the same denomination, which can happen only within the polar circles, the parallel of declination will not cut the horizon, and consequently the sun will circulate without setting. The same is true also with respect to other heavenly bodies. But if the latitude and declination be of different denominations, and the declination equal to, or greater than, the co-latitude, the sun and other heavenly bodies will circulate without rising.

Note 1. If the declination sensibly vary during the day, there must be two operations for the times of sunrise and sunset; that declination being used in each, which corresponds to the time estimated nearly.

NOTE 2. Of the four things in this Problem, namely, the *latitude of the place*, and the *sun's declination*, *amplitude*, and *ascensional difference*, any two being given, the rest may be found.

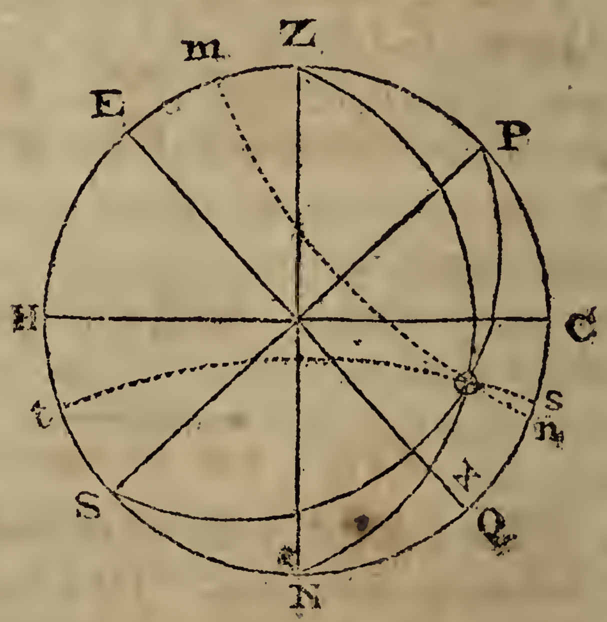
NOTE 3. The time of *apparent* rising is earlier, and that of *apparent* setting later, than the *true*, on account of the refraction of the atmosphere, which raises the heavenly bodies to the horizon, when they are about 33′ below it. And therefore to deduce the times of apparent rising and setting from the true, a correction, corresponding to this effect of refraction, must be applied. The correction is different for different places and declinations. For the preceding example, it is about 3′ 33″. See note to the next Problem.

PROBLEM VI.

Given the latitude of the place, and the sun's declination; to find the time, when the twilight begins and ends.

EXAMPLE. Suppose the latitude to be 42° 23′ 28″ N, and the sun's declination 23° 27′ 57″ N; required the time, when the twilight begins and ends.

PROJECTION.



Describe the primitive for the meridian, HO for the horizon, ZN for the prime vertical, and the parallel circle ts, at the distance of 18° below the horizon. Draw the axis PS, and the equator EQ perpendicular to it; describe the parallel circle mn, at the distance of 23° 27′ 57″

from EQ, cutting ts in ☉, the sun's place at the beginning or end of twilight. Through ☉ describe the vertical circle Z☉N, and the hour circle P☉S.

CALCULATION.

In the oblique spheric triangle ZP☉, ZP = the co-latitude, P☉ = the co-declination or polar distance, Z☉ = the zenith distance, ZP☉ = the hour angle from noon, which is required; for m☉ is the arc, described from noon to the end of twilight, which is measured by the arc Ex = the angle ZP☉.

PZ	47° $36'$ $32''$		
P☉	66 32 3		
Z☉	108		
	2)222 8 35		
½ sum	111 4 17·5		
PZ	47 36 32		
½ sum—PZ	63 27 45·5	sine	9·9516498
	111 4 17·5		
P☉	66 32 3		
½ sum—P☉	44 32 14·5	sine	9·8459498
Co-ar. sin. PZ	47° $36'$ $32''$		0·1316143
Co-ar. sin. P☉	66 32 3		0·0374897
			2)19·9667036
	74° $14'$ $11''\frac{2}{3}$	sine	9·9833518
	2		
ZP☉	148 28 23		

The ∠ZP☉, reduced to time, = 9h. $53'$ $53''$ $32'''$, either before or after noon. Thus the twilight begins at 2h. $6'$ $6''\frac{1}{2}$ in the morning, and ends at 9h. $53'$ $53''\frac{1}{2}$ in the evening, at Cambridge on the 21st of June.

COR. It is evident from the figure, that when the declination is greater than the excess of the co-latitude above 18°, the parallel of declination mn will not cut the parallel circle ts, which is 18° below the horizon, but n will be above s ; and therefore the twilight will then continue the whole night, or from sunset till sunrise.

NOTE. The time of *apparent* sunrise or sunset may be found in a similar manner, if the parallel circle ts be drawn at the distance of 33′ below the horizon HO.

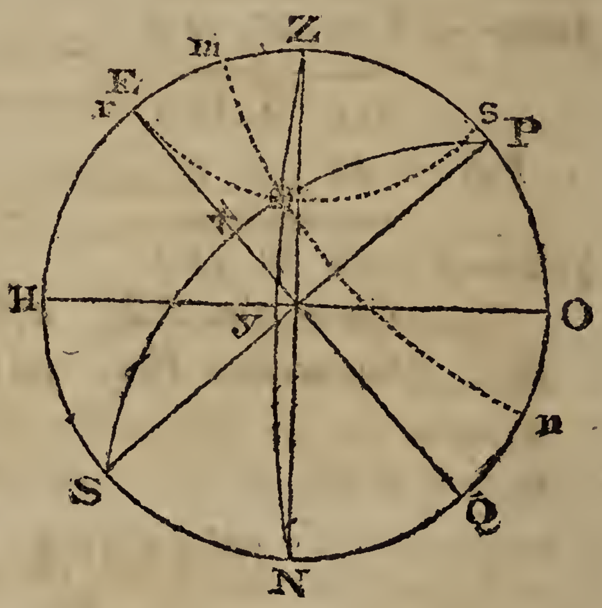
PROBLEM. VII.

Given the sun's altitude, and declination, with the latitude of the place ; to find the hour, and the sun's azimuth.

EXAMPLE. Suppose the sun's altitude, observed at Cambridge, to be 46° 20′, on the 21st of June, its declination being 23° 27′ 57″ N ; required the time, and the sun's azimuth.

PROJECTION.

Draw the meridian, horizon, prime vertical, axis, equinoctial, and parallel of declination, as before. Draw the parallel circle rs, at the distance of 46° 20′ above the horizon HO, and it will intersect mn in ☉. Through Z, ☉, N, describe the azimuth circle Z☉N, and through P, ☉, S, the hour circle P☉S.



CALCULATION.

In the oblique spheric triangle P☉Z, PZ = the co-latitude, Z☉ = the co-altitude, P☉ = the co-declination or polar distance, ☉PZ = the hour angle from noon, and ☉ZP = the azimuth angle.

The arc ⊙m is measured by Ex, the measure of the angle EPx, or ZP⊙; and yO measures the angle yZO, or ⊙ZP.

1. *To find the hour angle from noon* ⊙PZ.

PZ =	47° 36′ 32″				
P⊙ =	66 32 3				
Z⊙ =	43 40				
	2)157 48 35				
½ sum	78 54 17½				
	47 36 32				
½ sum—PZ	31 17 45½			sine	9·7155513
½ sum	78 54 17½				
P⊙	66 32 3				
½ sum—P⊙	12 22 14½			sine	9·3308919
		Co-ar. sin.	47° 36′ 32″		0·1316143
		Co-ar. sin.	66 32 3		0·0374897
					2)19·2155472
		23° 54′ 35″		sine	9·6077736
		2			
		47 49 10 = ∠⊙PZ.			

The hour angle 47° 49′ 10″, reduced to time, is 3h. 11′ 16″ 40‴. Therefore the time required is 8h. 48′ 43″½ in the fore-noon, or 3h. 11′ 16″½ in the afternoon, if the given declination be that of the sun at each of those times.

2. *To find the azimuth* ∠⊙ZP.

1st rem.	31° 17′ 45″½	sine	9·7155513
½ sum	78 54 17½		
Z⊙	43 40		
2d rem.	35 14 17½	sine	9·7611584

	Co-ar. sin. 47° 36′ 32″		0·1316143
	Co-ar. sin. 43 40		0·1608604
			2)19·7691844
50° 3′ 11″½		sine	9·8845922
2			
100 6 23 = the azimuth, reckoned from the north.			

NOTE 1. Either the time or the sun's azimuth being first found, the other may be determined from it by the proportion of the sines of the sides to those of the opposite angles; but each is determined separately and independently in the same manner.

NOTE 2. If the latitude and declination be of different denominations, the side P☉, extending beyond the equinoctial EQ, will be obtuse, and equal to the sum of the declination and 90°; whereas it is equal to the difference of the declination and 90°, when the latitude and declination are of the same denomination.

NOTE 3. The amplitude or azimuth of a celestial body, observed with a magnetic needle, is called the *magnetic amplitude* or *azimuth*. The true amplitude or azimuth being calculated for the same time, and compared with the magnetic, the difference is the *variation* or *declination of the needle*, which is reckoned from the north. The observer being supposed to look directly toward the point of true azimuth, if the magnetic azimuth be on the left, the variation is *easterly*; but if on the right, it is *westerly*.

Thus, if the magnetic azimuth at the time of the preceding true one be observed to be S 72° 54′ 37″ E

Then, N 107° 5′ 23″ E magnetic az.

S 79 53 37 E true az. supp. of 100° 6′ 23″.

6 59 0 Diff. = the variation, which is westward, because the magnetic azimuth is on the right.

Again, if the magnetic az. of the ⊙ be	S 45° 0′ W
and the true azimuth	S 62 7 W
The variation is	17 7,

which is eastward, because the magnetic azimuth is on the left.

NOTE 4. The five things, contained in this Problem, are the *latitude of the place*, and the *sun's altitude*, *declination*, and *azimuth*, and the *hour*; any three of which being given, the other two may be found by the solution of an oblique spheric triangle. Hence several useful Problems are formed, three of which are subjoined.

PROBLEM VIII.

Given the sun's altitude, the hour, and the latitude of the place; to find the sun's azimuth, and declination.

EXAMPLE. Suppose the sun's altitude to be 46° 20′ at Cambridge, in latitude 42° 23′ 28″ N, at 8h. 48′ 43″ ½ in the morning; required the sun's azimuth, and declination.

In this Problem the data are the side ZP = the co-latitude, the side Z⊙ = the co-altitude, and the opposite angle ZP⊙ = the hour from noon. The quæsita are P⊙ = the co-declination, and ⊙ZP = the azimuth.

[See the last figure.]

Ans. { Azim. = S 79° 53′ 37″ E
Dec. = 23 27 57 N.

PROBLEM IX.

Given the latitude of the place, the sun's azimuth, and the hour; to find the sun's altitude, and declination.

EXAMPLE. At Cambridge, in latitude 42° 23′ 28″ N, the sun's azimuth being S 79° 53′ 37″ E, at 8h. 48′ 43″½ in the morning; required the altitude, and declination.

In this example the data are ZP = the co-latitude, $\odot ZP$ = the azimuth, and $ZP\odot$ = the hour angle. The quæsita are $Z\odot$ = the co-altitude, and $P\odot$ = the co-declination.

[See the last figure.]

Ans. { Altitude = 46° 20′.
Dec. = 23 27 57″ N.

PROBLEM. X.

Given the latitude of the place, the sun's declination, and the hour; to find the sun's altitude, and azimuth.

Example. What is the sun's altitude and azimuth at Cambridge, in latitude 42° 23′ 28″, at 8h. 48′ 43″½ in the morning, the declination being 23° 27′ 57″ N ?

Here the data are ZP = the co-latitude, $P\odot$ = the co-declination, and the $\angle ZP\odot$ = the hour. The quæsita are $Z\odot$ = the co-altitude, and the $\angle PZ\odot$ = the azimuth.

[See the last figure.]

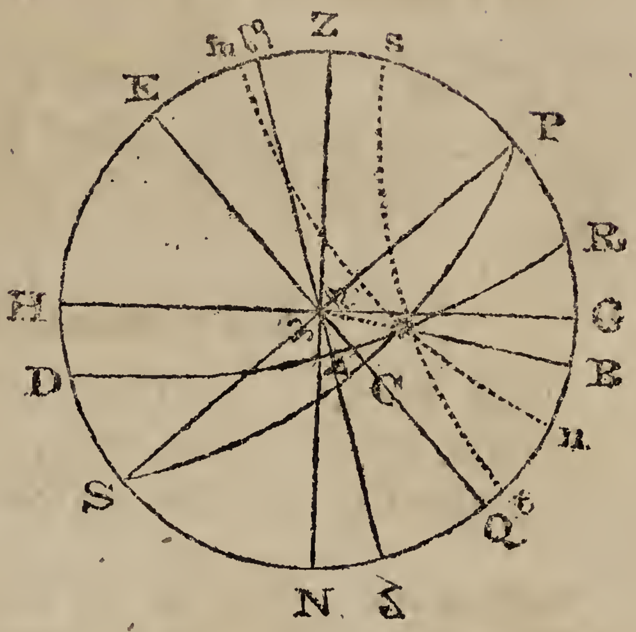
Ans. { Altitude = 46° 20′.
Azimuth = S 79 53 37″ E.

PROBLEM XI.

Given the obliquity of the ecliptic, the right ascension, and declination of a star; to find its longitude, and latitude.

Example. Suppose the obliquity of the ecliptic to be 23° 28′, the right ascension of Arcturus 211° 38′ 43″, and its declination 20° 13′ 25″ N; required the star's longitude, and latitude.

PROJECTION.



Draw the primitive for the solstitial, and PS for the equinoctial colure, EQ for the equinoctial, and ♋ ♑ for the ecliptic. P is the pole of the equinoctial, and R the pole of the ecliptic. Draw the parallel of declination mn, at the distance of 20° 13′ 25″ from EQ northward; set 31° 38′ 43″, the excess of the right ascension above 180°, from ♎ to C; draw the hour circle PCS, and it will intersect mn in ∗, the place of the star; and describe the circle of celestial longitude R ∗ D.

CALCULATION.

In the oblique spheric triangle P∗R, PR = 23° 38′, the obliquity of the ecliptic, P∗ = 69° 46′ 35″, the co-declination, ∠RP∗ = 58° 21′ 17″ = the complement of ♎C; ∠PR∗ = the longitude from ♋, or its excess above 90°, R∗ = the co-latitude, for V∗ = the latitude. Therefore two sides and the included angle are given, to find one of the other angles, and the third side.

Draw the perpendicular ∗B.

1. *To find the longitude.*

Cot. P∗	69° 46′ 35″	ar. co.	0·4336847
: R	90		10·
:: Cos. RP∗	58 21 17		9·7198770
: Tang. PB	54 55 30		10·1535617
PB — PR = RB =	31 27 30.		

Sin. RB	31° 27′ 30″	ar. co.	0·2824307
: Sin. PB	54 55 30		9·9129659
: : Tang. RP✳	58 21 17		10·2102121
: Tang. BR✳	68 32 43		10·4056087

180° — BR✳ = PR✳ = 111° 27′ 17″ ;
therefore 90° + 111° 27′ 17″ = 6s. 21° 27′ 17″ = the longitude.

2. *To find the latitude.*

Sin. PR✳	111° 27′ 17″	ar. co.	0·0311871
: Sin. RP✳	58 21 17		9·9300891
: : Sin. P✳	69 46 35		9·9723652
: Sin. R✳	59 7 36		9·9336414

90° — R✳ = 30° 52′ 24″ = the latitude N.

PROBLEM XII.

Given the obliquity of the ecliptic, the longitude, and latitude of a star; to find its right ascension, and declination.

This is the reverse of the last Problem. Let R✳D, [last Fig.] be the circle of longitude, and st the parallel of latitude, intersecting each other in ✳, the place of the star. Draw the meridian P✳S. Then in the oblique spheric triangle P✳R, the data are PR✳ = the longitude from ♋, PR = the obliquity of the ecliptic, and R✳ = the co-latitude. And the quæsita are RP✳ = the complement of the right ascension from ♎, or of ♎C, and P✳ = the co-declination.

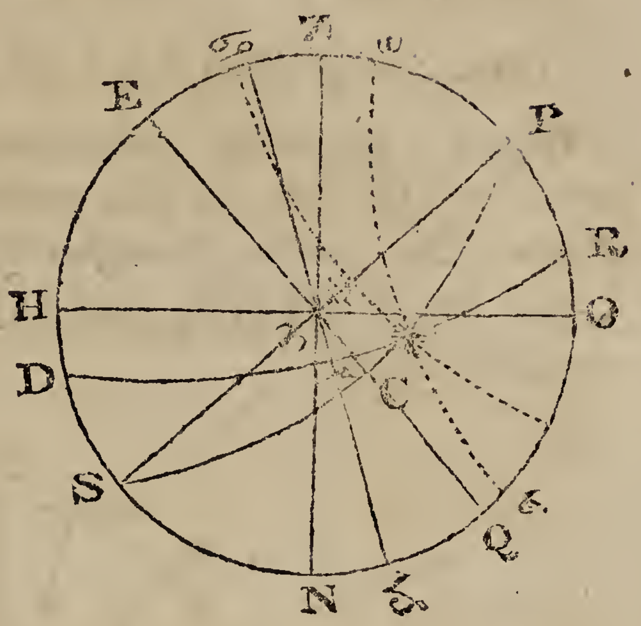
PROBLEM XIII.

Given the obliquity of the ecliptic, the right ascension, and latitude of a star; to find the angle of position.

Example. Suppose the obliquity of the ecliptic to be 23° 28′, the right ascension of Arcturus 211° 38′ 43″, and its latitude 30° 52′ 24″ N; required the angle of position.

PROJECTION.

Having drawn the colures, equinoctial, and ecliptic, as before; project the circle of right ascension P✳S, and the parallel of latitude st; their intersection ✳ is the place of the star. Describe the circle of longitude R✳D.



CALCULATION.

In the oblique spheric triangle P✳R, the data are RP✳= 58° 21′ 17″, the complement of ♎C, PR = 23° 28′, and R✳ = 59° 7′ 36″, the co-latitude; to find ∠P✳R, the angle of position.

Cos. lat.	30° 52′ 24″	ar. co.	0·0663590
: Cos. ♎P✳	31 38 43		9·9300891
: : Sin. PR	23 28		9·6001181
: Sin. P✳R	23 15 51		9·5965662

NOTE. The angle of position may also be found, if with the obliquity of the ecliptic there be given the longitude and latitude, right ascension and declination, or longitude and declination.

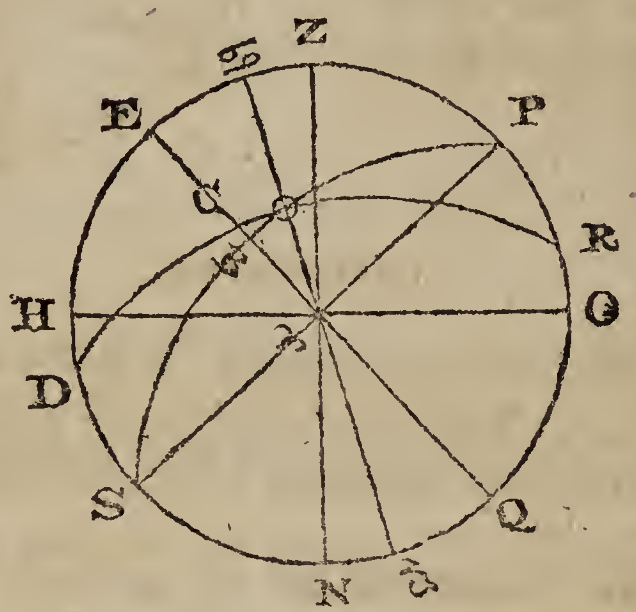
COR. 1. With respect to the sun, it having no latitude, and of course the co-latitude being equal to radius; the following will be the proportion.

If, as in the annexed figure, the obliquity of the ecliptic be 23° 28′, and the sun's right ascension 45° 30′;

R	90°	10·
: Cos. R. A. ♈B	45 30′	9·8456618
:: Sin.	23 28	9·6001181
: Sin. ∠ of position P☉R	16 12 26″	9·4457799

Cor. 2. Hence the complement of the angle of position = 73° 47′ 34″ is the angle, which the ecliptic makes with the meridian passing through the sun's centre.

For P☉R = C☉B, and C☉♈ — C☉B = B☉♈, the *meridian angle.*



NOTE. The meridian angle B☉♈ may also be found by solving the right-angled spheric triangle ♈☉B, having the right ascension of the sun and the obliquity of the ecliptic given; and its complement is the angle of position at the sun.

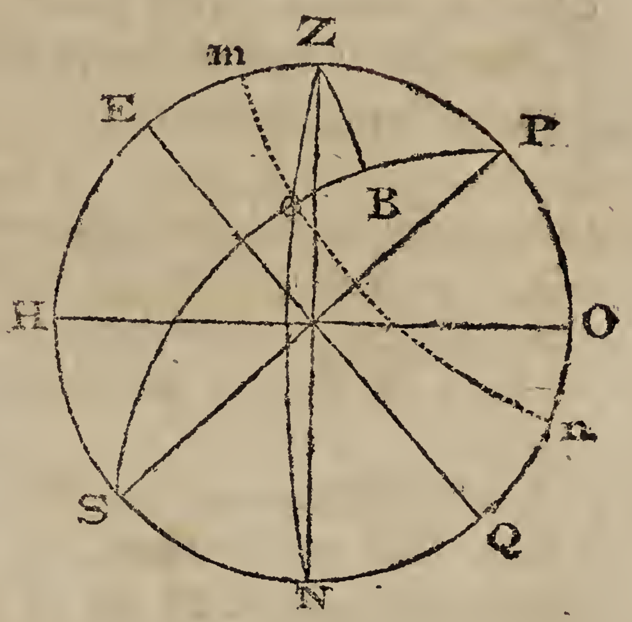
PROBLEM XIV.

Given the latitude of the place, the declination of a star, and the hour angle; to find the angle, formed at the star by the hour circle and vertical circle, called the parallactic angle.

EXAMPLE. Suppose the sun's declination to be 23° 27′ 57″ N, at 9 o'clock in the morning at Cambridge, in latitude 42° 23′ 28″ N; required the parallactic angle.

PROJECTION.

Having drawn the colures, equinoctial, horizon, prime vertical, and parallel of declination mn, describe the hour circle P☉S, making the angle EP☉= the hour angle from noon, and it will intersect mn in ☉; and through Z, ☉, N, draw the vertical Z☉N.



CALCULATION.

In the oblique spheric triangle Z☉P, ZP = 47° 36′ 32″, the co-latitude, P☉ = 66° 32′ 3″, the co-declination, the ∠ZP☉ = 45°, the hour angle; two sides and the included angle being given, to find one of the other angles, namely, Z☉P, the parallactic angle.

Draw the perpendicular ZB.

Cot. ZP	47° 36′ 32″ ar. co.	0·0396049
: R	90	10·
: : Cos. ZP☉	45	9·8494850
: Tang. PB	37 45 44	9·8890899

P☉—PB=B☉=28° 46′ 19″

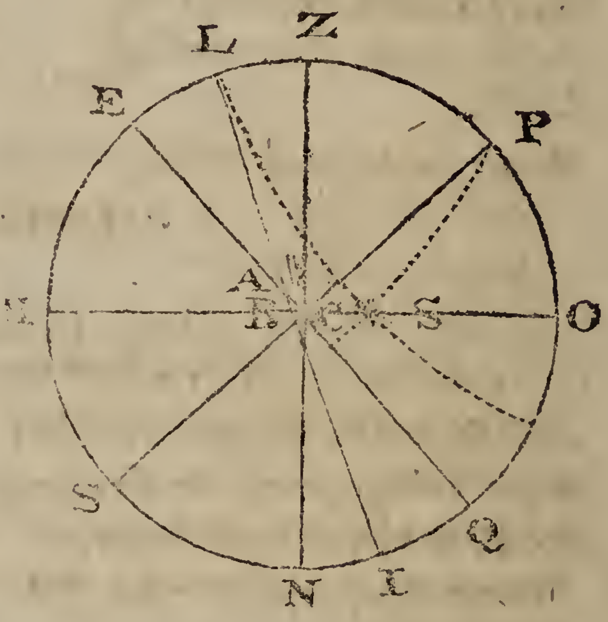
Sin. B☉	28° 46′ 19″ ar. co.	0·3175620
: Sin. PB	37 45 44	9·8870252
: : Tang. ZP☉	45	10·
: Tang. paral. ∠Z☉P	51 50	10·1045872

PROBREM XV.

Given the latitude of the place, the right ascension, and declination of a star; to find when it rises and sets cosmically, that is, at sunrise.

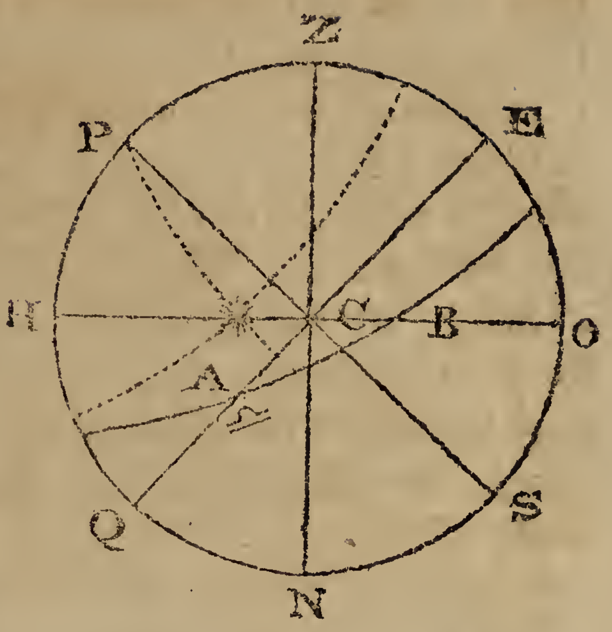
Find the ascensional difference. Then,

1. The sum or difference of the ascensional difference and right ascension, according as the declination is south or north, is the oblique ascension.



Let HZON be the meridian, HO the horizon, EQ the equinoctial, LI the ecliptic, and S the star. Then in the oblique spheric triangle ABC, the data are by the oblique ascension AC, the angle ABC, the supplement of ABH, the height of the equinoctial, or complement of the latitude, and the angle BAC, the obliquity of the ecliptic; to find the side AB, and consequently the point B of the ecliptic, that rises with the star. Therefore find the day, when the sun is in this point of the ecliptic, and the same day the star rises cosmically.

2. The sum or difference of the ascensional difference and right ascension, according as the declination is north or south, is the oblique descension.



Then in the oblique spheric triangle ABC, the data are AC, and the angles ACB, BAC, as before, to find AB; and B is the point setting with the star. Consequently the point of the ecliptic, opposite to B, is the point rising at the same time. Therefore, when the sun is in the opposite point to B, the star sets cosmically.

PROBLEM XVI

Given the latitude of the place, the right ascension and declination of a star; to find when it rises and sets acronycally; that is, at sunset.

1. Find, by the last Problem, the point of the ecliptic B, in which the sun is, when the star rises cosmically. Then it is evident, that the sun, being in the opposite point of the ecliptic, will set, when the star rises. Therefore find, when the sun is in that point; and on the same day the star rises acronycally.

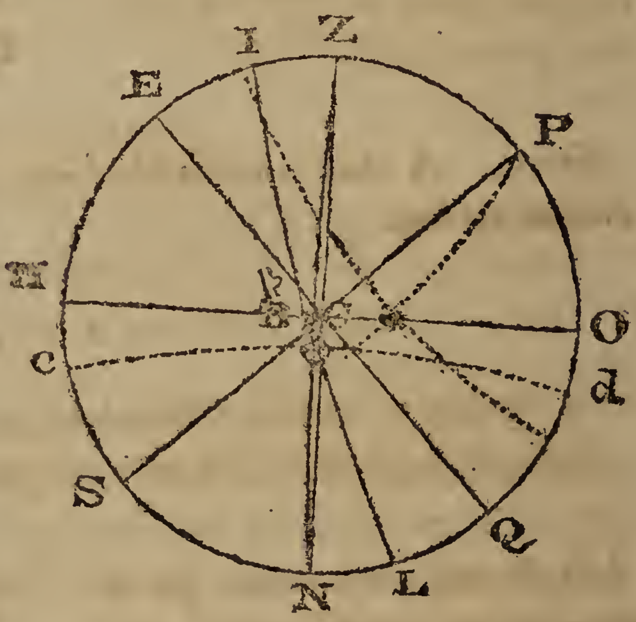
2. Also find, by the last Problem, the sun's place, when it sets with the star; and at the time, when it is in that point, the star sets acronically.

PROBLEM XVII.

Given the latitude of the place, the obliquity of the ecliptic, the right ascension, and declination of a star; to find its heliacal rising, and setting.

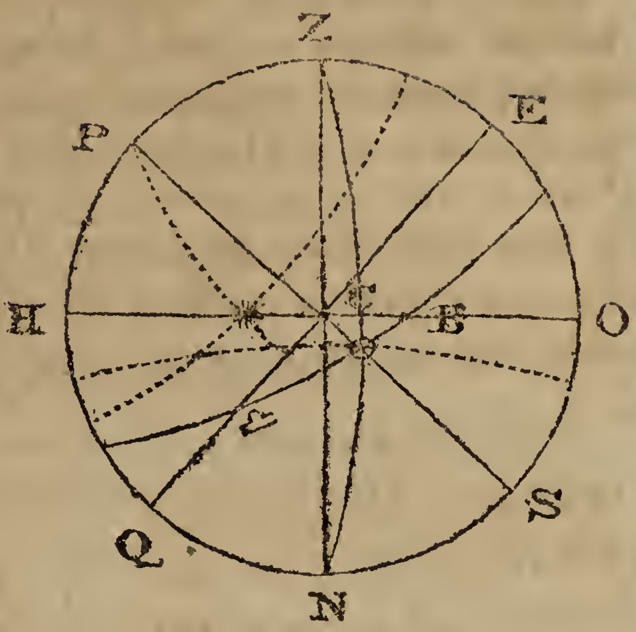
NOTE. The *heliacal rising* of a star is its emergence from the sun's beams, or its first appearance above the horizon before sunrise. And the *heliacal setting* of a star is its immersion in the solar beams, or disappearance near the horizon after sunset, on account of the proximity of the sun. As stars differ in magnitude or splendor, they rise or set heliacally, when the sun has different degrees of depression below the horizon. The depression allowed is 5° for the ☽, 5° for ♀, 10° for ☿, 10° for ♃, 11° for ♂, 11° for ♄, 12° for stars of the first magnitude, 13° for stars of the second magnitude, 14° for those of the third, 15° for those of the fourth, 16° for those of the fifth, and 17° for those of the sixth.

Let ∗ be the star's place when rising; draw the parallel circle cd, at the requisite distance below the horizon HO, the equinoctial EQ, and the ecliptic IL, by means of the star's oblique ascension. And describe the vertical circle Z☉N through Z, ☉, the point where IL intersects cd, and N.



In the right-angled spheric triangle BC☉, (which in this figure is very small) C☉ is given; CB☉, the angle the ecliptic makes with the horizon, being equal to the supplement of the ∠CB♎, may be found by means of the triangle CB♎, in which BC♎ is the co-latitude, C♎B the obliquity of the ecliptic, and C♎ is known from the oblique ascension. Whence ☉B may be determined. And ☉B being added to B♎, found in BC♎, we have ♎☉, the sun's distance from the equinoctial point ♎. Whence the time of the sun's being there may be found; and the same is the time, when the star rises heliacally.

And if $\odot$ be the sun's place, when the star sets heliacally, then B$\odot$, found as before, and subtracted from ♎B, gives the sun's place at $\odot$; and at the time, when it is there, the star sets heliacally.

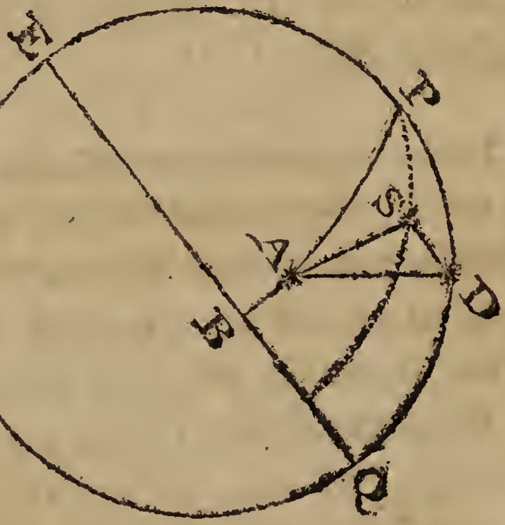


NOTE. A star is invisible from its heliacal setting till its heliacal rising.

PROBLEM XVIII.

Given the places of two stars, and the distance of a third star from each of them ; to find the place of the third star.

Let A and D be the given places of two stars, and S the place of the third. Then, if the places of the stars be defined by their longitude and latitude ; let BQ be an arc of the ecliptic, representing their difference of longitude, and P the pole of the ecliptic. Then in the shperic triangle PAD, the data are the sides PA, PD, the complements of the latitudes of the stars, and the included angle APD, or BQ, the difference of the longitudes of the stars. Whence the side AD, and $\angle$ PAD may be found.



In the triangle ASD, the three sides are then given, to find the angle SAD. Thence the $\angle$ PAS becomes known, being equal to PAD—SAD.

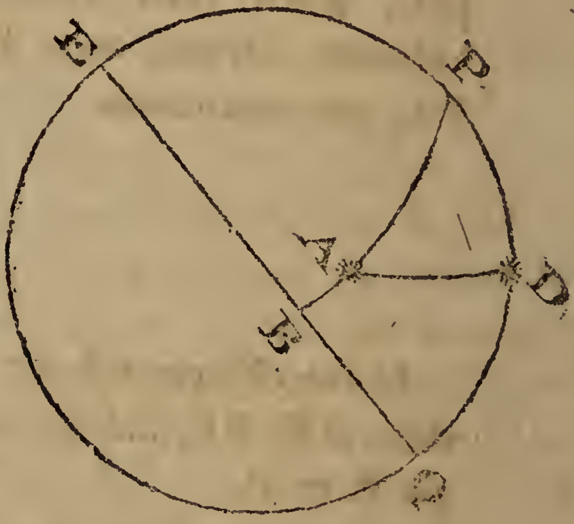
And in the triangle PAS, the data are the sides PA, AS, and the included ∠ PAS. Whence may be found the ∠APS, the difference of longitude from A; and the side PS, the complement of the latitude of the star S.

But if the places be defined by the right ascension and declination; let BQ be the arc of the equinoctial, representing the difference of their right ascension, and P the pole.

Then, by a similar process, may be found the ∠ APS, the difference of the right ascension of S from A; and SP, the complement of the declination of the star S.

PROBLEM XIX.

Given the right ascensions and declinations of two stars; to find their distance.



Let A and D be the places of the two stars, EQ the equinoctial, P the pole, AB, DQ, the declinations of the stars, and PA, PD, their complements. Then the right ascensions being given, their difference BQ is given. Hence in the oblique spheric triangle PAD, the data are the sides PA, PD, and the included angle APD, to find the distance AD.

Cor. 1. If the longitudes and latitudes of two stars be given, their distance may be found in a similar manner.

For then EQ may represent the ecliptic, APD their difference of longitude, and AP, DP, the complements of their latitudes.

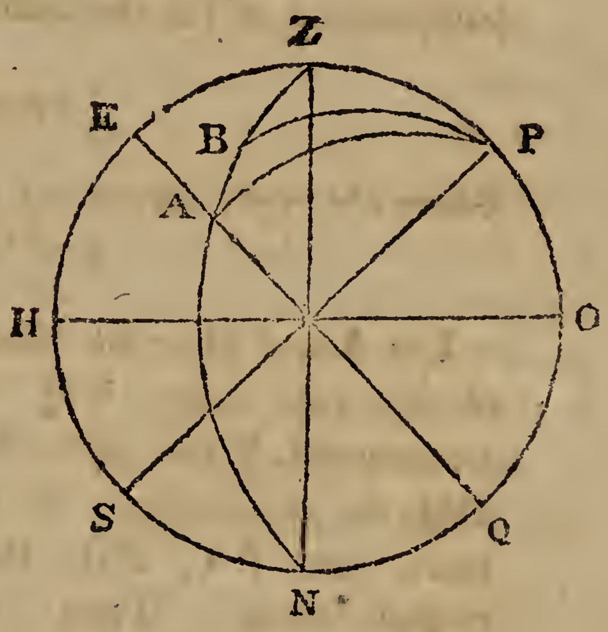
Cor. 2. If the right ascension and declination of one star, and the declination of another be given, with its distance from the first; its right ascension may be found.

For then, in the oblique spheric triangle PAD, the three sides AP, AD, and PD are given, to find the angle APD.

Cor. 3. A similar inference may be drawn with respect to the longitudes and latitudes of two stars.

PROBLEM XX.

Given the latitude of the place, the right ascensions and declinations of two stars; to find the time, when they will be in the same vertical circle, and their azimuth at that time.



Let A and B be the two stars, Z the zenith, P the pole, ZBA their common azimuth circle, and PB, PA, two meridians.

In the oblique spheric triangle APB, the data are the two sides AP, BP, and the included angle APB, to find the angle A or B.

Then in the triangle ZPA, the data are the sides ZP, AP, and the opposite angle A, to find the angle ZPA, or the hour angle of A; and the angle PZA, or the azimuth from the north.

Or in the triangle ZPB, the data are the sides ZP, BP, and the opposite angle B, to find the angle ZPB, the hour angle of B, and BZP, the azimuth.

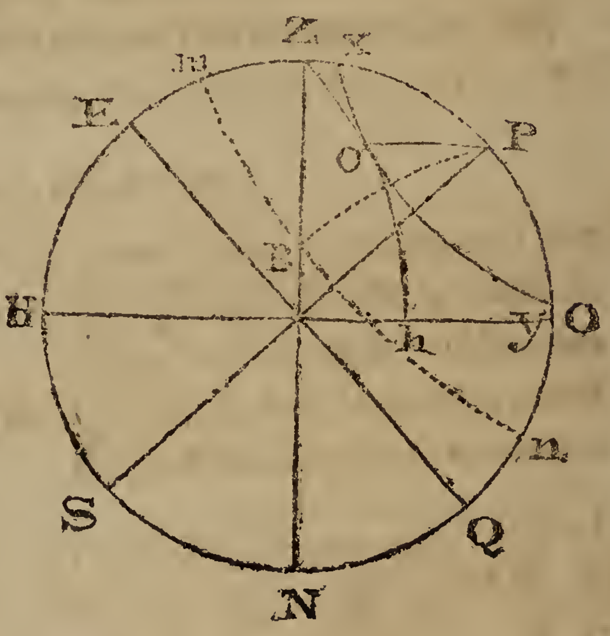
Cor. 1. In the same triangle ZAP, or ZBP, the co-altitude ZA, or ZB, at the same time, may be found.

Cor. 2. If the co-altitude ZA, or ZB be given instead of the latitude, the co-latitude ZP may be determined in the same triangle ZAP, or ZBP.

PROBLEM XXI.

Given the latitude of the place, and the declination of a star; to find the time, when its apparent diurnal motion is perpendicular to the horizon.

Let yx be the parallel, described by the star; and draw the vertical circle Zh, touching it at o. Now when the star is at o, its motion is perpendicular to the horizon. And in the right-angled spheric triangle ZPo, the data are ZP = the co-latitude, and Po = the co-declination, to find ∠ ZPo, the hour angle.



Therefore, as R : tang. oP :: cot. PZ : cos. ZPo; that is, R : cot. dec. :: tang. lat. : cos. ZPo; which, converted into time, is the time from the star's being on the meridian. Hence the time, when the star is on the meridian, being found, the required time may be determined.

Note 1. The time, when a fixed star is on the meridian, may be found thus:

Subtract the sun's right ascension for the given day from the star's right ascension, increased by 24 hours, if necessary; the remainder is the approximate time of the star's being on the meridian.

As 24h. : the daily change of the sun's right ascension :: the approximate time : a fourth number; which being subtracted from the approximate time, the remainder is the true time.*

Note 2. The time of a fixed star's being on the meridian, together with its declination and the latitude of the place being given, the time of its being on the prime vertical may be thus determined:

In the right-angled spheric triangle ZPB, as R : cot. BP :: tang. ZP : cos. ZPB ; that is, as R : tang. dec. :: cot. lat. : cos. of the distance from the meridian, or hour angle; which being converted into time and subtracted from the time of the meridian transit, the remainder is the time, when it is on the prime vertical eastward.

PROBLEM XXII.

Given the apparent semidiameter of the moon, of the earth's shadow at the distance of the moon, and the inclination of the moon's path to the ecliptic; to find the lunar ecliptic limits; that is, the distance from a node, at or beyond which a lunar eclipse cannot happen.

In the right-angled spheric triangle SM☊, the data are the ∠ ☊, the inclination of the path ☊M to the ecliptic ☊S, and SM = the sum of the semidiameters of the earth's shadow and the moon, to find ☊S, the distance of the centre of the shadow S from ☊, when the limb of the moon, being in opposition, just touches the shadow without entering it.

Cor. The solar ecliptic limits may be determined in a similar manner, by using, for the semidiameter of the earth's

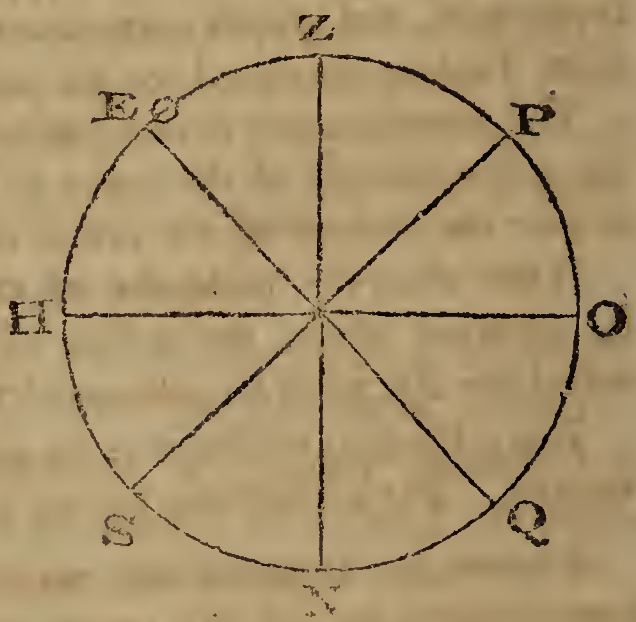
---

* Or let x = the sun's motion in right ascension in 24 hours, and T = the difference of right ascension, when the sun is on the meridian. Then the time of the star's being on the meridian $= T - \frac{Tx}{24}$.

shadow, the sun's semidiameter, or the sum of the sun's semidiameter and moon's horizontal parallax, according as regard is had to an eclipse at a particular place, or to a general eclipse.

PROBLEM XXIII.

Given the true meridian altitude of the sun or a fixed star; to find the latitude of the place.



Subtract the given meridian altitude from 90°, and the remainder is the zenith distance north or south, according as the zenith is toward the north or south of the sun or star; and find the declination of the body for the time of the altitude.

Then the sum or difference of the zenith distance and declination, according as they are of the same or different denominations, is the required latitude of the same name with the greater.

EXAMPLE. Suppose the true meridian altitude of the sun's centre to be 49′ 47′ 22″ at Cambridge, the zenith being northward, and its declination at the same time 2° 10′ 50″ N; required the latitude of Cambridge.

True altitude of ☉'s centre —	49°	47′	22″	
	90			
Zenith distance	40	12	38	N
Sun's dec.	2	10	50	N
Latitude	42	23	28	N.

COR. 1. Were a star situated in the elevated pole, its true altitude would be equal to the latitude of the place.

Cor. 2. If the sun or a star, revolving about the elevated pole without setting, be observed on the meridian below the pole; then the sum of its true meridian altitude and the complement of its declination is the latitude.

Note. In order to find the *true* altitude of a celestial body, the *apparent* altitude of the body, or with respect to the sun and moon the apparent altitude of the upper or lower limb, is observed with a quadrant or other suitable instrument, and proper corrections applied; which are taken from Tables, calculated for the purpose.

The semidiameter is to be added or subtracted, according as the altitude of the lower or upper limb is observed, to give the altitude of the centre of the sun or moon.

If the observer make use of a free horizon at sea, and be elevated above the surface of the water, a correction, called the *dip*, must be subtracted from the observed altitude.

The refraction is always to be added, and the parallax subtracted. But a fixed star has no parallax.

Example. Suppose the meridian altitude of the sun's lower limb to be 56° 49′½ by observation on the deck of a ship 21 feet high, and its declination at that time to be 22° 22′ 57″ N, the zenith being northward from the sun; required the true altitude, and latitude of the place.

Apparent alt. sun's l. l.		56°	49′	30″	
Dip	—		4	22	
		56	45	8	
Refraction	—			37	
		56	44	31	
Parallax in alt.	+			5	
		56	44	36	
Sun's ½ diameter	+		15	47	
True alt. centre		57	0	23	
		90			
Zenith distance		32	59	37	N
Declination		22	22	57	N
Latitude		55	22	34	N

PROBLEM XXIV.

Given two altitudes of the sun, with its declination, and the interval of time; to find the latitude of the place.

Let P be the pole, Z the zenith, S and T the places of the sun, when the observations were made.

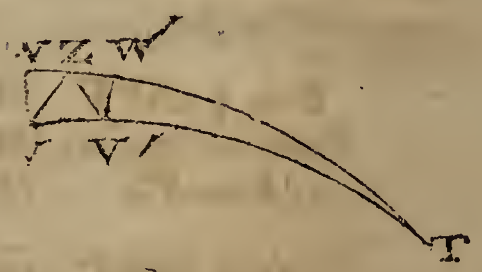
Then in the oblique spheric triangle SPT, the data are SP, TP, the complements of the sun's declination at S, T, and the angle TPS equal to the interval of time; to find ST, and the angle PST.

Hence in the triangle SZT, the data are ST, and SZ, TZ, the complements of the two altitudes; to find the angle TSZ; whence the angle ZSP will become known.

Therefore in the triangle SPZ, the data are SZ, SP, and the angle ZSP; to find ZP, the complement of the latitude.

In this solution the observations are supposed to be both made at the same place; but at sea, the ship changing its place in the interval between the observations, it is necessary to reduce one altitude to what it would have been, if taken at the place of the other observation.

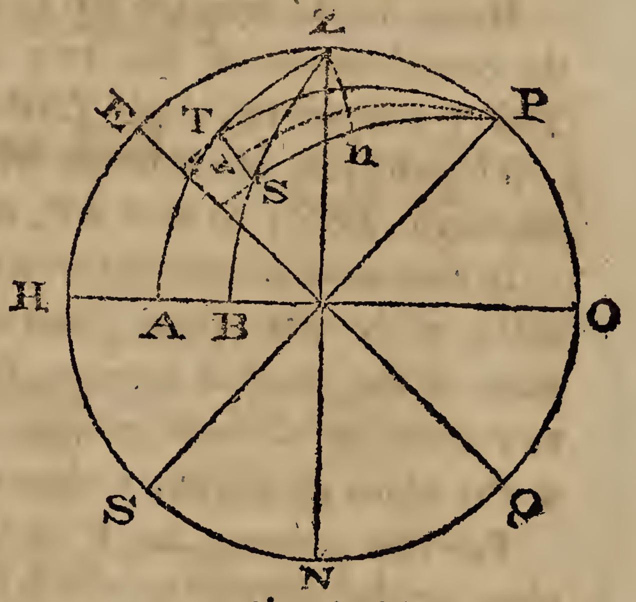
For this purpose, let T be the place on the earth, to which the sun was vertical at the time of the first observation, Z the observer's place at that time, ZV, or ZV′, the length of the run in the interval between the observations.



Then TV, or TV′, would have been the zenith distance by the first observation, if it had been made at the place of the second. Draw VW, V′W′, perpendicular to ZT. Then, as the angle T is small, TV is very nearly equal to TW, and TV′ to TW′; and therefore they may be considered as respectively equal. Hence ZW, or ZW′, may be considered as the difference of the zenith distances; the former increasing the distance, and the latter decreasing it. To find this difference, observe the angle VZT, or V′ZT, formed by the rhumb line, on which the ship sails, and the sun's bearing. Then, in the right-angled plane triangles VZW, V′ZW′, the data are the angles, and the side ZV, or ZV′, the distance or length of the run; to find ZW, or ZW′.

Example. Given the sun's declination 14° 30′ N, its true altitude 47° 30′ by the former of two observations, both taken in the forenoon, and 56° 30′ by the latter; the interval of time being 1h. 44′; required the latitude of the place.

PS = PT =	75° 30′ =	co-dec.
ZS =	42° 30′ =	1st co-alt.
ZT =	33 30 =	2d co-alt.
∠SPT =	26 =	interval of time.



Draw the perpendiculars Pr and ZN, in the triangles PST and and PZS. PST being an isosceles triangle, Pr bisects the angle SPT, and the side ST.

Cot. ½ ∠SPT	13°	ar. co. 9·3633641
: R	90	10·
:: Cos. PS	75 30′	9·3985996
: Cot. ∠PST	86 41 30″	8·7619637
R	90°	10·
: Sin. PS	75 30′	9·9859416
:: Sin. ½ ∠SPT	13	9·3520880
: Sin. Sr.	12 34 44″	9·3380296
	2	
ST	25 92 8	

ZS =	42°	30′				
ZT =	33	30				
ST =	25	9	28″			
2)101		9	28			
	50	34	44			
ZS	42	30				
	8	4	44	1st. rem.	sine	9·1477889
	50	34	44			
ST	25	9	28			
	25	25	16	2d rem.	sine	9'6327285

ZS	42°	30′		co-arc	sine	0.1703167
ST	25	9	28″	co-arc	sine	0'3714962
						2)19'3223303
½ ∠ZST	27°	16′	43″		sine	9'6611651
			2			
ZST	54	33	26			

PST	86°	41′	30″
ZST	54	33	26
PSZ	32	8	4

Cot. ZS	42°	30′		ar. co.	9'9620525
: R	90				10'
: : Cos. ZSn	32	8	4″		9'9277820
: Tang. Sn	37	48	35		9'8898345

SP — Sn = Pn = 37° 41′ 25″.

Cos. Sn	37°	48′	35″	ar. co.	0'1023449
: Cos. SZ	42	30			9'8676309
: : Cos. Pn	37	41	25		9'8983561
: Cos. PZ*	42	23	56		9'8683319

* R : cos. nZ : : cos. Sn : cos. SZ } By the Catholic Pro-
R : cos. nZ : : cos. Pn : cos. PZ } position.

∵ cos. Sn : cos. SZ : : cos. Pn : cos. PZ; That is, the cosines of the segments of the base are as the cosines of the sides.

Therefore the latitude $= 90^\circ - 42^\circ\ 23'\ 56'' = 47^\circ\ 36'\ 4''$ N, the two observations being supposed to have been made at the same place.

Note. In the triangle PSZ may be found the angle PZS, the azimuth of the sun from the north, when the first observation was taken ; and the angle SPZ, the hour angle from noon, or the time of the observation.

PROBLEM XXV.

Given the apparent altitudes of the moon and the sun or a fixed star, their apparent distance, and the time ; to find the longitude of the place.

The method of finding the longitude of a place by the moon's distance from the sun or a fixed star consists, beside observations of the altitudes of the moon and sun or star, and their distance, and the determination of the time of the observations,

1. In computing the *true* distance of the moon from the sun or star.

2. In determining the time at a known meridian corresponding to the same distance. The difference between this time and that of the observations being the required difference of longitude.

1. *To find the true lunar distance.*

The observations determine the three sides of a spheric triangle, which are the apparent co-altitudes of the two bodies, and their apparent distance from each other. The calculation consists in finding the true altitudes, and the angle at the zenith ; and then, with the corrected or true co-altitudes and the angle at the zenith, in finding the third corrected side, which is the true distance. Therefore the calculation is the solution of two spheric triangles ; in the first of which the three sides are given, to find an angle ; and in the second, two sides and the included angle are given, to find the third side.

The former solution may be performed by the rules under the fifth case of Oblique Spheric Trigonometry, and the latter by the Method, page 454. Or the whole operation of correcting the observed altitudes, except for dip and semidiameter, and determining the true distance may be performed according to the following rule, in which the said methods are combined.

RULE.

1. To the sine of the moon's horizontal parallax add the sine of its apparent zenith distance, the sum, 10 in the index being rejected, is the sine of its parallax in altitude, which, diminished by the refraction, is the correction, and must be subtracted from the moon's apparent zenith distance, to give its true zenith distance.

2. The sun's parallax being subtracted from the refraction at the altitude of that luminary, add the remainder to the sun's apparent zenith distance; the sum is the true zenith distance.

3. Add together the apparent distance and the apparent zenith distances, and subtract each of the said zenith distances from the half sum.

4. Add together the sines of these two remainders, the co-arcs of the sines of the apparent zenith distances, and the sines of the true zenith distances.

5. From half the sum of these six logarithms subtract the sine of half the difference of the true zenith distances, and the remainder is the tangent of an arc.

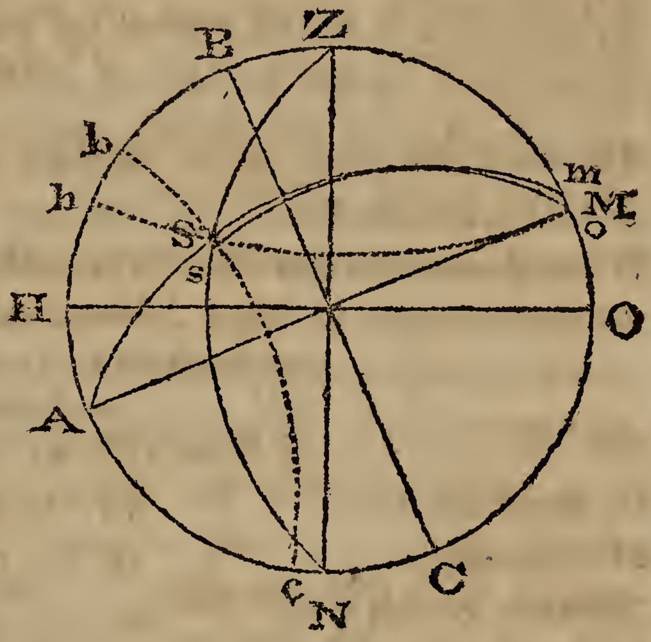
6. Subtract the sine of this arc from the said half sum, and the remainder is half the true distance.

2. *To find the time at a known meridian corresponding to the same distance, and the difference of longitude.*

In the Nautical Almanac the true distance of the moon from the sun and certain fixed stars is calculated for every three hours at the meridian of Greenwich ; and for any distance intermediate to two of these, the corresponding time is found by taking the proportional part of three hours, and adding it to the time of the next preceding distance.

EXAMPLE. On the 7th of June, 1795, at 18h. 20′ 10″, the apparent altitude of the moon was 22° 15′, that of the sun 21° 35′, their apparent distance 119° 20′ 34″, and the moon's horizontal parallax 58′; required the true distance, and the longitude of the place.

PROJECTION.



Describe the primitive HZON, and draw the vertical ZN, and the horizon HO. Set the ☽'s altitude from O to M, and M is the apparent place of the moon. Draw the diameter MA, and BC perpendicular to it. Describe above HO the parallel circle ho, at the height Hh = the sun's apparent altitude. Describe also the parallel circle bc, at a distance from A equal to the supplement of the apparent distance of the moon from the sun. S, the intersection of bc and ho, is the apparent place of the sun. Through the three points N, S, Z, draw the circle NSZ ; and through M, S, A, draw the circle MSA. Then ZSM is the first triangle.

Set the correction of the moon's altitude from M to m, and m is the ☽'s true place. Set the correction of the sun's altitude from S to s, and s is the sun's true place. Through the points m and s draw the arc of a great circle. Then will sm represent the true distance, and Zsm be the second triangle.

CALCULATION.

1. *For the true distance.*

☽'s horizontal parallax	0°	58′	0″		S 8·2271335
☽'s apparent zen. dist.	67	45	0		S 9·9663954
☽'s parallax in alt.	0	53	41		S 8·1935289
☽'s refraction	0	2	19		
☽'s correction	0	51	22	True alt.	23° 6′ 22″
☉'s correction	0	2	16	True alt.	21 32 44
Apparent distance	119	20	34		
☽'s app. zenith dist.	67	45	0	Co-ar.	S 0·0336046
☉'s app. zenith dist.	68	25	0	Co-ar.	S 0·0315714
Sum	255	30	34		
½ Sum	127	45	17		
1st remainder	60	0	17		S 9·9375513
2d remainder	59	20	17		S 9·9345950
☽'s true zenith dist.	66	53	38		S 9·9636838
☉'s true zenith dist.	68	27	16		S 9·9685417
				Sum	39·8695478
				½ Sum	19·9347739
½ diff. true zen. distances	0	46	49		S 8·1341132
Tangent of arc	89	5	36		11·8006607
Same arc	89	5	36		S 9·9999456
½ true distance	59	23	24		S 9·9348283
			2		
True distance	118	46	48		

2. *For the longitude.*

June 7, 1795.

Lun. dist. from the sun at midnight	119°	27′	4″	per N. A.
15 hours	117	56	14	
Diff.	1	30	50	

First of these	119	27	4
True distance by observation	118	46	48
Diff.	0	40	16

As 1° 30′ 50″ : 3h. : : 40′ 16″ : 1h. 19′ 47″
12

Time, when the distance was the same at Greenwich	13	19	47

Apparent time at Greenwich	13h.	19′	47″
Apparent time of the observation	18	20	10

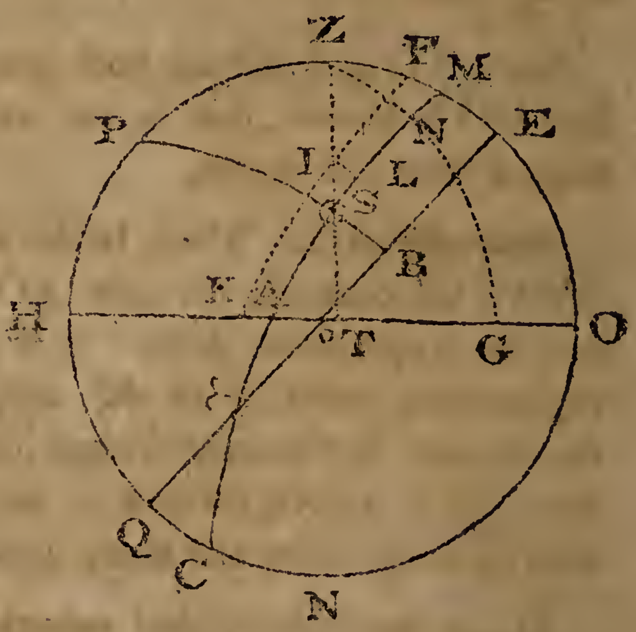
Difference in time 5 0 23, which, converted into degrees, is equal to 75° 5′ 45″ = the longitude east from Greenwich, because the time is later.

NOTE. The time of the observation, if not shown by a time keeper, may be calculated from the latitude of the place, the sun's altitude and declination.

PROBLEM XXVI.

Given the latitude of the place, the obliquity of the ecliptic, the sun's longitude, and the hour ; to find the angle, formed by the ecliptic and horizon, or the height of the nonagesimal degree, the ascending point of the ecliptic, the point of the nonagesimal degree, the azimuth of the ascending point, the culminating point, its altitude, &c.

Let HZON be the meridian, HO the horizon, Z the zenith, N the nadir, EQ the equinoctial, P the pole, CAM the ecliptic, ♈ the first point of aries, S the sun, PSB an hour circle, A the descending point of the ecliptic, AN 90°, N the highest point of the ecliptic, or the nonagesimal degree, and ZNG a vertical circle.



In the right-angled spheric triangle ♈SB, the data are the sun's longitude ♈S, and the obliquity of the ecliptic S♈B, to find the sun's right ascension ♈B.

Hence, the hour being given, the distance of the meridians, or the arc of the equinoctial BE, is known, and consequently its complement Bo. And hence ♈ o = ♈ B—Bo is determined.

Therefore in the oblique spheric triangle A♈o, the data are A♈o, the obliquity of the ecliptic, Ao♈, the complement of the latitude, and ♈o, to find the rest.

1. The angle ♈ Ao, and consequently its supplement SAo, which is the angle, formed by the ecliptic and horizon ; or,

which is the same, the arc NG, the height of the nonagesimal degree.

2. The side A♈, and consequently the point A, which is the descending point of the ecliptic, and the opposite or ascending point.

3. The side Ao, which is the azimuth of the descending point in the west, or of the ascending point in the east.

By the addition of AN, or 90°, to A♈ the arc ♈N is found, or N the highest point of the ecliptic, or point of the nonagesimal degree.

Since AG and oO are both quadrants, GO = Ao; therefore GO is known, that is, the azimuth of the nonagesimal degree from the south.

Ao+oO=AO. Then, in the right-angled spheric triangle AOM, the data are the side AO and the angle MAO, to find the hypotenuse AM. ♈A + AM=♈M; whence the culminating point M of the ecliptic is konwn. And MO the altitude of the culminating point of the ecliptic, together with the ∠ AMO, formed by the ecliptic and meridian, may also be found from the same data.

Or, in the right-angled spheric triangle ♈EM, ♈E and E♈M are given, to find ♈M, ♈ME, and ME. Whence MO is known.

In the right-angled spheric triangle ZSN, the sides ZN, NS, are given; for ZN is the complement of NG, and NS is the complement of AS, or of ♈S — ♈A. Whence the angle ZSN may be found, that is, the angle, formed by the ecliptic and vertical circle passing through the centre of the sun; and ZS, the sun's zenith distance. These may also be found in the oblique spheric triangle ZMS. For ZM=90°—MO, ZMS =180°—AMO, and MS = ♈M—♈S, are given; to find ZSM, and ZS.

Cor. Hence may be found the angle ZIF, formed by the vertical circle ZS and the small circle FIK, parallel to the ecliptic.

From the angular point I draw IL, perpendicular to MSA. Then in the right-angled spheric triangle ISL, the data are the angle ISL, found before, and the side IL, the distance of the parallel from its great circle; to find the angle SIL, the complement of KIS or ZIF.

NOTE. The right ascension of the meridian, medium Cœli, or mid-heaven, at any given time, is equal to the apparent time, reckoned astronomically in degrees, added to the sun's right ascension.

Or, the mean longitude of the sun, added to the mean time in degrees, is equal to the right ascension of the mid-heaven.

FINIS.