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Development of Fraction Understanding

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Consider Julie (described in Mack, 1990, p. 22), a sixth grader who knows little about fractions. Julie knows that when you have two pizzas of the same size, and one is cut into six equal-sized pieces but the other cut into eight equal-sized pieces, she'll get more pizza if she chooses a slice from the first pizza, rather than the second (see Figure 1). However, when asked which fraction is bigger, 1/6 or 1/8, she says, "One eighth is bigger. [Eight] is a bigger number I think. [Eight] is bigger than [six]."

Like Julie, many children have sound intuitions about the mathematical patterns in our world. These mathematical intuitions enable them to predict, for example, that dividing an amount into two groups will result in two, smaller groups that are approximately one half of the original amount (McCrink, Spelke, Dehaene, & Pica, 2013) and that combining two quantities will result in a bigger quantity that is approximately equal to the sum of the parts (Knops, Viarouge, & Dehaene, 2009; Mix, Levine, & Huttenlocher, 1999; Wynn, 1998). As children's mathematics knowledge develops in educational and informal settings, they link these intuitions about numbers and mathematical relationships to formal symbolic mathematics, extend their early formal knowledge to support more complex mathematical thinking, and develop new intuitions and concepts about the relations among symbolic quantities.

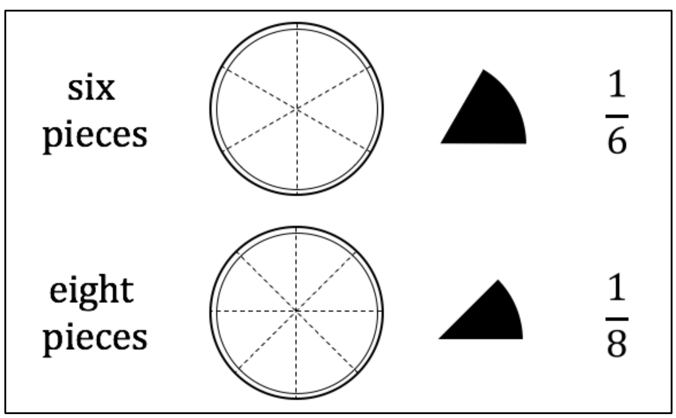

Figure 1. While children may have intuitions about the size of pieces, they often have difficulty reasoning about the formal symbolic fractional notation used to represent those magnitudes. In this example, a child who can reason about two pizzas cut into different numbers of slices may be unable to accurately reason about fraction symbols meant to represent the size of each slice. One goal of mathematics education is to support students' reasoning about the formal mathematical symbols.

In this chapter, we review the empirical research on children's understanding of fraction concepts, symbols, and procedures. In the first two sections, we argue for the practical and theoretical importance of examining children's fraction understanding. In the third section, we describe children's and adults' fraction reasoning, while highlighting the common methodologies in this literature, and present the developmental picture painted by current research across the lifespan. In the fourth section, we explore the educational implications of developmental research on children's fraction reasoning. Finally, we conclude by considering the directions for future research.

Mathematical Cognition and Education

Mathematical thinking has long been an important and productive area of cognitive and developmental research, for both practical and theoretical reasons. Practically, understanding how children learn mathematics—particularly which experiences are most beneficial for robust learning—helps us to improve mathematics education in the US and abroad. Development of mathematical thinking is an essential aspect of children's cognitive development. As adults, we reason about numbers and mathematics relationships every day. We reason with decimal numbers when calculating how much something costs or how much change we should receive. We measure, add, and transform fractions of ingredients while cooking. We consider and compare percentages when making decisions about interest rates at the bank and false positive rates at the doctor's office. In addition to using mathematics in everyday contexts such as these, many professions require advanced mathematical skills.

Given the critical role of mathematical thinking across a variety of contexts, the vast variability in children's mathematical knowledge across cultures--even among economically developed nations--is concerning. For example, children in the US have been lagging behind those in many other developed countries on international assessments of mathematics achievement, despite sufficient access to public education. Among the 72 countries participating in the OECD's Programme for International Student Assessment (PISA), a survey of science, reading, and mathematics literacy among 15 year olds, the US scored below average in mathematics in the 2015 survey, with a significantly *lower* average score than the previous 2012 survey (OECD, 2016). The top scoring countries included Singapore, China, and Japan. Furthermore, the survey showed that only about 1% of students in the US reached the most advanced levels of mathematics reasoning, compared to about 10% in higher achieving countries. This raises a variety of questions about what facets of students' educational experience in higher-performing countries result in more mathematical proficiency, and which of these could be implemented in other, under-performing countries. Cross-cultural comparisons of American and East Asian classrooms have uncovered a variety of differences that may contribute to differential mathematics knowledge, including differences in teacher preparation (e.g., Ma, 1999), classroom practices (e.g., Richland, Zur, & Holyoak, 2007), and attitudes towards mathematics learning and practice (e.g., Stevenson, Chen, & Lee, 1993). One goal of research in mathematics cognition and development is to uncover the psychological mechanisms that propel development and shape children's emerging ideas and to identify experiences that enhance children's understanding in order to inform the best practices for mathematics education in the US (e.g., Siegler et al., 2010, an IES Practice Guide) and abroad.

The development of children's fraction knowledge also appears to play an important role in the development of mathematical cognition. Children's understanding of fraction magnitudes is highly correlated with their current standardized mathematics achievement scores (Siegler, Thompson, & Schneider, 2011) and is a better predictor for students' readiness for learning algebra (i.e., knowledge of symbols and proficiency with solving equations and word problems) than their understanding of whole number magnitudes (Booth & Newton, 2012). More strikingly, fifth-grade students' fraction knowledge predicts later high school mathematics achievement, even after controlling for children's other early mathematics knowledge, domain general capacities (i.e., verbal IQ, nonverbal IQ, and working memory), and social factors (i.e., family education and family income; Siegler et al., 2012).

Despite its critical importance in mathematics education, and therefore educational success more broadly, American children continue to struggle to understand the magnitudes associated with fraction symbols (e.g., NCTM, 2007), procedures for solving fraction problems (e.g., Siegler et al., 2011; Siegler & Pyke, 2013), and the concepts underlying fraction operations (e.g., Richland & Hansen, 2013; Sidney & Alibali, 2015). For example, in a recent NAEP survey, half of the tested eighth graders incorrectly ordered three fractions (2/7, 1/12, and 5/9) from smallest to largest (NCTM, 2007). This is particularly striking given that current standards for mathematics education recommend introducing symbolic fractions as early as 3rd grade (NGA & CCSSO, 2010). Indeed, in a survey of 1,000 US Algebra teachers evaluating their students' preparation for Algebra 1, teachers reported that students' fraction understanding was the second "biggest problem", out of 15 possible areas, in their algebra preparation (Hoffer, Venkataraman, Hedberg, & Shagle, 2007). Even elementary school *teachers* often struggle with fraction concepts (e.g., Lo & Leu, 2012; Ma, 1999). For example, in one cross-cultural study of American and Chinese early mathematics teachers, nearly all of the 23 American teachers who were interviewed struggled to accurately solve and generate word problems about fraction division (Ma, 1999). In contrast, all of the Chinese teachers could not only accurately solve fraction division problems, but they were also able to describe multiple ways of conceptualizing fraction division and multiple strategies for approaching this topic with their students. Given the alarming gaps in students' knowledge, it is crucial to examine the ways in which children's fraction knowledge, and their educational success more generally, can be improved.

Mathematical Cognition and Developmental Theory

In addition to the practical applications of research to education, research on mathematical cognition often sheds light on a variety of fundamental questions about cognition generally. What is the nature of our *mental representations* of quantity? How do these mental representations *change over time*? What kinds of *environmental experiences* have formative and lasting effects? To what extent is mathematics cognition supported by *domain specific* knowledge about mathematics or by *domain general* competencies and processes? Which aspects of children's cognition are *innate* or very early emerging? Historically, many of these questions have been primarily addressed within children's mathematics cognition with respect to children's developing understanding of natural, whole numbers. As we will show, many findings from this research generalize to development of fraction knowledge.

Representing Whole Numbers

Researchers have long been interested in how children represent and estimate the numerosity of sets and the magnitudes of symbolic numbers. Using numerosity comparison tasks in which participants are asked to choose the more numerous of two sets of dots (see Figure 2, Panel A), a variety of studies have shown that infants to adults can rapidly choose which set has more, without the need for counting (e.g., Barth et al., 2003; Halberda, Mazzocco, & Feigenson, 2008; Xu & Spelke, 2000; Xu, 2003). Importantly, the *ratio* between two numerosities (or the difference in their *logarithms*) governs our ability to distinguish them, such that lower ratios between sets (i.e., 1:2) are more easily distinguished than higher ratios (i.e., 7:8), regardless of the number of items in each set. This ability to represent the numerosity of items has been attributed to an approximate number system (ANS) that supports estimating the numerosity of sets in an inexact way, with decreasing precision with increasing numerosity, and appears to be present in even very young children. Similar magnitude estimation abilities have also been documented in other primates (e.g., Brannon & Terrace, 1998) and rats (Meck & Church, 1983), suggesting that the ability to extract quantity information from nonsymbolic sets is an evolutionarily primary, and perhaps innately specified, ability.

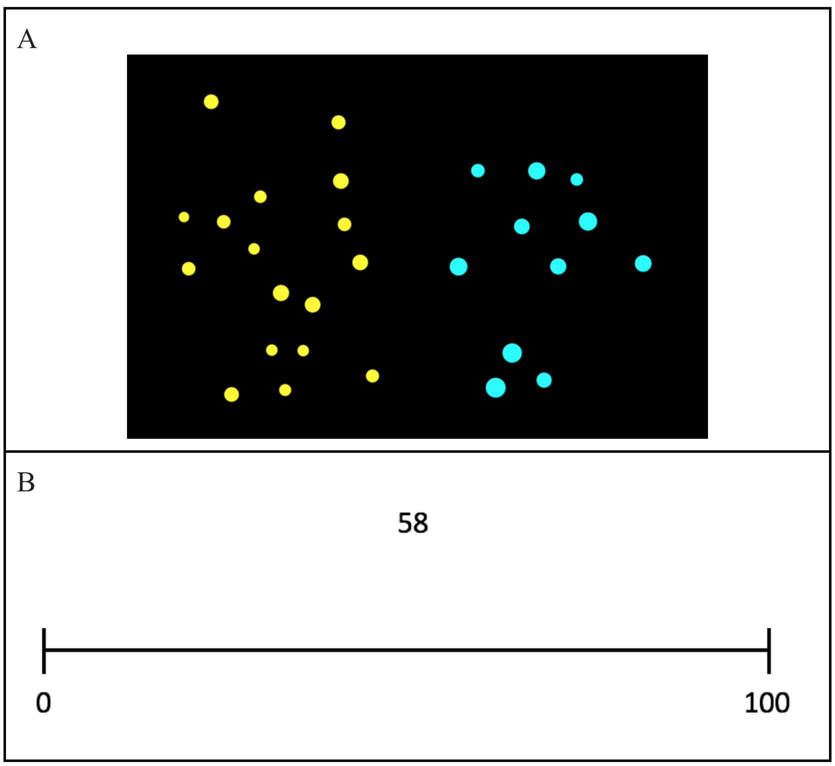

Figure 2. Tasks for assessing knowledge of natural, whole number magnitudes. Panel A shows an example of the dot comparison task, which is used to measure nonsymbolic whole number magnitude comparison (e.g., Halberda et al., 2008). In this task, people are asked to choose the larger number of dots without sufficient time to count them. Panel B shows an example of the number line task, which is used to measure symbolic whole number magnitude estimation (e.g., Siegler & Opfer, 2003). In this task, people are asked

to mark the number line to show where 58 is located.

This early emerging ability to represent the magnitude of nonsymbolic sets appears to support the more advanced ability to represent numeric magnitudes with numeric symbols, such as representing the numerosity of a set of two with the numeral 2. Using a number line task, in which participants are asked to locate a number on a given number line (see Figure 2, Panel B), Siegler and colleagues (e.g., Siegler & Opfer, 2003) have demonstrated that children's early magnitude judgments of symbolic whole numbers follow similar, ratio-governed patterns as their judgments of nonsymbolic magnitudes – unit differences between smaller numbers are overestimated while unit differences between larger numbers are underestimated. With experience, children's mental representations of numerical magnitude become increasingly linear (Siegler & Opfer, 2003; Siegler, Thompson, & Opfer, 2009, for a review). Siegler and Opfer (2003) have argued that as children gain experience with whole number symbols and their relative magnitudes, they develop linear representations of number akin to a mental number line, on which small numbers are represented on the left and large numbers on the right, and unit differences (i.e., differences of one) are represented in the same way across the entire scale.

Development of nonsymbolic and symbolic number representations appear to occur concurrently and as a result of numerical experience. The ANS acuity, or highest ratio of sets that can be compared accurately, increases over developmental time (Halberda & Feigenson, 2008) and in response to practice (DeWind & Brannon, 2012). Children's number line representations also become linear across increasingly large scales as they gain practice with larger numbers (e.g., Siegler & Opfer, 2003; Siegler & Booth, 2004), with experiences that link symbolic numbers with linear, number-line like representations of numerical magnitude (e.g., Ramani & Siegler, 2008), with feedback about critical magnitudes (Opfer & Siegler, 2007), and via analogies between smaller number scales and larger number scales (Thompson & Opfer, 2010).

It remains somewhat controversial whether these mental representations are domain-specific representations of numerical magnitude or are based on domain-general representations of magnitude that also support the measurement of other nonsymbolic magnitudes, such as area and time. The ANS has been described as a domain specific representation (e.g., Feigenson, Dehaene, & Spelke, 2004). Alternatively, the classic view is that the brain uses a general analog magnitude system to represent number (Dehaene, 2003; Meck & Church, 1983; Moyer & Landauer, 1967). Still a third view was recently proposed by Leibovich and colleagues (Leibovich, Katzin, Harel, & Henik, in press; Leibovich, Kallai, & Itamar, 2016), who have suggested that numerical estimation of nonsymbolic sets is supported by a domain general, approximate magnitude system (AMS), which becomes specialized to support numerical reasoning. Regardless of the specific nature of the magnitude representations, any truly domain general process, such as working memory (Namkung & Fuchs, 2016) and inhibitory control (e.g., Fuchs & McNeil, 2012), must also play a role in children's ability to map between symbolic numbers and magnitude representations.

Understanding Fractions

One challenge of research in children's mathematical thinking is to build a theory of mathematical development that incorporates children's thinking about numbers more broadly (Sidney, Thompson, Matthews, & Hubbard, in press; Siegler, Fazio, Bailey, & Zhou, 2013; Siegler et al., 2011). Under this goal, the study of children's fraction reasoning has emerged as an important topic in children's mathematical thinking and development (Siegler, 2016; Siegler & Lortie-Forgues, 2014; Siegler et al., 2013; Siegler et al., 2011). Natural, whole numbers comprise a very small subset of the kinds of numbers we use in complex mathematics, and even in our everyday lives. As illustrated by the examples given at the beginning of the chapter, in addition to reasoning about whole number sets, we also reason about ratios and proportions represented with rational numbers, such as fractions, decimals, and percentages.

Examining children's developing understanding of fraction magnitudes, symbols, and operations allows us to broaden our characterization of mathematical development. Critically, fractions can serve as an illuminating test case for theories of mathematical development that have been based primarily on studies of children's whole number reasoning. Studies of fraction reasoning allow us to examine whether these theories generalize. For example, as we will discuss further in the next section, Siegler, Thompson, and Schneider (2011) examined whether the mental representations that support symbolic whole number reasoning (i.e., the mental number line) are similar to those that support symbolic fraction reasoning. Fractions also allow us to evaluate assumptions about the relationships between early emerging whole number competencies and later complex mathematics. For example, many researchers have argued that whole number reasoning is early emerging because the mind is innately equipped to reason about natural whole numbers, whereas fraction representations must be constructed from whole number representations. Recently, some researchers (Lewis, Matthews, & Hubbard, 2015; Matthews & Hubbard, 2016) have pointed to evidence for early competence in ratio reasoning, arguing for an intuitive ratio processing system (RPS) that should support early fraction reasoning as well. This work will also be further discussed in the next section. Furthermore, researchers who argue against the domain specificity of our understanding of numerical magnitude, such as Leibovich and colleagues (Leibovich et al., in press; Leibovich et al., 2016) who argue for the AMS rather than ANS, leave room for whole number and fraction reasoning to develop in parallel rather than in sequence. As these examples illustrate, examining the development of fraction reasoning opens up several intriguing questions about the relationship between natural, whole number reasoning and more complex mathematics.

In addition to these questions about mathematical development, examining the development of fraction reasoning also allows us to test key theories of general cognitive development, in particular those concerning the development of *relational reasoning* and *transfer*. Fractions are fundamentally a relational concept. Their meaning is not derived from a single component, either the numerator or denominator alone, but from the ratio relationship between these two components. The ability to represent relations among elements requires representing individual elements and thus develops later than the ability to represent the individual elements. For example, the ability to match sets based on relational patterns across elements within sets (e.g., small-medium-large or A-B-A patterns) rather than matching based on the perceptual details of the elements increases with age (e.g., Gentner, 1988). Furthermore, children's attention to relational structure can be supported by using relational language (e.g., referring to a small-medium-large pattern as “Baby, Mommy, Daddy”; Rattermann & Gentner, 1998), making an analogy to relationships in more familiar contexts (e.g., Goswami & Brown, 1990; Goswami, 1995), and by reducing the working memory demands of tasks relying on relational reasoning (e.g., Richland & McDonough, 2010; Thompson & Opfer, 2010). Although only a small subset of research on children’s relational reasoning is situated in mathematical contexts, many mathematical concepts, such as fraction magnitudes, are inherently relational. Therefore, by examining whether these cognitive supports also enhance children’s ability to reason about fractions, we can test the generalizability of these prior findings.

Similarly, by examining the relationship between children’s emerging fraction understanding and their prior knowledge of whole number magnitudes and operations, we can test and further illuminate our theories of transfer. Children’s whole number knowledge sometimes appears to negatively bias their fraction concepts (e.g., Ni & Zhou, 2005), suggesting that children’s prior numerical knowledge transfers to new fraction concepts in ways that are unhelpful for new learning. Yet, several aspects of children’s whole number and fraction knowledge are strongly correlated (e.g., Bailey, Siegler, & Geary, 2014) and their knowledge of arithmetic operations with whole numbers can directly support learning about fraction arithmetic through analogical transfer (e.g., Richland & Hansen, 2013; Sidney & Alibali, 2015). Further investigation of when and how children spontaneously transfer from their prior knowledge of whole numbers when making sense of fractions may further illuminate the mechanisms of such transfer and contextual features of instruction that would further support appropriate transfer from students’ prior knowledge. The relationship between children’s knowledge of whole numbers and their developing understanding of fractions will be further discussed in the next section.

The Development of Fraction Skills

In this section, we review the current state of evidence of children’s and adults’ understanding of fractions, as well as some comparative evidence. First, we describe the nature of children’s nonsymbolic proportion and ratio reasoning, as children typically show early competence with matching and comparing nonsymbolic fractions, similar to that of non-human animals. Next, we describe the nature of children’s and adults’ reasoning about the magnitudes of symbolic fractions, which are first introduced in early childhood, and appear to cause difficulty for some students. Then, we will turn to more complex symbolic reasoning (e.g., arithmetic operations with fractions), which often depends on having adequate knowledge of fraction magnitudes. Along the way, we will discuss the methodology used to examine children’s nonsymbolic and symbolic fraction understanding. Finally, we will consider the developmental pathways to fraction understanding including the evidence for relationships between nonsymbolic and symbolic reasoning, and the relationship between facets of children’s fraction knowledge and other mathematics knowledge, as well as the role of domain general processes in the development of children’s fraction skills.

Nonsymbolic Proportions and Ratios

Children, adults, infants, and even non-human animals are able to reason about proportions and ratios in nonsymbolic contexts. Children as young as 4 years old can easily match visual stimuli representing the same proportions, despite different overall size (Duffy, Huttenlocher, & Levine, 2005; Sophian, 2000; Spinillo & Bryant, 2001) or even vastly different perceptual details (Singer-Freeman & Goswami, 2001). Furthermore, children as young as 4 years old can predict the outcome of simple addition and subtraction of nonsymbolic proportions, represented as parts of a circle, and this ability appears to develop from 3 to 5 years in parallel with the ability to add and subtract discrete, whole number, sets of objects (Mix et al., 1999). One limitation of these earlier studies is that they leave open the possibility that children may be completing the tasks via mechanisms other than attending to proportion, such as object-matching or attending to overall amount rather than proportion per se; similar criticisms have also been leveled at studies of nonsymbolic whole number reasoning (see Gebuis & Reynvoet, 2012; Leibovich et al., in press; Mix, Huttenlocher, & Levine, 2002).

More recently, interest in children's fraction reasoning has led many researchers to examine children's and adults' abilities to compare one *ratio* of two numerosities to another *ratio* of two numerosities, in magnitude comparison tasks that are parallel to those used to investigate whole number reasoning (i.e., the dot comparison tasks used in studies of the ANS, see Figures 2 and 3) and tasks that better reflect symbolic fractional notation (e.g., two sets of dots separated by a fraction bar, with one taking the place of the numerator and the other taking the place of the denominator; see Figure 3, Panel B). These studies more closely target participants' sense of fractions as a ratio between two numbers or quantities.

In one such study, Fazio, Bailey, Thompson, and Siegler (2014) administered a variety of fraction and whole number tasks to fifth-grade children, including a nonsymbolic fraction magnitude comparison task. In this task, children viewed displays with two sets of yellow and blue dots, intermixed, of varying sizes (see Figure 3, Panel A). Each set represented a specific ratio (e.g., 3:8 was represented with three blue dots and five yellow dots). Children were told that the dots represented candies, and the blue dots taste the best, therefore, they should choose the side that would give them the best chance of picking a blue candy. Fazio and colleagues found that children were fairly successful at this task, choosing the larger ratio, on average, on 70% of trials. Similar to studies of whole number magnitude comparison, children were more accurate on the largest ratio differences (76%) than on the smallest ratio differences (65%).

A	B	C	D

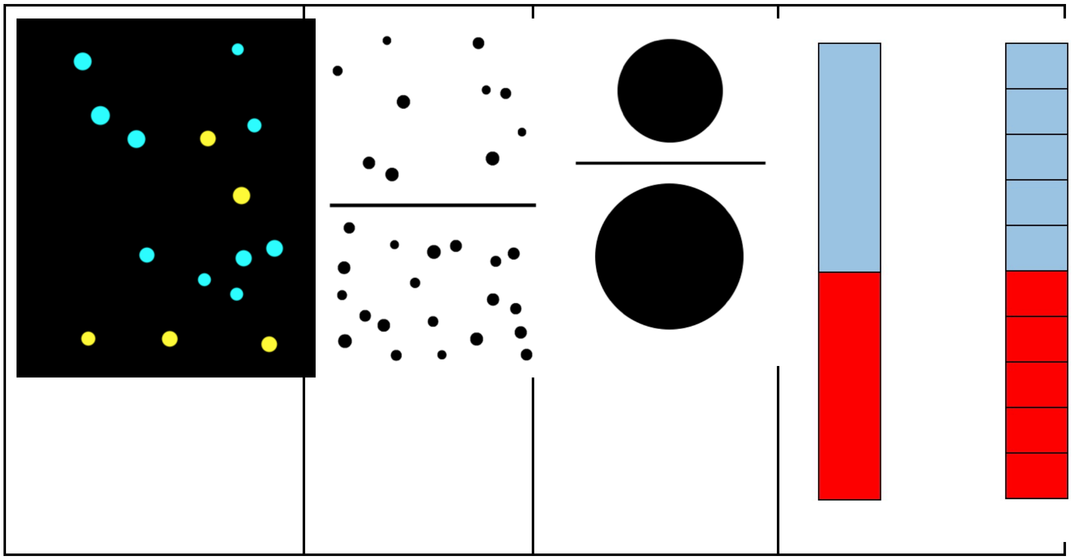

Figure 3. Example stimuli for tasks assessing infants', children's, and adults' ability to represent and compare nonsymbolic ratios and proportions. Panel A shows an example of a dot ratio stimulus in which the component numerosities are intermixed, as in Fazio et al. (2014) and McCrink & Wynn (2007). Panel B shows an example of a dot ratio stimulus in which the component numerosities are represented separately, in a fraction format, as in Matthews & Chesney (2015). Panel C shows an example of an area ratio stimulus, as in Matthews & Chesney (2015). Panel D shows two examples of area proportion stimuli, one continuous and one discretized, as in Boyer et al. (2008). Each panel shows an example of a 1:2 ratio.

Adults are also adept at perceiving and comparing nonsymbolic ratios and proportions (e.g., Matthews & Chesney, 2015; Meert, Grégoire, Seron, & Noël, 2012). In their experiments with adult participants, Matthews and Chesney used symbolic fractional notation with sets of dots to represent component numerosities (i.e., one set of dots in the numerator position of the fraction and another set of dots in the denominator) or differently sized circular areas to represent component magnitudes (i.e., a smaller circle in the numerator position and a larger circle in the denominator; see Figure 3, Panels B and C). In their tasks, adults were able to accurately choose the greater ratio when comparing symbolic fractions to nonsymbolic dot fractions, symbolic fractions to nonsymbolic area fractions, and nonsymbolic dot fractions to nonsymbolic area fractions. In all cases, participants demonstrated *the distance effect*: they were faster and more accurate for more distant ratios and slower and less accurate for closer ratios, in parallel to earlier studies of adults' whole number comparisons. Furthermore, in a supplemental experiment, Matthews and Chesney found that adults were slower to make comparisons across two symbolic fractions than two nonsymbolic fractions in the same format, making it improbable that adults were converting nonsymbolic ratios to symbolic fractions in order to make the comparisons across ratios. Thus, adults can automatically represent ratio magnitudes in fractional formats, and the mental representations of these ratio magnitudes show an important similarity to adults' mental representations of whole number magnitudes -- the comparison process itself is affected by the distance between ratios.

Furthermore, the ability to compare ratio magnitudes appears to be quite early emerging. McCrink and Wynn (2007) found that infants as young as 6 months old were sensitive to changes in ratios of items, as long as the ratios differed by at least a factor of 2. Their study used a habituation paradigm to test infants' ratio perception, in which infants were first habituated to stimuli with one ratio and then tested with stimuli that either matched or did not match the ratio of the habituated stimuli. Critically, habituation paradigms are used to test whether participants can perceive differences between habituated stimuli and test stimuli. If infants *dishabituate*, or begin to look longer at the new stimuli, during the test phase, researchers can infer that infants perceive the test stimuli as being different from habituated stimuli. In contrast, if infants do not look longer at the new stimuli during the test phase, then researchers can infer that infants *do not* perceive the new stimuli as being different from the habituated stimuli.

During the habituation phase, infants viewed a series of displays with yellow Pac-Men and blue pellets in different numbers, but constant ratio in each display (e.g., 8 blue pellets and 4 Pac-Men and 14 blue pellets and 7 Pac-Men are both a 2:1 ratio), until habituated. As in the adult work, the size of the dots varied within and across displays. To test whether infants can perceive ratio, McCrink and Wynn (2007) then tested whether infants would dishabituate to new ratios of items. The infants were assigned to one of two groups, a distant ratio group and a close ratio group (see Table 1). In the distant ratio group, the new ratio differed from the old ratio by a factor of 2. For example, some infants were habituated to displays with a 2:1 ratio and tested with displays with a 4:1 ratio. In the close ratio group, the new ratio differed from the old ratio by a factor of 1.5. For example, some infants were habituated to displays with a 2:1 ratio and tested with displays with a 3:1 ratio. Importantly, the numbers of items in the testing displays were different than in the habituation items seen by both groups.

McCrink and Wynn (2007) found that infants in the distant ratio group, on average, paid more attention to new ratios that differed by a factor of 2, whereas infants in the close ratio group did not pay attention to the new ratios that differed by a factor of 1.5. In other words, when the ratios of objects in the habituated and tested displays were very different, infants perceived them to be different. When the ratios of objects was less different, the infants did not appear to perceive the difference, even though the *number* of items was different in both conditions. Their findings suggest that infants can perceive differences in ratios, when they are sufficiently large, and thus must also be able to perceive the individual ratios between numbers of objects.

Table 1

Experimental Design and Findings from McCrink & Wynn (2007)

Group	Habituated Ratio	Test Ratios	Difference Factor[b]	Outcome
Distant Ratio[a]	Infants habituated to 2:1 (8:4, 38:19, 22:11, 14:7, 30:15)	2:1 (20:10, 32:16)	1	Infants looked longer at new, 4:1 test stimuli
		4:1 (40:10, 32:8)	2	
	Infants habituated to 4:1 (12:3, 48:12, 28:7, 24:6, 36:9)	2:1 (20:10, 32:16)	2	Infants looked longer at new, 2:1 test stimuli
		4:1 (40:10, 32:8)	1	
Close Ratio[c]	Infants habituated to 2:1 (8:4, 38:19, 22:11, 14:7, 34:17)	2:1 (20:10, 32:16)	1	Infants *did not* look longer at new 3:1 test stimuli
		3:1 (18:6, 30:10)	1.5	
	Infants habituated to 3:1 (9:3, 48:16, 27:9, 15:5, 36:12)	2:1 (20:10, 32:16)	1.5	Infants *did not* look longer at new 2:1 test stimuli
		3:1 (18:6, 30:10)	1	

[a] The distance, or difference factor, between the habituated ratio and the new test ratio is greater. [b] The difference factor indicates the magnitude of the difference between the habituated and test ratios. A 2:1 ratio differs from a 4:1 ratio by a factor of 2 and differs from 3:1 by a factor of 1.5. [c] The distance, or difference factor, between the habituated ratio and new test ratio is smaller.

This work with infants has at least two implications for the development of nonsymbolic ratio reasoning. First, it demonstrates that the ability to represent and discriminate between ratios across numbers is early emerging, and thus likely supported by innate cognitive capacities. This idea is further supported by many similar studies with animals demonstrating sensitivity to differences or changes in proportions (Drucker, Rossa, & Brannon, 2016; Emmerton, 2001; Harper, 1982; McComb, Packer, & Pusey, 1994; Vallentin & Nieder, 2008). For example, recently, Drucker and colleagues (2016) demonstrated that rhesus macaques were able to choose a display with the larger ratio for a candy reward, and resembling the 6-month-old infants, their accuracy was modulated by the differences in ratio. Second, the role of distance in infants', adults', and other animals' ability to discriminate proportions bears a striking similarity to the role of distance in infants' ability to discriminate numerosities of sets. As McCrink and Wynn (2007) discuss, 6-month-old infants' approximate representations of numerosities appear parallel to their approximate representations of proportions, as in both contexts, infants can discriminate sets that differ by a factor of 2 (20 dots *vs.* 40 dots, 2:1 dot ratio *vs.* 4:1 dot ratio), but not sets that differ by a factor of 1.5. Their work suggests a similar mechanism for numerical (whole number) discrimination and ratio (fraction) discrimination, although the exact mechanism remains unclear.

Despite these early competencies with nonsymbolic ratios and proportions, older children's and adults' nonsymbolic fraction reasoning sometimes appears to be sensitive to the nature of the proportional stimuli, specifically, whether the fraction is represented as a ratio of discrete, countable segments or a ratio of continuous amounts (Boyer & Levine, 2015; Boyer, Levine, & Huttenlocher, 2008; DeWolf, Bassok, & Holyoak, 2015). For example, Boyer and colleagues (2008) asked first, second, third, and fourth grade students to choose a mix of juice and water that matched a sample "recipe" (i.e., in the right proportions) to give to a very particular teddy bear. Children were shown a picture of a juice and water mixture in one of four conditions, which varied on how the "recipe" sample and the target proportion were displayed, either as continuous amounts of juice and water (e.g., as in an unmarked cylinder) or discretized, countable sections of juice and water (e.g., as in a graduated cylinder; see Figure 3, Panel D). Then, they were asked to choose between a target which matched on juice:water ratio, but not the overall amount of fluid, or a foil which matched on the amount of juice or overall amount of fluid, but not the juice:water ratio. When both proportions were represented in a discretized way, children were less likely to choose the correct, proportion-match, and instead, likely to choose a foil that matched the number of juice "sections". Thus, although children are able to reason about proportions and ratio when the components are continuous or not easily countable (e.g., in the dot ratio tasks), when ratio components are easily countable, children appear to rely on independent whole number components, rather than ratio, to match. Studies of children's symbolic fraction reasoning have uncovered parallel strategies during symbolic fraction comparison, and this reliance on only one ratio component has been attributed to a *whole number bias* (e.g., Ni & Zhou, 2005), which we will discuss further in the following section.

Symbolic Fractions

Given an early emerging ability to discriminate ratios, it might be expected that symbolic fractions would pose no special problems for young children. However, a wealth of research demonstrates that children, in particular those educated in the US, have difficulty understanding the magnitudes and ratios expressed by symbolic fractions. For example, in interviews with elementary-aged children, Mack (1990, 1995) found that children learning about symbolic fractions in school displayed a range of misconceptions about the meanings of various fractions, many based on incorrect application of counting strategies to assess magnitude. For example, one young student reported 3/5 as meaning: "Oh, three fifths, that's three whole pumpkin pies with five pieces in each pie" (Mack, 1995, p. 431). As the example at the beginning of this chapter illustrates, even when children have informal understanding of the magnitudes of fractional parts, this intuition is not always connected to their symbolic reasoning. Instead, often children, and sometimes adults, focus on the magnitudes of individual, whole number components of symbolic fractions in order to make sense of fraction symbols.

Children's systematic tendency to interpret fractions in a way that reflects robust transfer of children's early counting and whole number knowledge is often referred to as the *whole number bias* (e.g., Ni & Zhou, 2005). One common example of the whole number bias occurs when children are comparing fraction magnitudes and rely only on the denominator or numerator component (i.e., the independent whole number component) to judge relative size. For example, children will often judge 1/3 as less than 1/4, because 3 is less than 4 (Behr, Wachsmuth, Post, & Lesh, 1984) or judge 2/2 as less than 3/4, because 2 is less than 3 and 4 (Harnett & Gelman, 1998). Even adults' fraction comparison is sometimes influenced by whole number components. Bonato, Fabbri, Umilta, and Zorzi (2007) found that even adults rely on whole number magnitude representations while comparing unit fractions, fractions with 1 in the numerator (e.g., ⅕ *vs*. ¼). When asked to decide which of two symbolic fractions is bigger, Bonato and colleagues found that distances between *denominator components* significantly influenced the comparison speed (e.g., 5 - 4, for ⅕ *vs*. ¼), such that participants were faster when the distances between denominator components was larger. The distances between the ratios (e.g., .20 - .25, for ⅕ *vs*. ¼) did not significantly predict comparison speed, as it does in many nonsymbolic fraction comparison tasks, suggesting that adults were only attending to denominator components to make the comparison.

However, adults' reliance on independent whole number components during fraction comparison tasks is often constrained to pairs for which relying on either the numerator or denominator to make comparisons is an efficient strategy. For example, when comparing two unit fractions, one can rely on their knowledge of the heuristic that larger denominators indicate smaller fractions, thus a comparison across denominator components will always result in a quick, accurate answer (see Table 2). Schneider and Siegler (2010) demonstrated a distance effect based on overall ratio, rather than components, when more complex, non-unit fractions (e.g., 7/9 *vs*. ⅗) were included in the task, preventing strategies based on independent components, alone. This distance effect, based on the decimal distance between the two to-be-compared fractions, was taken as evidence that adults could reason about the holistic magnitude of fractions. Adults were more accurate when they compared more distant fractions.

Table 2

Common Strategies for Comparing Fraction Magnitudes among Adults

Strategy Type	Strategy	Description	Example Pairs[a]	
Strategies that rely on only the numerator or denominator magnitude	Equal Denominators	When the denominators are equal, the fraction with the larger numerator is larger	2/5	3/5
	Equal Numerators	When the numerators are equal, the fraction with the smaller denominator is larger	2/7	2/5
	Multiply for a Common Denominator	Multiply the numerator and denominator of one fraction by a whole number so that the denominators become equal, deploy *Equal Denominators* strategy	2/5	7/10
	Multiply for a Common Numerator	Multiply the numerator and denominator of one fraction by a whole number so that the numerators become equal, deploy *Equal Numerators* strategy	2/5	4/7
Strategies that rely on considering the numerator and denominator magnitudes	Larger Numerator and Smaller Denominator	Fractions with larger numerators AND smaller denominators are larger in magnitude	2/5	3/4
	Difference between numerator and denominators	Fractions with smaller differences between denominators and numerators are larger	2/5	8/9
Strategies that rely on considering the ratio between numerator and denominator	Halves Reference	When one fraction is less than 1/2, the fraction that is greater than ½ is larger	2/5	6/11
	Numerator goes into the denominator fewer times	Divide the denominator by the numerator; the smaller quotient is the larger fraction	2/5	7/17

Note. This strategy table is adapted from Fazio, DeWolf, and Siegler (2016). They found that adults rely on a range of strategies for magnitude comparison. These strategies can lead to accurate comparisons when used effectively.

[a]Larger fractions appear on the right.

Although relying on independent components to judge whether one fraction is larger than another can be advantageous in the magnitude comparison task, it should be noted that relying on only the denominator to estimate magnitude more precisely (e.g., when estimating the size of 1/60 relative to 1/1 and 1/1440) can decrease the accuracy of the estimate (Opfer & DeVries, 2008; Thompson & Opfer, 2008). Thus, by adulthood, people can accurately reason about the magnitude of symbolic fractions, using a variety of strategies, although they still have difficulty representing their magnitudes precisely. Given that fraction magnitude estimation is often a strategic, rather than automatic process, adults are slower to process fraction magnitudes than other kinds of numbers (e.g., decimals; DeWolf, Grounds, Bassok, & Holyoak, 2014).

Siegler, Thompson, and Schneider (2011) demonstrated that, like adults, children also rely on a variety of strategies for estimating the magnitude of symbolic fractions. Siegler and colleagues examined sixth and eighth grade students' fraction understanding in a variety of tasks, including a fraction number line task. The fraction number line task is adapted from the original number line task (see Figure 2, Panel B) used to assess the nature of people's whole number magnitude representations (e.g., Siegler & Opfer, 2003). This task assesses symbolic fraction magnitude estimation by requiring participants to place fractions (e.g., ⅗) on a number line with given endpoints, often 0 and 1 or 0 and 5. Accuracy is measured with percentage absolute error (PAE), an index of the measured difference between the participant's placement of the fraction and the correct location of the fraction relative to the total length of the number line, such that higher PAE indicates lower accuracy.

Children use a wide variety of strategies in order to estimate fraction magnitudes on the fraction number line task (Siegler, et al., 2011; Siegler & Thompson, 2014), including highly advantageous strategies, such as transforming less familiar fractions into more familiar fractions and comparing fractions to subjective landmarks on the number line (e.g., using their knowledge of ½, the midpoint of the 0-1 line, as a landmark for estimating other fraction magnitudes), and less advantageous strategies, such as using the numerator or denominator magnitude to guide magnitude estimation (e.g., placing 3/7 near 0, since 3 is a small number). The highly advantageous transformation and landmarks strategies are common and often significantly related to lower PAE. As a further indicator of children's reliance on their strategic knowledge to estimate fraction magnitudes, children's estimations are often time-consuming compared to the time it typically takes younger children to estimate whole number magnitudes, and their estimates of fraction magnitudes are less accurate overall than whole number magnitudes (Fazio et al., 2014).

Taken together, this research (Bonato et al., 2007; DeWolf, Grounds, Bassok, & Holyoak, 2014; Schneider & Siegler, 2010; Siegler et al., 2011; Siegler & Thompson, 2014) demonstrates that children's and adults' estimation of the magnitude of symbolic fractions requires them to process the ratio across component numbers, rather than automatically perceive that ratio, and both children and adults often rely on strategic knowledge to make estimations and comparisons. Importantly, this contrasts with the research on nonsymbolic fraction understanding, which likely relies on automatic processes.

The variability in children's and adults' strategy use across tasks and stimuli can account for the variability in evidence for the whole number bias. On the basis of this research, Alibali and Sidney (2015) have argued for a *dynamic strategy choice* account of the whole number bias, suggesting that people's strategy choices are guided by the strength of their magnitude representations for the numbers in the problems (e.g., how easily a person can directly estimate the magnitude of a number), their repertoire of available strategies, and the context or affordances of the task at hand. For example, among children who are still learning about fraction symbols, fraction magnitude representations and strategic knowledge are not well developed. In contrast, their knowledge of whole number magnitude representations and strategies may be quite well developed, and automatically activated, causing children to rely on their whole number knowledge and often resulting in the whole number bias. Similarly, adults may rely on whole number magnitude knowledge when the task stimuli afford whole number-based reasoning (e.g., as in Bonato et al., 2007) or when their fraction magnitude representations are not precise enough to support a direct comparison (e.g., when comparing fractions that are close in magnitude, e.g., 4/5 and 7/9). Among older children and adults, having rich strategic knowledge helps them to leverage their whole number knowledge and avoid the pitfalls of the whole number bias.

Advanced Fraction Concepts

Children's knowledge of fractions extends beyond understanding the magnitudes of fraction symbols. For example, children also learn about other fraction properties, such as the idea that unlike natural, whole numbers, fractions cannot be counted in a sequence and that there are an infinite number of fractions between any two fractions. This latter idea is referred to as *numerical density,* and children's understanding of numerical density lags behind their understanding of fraction magnitudes (e.g., McMullen, Laakkonen, Hannula-Sormunen, & Lehtinen, 2015). Even when children understand the idea of numerical density in the context of whole numbers (i.e., there are an infinite number of numbers between any two whole numbers), they are less likely to understand this same idea in the context of fractions (i.e., there are an infinite number of numbers between any two fractions; e.g., Vamvakoussi & Vosniadou, 2010). Instead, seventh, ninth, and even eleventh graders may claim, for example, that there are no fractions between ⅗ and ⅘ because there are no whole numbers between 3 and 4. Children may have difficulty reasoning about numerical density between fractions in part because of poorly developed or imprecise mental representations of fraction magnitudes or strategies for thinking about fraction magnitudes, thus relying on their knowledge of countable, whole number sequences to reason about density (see Alibali & Sidney, 2015 for a discussion). Even adults show evidence of a whole number bias, for example, responding that there is only one number between ½ and ¼ (e.g., ⅓), particularly when the experimenter highlights this possibility by gesturing once between ½ and ¼ printed on paper (Brown, Donovan, & Alibali, 2016).

Furthermore, a great deal of research has examined children's ability to calculate and understand addition, subtraction, multiplication, and division with fractions. Children, and even adults, show deficits in their ability to apply fraction arithmetic procedures to symbolic problems. For example, Siegler and colleagues (Siegler et al., 2011; Siegler & Pyke, 2013) found that sixth and eighth grade children showed poor accuracy on fraction arithmetic problems across all four operations. Children often implemented incorrect strategies, such as operating on the numerators and denominators independently (e.g., ⅗ + ½ = 4/7) or inappropriately applying a problem-solving strategy meant for one operation on another, such as only operating on the numerators when the denominators are the same (e.g., ⅗ * ⅖ = 6/5) a correct procedure for fraction addition, but not fraction multiplication. Both types of errors suggest spontaneous transfer for children's prior knowledge to these difficult problems. Furthermore, children's knowledge of fraction division appears to be especially poor. Even at eighth grade, long after fraction arithmetic instruction typically occurs in 5th and 6th grade (NGA & CCSSO, 2010), children's proficiency with fraction division calculation is quite low (46% correct, Siegler et al., 2011).

In addition, children often have poor conceptual understanding of fraction operations, and most notably, fraction division (e.g., Sidney & Alibali, 2015; Richland & Hansen, 2013). As described in an earlier section, even American elementary teachers are unsure of how fraction division might be represented in a "real world" context (e.g., Ma, 1999). Studies assessing conceptual knowledge of fraction division often employ tasks in which people are asked to generate a story (e.g., Ball, 1990; Ma, 1999; Sidney & Alibali, 2015; Sidney, Hattikudur, & Alibali, 2015) or a diagram (e.g., Richland & Hansen, 2013; Sidney, 2016; Sidney et al., 2015) to represent a fraction division problem (see Figure 4). Similar to children's errors on solving symbolic fraction problems (e.g., Siegler & Pyke, 2013), children's errors on conceptual items often also reflect inappropriate transfer from their prior knowledge of other fraction operations. For example, children may write stories that represent multiplication and subtraction rather than division (e.g., for 5 ÷ ¼, representing ¼ of five units taken away from five whole units or 5 * (5 - ¼), see Figure 4 Panel B). Such errors are common (e.g., Sidney & Alibali, 2017) and are consistent with the misconception that "*division makes smaller*" (Fischbein, Deri, Nello, & Marino, 1985). However, children's conceptual understanding of fraction division is improved by instruction that draws on children's more familiar, whole number division concepts (e.g., Richland & Hansen, 2013; Sidney, 2016; Sidney & Alibali, 2015). Such educational interventions will be further discussed in the Implications for Instruction section.

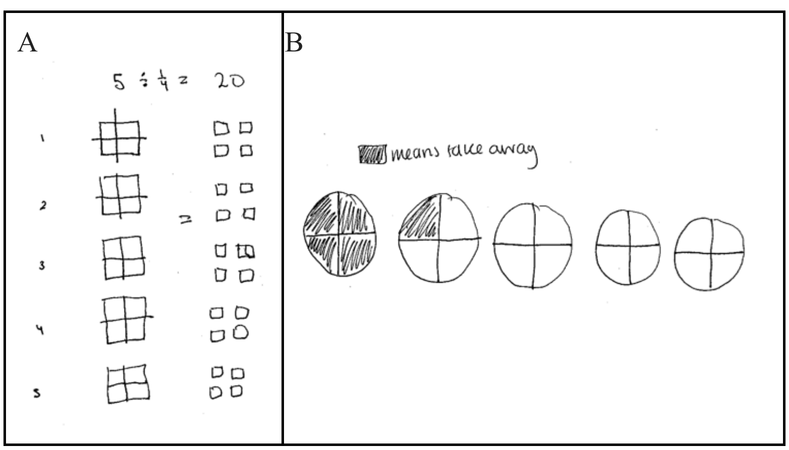

Figure 4. In some studies, children's and adults' understanding of the conceptual structure of fraction division is assessed by asking participants to draw a diagram to represent a fraction division problem. Panels A and B are examples of participants' drawings to represent 5 ÷ ¼ from Sidney and Alibali (2012). Panel A shows accurate

reasoning; 5 is divided in 20 subsections of ¼. Panel B reflects the “division makes smaller” misconception, and represents a combination of multiplication and subtraction instead of division. Sidney and Alibali (2017) also documented this error using an object modeling task assessing fraction division concepts.

Developmental Pathways

Which aspects of children’s early fraction understanding support children’s later fraction understanding as well as their later mathematics achievement? Although the complete developmental picture is still emerging, we examine several correlational and longitudinal studies that shed light on what these pathways may be. First, we discuss the role of symbolic fraction magnitude understanding in fraction arithmetic and mathematics achievement. Second, we discuss the role of nonsymbolic fraction understanding in fraction development, and mathematics development more broadly. Finally, we consider the role of whole number knowledge and general cognitive predictors in children’s fraction development.

As discussed in a previous section, much of the research on children’s numerical development has occurred in the context of their whole number reasoning. Children’s estimation and comparison of whole number magnitudes in nonsymbolic (e.g., Halberda et al., 2008; Schneider et al., 2016) and symbolic tasks (e.g., Booth & Siegler, 2008; Schneider et al., 2016) are correlated to mathematics achievement more broadly. In particular, children’s whole number magnitude estimation on the number line task appears to be important for later success in mathematics. Children who are better able to estimate the magnitude of symbolic whole numbers on a number line are more proficient with whole number arithmetic (e.g., Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007; Geary, Hoard, Nugent, & Byrd-Craven, 2008) and have better memory for numbers (e.g., Thompson & Siegler, 2010). Furthermore, providing training that increases children’s accuracy on a number line estimation task (e.g., by playing a board game in a left-to-right orientation, in parallel with the hypothesized mental number line) results in improvements in counting ability (Ramani & Siegler, 2008; Whyte & Bull, 2008) and arithmetic proficiency (e.g., Booth & Siegler, 2008; Siegler & Ramani, 2009). These findings suggest that the ability to precisely estimate the magnitude of numbers, as measured by the number line task, is a fundamental mathematics development.

Siegler, Thompson, and Schneider (2011) proposed that estimating the magnitude of symbolic fractions on a number line is similarly central to later mathematics understanding. Their *integrated theory of numerical development* posits an important role for the number line representation, in general. Thinking about numbers as represented on a continuous number line can support children’s thinking about magnitudes of both whole numbers and fractions relative to 0, 1, and other numbers as well as children’s thinking about other important numerical concepts such as numerical density between whole numbers and fractions.

In support of their hypothesis, and in parallel with the research of children’s whole number development, both correlational and longitudinal studies have demonstrated that children’s fraction magnitude understanding indeed supports more complex mathematics skills. Children’s fraction magnitude understanding is correlated with both their fraction arithmetic skills (Hecht & Vagi, 2010; Siegler et al., 2011), and children’s understanding of fraction magnitudes and their fraction arithmetic skills are both, independently correlated with their concurrent mathematics achievement (Fazio et al., 2014; Siegler et al., 2011; Torbeyns et al., 2015). Knowledge of fraction magnitudes is necessary, although not sufficient, for understanding the density of rational numbers (McMullen et al., 2015). Furthermore, children's fraction knowledge predicts later, high school mathematics achievement, over and above their knowledge of whole number arithmetic, IQ, working memory, and family education and income (Siegler et al., 2012). These findings clearly point to a central role of children's understanding of the magnitude of fraction symbols in numerical development more generally. However, the role of children's understanding of nonsymbolic ratios and proportions in their understanding of symbolic fractions, advanced fraction concepts, and mathematics achievement is less clear.

Children's and adults nonsymbolic ratio understanding is correlated with symbolic fraction understanding (Fazio et al., 2014; Matthews, Lewis, & Hubbard; 2016; Möhring, Newcombe, Levine, & Frick, 2016), algebra proficiency (Matthews, Lewis, & Hubbard, 2016), and general mathematics achievement (Fazio et al., 2014), although the relationships between symbolic fraction knowledge and achievement appears to be stronger than the correlations between nonsymbolic ratio understanding and achievement (Fazio et al., 2014). Furthermore, in a mediation model that included both whole number and fraction nonsymbolic and symbolic tasks, Fazio and colleagues examined the hypothesis that symbolic understanding of magnitudes mediates the relationship between nonsymbolic understanding and mathematics achievement, in other words, that nonsymbolic understanding is a precursor to symbolic magnitude understanding, which in turn affects children's general mathematics knowledge. They did not find support for this hypothesis. Instead, they found that nonsymbolic and symbolic magnitude understanding independently correlated with mathematics achievement, with a stronger correlation between symbolic knowledge and achievement, consistent with a recent meta-analysis (Schneider et al., 2016). However, their conclusions may be in part due to a reliance on measures of nonsymbolic and symbolic magnitude knowledge that include both whole number and fraction measures. Matthews and colleagues (2016) found that nonsymbolic ratio understanding, but *not* nonsymbolic whole number magnitude knowledge, was associated with algebra proficiency. Likewise, recent longitudinal work suggests that the relationships between nonsymbolic whole number magnitude comparison and later mathematics proficiency may be quite small when controlling for other general cognitive competencies (e.g., Sullivan, Frank, & Barner, 2016). Therefore, it remains unclear whether nonsymbolic ratio understanding directly affects the development of non-fraction mathematical knowledge, or whether students' understanding of symbolic fractions, specifically, mediates this relationship.

In addition to nonsymbolic ratio reasoning, facets of children's early mathematics knowledge of whole numbers as well as several domain general processes support the development of children's fraction concepts and skills. For example, knowledge of whole number magnitudes is correlated with knowledge of fraction magnitudes in fifth grade (e.g., Fazio et al., 2014) and children's whole number division knowledge predicts concurrent fraction arithmetic proficiency in sixth and eighth grade (Siegler & Pyke, 2013). Furthermore, children's analogical reasoning ability supports their estimation of whole number magnitudes (Alvarez et al., 2017; Sullivan & Barner, 2014; Thompson & Opfer, 2010) and is related to their ability to complete number analogies (i.e., 30:60 is like 50:__) akin to symbolic fraction matching (Alvarez et al., 2017).

A handful of longitudinal studies have begun to reveal some of the developmental precursors of children's early symbolic fraction reasoning in fourth grade (Jordan et al., 2013; Vukovic et al., 2014). Children's early symbolic fraction knowledge in fourth grade is related to a variety of domain general processes in third grade, including their attentive behaviors in the classroom, language ability, nonverbal reasoning, and working memory (Jordan et al., 2013), and in first grade, including their language ability, attentive behaviors, and visual-spatial memory (Vukovic et al., 2014). However, among the strongest predictors of children's fourth grade fraction understanding is children's second (Vukovic et al., 2014) and third grade (Jordan et al., 2013) whole number magnitude knowledge, as measured by the number line task. Moreover, Vukovic and colleagues (2014) found that children's number line estimation and whole number arithmetic fluency in second grade fully mediated the relationships between fourth grade fraction knowledge and first grade domain general skills.

Taken together, this research suggests that children's language ability, their ability to attend to instruction in the classroom, their working memory, and their analogical skills support mathematics learning more generally, and that children's whole number magnitude understanding is positively related to their fraction knowledge. This latter point is especially important in the context of research on the whole number bias. Although some aspects of children's whole number knowledge appear to interfere with fraction reasoning, understanding how to map whole numbers symbols to magnitudes on a number line provides an advantage for understanding fraction magnitudes as well.

Implications for Instruction

As we discussed at the beginning of the chapter, students' understanding of fraction concepts, such as the magnitudes associated with fraction symbols, and their understanding of fraction procedures, such as those for adding, subtracting, multiplying, and dividing fractions, is highly, and often uniquely, predictive of more complex mathematics and later mathematics achievement. Yet, many students struggle with fractions, more so than other areas of mathematics (e.g., Hoffer et al., 2007). In this section, we consider the implications of the research on children's and adults' fraction reasoning for improving classroom instruction of this important area of early mathematics. First, we consider the aspects of children's fraction understanding on which educational interventions may have the broadest impact.

Areas for Intervention

The psychological research on children's ratio and fraction reasoning, as well as the developmental theory, points to children's mental representations of symbolic fractions as particularly important in mathematics education. As we have described in earlier sections, the accuracy of children's magnitude representations for symbolic fractions, and more generally, their ability to understand the magnitudes, ratios, and proportions to which symbolic fractions refer, critically supports both later fraction reasoning as well as later mathematics skills, such as algebra reasoning. Although children's nonsymbolic reasoning and fraction arithmetic proficiency are correlated with later mathematics achievement, these correlations tend to be considerably weaker. One implication of these weaker correlations, coupled with the evidence that children appear to have competency representing and comparing nonsymbolic ratios quite early on, is that intervening on children's nonsymbolic reasoning may not be optimally productive. In contrast, although fraction arithmetic is not always correlated with later mathematics achievement, competency with fraction arithmetic is a practical skill and an important component of school mathematics (NGA & CCSSO, 2010) as well as the standardized tests used to measure mathematics proficiency (e.g., the NAEP and PISA, see Mathematical Cognition and Education section). Thus, the research we have discussed in this chapter points to two areas in which educational interventions may be most impactful: children's understanding of symbolic fraction magnitudes and children's understanding of fraction arithmetic concepts and procedures.

Improving Magnitude Representations

Children's mental representations of symbolic fractions might be improved through several avenues. One possibility is to improve the accuracy and precision of children's nonsymbolic ratio representations in order to provide a better basis for understanding the magnitudes of symbolic fractions. This logic has been applied to children's whole number reasoning, as well, and some researchers have suggested that improving children's ANS acuity through training improves their ability to estimate the magnitudes of symbolic whole numbers (e.g., Park & Brannon, 2013). However, although nonsymbolic ratio reasoning is correlated with symbolic fraction reasoning, it remains unclear whether or how nonsymbolic ratio reasoning may be improved and whether greater accuracy at matching or estimating nonsymbolic ratios would necessarily, and spontaneously, result in substantive improvements in symbolic fraction magnitude estimation.

There may be more promise in interventions that target the links between children's nonsymbolic magnitude estimations and their symbolic reasoning. Tasks such as the number line estimation task directly address the link between the symbolic fraction that is placed on the number line and a nonsymbolic ratio (e.g., the ratio between the line length between 0 and the fraction and the total line length between 0 and 1; see Sidney et al., in press, for discussion). Indeed, as discussed in an earlier section of this chapter, Siegler and colleagues' (2011) *integrated theory of numerical development* suggests that the number line is a powerful tool for representing magnitudes of all rational numbers, including both whole numbers and fractions, and children's ability to represent numbers on a number line is highly predictive of later mathematics success. Furthermore, similar work on children's whole number understanding has shown that experience with linear, continuous representations of whole number magnitudes is causally related to improvements in children's understanding of the magnitudes that underlie symbolic numbers (e.g., Ramani & Siegler, 2008) as well as improvements in arithmetic (e.g., Siegler & Ramani, 2009).

One critical characteristic of the number line as a visual representation for understanding the magnitudes of symbolic fractions may be its continuous nature. Often, early fraction education in the US includes area models of fractions that are discrete in nature (NGA & CCSSO, 2010). For example, in an area model, the fraction $\frac{3}{5}$ might be represented as a circle with five sections, three of which are shaded. In contrast, on a number line, the fraction $\frac{3}{5}$ might be represented as a continuous length that is $\frac{3}{5}$ of the line length from 0 to 1 (see Figure 5). Boyer and colleagues' (Boyer & Levine, 2015; Boyer et al., 2008) studies of children's nonsymbolic ratio comparison suggest that children are better able to compare ratios across continuous representations than discretized, countable representations. Discrete representations afford counting, which in turn can cause whole number bias-type errors.

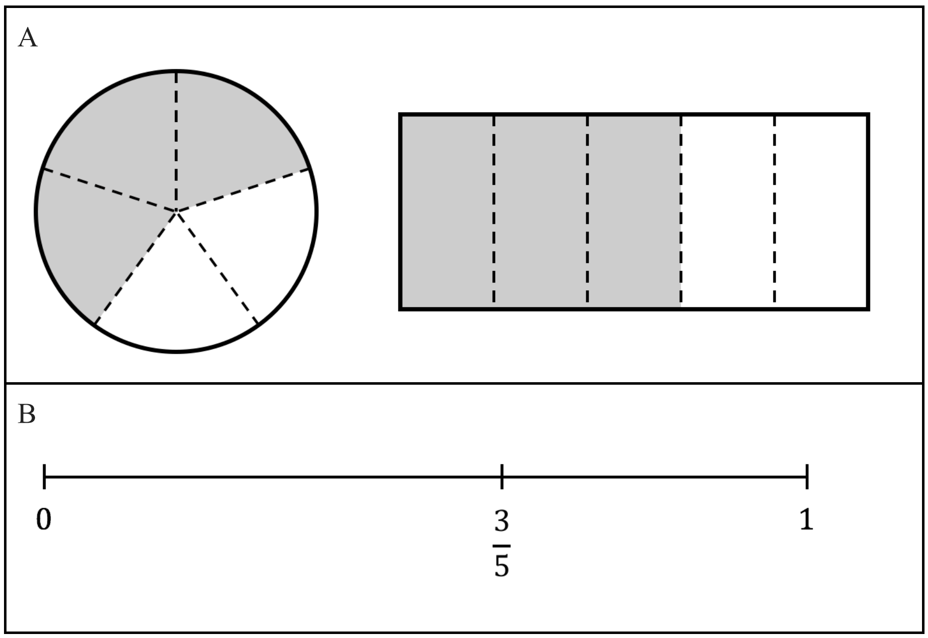

Figure 5. Fractions can be represented using area models (Panel A) and number lines (Panel B). Research suggests that number lines are powerful tools for reasoning about the magnitudes of number symbols (e.g., Hamdan & Gunderson, 2016; Siegler, Thompson, & Schneider, 2011). Panel A shows a circular area model and rectangular area model, also known as a *tape diagram*, for the fraction 3/5. Panel B shows 3/5 represented on a continuous 0 to 1 number line.

Indeed, some educational interventions have demonstrated that including number line representations in instruction improves children's understanding of fraction magnitudes (e.g., Cramer, Post, & delMas, 2002). In their *Rational Number Project* (RNP), Cramer and colleagues designed fraction instruction that represented fractions with both area models and number line models to introduce fraction symbols and their magnitudes. The intervention also included opportunities to compare and contrast across different fraction representations. Cramer and colleagues found that RNP instruction was more effective in helping students learn about the magnitudes of fraction symbols, which in turn supported their understanding of fraction arithmetic, in comparison to fraction instruction that only included area models. This work provides some evidence that more continuous, number line representations are beneficial for children's fraction learning. However, there were many differences between the activities in the intervention and those in the control instruction, rendering it difficult to make a strong conclusion about the number line, per se.

In one recent experimental study, Hamdan and Gunderson (2016) directly, and rigorously, compared children's learning about fraction magnitudes from number line and area models of fractions. In this study, second- and third-grade children were introduced to fraction symbols for the first time using continuous, number line representations or discrete, area representations of their magnitudes. After the lesson, children who learned about fraction magnitudes using number line representations were not only better able to represent fractions on a number line than children who learned about fractions using area representations, but also more accurate at comparing symbolic fractions, a task that had not been introduced in the lesson. Furthermore, children in the number line condition displayed less evidence of errors stemming from a whole number bias. Thus, learning about the magnitudes represented by fraction symbols using number line representations directly supports children's ability to estimate and compare symbolic fractions.

Number line representations may also be advantageous to children's fraction learning because they are *abstract* representations of numerical magnitude. Mathematics textbooks often represent number concepts with real objects, and sometimes visually detailed illustrations. Researchers across psychology, educational psychology, and mathematics education have questioned whether concrete representations of mathematical concepts benefit or limit students' learning about those mathematical concepts, in comparison to more abstract representations. Although concrete representations can sometimes constrain children's errors during problem solving and learning, by providing familiar contexts for thinking about abstract concepts (e.g., Koedinger, Alibali, & Nathan, 2008; McNeil, Uttal, Jarvin, & Sternberg, 2009), learners are better able to generalize their knowledge to new problems when they learn with abstract representations (e.g., Kaminski, Sloutsky, & Heckler, 2008). The abstract nature of the number line representation, compared to representing fractions as pies or pizzas for example, may be yet another reason why learning about fractions on number lines results in generalizable knowledge.

In addition to including number line representations in instruction, children's understanding of fraction magnitudes may also be supported by drawing on their knowledge of whole number magnitudes. Despite evidence of the whole number bias in children's fraction reasoning, the developmental research makes clear that children's whole number magnitude estimation is correlated with their fraction magnitude estimation. Thus, interventions that focus on integration of children's fraction understanding with their prior knowledge of whole number magnitudes may also prove successful.

In one such intervention, Moss and Case (1999) developed a rational number curriculum in which children's experiences with fractions drew on earlier lessons including decimal and whole number magnitudes. In their instruction, children were first shown how to connect their understanding of whole numbers between 0 and 100 to representations of percentage (i.e., 0% to 100%). Then, the instructor linked between percentages and decimals (i.e., 0.00 to 1.00). Finally, the instructors built on children's understanding of decimals between 0 and 1 to make sense of fractions between 0 and 1. Along the way, the instructor used continuous linear representations, akin to the number line, to ground students' understanding of numerical magnitude. Children who received the experimental instruction, in which their whole number knowledge was leveraged to make sense of rational number magnitudes, showed a better understanding of fraction magnitudes and fewer whole number bias errors than children who learned via typical instruction. Although, as in the RNP curriculum (Cramer et al., 2002), any of the components of the instruction may have contributed to student learning, this study provides converging evidence with that of the longitudinal studies that children's whole number magnitude knowledge underlies their understanding of numerical magnitude, more generally.

Improving Fraction Arithmetic

Few experimental interventions are specifically aimed at improving children's fraction arithmetic skills, rather than at improving children's arithmetic skill through their understanding of symbolic fraction magnitudes. Indeed, competence at estimating the magnitude of symbolic fractions is correlated with fraction arithmetic skill, and improvements in magnitude estimation are likely causally related to improvements in addition and subtraction of fractions. In children's whole number reasoning, providing support for children's visualization of whole number magnitudes improves their ability to add whole numbers (Booth & Siegler, 2008). Similarly, being able to visualize or estimate the magnitude of a fraction may also support children's ability to estimate or predict the answer to fraction addition and subtraction problems.

Although children's fraction magnitude knowledge may help them better estimate or predict the answer to fraction addition or subtraction problems, children's knowledge of the multiplicative operations, and of division specifically, tends to be much less robust than their knowledge of addition and subtraction (Dixon, Deets, & Bangert, 2001). Thus, improving children's knowledge of fraction magnitudes may not be sufficient to improve children's understanding of fraction division, without also supporting children's understanding of the division concepts.

A handful of studies suggest that drawing on students' prior knowledge of whole number division concepts during fraction division instruction can improve students' fraction division learning (Richland & Hansen, 2013; Sidney, 2016; Sidney & Alibali, 2015). In these studies, the instructor first reminds fifth- and sixth-grade children about what it means to divide by a whole number (e.g., division can be construed as dividing into groups as big as the divisor), and then introduces the analogous concept in the context of fraction division. Reminding children about their relevant prior knowledge of whole number division before the lesson improves children's mental models for fraction division (Richland & Hansen, 2013; Sidney, 2016; Sidney & Alibali, 2015) and their problem solving accuracy (Sidney & Alibali, 2015).

However, these studies have mixed recommendations on the extent to which this link should be made explicitly for students. Richland and Hansen (2013) found that making the analogy between whole number division and fraction division with a high degree of instructional support (e.g., using similar visual representations, using gesture to indicate corresponding features across whole number and fraction division) was beneficial for student learning. In contrast, Sidney and Alibali (Sidney, 2016; Sidney & Alibali, 2015) have found that reminding students about whole number division prior to fraction division instruction, but not explicitly making analogies between division problems, results in fewer misconceptions and better transfer to novel problems than making explicit analogies either with or without a high degree of instructional support. Although these studies highlight the benefits of drawing on children's prior knowledge of whole numbers during fraction learning, the specific aspects of instruction that might best support this link remain unclear. In the next section, we outline future directions, both for applied research on fraction instruction and basic research on the development of children's mathematical reasoning.

Future Directions

Recently, several experts across the fields of mathematics education, educational psychology, psychology, and neuroscience engaged in a collaborative exercise to outline the most important questions about children's mathematical cognition, given the current state of research and theory (Alcock et al., 2016). Their agenda included questions pertaining to the developmental pathways of mathematical cognition (e.g., "What are reliable early and later longitudinal predictors of the development of number skills, arithmetic, and other aspects of mathematics?", p. 26) as well as to the interactions between developmental trajectories (e.g., "How are different mathematical skills (including representing number, counting, performing arithmetic, using fractions) and their developmental trajectories related to each other?", p. 27). They also challenged mathematics cognition researchers to shed light on educational interventions (e.g., "Which domain-specific foundational competencies are most malleable and when in developmental time? And does their malleability impact on other aspects of mathematical performance?") and develop measures that would allow for better comparison across studies. As we have discussed in this chapter, many researchers have addressed these questions in the context of children's fraction development. Still, we see several avenues for future research that would contribute to our understanding of both children's fraction development and mathematics development more generally.

First, although longitudinal and other correlational studies are beginning to uncover the precursory whole number competencies for later fraction reasoning, further research is needed to provide a more comprehensive account of the relationship between the development of children's whole number reasoning and the development of children's fraction reasoning. Young children demonstrate early competence in reasoning about both nonsymbolic sets and ratios, and symbolic whole number magnitude estimation is correlated with symbolic fraction magnitude estimation. However, the specific relationships between nonsymbolic and symbolic whole number and fraction competencies, and the interactions between their developmental trajectories, remain unclear. For example, what is the relationship between children's ability to estimate numerosity of sets and their ability to estimate ratios; do these competencies develop independently or in parallel? Is it advantageous for children to develop symbolic knowledge of whole number prior to understanding symbolic fractions, or would it be more advantageous to introduce fractions earlier, to prevent a strong whole number bias?

Second, although several researchers have found positive correlations across fraction tasks and between fraction tasks and general mathematics achievement, some of the evidence shows mixed results, for example, the evidence for relationships between nonsymbolic ratio reasoning and mathematics achievement. Furthermore, much of this research is correlational, therefore, it remains unclear which pathways are causal and open to intervention.

One barrier to both of these goals is that mixed findings across studies may be due in part to differences across tasks used to measure ostensibly the same underlying competency. For example, some nonsymbolic tasks represent fractions as ratios across dots whereas others represent ratios across line lengths (see Matthews & Chesney, 2015). In some studies, the components are separated and in others they are joined (see Möhring et al., 2016). Future research is needed to further explore the consequences of differences across tasks and create standardized measures to facilitate comparison across studies and research labs.

In addition to these questions about the developmental pathways towards fraction understanding, current research on children's fraction development has opened up several new questions about children's analogical and relational reasoning. For example, given the relationships between children's whole number and fraction reasoning, more research is needed to understand the roles of analogical mapping and transfer in integrating children's knowledge about whole numbers, fractions, and other kinds of numbers. Do the same types of instructional features (i.e., visual representations, familiar source domain, shared labels) that support children's analogical and relational reasoning also support learning about fraction concepts? What is the relationship between the development of children's relational reasoning, the general cognitive competencies that support relational reasoning (e.g., executive functioning), and their understanding of fractions as a relational concept?

Finally, there are many more applied questions to pursue. Here, we have argued for the number line as critically important for grounding children's mental representations of fraction magnitudes. Although Hamdan and Gunderson (2016) have empirically demonstrated the promise of number line representations for children's learning, more research is needed to examine the instructional activities that would support further learning, such as learning about arithmetic operations with fractions. Furthermore, several of the theoretical questions we have posed here may also further efforts to design fraction instruction that fully supports children's early fraction concepts, as well as the specific aspects of fraction competencies that underlie students' success with algebra and more advanced mathematics concepts.

Conclusions

Children's understanding of fractions, including their symbols, concepts, and arithmetic procedures, is an important facet of both developmental research on mathematics cognition and mathematics education. Research on infants', children's, and adults' fraction and ratio reasoning allows us to test a range of proposals about the development of numerical cognition that have largely been developed with natural, whole numbers in mind. As with whole numbers, even young infants can reason about nonsymbolic ratios, people's ability to compare nonsymbolic ratios is governed by the ratio between ratios, and children's ability to accurately place fractions on a number line predicts later competencies and mathematics achievement on standardized measures. However, as with whole numbers, the causal relationships between early emerging nonsymbolic competencies and later symbolic competencies remain underspecified. The developmental research opens several avenues for improving mathematics education. Here, we have focused on the inclusion of the number line representation and analogies between children's prior knowledge of whole numbers and related fraction concepts as two beneficial facets of fraction instruction. Although children's fraction understanding has recently received a great deal of attention, this work has opened up many additional questions, and more empirical research on the relationships between nonsymbolic and symbolic whole number and fraction tasks is necessary to paint a clearer picture of the development of children's understanding of fractions.

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