Siegler, R. S., Im, S., Schiller, L. K., Tian, J., & Braithwaite, D. W. (2020). The sleep of reason produces monsters: How and when biased input shaped mathematics learning. *Annual Review of Developmental Psychology, 2,* 413–435. https://doi.org/10.1146/annurev-devpsych-041620-031544

The Sleep of Reason Produces Monsters:
How and When Biased Input Shapes Mathematics Learning

Robert S. Siegler[1, 2], Soo-hyun Im[3], Lauren Schiller[1], Jing Tian[4], and David W. Braithwaite[5]

Children's failure to reason often leads to their mathematical performance being shaped by spurious associations from problem input and overgeneralization of inapplicable procedures rather than by whether answers and procedures make sense. In particular, imbalanced distributions of problems, particularly in textbooks, lead children to create spurious associations between arithmetic operations and the numbers they combine; when conceptual knowledge is absent, these spurious associations contribute to the implausible answers, flawed strategies, and violations of principles characteristic of children's mathematics in many areas. To illustrate mechanisms that create flawed strategies in some areas but not others, we contrast computer simulations of fraction and whole number arithmetic. Most of their mechanisms are similar, but the model of fraction arithmetic lacks conceptual knowledge that precludes strategies that violate basic mathematical principles. Presenting balanced problem distributions and inculcating conceptual knowledge for distinguishing flawed from legitimate strategies are promising means for improving children's learning.

Keywords: mathematics, fractions, decimals, textbooks, fraction arithmetic, decimal arithmetic

In 1799, Francisco de Goya y Lucientes created *El sueño de la razon produce monstruos,"* a title that roughly translates to, "The sleep of reason produces monsters" (Figure 1). Although Goya's painting has been interpreted in many ways, a common theme is that failure to reason produces nightmarish consequences (Huxley 1960).

Correspondence should be addressed to Robert S. Siegler, rss2169@tc.columbia.edu.; This article was published Online First October 9, 2020.

This research was funded by the Institute of Education Sciences, US Department of Education, through Grant R305A180514 to Columbia University; the Siegler Center for Innovative Learning and Advanced Technology Center, Beijing Normal University; and National Science Foundation Grant No. 1844140 to David W. Braithwaite and Robert S. Siegler. Any opinions, findings, conclusions, and recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the agencies that supported the research.

[1] Department of Human Development, Teachers College, Columbia University, New York, NY 10027, USA
[2] The Siegler Center for Innovative Learning (SCIL), Beijing Normal University, Beijing 100875, China
[3] Department of Education, Hanyang University, Seoul, 04763, South Korea
[4] Department of Psychology, Temple University, Philadelphia, PA 19122, USA
[5] Department of Psychology, Florida State University, Tallahassee, FL 32306, USA

This disquieting image might seem far removed from children's mathematics learning. However, we believe that the origins of the owls, bats, and malign cats in the sleeper's dream are not so different from the origins of common errors in children's mathematics. In both, failure to reason opens the door to irrational thoughts.

The central argument of this article is that children's failure to reason often leads to their mathematics performance being shaped by factors other than the plausibility of answers and procedures. We find that mathematically irrelevant aspects of learning environments, in particular distributions of problems in textbooks, contribute to children's weak performance and shape the errors they make and the flawed strategies they use. We illustrate these points primarily in the context of fraction arithmetic, though we briefly describe how subtle features of the learning environment influence other areas of mathematics learning as well. To explain how biased problem input exercises its effects, we present a computer simulation of the way that problem distributions in textbooks, together with standard learning mechanisms of association and generalization, give rise to the specifics of children's fraction arithmetic. We conclude by discussing how improved conceptual understanding can promote better mathematics learning and minimize the influence of mathematically irrelevant factors.

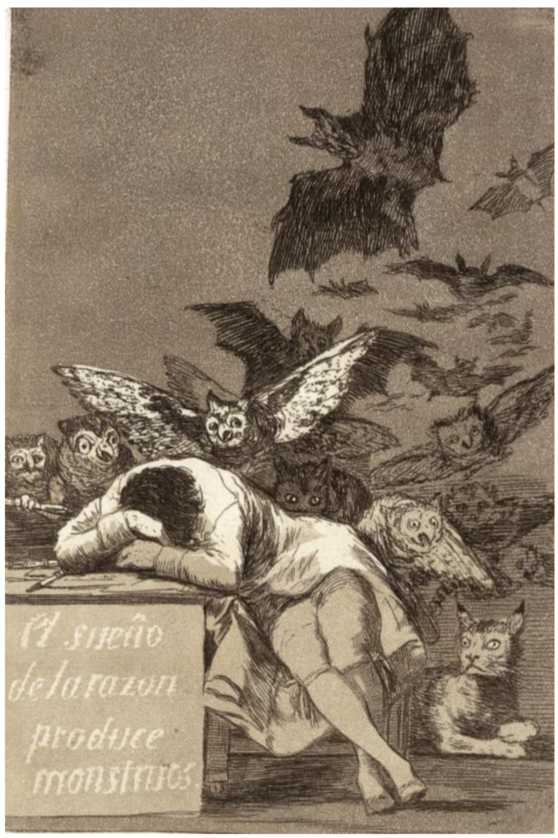

Figure 1. Francisco Goya (Francisco José de Goya y Lucientes), *Plate 43 from 'Los Caprichos': The Sleep of Reason Produces Monsters* (*El sueño de la razon produce monstruos*), 1799, The Metropolitan Museum of Art, New York, USA (Gift of M. Knoedler & Co., 1918), Figure reproduced from https://www.metmuseum.org/art/collection/search/338473 (public domain).

BACKGROUND

Research on rational number arithmetic presents endless examples of children making errors such as "1/2+3/4=4/6" that are implausible, violate basic mathematical principles, or both (Mack 1995; Ni & Zhou 2005). The answer 4/6 is implausible, because it is much too small to be correct; it also violates the mathematical principle that adding positive numbers must yield answers larger than any of the addends.

Such rational number arithmetic errors take many forms. When thousands of US 8th graders were asked on the 1978 National Assessment of Educational Progress (NAEP) whether 12/13+7/8 was closest to 1, 2, 19 or 21, only 24% answered "2" (Carpenter et al. 1980). The most common answers were "19" and "21." This and similar findings triggered a variety of reform efforts to improve mathematics education, culminating in the Common Core State Standards. However, Lortie-Forgues et al. (2015) found that when the same problem was presented to eighth graders in 2014, the percentage who answered correctly had increased only from 24% to 27% over the more than three intervening decades.

Lack of understanding of rational number arithmetic is not limited to fractions but rather is general across rational number notations. When seventh graders were presented the seemingly simple decimal arithmetic problem 6 + .32, more than half answered incorrectly, with the most common error being .38 (Hiebert & Wearne 1985). Understanding of arithmetic with percentages is similarly flawed: for example, when seventh and eighth graders were asked to judge whether 87% of 10 is greater than 10, only 45% answered correctly (Gay & Aichele 1997).

Such errors might be interpreted as implying that children failed to learn correct rational number arithmetic procedures, but that interpretation is only partially correct. The same children who use flawed strategies and generate implausible errors on some trials use correct strategies and answer correctly on other trials. For example, Siegler & Pyke (2013) found that most sixth and eighth graders who were presented pairs of virtually identical fraction arithmetic problems (e.g., 3/5 × 1/5 and 3/5 × 4/5) used different strategies on at least one pair of the highly similar problems; 65% of such differing pairs of strategies included both a correct strategy and an incorrect one. Equally striking, children were not much more confident in their correct than in their incorrect answers. Together, these findings suggest that children learn both correct and incorrect strategies but are unable to identify through reasoning which are correct, leading to a competitive retrieval process without a reliable filter for rejecting incorrect strategies when they are retrieved.

The weak understanding of rational numbers extends beyond arithmetic. For example, when asked on the 2004 NAEP to order the three fractions 5/9, 2/7, and 1/2, 50% of eight graders failed to do so (Martin et al. 2007). Similarly, few elementary, middle, and high school students know that there are an infinite number of numbers between pairs of decimals such as 0.7 and 0.8 and pairs of fractions such as 1/3 and 1/4 (Hansen et al. 2017, Vamvakoussi & Vosniadou 2010). Unsurprising, given this weak understanding of rational numbers, majorities of both children and adults report far more negative attitudes toward dealing with fractions than whole numbers (Sidney et al. 2019).

These findings matter because good understanding of rational numbers is crucial for later success both in and out of school. Consistent with the view that such knowledge is important for success in school, fifth graders' knowledge of fractions predicts tenth graders' overall math achievement in both the United States and the United Kingdom, even after statistically controlling for IQ, reading comprehension, working memory, whole number knowledge, socioeconomic status, race, and other variables (Siegler et al. 2012). Consistent with the view that such knowledge is important beyond school, 68% of adults working in upper- and lower-level blue-collar and white-collar jobs report using rational numbers at work (Handel 2016), and many employees fail at their jobs due to poor knowledge of rational numbers (McCloskey 2007).

Children's (and adults') difficulty understanding rational numbers and rational number arithmetic has many sources. Here, we focus on one source that has been recognized only recently: spurious associations between problems and procedures that have been formed largely in response to biased distributions of problems in textbooks. One reason for focusing on the role of textbook problem distributions in children's difficulty is that this source of difficulty could be remedied far more easily than many others, such as socioeconomic inequalities, uneven societal valuation of the importance of learning math, limited understanding of math by teachers, and weak motivation among many students. Indeed, changing from less to more effective textbooks has been found to be more cost-effective for improving student achievement than alternatives such as teacher professional development and class-size reductions (Chingos & Whitehurst 2012, Koedel & Polikoff 2017).

Several other considerations also recommend studying textbooks to better understand children's mathematics learning. First, textbooks are an ecologically valid part of the learning environment used by millions of children each year. Second, textbooks indicate not only which problems are presented but also the order in which they are presented, a factor that might influence the development of mathematical knowledge. A third advantage of studying textbooks is that the raw data are widely available; this makes it easy to replicate analyses of textbook problems and per- form new analyses to test alternative interpretations. A fourth advantage is that parallel analyses of textbooks can easily be done cross-nationally; textbooks are used throughout the world, and many features are easy to compare. In a survey of fourth and eighth graders from more than 20 countries who were surveyed as part of the 2011 TIMSS (Trends in International Mathematics and Science Study), 75% of students reported that their teachers primarily used textbooks for mathematics instruction (Horsley & Sikorová 2014). Thus, analyzing textbook content is a promising means for assessing the environments within which children learn math and therefore for understanding the learning process itself.

The article is organized into five main sections:

1) Descriptions of main phenomena in children's fraction arithmetic
2) Characteristics of fraction arithmetic problem distributions in textbooks and classroom assignments
3) A computational model of fraction arithmetic
4) Analyses of relations between input problems and children's performance in other mathematical domains: decimal arithmetic, the measurement interpretation of fractions, geometric shapes, counting, mathematical equality, and order of operations
5) Conclusions regarding the roles of input problems and conceptual understanding in determining when and how spurious associations influence mathematical performance, and how instruction can reduce their influence.

MAIN PHENOMENA IN CHILDREN'S FRACTION ARITHMETIC

At least eight consistent phenomena have emerged from studies of children's fraction arithmetic (Byrnes & Wasik 1991, Hecht & Vagi 2010, Jordan et al. 2013, Ni & Zhou 2005). As noted by Braithwaite et al. (2017), all eight were present in a study by Siegler & Pyke (2013); we illustrate the phenomena with data from that study to demonstrate that all the phenomena can be observed in a single study and cite converging findings from other studies that used different procedures and problems to illustrate the generality of the phenomena.

The children observed by Siegler & Pyke (2013) were sixth and eighth graders, half from schools in a predominantly low-income school district and half from schools in a predominantly middle-income district. They were presented eight types of problems: four arithmetic operations, each with equal or unequal denominators, and two items of each problem type for a total of 16 items. To maximize the comparability of items across operations, the same four pairs of operands— 3/5 and 1/5, 3/5 and 1/4, 3/5 and 2/3, and 4/5 and 3/5—were presented with each of the four arithmetic operations. Children were given pencil and paper, but not calculators, to solve the problems.

The following eight phenomena of fraction arithmetic were observed in Siegler & Pyke (2013) and other studies:

1) **Low overall accuracy.** Many studies in Europe and North America have found that fourth to eighth graders' fraction arithmetic is highly inaccurate (e.g., Hecht & Vagi 2010, Newton et al. 2014, Torbeyns et al. 2015). Accuracy improves beyond eighth grade, but to a low asymptotic level; both high school and community college students are quite inaccurate (Brown & Quinn 2006, Richland et al. 2012). Consistent with these findings, the sixth and eighth graders observed by Siegler & Pyke (2013) correctly answered only 52% of items.
2) **Especially low accuracy on division problems.** Children's accuracy tends to be especially low on fraction division problems (Siegler et al. 2011). Siegler & Pyke (2013) found only 20% of division answers were correct.
3) **Variable responses within individual problems.** Children generate multiple answers on each problem (Hecht 1998, Newton et al. 2014). In Siegler & Pyke (2013), on the problem 4/5 ÷ 3/5, children advanced on at least 5% of trials these answers: 1/5 (21% of trials), 20/15 or 4/3 (20%), 15/20 or 3/4 (7%), 1.3/5 or 1.33/5 (7%), 1 (7%), and 1.3 or 1.33 (6%).
4) **Variable strategy use by individual children.** Variable strategy use is not solely due to different children using different strategies; rather, the same child often uses different strategies on closely similar problems. This strategic variability is a widespread phenomenon (Siegler 2006), and rational numbers are no exception: Most children studied by Siegler & Pyke (2013) used different strategies on at least one pair of closely similar problems (e.g., 3/5×1/5 and 3/5×4/5).
5) **More strategy errors than execution errors.** Mathematical errors are of two types: strategy errors, where the intended strategy is incorrect, and execution errors, where the intended strategy is correct but executed incorrectly. In fraction arithmetic, strategy errors are far more common than execution errors (Gabriel et al. 2012, 2013; Hecht 1998); 91% of errors observed by Siegler & Pyke (2013) were strategy errors.
6) **The most common errors are wrong-fraction-operation and independent-whole-number errors.** The best-documented type of fraction arithmetic error involves treating numerators and denominators as independent whole numbers (e.g., Gelman 1991, Ni & Zhou 2005). These independent-whole-number errors involve applying the arithmetic operation independently to numerators and denominators, as when claiming that 3/5 + 2/3 = 5/8. However, Siegler & Pyke (2013) found that wrong-fraction-operation errors are at least as common. These errors involve overgeneralization of

procedures for solving other fraction arithmetic operations. For example, on a fraction multiplication problem, a child might apply the fraction addition procedure of performing the operation on the numerators and passing through the denominator, resulting in errors such as 3/5×4/5=12/5. Failure to detect wrong-fraction-operation errors in previous studies seems due to the problems in those studies not including equal denominator multiplication and division items, where such errors are most common.

7) **Equal denominators increase addition/subtraction accuracy but decrease multiplication accuracy.** Problems with equal denominators elicit more accurate addition and subtraction performance but less accurate multiplication and division performance (Gabriel et al. 2013, Siegler et al. 2011). Siegler & Pyke (2013) found that, relative to unequal denominators, equal denominators elicited more accurate addition and subtraction answers (80% versus 55% correct) but less accurate multiplication and division answers (37% versus 58% correct).
8) **The most frequent type of error on each operation varies with denominator equality.** When adding and subtracting fractions, children make independent-whole-number errors more often on problems with unequal denominators than on problems with equal denominators (e.g., they more often claim that 3/5 + 2/3 = 5/8 than that 3/5 + 4/5 = 7/10) (Gabriel et al. 2013, Newton et al. 2014, Siegler & Pyke 2013). In contrast, when multiplying and dividing fractions, they more often make wrong-fraction-operation errors on equal-denominator problems than on unequal-denominator problems [e.g., children more often respond that 3/5 × 4/5 = 12/5 than that 3/5 × 1/4 = 60/20 or 3/1; they reach the latter incorrect answer through calculating that 3/5 × 1/4 = (4/4 × 3/5) × (5/5 × 1/4) = 12/20 × 5/20 = 60/20 or 3/1] (Siegler & Pyke 2013).

To understand the genesis of these phenomena, we examined the problems that children encounter while learning fraction arithmetic.

PROBLEM INPUT

A Basic Assumption: Textbooks Are a Major Source of Input

Understanding any aspect of development requires understanding the input that shapes development in that domain. In arithmetic, textbooks provide a major part of that input (Cai 2014, Moseley et al. 2007, Valverde et al. 2002). In an international survey of 20 countries, the number of eighth-grade textbook pages devoted to a given topic and the number of class periods that eighth-grade teachers reported teaching the topic were strongly correlated; in the United States, the correlation was r=0.95 (Schmidt 2002).

Textbooks also provide the majority of examples that teachers assign (e.g., Horsley & Sikorová 2014). For example, a recent large-scale survey of math teachers found that 93% of teachers reported using textbooks in more than half of their lessons for purposes such as selecting examples (Blazar et al. 2019). The present article focuses primarily on textbook input in the context of fraction and decimal arithmetic, but similar analyses of input are possible in all areas of mathematics learning (e.g., Geary 1996, Hamann & Ashcraft 1986).

This section presents research on the practice problems that children receive in learning rational number arithmetic. We focus primarily on problems from textbooks, which have the advantages of being used by millions of students and of being publicly available. We also devote some attention to problems assigned by teachers, which addresses the issues that teachers do not assign all textbook problems and they assign problems from sources other than textbooks. In our analyses of input problems, we only coded items that were presented without a word problem context, due to the impossibility of knowing the operation children would use to solve problems when the operation was not specified.

Textbook Problems

To assess the fraction arithmetic problems in textbooks, Braithwaite et al. (2017) coded all symbolic rational number arithmetic problems presented in the fourth through sixth grade volumes of three popular U.S. mathematics textbook series: Pearson Education's *enVisionMATH* (Charles et al. 2012), Houghton Mifflin Harcourt's *GO MATH!* (Dixon et al. 2012a, 2012b), and McGraw Hill Education's *Everyday Mathematics* (University of Chicago School Mathematics Project 2015a, 2015b, 2015c). The problems were all those that a) had two operands, at least one of which was a fraction or mixed number, b) were in symbolic form (i.e., not word problems), and c) required exact answers (i.e., not estimates). Problems with these characteristics constituted the large majority of problems in all three textbooks that we analyzed, as well as in three other textbook series analyzed by Cady et al. (2015). A survey by Opfer et al. (2018) indicated that that the textbooks examined by Braithwaite et al. (2017) were three of the four most widely used textbook series.

The analyses revealed strikingly nonrandom relations between arithmetic operations and the operands (numbers) in the problems. First, consider fraction arithmetic problems involving two fractions. As shown in Table 1, in the fourth- through sixth-grade volumes of the three textbook series cited above, only 4% of multiplication and division problems had equal denominators (e.g., 3/5×4/5). In contrast, in the same textbooks, 50% of addition and subtraction problems had equal denominators (e.g., 3/5+4/5).

Table 1. Percent of problems with two fraction operands classified by arithmetic operation and denominator equality from combined items in three textbooks[a]

	Arithmetic Operation			
Denominator equality	Addition	Subtraction	Multiplication	Division
Equal denominators	12	13	1	1
Unequal denominators	13	12	29	19

[a] The textbooks are Pearson Education's *enVisionmath* (Charles et al. 2012), Houghton Mifflin Harcourt's *GO MATH!* (Dixon et al. 2012a,b), and McGraw Hill Education's *Everyday Mathematics* (Univ. Chic. Sch. Math. Proj. 2015a–c).

Table 2. Percent of problems classified by arithmetic operation and operand type (two fractions or one fraction and one whole number) from combined items in three textbooks[a]

	Arithmetic Operation			
Operand number type	Addition	Subtraction	Multiplication	Division
Fraction-fraction	25	23	13	8
Whole number-fraction	0	2	17	13

[a] The textbooks are Pearson Education's *enVisionmath* (Charles et al. 2012), Houghton Mifflin Harcourt's *GO MATH!* (Dixon et al. 2012a,b), and McGraw Hill Education's *Everyday Mathematics* (Univ. Chic. Sch. Math. Proj. 2015a–c). Percentages may not sum up to 100% because of rounding.

Other types of imbalances were also present in the distributions of fraction arithmetic problems in the textbooks. Consider the distribution of problems having one fraction and one whole number (Table 2). Only 4% of addition and subtraction problems in the textbooks with at least one fraction operand also included a whole number operand (e.g., 6–3/5). In contrast, 59% of multiplication and division problems with at least one fraction operand also had a whole number operand (e.g., 6×3/5).

These imbalanced problem distributions do not have any apparent mathematical justification. Learners need to be able to multiply fractions with identical denominators, just as they need to be able to multiply fractions with unequal denominators. Learners also need to be able to add and subtract whole numbers and fractions, just as they need to be able to multiply and divide them.

Problems Used in Instruction

The fact that problems appear in textbooks does not guarantee that children encounter them. Teachers do not typically present all problems in textbooks; they also might compensate for the paucity of certain types of problems in textbooks by emphasizing them in class or homework assignments.

To test the assumption that textbook problems reflect the input children receive, J. Tian, E.R. Leib, C. Griger, C. Oppenzato, R. Allatas, et al. (manuscript in preparation) asked 14 fourth-, fifth-, and sixth-grade math teachers from five school districts in the greater Pittsburgh area to provide all problems that they presented to students in math class or as homework during the 2017–2018 school year. The problems were coded as by Braithwaite et al. (2017).

One main finding was that 73% of the in-class and homework assignments came from textbooks; most of the other 27% of problems were teacher created. Another significant finding was that the fraction arithmetic problems that teachers presented and assigned showed very similar distributions as those in the math textbooks in Tables 1 and 2. This was true both for the textbook problems that teachers assigned and for the problems from other sources. These findings supported our assumption that textbook problems are a good proxy for the problems that children encounter.

Do Children Learn Characteristics of Problem Input?

The fact that distributions of textbook and homework problems are biased does not mean that children learn the biases. Indeed, there was reason to believe that they would not. Mathematics instruction emphasizes general principles and procedures, not distributions of problems; also, there would be no obvious reason for textbooks or teachers to call students' attention to imbalanced distributions of problems.

To determine whether children learned the distributions of problems in their textbooks, Braithwaite & Siegler (2018) presented sixth and eighth graders with two complementary types of problems. Choose-operation problems specified operands and asked children to choose an arithmetic operation that was likely to accompany them (e.g., 3/5 • 2/5). Generate-operand problems specified an arithmetic operation and asked children to choose two numbers that were likely to accompany it (e.g., • × •). Children were told that the two numbers should be two fractions on half of the problems and a fraction and a whole number on the other half.

Children clearly learned the spurious operator–operand associations that were present in textbooks. On the generate-operands task, when the specified operation was addition or subtraction, children usually generated pairs of fractions with equal denominators. When the specified operation was multiplication or division, they usually generated operand pairs with a whole number and a fraction. Similarly, on the choose-operation task, when presented two fractions with equal denominators, children chose addition or subtraction more often than multiplication or division; when presented a whole number and a fraction, they chose multiplication or division more often than addition or subtraction. Children even learned the particular fractions (e.g., 3/4, 7/8, or 2/3) that were most likely to appear. The frequency with which each fraction appeared in textbooks and the frequency with which children generated that fraction on the generate-operand problems correlated r=0.78. Thus, children are exceptionally good at learning mathematically irrelevant characteristics of instructional input, such as relations between operations and operands and frequencies of particular fractions. Unfortunately, they are much less apt at learning desired procedures and concepts.

A COMPUTATIONAL MODEL OF FRACTION ARITHMETIC

To illustrate the mechanisms through which textbook input could give rise to children's fraction arithmetic performance, Braithwaite et al. (2017) generated a computer simulation, FARRA (Fraction Arithmetic Reflects Rules and Associations). As input, FARRA received all fraction arithmetic items from the textbook series *enVisionmath* in the order in which the problems appeared in the fourth-, fifth-, and sixth-grade volumes of the series. As output, FARRA produced patterns of strategy choices, accuracies, and particular errors for all four fraction arithmetic operations on problems with equal and unequal denominators. FARRA reflected three main hypotheses:

1) Imbalances in the distribution of input problems that children receive from textbooks impair their learning of fraction arithmetic, particularly on the underrepresented problems.
2) Children use statistical associations between problem features and solution procedures to guide their strategy choices. Such associative learning is beneficial in many situations, but it can be harmful in mathematics learning where correct performance usually depends on explicit rules rather than statistical associations. In particular, if children receive biased distributions of practice problems, the children's choices of strategies will reflect the biases.
3) Conceptual knowledge plays little, if any, role in most children's learning of fraction arithmetic. Because most children lack a conceptual basis for determining which procedures to use for which problems, they often commit overgeneralization errors—that is, they use procedures that are correct for some types of problems to solve problems on which those procedures are inappropriate.

Relevant to the third hypothesis, FARRA provides a test of whether a model devoid of conceptual knowledge can generate and explain the development of fraction arithmetic. FARRA lacks conceptual knowledge not because we believe that no children have such knowledge—some clearly do—but rather because the data indicate that most children have little or no conceptual understanding of fraction arithmetic or at minimum do not use any conceptual knowledge that they have, leading them to routinely violate basic mathematical principles when solving fraction arithmetic problems (e.g., Siegler & Lortie-Forgues 2015).

How the Simulation Operates

FARRA is a production system that includes both correct and flawed strategy rules, as well as rules for implementing the strategies (execution rules). As with other production systems, each rule is a condition–action pair that includes both a set of conditions under which it can fire and a set of actions that are taken when it fires. Correct rules are standard fraction arithmetic procedures; all but one of FARRA's flawed strategy rules are overgeneralized versions of the correct procedures in which the arithmetic operation is not specified. The flawed rules lead to some correct answers (when the rule happens to be used on a problem for which it is appropriate) but also to overgeneralization errors (when the rule is used on a problem for which it is not appropriate). For example, the correct rule for adding fractions with equal denominators involves executing the operation on the numerators and passing through the denominator (e.g., 3/5 + 4/5 = 7/5). However, this rule is often overgeneralized to multiplication, resulting in errors such as 3/5 × 4/5 = 12/5. FARRA learns the strong association present in textbooks that when operands have equal denominators, the addition/subtraction rule is appropriate. This leads to frequent overgeneralization of the addition/subtraction rule to multiplication and division on items involving equal denominators. The blocked presentation of fraction arithmetic problems of a given type may contribute to the overgeneralization by reducing attention to the operation. If the last N problems could be solved by executing the operation on the numerators and passing through the denominator, the next problem almost certainly can be solved in the same way (Rohrer et al. 2020).

FARRA also includes execution rules, which are procedures for implementing the strategies. The execution rules involve whole number arithmetic operations, such as the multiplication needed to create common denominators on fraction addition and subtraction problems that do not initially have them. Most execution rules produce correct answers, but three do not: incomplete execution (e.g., leaving numerators unchanged when multiplying to establish a common denominator), changing the operation to multiplication but not inverting either operand on division problems, and inverting a random operand rather than the correct one on division problems.

During the problem-solving process, FARRA often needs to choose which of two or more applicable rules to use. To choose, the model assumes stochastic rule selection combined with a reinforcement learning mechanism in which increases in the strength of a rule are greater when the rule is part of a sequence leading to a correct rather than an incorrect answer.

Input to the Simulation

In Braithwaite et al.'s (2017) study 1, FARRA received 659 input problems in the order in which the problems appeared in the fourth- through sixth-grade volumes of *enVisionmath* (Charles et al. 2012). That textbook series was chosen as the learning set because it was intermediate between the other two series in the number of problems it included.

After FARRA received the learning set of 659 textbook problems, it was presented a test set of the problems Siegler & Pyke (2013) presented to sixth and eighth graders. The test set included 16 items: two items each for the four arithmetic operations with equal and unequal denominators. A set of 1,000 simulated students was created by randomly choosing values for FARRA's three free parameters (learning rate, error discount, and decision noise) and presenting the learning set to FARRA using the values for each simulated student. The learning rate parameter determined the amount of reinforcement (increase in strength) that a correct answer produced in the productions that fired on the way to generating it. The error discount parameter specified how much less reinforcement the productions receive when the answer was wrong than when it was right. The decision noise parameter introduced random variability from trial to trial. More details about the impact of these parameters and the simulation generally are available elsewhere (Braithwaite et al. 2017, 2019).

FARRA's Performance and Its Relation to Children's Performance

FARRA generated all eight phenomena of children's performance noted above, with values quite close to those of the children in Siegler & Pyke (2013) on the same problems:

Low overall accuracy. FARRA's percent correct was 52%, exactly equal to the 52% accuracy of children observed by Siegler & Pyke (2013).

Especially low accuracy on division problems. Like children, FARRA was far less accurate on division than on the other arithmetic operations (20% correct for children; 26% for FARRA). This lower accuracy reflected less practice with division, interference from overgeneralized procedures used on earlier-presented operations, and frequent incorrect executions of the correct rule.

Variable responses within individual problems. FARRA, like children, generated varied responses on each problem. The variation is illustrated in Table 3, which displays the responses most frequently advanced by FARRA and children on 4/5 × 3/5 (for parallel data for other items, see Braithwaite et al. 2017, table 6). Most answers generated by children were also generated by the simulation, with percentages resembling those of children. Over all problem–answer pairs (N = 391), answer frequencies between the experimental and model data sets correlated r = 0.96. In part, this strong correlation reflected correct answers being relatively common for both children and FARRA, but even when correct answers were excluded, the frequency of errors correlated very highly (N = 354, r = 0.88).

Variable strategy use by individual children. Like children, FARRA generated variable strategies on virtually identical problems. On almost all simulation runs (99%), it generated different strategies on at least one pair of virtually identical problems in the test set, such as 3/5 + 1/5 and 3/5 ÷ 1/5.

More strategy errors than execution errors. Strategy errors comprised 91% of children's errors and 93% of FARRA's errors.

The most common errors were wrong-fraction-operation and independent-whole-numbers errors. As among children, almost all strategy errors generated by FARRA (93%) were wrong-fraction-operation or independent-whole-number errors. Wrong-fraction-operation procedures constituted 64% of FARRA's strategy errors; independent-whole-number procedures constituted 29% of strategy errors.

Table 3 Frequencies of common answers on 4/5 × 3/5 from children and FARRA

Problem	Answer	Frequency (% of responses)	
		Children[a]	FARRA[b]
4/5 × 3/5	12/25	40.0	38.7
	12/5	36.7	40.9
	15/20	4.2	5.5
	20/15	3.3	9.6

[a]Data in Children column from Siegler & Pyke (2013).

[b]Data in FARRA (Fraction Arithmetic Reflects Rules and Associations) column from Braithwaite et al. (2017).

Equal denominators increase addition and subtraction accuracy but decrease multiplication accuracy. Like children, FARRA was more accurate on addition and subtraction problems when operands in a problem had equal denominators (81% correct) than when problems had unequal denominators (46% correct). Also like children, FARRA showed the opposite pattern on multiplication problems: lower accuracy when operands had equal denominators (40% correct) than when operands had unequal denominators (62% correct). Division did not follow the anticipated pattern for either children or FARRA because performance on the rarely presented equal denominator division problems was better than expected. The reason appeared to be use of an incorrect strategy that happened to yield a correct answer on one of the two equal-denominator division problems (for details, see Braithwaite et al. 2017).

The most frequent type of error on each operation varies with denominator equality. FARRA overgeneralized the addition and subtraction strategy more often on equal denominator multiplication and division problems than on unequal denominator multiplication and division problems (40% versus 18% of trials). In contrast, it overgeneralized the multiplication strategy more often on unequal denominator addition and subtraction problems than on equal denominator addition and subtraction problems (24% versus 10% of trials). Children showed the same pattern (41% versus 25% and 26% versus 14% of trials, respectively). Again, the phenomena appeared to stem from children and FARRA learning the statistical relations between denominator equality and arithmetic operation in the input problems.

Subsequent analyses of FARRA's performance in Braithwaite et al. (2017; Study 2) demonstrated that the simulation's success in modeling children's performance was equally apparent with learning set problems from a different textbook series (*GO MATH!)* that had more problems and with a test set of problems from a different study that included the same eight types of problems (Siegler et al. 2011).

Of particular interest were the results of Braithwaite et al. (2017; Study 5) in which we tried to optimize FARRA's performance (within the bounds of plausibility). Tripling the number of learning set problems, presenting each of the eight types of problems equally often in the learning set, and improving the three free parameter values of the simulation led to substantial improvements in FARRA's accuracy (from 52% to 80% correct). Note, however, that the improved learning was still well short of 100% accuracy.

EFFECTS OF PROBLEM INPUT ON LEARNING IN OTHER AREAS OF MATHEMATICS

Similar relations between children's performance and distribution of problems in textbooks and other printed material have been found in a number of other areas of mathematics. Among these areas are decimal arithmetic, measurement interpretation of fractions, geometric shapes, counting, mathematical equality, and order of operations.

Decimal Arithmetic

Although usually taught separately, decimals and fractions are integrally related; indeed, decimals are equivalent to the subset of fractions whose denominators are powers of 10. This relation led Tian et al. (in press) to hypothesize that textbook distributions of decimal arithmetic problems would show similar imbalances to those in fraction arithmetic.

Textbook problems. To obtain a comprehensive and representative database of decimal arithmetic problems in US textbooks, Tian et al. (in press) coded all decimal arithmetic problems from the same three textbook series that Braithwaite et al. (2017) selected to code fraction problems. Tian et al.'s coding distinguished between problems that had two decimal operands (e.g., 0.12×0.34) and problems that had a whole number operand and a decimal operand (e.g., 5×0.6); it also distinguished between problems with two decimal operands that had equal numbers of decimal digits (e.g., $1.23 + 4.56$) and ones that had two decimal operands with unequal numbers of decimal digits (e.g., $4.5 - 1.23$).

Across the three textbook series, 98% of items with a whole number operand and a decimal operand involved multiplication or division. In contrast, addition and subtraction problems far more frequently involved two decimal operands than a whole number and a decimal (95% versus 5%). Among problems that had two decimal operands, addition and subtraction items more often had equal numbers of decimal digits than unequal numbers of decimal digits (71% versus 29%), whereas operands on multiplication and division problems similarly often involved equal and unequal numbers of decimal digits (51% versus 49%).

Problems used in instruction. As with fractions, the distribution of decimal arithmetic problems that teachers presented in class or as homework paralleled the distribution in the textbooks (J. Tian, E.R. Leib, C. Griger, C. Oppenzato, R. Allatas, et al., manuscript in preparation). The addition and subtraction problems assigned by teachers far more often involved two decimals than a whole number and a decimal, whereas with multiplication and division, the difference was in the opposite direction. Moreover, addition and subtraction problems with two decimal operands more often had equal numbers of decimal digits than unequal numbers of decimal digits, whereas there was no difference on multiplication and division problems. These findings again supported the assumption that textbook problem distributions are a good proxy for the problems children encounter in school.

Relations of textbook input to children's performance. Based on the textbook input, Tian et al. (in press) predicted that the textbook distributions of decimal arithmetic problems would predict children's decimal arithmetic performance. They tested this prediction against children's performance in (a) an experiment published more than 30 years ago by researchers who had never been affiliated with our lab (Hiebert & Wearne 1985), (b) an unpublished data set obtained in 2019 from a large-scale, web-based learning platform (ASSISTments, described in Heffernan & Heffernan 2014), and (c) data from a recent controlled experiment in our lab. The goal was to examine the generality of the findings over labs (the Hiebert and Wearne lab versus our own), time of data collection (before 1985 versus 2013–2019), and data source (web-based platform versus controlled experiment).

The biases in the textbook problem distributions predicted accuracy of children's decimal arithmetic in all three data sets. For example, addition and subtraction problems in textbooks rarely included a whole number and a decimal; children were surprisingly inaccurate on seemingly simple whole-number-plus-decimal problems such as $6 + .32$; children were less accurate on those types of problems than on problems involving addition or subtraction of two decimals. In contrast, multiplication and division problems in textbooks frequently included a whole number and a decimal; children were more accurate on these problems than on problems involving multiplication and division of two decimals. Thus, in decimal arithmetic as in fraction arithmetic, differences in presentation frequency of various types of problems in textbooks predict corresponding differences in children's accuracy.

The Measurement Interpretation of Fractions

Both correlational and causal evidence indicate that textbook problem input is related to learning of the measurement interpretation of fractions—the interpretation that fractions are measures of magnitude that can be placed and ordered on number lines. On the input side, textbooks emphasize the part–whole interpretation of fractions far more than the measurement interpretation (Cady et al. 2015, Charalambous et al. 2010, Hansen et al. 2019). On the output side, children are far more accurate on fraction problems that can be solved via the part–whole interpretation (e.g., problems on which units corresponding to the numerator and denominator can be counted) than on problems that require a measurement interpretation (e.g., estimation on a number line with only the endpoints marked) (Charalambous & Pitta-Pantazi 2007, Hannula 2003, Tunç-Pekkan 2015).

These findings are correlational, but results of interventions in which children were randomly assigned to conditions suggest that causal relations are also present. Interventions that emphasized the measurement interpretation have yielded greater improvement in children's fraction knowledge than conditions that emphasized the part–whole interpretation (e.g., Braithwaite & Siegler 2020, Fuchs et al. 2013, Gunderson et al. 2019, Hamdan & Gunderson 2017, Moss & Case 1999). For example, Barbieri et al. (2020) found that relative to instruction emphasizing the part–whole interpretation of fractions, instruction emphasizing the measurement interpretation led to greater improvement in number line estimation and magnitude comparison among at-risk students. Similar findings have emerged with typical students (Saxe et al. 2013).

Geometric Shapes

Resnick et al. (2016) analyzed geometric input from preschoolers' books, games, and apps. They found that circles appeared in 93% of books, 85% of games, and 95% of apps, whereas rectangles appeared in 72% of books, 20% of games, and 65% of apps. Canonical versions of the shapes (e.g., equilateral triangles) were consistently more common than non-canonical versions. Parallel to these relative input frequencies, preschooler's shape identification was considerably more accurate for circles than rectangles (Clements et al. 1999) and for canonical than non-canonical shapes (Satlow & Newcombe 1998).

Counting

Similar parallels between imbalances in the input to which children are exposed and children's performance have been found for counting. Counting competence includes knowing how to recite the numbers in order and understanding the cardinality principle, which states that when counting from 1, the last item in the count is the number of items in the set. Two recent studies of children's counting books (Powell & Nurnberger-Haag 2015, Ward et al. 2017) indicated that more than 70% of the books analyzed presented numbers in order, starting with 1. In contrast, the cardinality principle was included in fewer than 10% of the books in both studies. Moreover, in a study of parents reading counting books to their preschoolers, parents rarely provided cardinal labels after the count (Mix et al. 2012). Thus, the protracted development of the cardinality principle (Geary & vanMarle, 2018) may reflect a lack of input that calls attention to the principle.

Mix et al. (2012) provided causal evidence for this conclusion. Training randomly chosen preschoolers in the cardinality principle by labeling the set size of a display and then counting the objects in it (i.e., saying "Three crackers, count them, 1, 2, and 3") led to better understanding of the cardinality principle than training randomly chosen peers in only labeling the set size or only counting the objects.

Mathematical Equality

Textbooks rarely present equations with operations on both sides of the equal sign. For example, McNeil et al. (2006) found that the majority of problems in four middle-school textbooks presented the operation and operands to the left of the equal sign and a blank for the answer to its right (e.g., 4+5 =__). Only 5% of problems had operations on both sides of the equation (e.g., 4+5= 2+_). A similar pattern is present in elementary school textbooks (Powell, 2012).

Lack of experience with problems that depart from the usual format leaves openings for children to misinterpret the equal sign. For example, as late as fourth grade, most children answer incorrectly when presented problems with operations on both sides of the equal sign, such as by answering “12” or “17” to “8+4 =• +5” (Falkner et al. 1999). These incorrect answers appear to reflect misinterpreting the equal sign as a signal to add all numbers to the left of the equal sign or to add all numbers in the problem rather than as expressing a relation of equality between the left and right sides of an equation.

McNeil et al. (2015) tested whether a modified workbook that included a broader range of problems than standard workbooks helped second graders form a relational understanding of the equal sign. The modified workbook had the same total number of problems as the control workbook, but included items that were absent from typical workbooks, for example, problems with operations on the right side of the equal sign (e.g., __=4+3) and problems that replaced the equal sign with the words “is the same amount as.” Children who were randomly assigned to use the modified workbook displayed greater understanding of mathematical equivalence than peers who used a standard workbook on both an immediate posttest and a delayed posttest 5 to 6 months later.

Order of Operations

Biased distributions of problems are not the only mathematically irrelevant feature of input that influences mathematics performance. Even typographical features, such as internal spacing on a page or screen of problems involving both addition and multiplication (e.g., 2 + 3 × 4), influence speed and accuracy in solving problems (Landy & Goldstone 2007a,b, 2010). In particular, narrower spacing between the operation and the surrounding operands increases the probability of performing that operation first, regardless of the formal rules for ordering operations. In the above problem, narrower spacing between 2 + 3 than between 3 × 4 increases the likelihood of answering 20, due to the narrower spacing leading students to add 2 + 3 and then multiply 5 × 4. One reason for such errors may be the spacing previously encountered in textbooks. In textbook presentations of arithmetic and algebra, multiplication problems tend to be written closer together than addition problems, increasing the likelihood of students answering correctly even among students who do not know the order of operations (Landy & Goldstone 2007a, 2010).

Thus, although mathematics involves abstraction over irrelevant features, this does not mean that learners abstract over those features. Rather, mathematically irrelevant characteristics of input influence learning in a wide range of contexts.

CONCLUSIONS

The Role of Textbook Problems

Distributions of textbook problems shape children’s mathematical performance. Across many areas, including fraction arithmetic, decimal arithmetic, counting, and identification of geometric shapes, performance on rarely encountered types of problems lags behind performance on frequently presented types of problems.

FARRA demonstrates that presenting fraction arithmetic problems from textbooks to a computer simulation with standard correct fraction arithmetic procedures, overgeneralized versions of those procedures, stochastic strategy choice mechanisms, and reinforcement learning mechanisms produces performance that closely resembles children's performance. Presenting FARRA a greater proportion of underrepresented problems improves the model's performance. Similarly, presenting greater numbers of rarely presented problems to randomly selected children produces gains in their understanding of other mathematical concepts. Balancing the distribution of textbook problems would be far simpler than addressing other sources of poor math achievement, such as socioeconomic inequality, racism, inconsistent values among US families for math learning, and inconsistent knowledge of mathematics among US teachers. Thus, presenting more balanced distributions of problems in mathematics textbooks is a promising way to improve children's mathematics learning.

The Importance of Conceptual Knowledge

A major reason textbook problem distributions can strongly influence rational number arithmetic and numerous other areas of mathematics is that many children lack conceptual understanding of these areas. If children possessed such understanding, it could shield them from the influence of spurious associations, but they do not. The impact of this absence can be seen by contrasting children's performance in whole number and rational number arithmetic.

Children almost never make errors such as $3 \times 5 = 3$, but they often make errors such as $3/5 \times 1/5 = 3/5$. Why is it that implausible errors are rare in some contexts, such as whole number multiplication, but common in others, such as fraction multiplication?

A major difference between whole number and rational number arithmetic is that in at least some areas of whole number arithmetic, children employ a goal sketch that reduces use of flawed strategies. Goal sketches are domain-specific mechanisms for evaluating the plausibility and potential usefulness of strategies in a domain. In mathematics, goal sketches include requirements for legitimate strategies and principles, as well as estimation processes for evaluating the plausibility of answers. A goal sketch for fraction multiplication, for example, would include the information that multiplying two positive fractions less than 1 must result in an answer less than either multiplicand; any strategy that violated that principle would be rejected. This goal sketch would lead children to reject 3/5 as a potential answer to $3/5 \times 1/5$ because that answer would be larger than one of the operands and equal to the other. Such evaluations could result in children turning to the other main fraction multiplication strategy they know, the correct strategy, and thereafter choosing it increasingly because it produced answers that met the requirements of the goal sketch and received reinforcement.

The functions served by goal sketches resemble those of the System 2 reasoning described by Stanovich & West (2000) and Kahneman (2011), among others. However, the quick and seemingly effortless evaluations of both familiar and unfamiliar strategies by the kindergartners studied by Siegler & Crowley (1994) suggest a process more like System 1 reasoning. Perhaps when goal sketches are first formed, their application is slow and effortful, but with use they become automatic.

In this concluding section, we review evidence that young children possess considerable conceptual understanding of whole number addition, describe the SCADS (Strategy Choice and Discovery Simulation) computer simulation and how its goal sketch prevents use of flawed whole-number-arithmetic strategies, compare empirical data and computational models for whole number arithmetic to those for rational number arithmetic, and explore how helping children form goal sketches for rational number arithmetic could improve their learning.

Children's understanding of whole number addition. Preschoolers have considerable understanding of whole number arithmetic (Gilmore et al. 2018). For example, they choose adaptively among the varied addition strategies they use, in the sense of using each approach most often on problems on which it yields favorable combinations of accuracy and speed (Siegler & Shrager 1984). In particular, preschoolers predominantly use retrieval, the fastest strategy, when they can execute it accurately; they predominantly use slower strategies, such as counting from 1, on problems where such strategies are necessary for accurate performance. Adaptive strategy choices, along with the almost total absence of implausible answers such as $3 + 4 = 2$ or $3 + 4 = 22$, reflect a kind of implicit understanding of basic addition.

Preschoolers' understanding of whole number addition extends to discovery of new strategies. Siegler & Jenkins (1989) identified 4- and 5-year-olds who, on a pretest, solved problems by counting from 1 but never counted-on from the larger addend, even on problems such as $2 + 9$ where counting-on could have been advantageous. The children were presented many addition problems, with feedback about the answer's correctness following each problem. Solving problems led almost all the preschoolers to discover the counting-on strategy, though some took more than 200 problems to do so. Most children also discovered another correct strategy that was intermediate between counting from 1 and counting-on from the larger addend. Perhaps most striking, no preschooler ever tried a conceptually flawed strategy, such as counting the first addend twice or only counting the second addend.

Beyond this implicit understanding, young children also possess some explicit understanding of whole number addition. On the trial during which they discovered the counting-on strategy, some preschoolers explicitly noted its superiority to counting from 1 because, as one child put it, when you count-on, "You don't have to count a very long way" (Siegler & Jenkins 1989, p. 66). Moreover, when kindergartners in another study were asked to judge whether a strategy that an experimenter demonstrated was very smart, kind of smart, or not smart, the kindergarteners judged counting-on, which they had not used on the pretest, to be much smarter than the conceptually flawed strategy of counting the first addend twice, which they also had not used (Siegler & Crowley 1994).

A computer simulation of preschoolers' whole number addition. The cognitive processes that generate preschoolers' adaptive strategy choices and discovery of useful new whole-number-addition strategies without use of flawed approaches were modeled in Shrager & Siegler's (1998) computer simulation, SCADS. Like FARRA, SCADS generated numerous changes in performance that closely resembled those of children. The learning mechanisms and strategy choice procedures in the two simulations were also highly similar.

What SCADS possessed and FARRA lacked, however, was a goal sketch that guided strategy discovery toward useful new strategies and away from flawed ones. SCADS generated between 15 and 21 strategies on different runs. However, many of these strategies were rejected without being tried because they violated the requirements of the goal sketch that legitimate strategies must quantify each addend once and only once. In terms of the metaphor with which we began this article, goal sketches protect SCADS (and children) from the monsters.

The potential value of goal sketches for rational number arithmetic. FARRA does not include a goal sketch because there is no evidence that children evaluate the plausibility of rational number arithmetic strategies or the answers they yield. Indeed, there is considerable evidence that children do not use goal sketches for rational number arithmetic. If children evaluated the plausibility of answers and the strategies that generated them, they would not claim that 19 was the closest answer to 12/13 + 7/8, that 6 + 0.32 = 0.38, or that 3/5 × 4/5 = 12/5.

More frequently presenting underrepresented types of problems improved FARRA's performance, and it probably would improve children's performance as well. However, the improvement would almost certainly be greater if balanced textbook presentation of problems were supplemented by goal sketches. For example, a fraction multiplication goal sketch would include the requirement that multiplying two positive fractions less than 1, such as 3/5 × 4/5, must result in an answer less than either multiplicand. This knowledge would guide children to reject 12/5 as a potential answer because it is larger than both 3/5 and 4/5. Such evaluations could lead children to turn to the other main strategy they know, the correct strategy, and choose it increasingly because it would produce answers that meet the requirements of the goal sketch and elicit positive reinforcement. Similarly, a goal sketch for fraction addition would guide children to reject answers such as 1/2 + 1/2 = 2/4 because they violate the requirement that adding positive numbers must produce answers that exceed all addends. Children would again likely turn to the correct strategy, which most also know, and use it increasingly for the same reasons.

This analysis raises the issue of why, after years of extensive experience with rational number arithmetic, children do not form goal sketches for it. We suspect that weak knowledge of the magnitudes of individual rational numbers and weak understanding of the meaning of arithmetic operations with rational numbers interfere with formation of such goal sketches. Both weaknesses were evident in research by Braithwaite et al. (2018) in which sixth and seventh graders were asked to estimate the positions of individual fractions and sums of fractions on a 0–1 number line and to estimate the positions of individual whole numbers and sums of whole numbers on a 0–1,000 number line. As expected, estimation accuracy was greater for individual whole numbers than individual fractions, thus demonstrating greater knowledge of whole number magnitudes. More striking, however, were the much larger differences in estimation accuracy between fraction and whole number sums. Estimation of fraction sums was so inaccurate that it would have improved if children had placed every estimate at the middle of the number line, regardless of the correct sum; on half of trials, estimates of the magnitudes of one or both addends were greater than estimates of their sum. These and other findings indicate that helping children create goal sketches for fraction arithmetic will require improving their understanding of how arithmetic operations work in the context of fractions, as well as improving children's understanding of the magnitudes of individual fractions. Braithwaite & Siegler (2020) describe an effective intervention based on these ideas.

There probably is no way to prevent spurious associations in textbooks from influencing children's mathematics. Even professional mathematicians show influences of spurious associations under some circumstances (Obersteiner et al. 2013). However, more balanced presentation of textbook problems can mitigate the difficulty to an extent, inculcating conceptual understanding like that in goal sketches can mitigate the difficulty further, and the two together can help keep the monsters at bay.

SUMMARY POINTS

1. Imbalanced distributions of problems in math textbooks contribute to children's difficulty learning mathematics.
2. Children learn spurious associations from the statistical relations present in textbooks; these associations lead children to choose inappropriate strategies.
3. The negative influence of imbalanced problem distributions extends to many areas, including fraction and decimal arithmetic, counting, geometric shapes, and the concept of mathematical equality.
4. Presenting balanced problem distributions improves mathematics learning in many areas.
5. In domains where they lack conceptual understanding, children are especially vulnerable to the negative influences of imbalanced problem distributions.
6. Children have particularly little understanding of fraction and decimal arithmetic.
7. The FARRA computer simulation, which is totally devoid of conceptual knowledge, closely approximates children's fraction arithmetic performance.
8. Improving children's understanding of the difference between legitimate and flawed strategies can help children avoid irrational errors and reduce the influence of irrelevant problem features on performance.

ACKNOWLEDGMENTS
We want to thank for research support the Institute of Education Sciences, US Department of Education, through Grant R305A180514 to Columbia University; the Siegler Center for Innovative Learning and Advanced Technology Center, Beijing Normal University; and National Science Foundation Grant No. 1844140 to David W. Braithwaite and Robert S. Siegler. Any opinions, findings, conclusions, and recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the agencies that supported the research. We also thank David Geary and James Staszewski, who provided invaluable comments on an earlier draft of the chapter, and Sean Ling and Jiwon Ban, who helped greatly with manuscript preparation.

LITERATURE CITED

Blazar D, Heller B, Kane T, Polikoff M, Staiger D, et al. 2019. *Learning by the book: comparing math achievement growth by textbook in six common core states*. Res. Rep., Cent. for Educ. Policy Res., Harvard Univ. Cambridge, MA

Barbieri CA, Rodrigues J, Dyson N, Jordan NC. 2020. Improving fraction understanding in sixth graders with mathematics difficulties: effects of a number line approach combined with cognitive learning strategies. *J. Educ. Psychol.* 112:628–48

Braithwaite DW, Leib ER, Siegler RS, McMullen J. 2019. Individual differences in fraction arithmetic learning. *Cogn. Psychol.* 112:81–98

Braithwaite DW, Pyke AA, Siegler RS. 2017. A computational model of fraction arithmetic. *Psychol. Rev.* 124:603–25

Braithwaite DW, Siegler RS. 2018. Children learn spurious associations in their math textbooks: examples from fraction arithmetic. *J. Exp. Psychol. Learn. Mem. Cogn.* 44:1765–77

Braithwaite DW, Siegler RS. 2020. Putting fractions together. *J. Educ. Psychol.* https://doi.org/10.1037/edu0000477

Braithwaite DW, Tian J, Siegler RS. 2018. Do children understand fraction addition? *Dev. Sci.* 21:e12601

Brown G, Quinn RJ. 2006. Algebra students' difficulty with fractions: an error analysis. *Aust. Math. Teach.* 62:28–40

Byrnes JP, Wasik BA. 1991. Role of conceptual knowledge in mathematical procedural learning. *Dev. Psychol.* 27:777–86

Cady JA, Hodges TE, Collins RL. 2015. A comparison of textbooks' presentation of fractions. *Sch. Sci. Math.* 115:105–16

Cai J. 2014. Searching for evidence of curricular effect on the teaching and learning of mathematics: Some insights from the LieCal project. *Math. Educ. Res. J.* 26:811–31

Carpenter TP, Corbitt M, Kepner H, Lindquist M, Reys R. 1980. Results of the second NAEP mathematics assessment: secondary school. *Math. Teach.* 73:329–38

Charalambous CY, Delaney S, Hsu H-Y, Mesa V. 2010. A comparative analysis of the addition and subtraction of fractions in textbooks from three countries. *Math. Think. Learn.* 12:117–51

Charalambous CY, Pitta-Pantazi D. 2007. Drawing on a theoretical model to study students' understandings of fractions. *Educ. Stud. Math.* 64:293–316

Charles R, Caldwell J, Cavanagh M, Chancellor D, Copley J, et al. 2012. *EnVisionMATH* (Common Core ed.). Glenview, IL: Pearson Education

Chingos MM, Whitehurst GJ. 2012. *Choosing Blindly: Instructional Materials, Teacher Effectiveness, and the Common Core*. Washington, DC: Brookings

Clements DH, Swaminathan S, Hannibal MAZ, Sarama J. 1999. Young children's concepts of shape. *J. Res. Math. Educ.* 30:192–212

Dixon JK, Adams TL, Larson M, Leiva M. 2012a. *GO MATH!* (Common Core ed.). Orlando, FL: Houghton Mifflin Harcourt Publishing Company

Dixon JK, Adams TL, Larson M, Leiva M. 2012b. *GO MATH! Standards Practice Book* (Common Core ed.). Orlando, FL: Houghton Mifflin Harcourt Publishing Company

Falkner KP, Levi L, Carpenter TP. 1999. Children's understanding of equality: a foundation for algebra. *Teach. Child. Math.* 6:232–36

Fuchs LS, Schumacher RF, Long J, Namkung J, Hamlett CL, et al. 2013. Improving at-risk learners’ understanding of fractions. *J. Educ. Psychol.* 105:683–700

Gabriel F, Coché F, Szucs D, Carette V, Rey B, Content A. 2012. Developing children's understanding of fractions: An intervention study. *Mind Brain Educ.* 6:137–46

Gabriel FC, Szucs D, Content A. 2013. The development of the mental representations of the magnitude of fractions. *PLOS ONE* 8:e80016

Gay AS, Aichele DB. 1997. Middle school students' understanding of number sense related to percent. *Sch. Sci. Math.* 97:27–36

Geary DC. 1996. The problem-size effect in mental addition: developmental and cross-national trends. *Math. Cogn.* 2:63–94

Geary DC, vanMarle K. 2018. Growth of symbolic number knowledge accelerates after children understand cardinality. *Cognition* 177:69–78

Gelman R. 1991. Epigenetic foundations of knowledge structures: initial and transcendent constructions. In *The Epigenesis of Mind: Essays on Biology and Cognition*, ed. S Carey, R Gelman, pp. 293–322. London, UK: Psychology Press

Gunderson EA, Hamdan N, Hildebrand L, Bartek V. 2019. Number line unidimensionality is a critical feature for promoting fraction magnitude concepts. *J. Exp. Child Psychol.* 187:104657

Gilmore C, Göbel SM, Inglis M. 2018. *An Introduction to Mathematical Cognition*. New York: Routledge

Hamann MS, Ashcraft MH. 1986. Textbook presentations of the basic addition facts. *Cogn. Instr*. 3:173–202

Hamdan N, Gunderson EA. 2017. The number line is a critical spatial-numerical representation: evidence from a fraction intervention. *Dev. Psychol.* 53:587–96

Handel MJ. 2016. What do people do at work? *J. Labour Mark. Res.* 49:177–97

Hannula MS. 2003. Locating fraction on a number line. In *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education*, pp. 17–24. Berlin, Germany: IGPME

Hansen N, Jordan NC, Rodrigues J. 2017. Identifying learning difficulties with fractions: a longitudinal study of student growth from third through sixth grade. *Contemp. Educ. Psychol.* 50:45–59

Hansen N, Rodrigues J, Kane B. 2019. Opportunities for students with learning disabilities to develop representational ability with fractions: a textbook analysis. *N. J. Math. Teach.* 77:24–37

Hecht SA. 1998. Toward an information-processing account of individual differences in fraction skills. *J. Educ. Psychol.* 90:545–59

Hecht SA, Vagi KJ. 2010. Sources of group and individual differences in emerging fraction skills. *J. Educ. Psychol.* 102:843–59

Heffernan NT, Heffernan CL. 2014. The ASSISTments ecosystem: building a platform that brings scientists and teachers together for minimally invasive research on human learning and teaching. *Int. J. Artif. Intell in Educ.* 24:470–97

Hiebert J, Wearne D. 1985. A model of students' decimal computation procedures. *Cogn. Instr.* 2:175–205

Horsley M, Sikorová Z. 2014. Classroom teaching and learning resources: international comparisons from TIMSS−a preliminary review. *Orb. Sch.* 8:43–60

Huxley A. 1960. *On Art and Artists*. New York: Harper & Bros

Jordan NC, Hansen N, Fuchs LS, Siegler RS, Gersten R, Micklos D. 2013. Developmental predictors of fraction concepts and procedures. *J. Exp. Child Psychol.*116:45–58

Kahneman D. 2011. *Thinking, Fast and Slow*. New York: Farrar, Straus and Giroux

Koedel C, Polikoff M. 2017. Big bang for just a few bucks: the impact of math textbooks in California. *Evidence Speaks Rep.* 2:1–7

Landy D, Goldstone RL. 2007a. Formal notations are diagrams: evidence from a production task. *Mem. Cogn.* 35:2033–40

Landy D, Goldstone RL. 2007b. How abstract is symbolic thought? *J. Exp. Psychol. Learn. Mem. Cogn.* 33:720–33

Landy D, Goldstone RL. 2010. Proximity and precedence in arithmetic. *Q. J. Exp. Psychol.* 63:1953–68

Lortie-Forgues H, Tian J, Siegler RS. 2015. Why is learning fraction and decimal arithmetic so difficult? *Dev. Rev.* 38:201–21

Mack NK. 1995. Confounding whole-number and fraction concepts when building on informal knowledge. *J. Res. Math. Educ.* 26:422–41

Martin WG, Strutchens ME, Elliot PC 2007. *The Learning of Mathematics, 69th Yearbook.* Reston, VA: National Council of Teachers of Mathematics

McCloskey M. 2007. Quantitative literacy and developmental dyscalculias. In *Why is Math so Hard for Some Children? The Nature and Origins of Mathematical Learning Difficulties and Disabilities,* ed. DB Berch, MMM Mazzocco, pp. 415–429. Baltimore, MD: Paul H. Brookes Publishing

McNeil NM, Fyfe ER, Dunwiddie AE. 2015. Arithmetic practice can be modified to promote understanding of mathematical equivalence. *J. Educ. Psychol.* 107:423–36

McNeil NM, Grandau L, Knuth EJ, Alibali MW, Stephens AC, et al. 2006. Middle-school students' understanding of the equal sign: the books they read can't help. *Cogn. Instr.* 24:367–85

Mix KS, Sandhofer CM, Moore JA, Russell C. 2012. Acquisition of the cardinal word principle: The role of input. *Early Child. Res. Q.* 27:274–83

Moseley BJ, Okamoto Y, Ishida J. 2007. Comparing US and Japanese elementary school teachers' facility for linking rational number representations. *Int. J. Sci. Math. Educ.* 5:165–85

Moss J, Case R. 1999. Developing children's understanding of the rational numbers: a new model and an experimental curriculum. *J. Res. Math. Educ.* 30:122–47

Newton KJ, Willard C, Teufel C. 2014. An examination of the ways that students with learning disabilities solve fraction computation problems. *Elem. Sch. J.* 115:1–21

Ni Y, Zhou Y-D. 2005. Teaching and learning fraction and rational numbers: the origins and implications of whole number bias. *Educ. Psychol.* 40:27–52

Obersteiner A, Van Dooren W, Van Hoof J, Verschaffel L. 2013. The natural number bias and magnitude representation in fraction comparison by expert mathematicians. *Learn. Instr.* 28:64–72

Opfer VD, Kaufman JH, Pane JD, Thompson LE. 2018. *Aligned curricula and implementation of Common Core State Mathematics Standards: findings from the American Teacher Panel.* Rep. RR-2487-HCT, RAND Corp., Santa Monica, CA. https://www.rand.org/pubs/research_reports/RR2487.html

Powell SR. 2012. Equations and the equal sign in elementary mathematics textbooks. *Elem. Sch. J.* 112:627–48

Powell SR, Nurnberger-Haag J. 2015. Everybody counts, but usually just to 10! a systematic analysis of number representations in children’s books. *Early Educ. Dev.* 26:377–98

Resnick I, Verdine BN, Golinkoff R, Hirsh-Pasek K. 2016. Geometric toys in the attic? a corpus analysis of early exposure to geometric shapes. *Early Child. Res. Q.* 36:358–65

Richland LE, Stigler JW, Holyoak KJ. 2012. Teaching the conceptual structure of mathematics. *Educ. Psychol.* 47:189–203

Rohrer D, Dedrick RF, Hartwig MK. 2020. The scarcity of interleaved practice in mathematics textbooks. *Educ. Psychol. Rev.* https://doi.org/10.1007/s10648-020-09516-2

Satlow E, Newcombe N. 1998. When is a triangle not a triangle? young children's developing concepts of geometric shape. *Cogn. Dev.* 13:547–59

Saxe GB, Diakow R, Gearhart M. 2013. Towards curricular coherence in integers and fractions: a study of the efficacy of a lesson sequence that uses the number line as the principal representational context. *ZDM* 45:343–64

Schmidt W. 2002. The benefit to subject-matter knowledge. *Am. Educ.* 26:18

Shrager J, Siegler RS. 1998. SCADS: a model of children’s strategy choices and strategy discoveries. *Psychol. Sci.* 9:405–10

Sidney PG, Thompson CA, Fitzsimmons C, Taber JM. 2019. Children's and adults' math attitudes are differentiated by number type. *J. Exp. Educ.* https://doi.org/10.1080/00220973.2019.1653815

Siegler RS. 2006. Microgenetic analyses of learning. In *Handbook of Child Psychology: Vol. 2. Cognition, Perception, and Language* (6th ed.), ed. D Kuhn, RS Siegler, W Damon, RM Lerner (Eds.), pp. 464–510. Hoboken, NJ: Wiley

Siegler RS, Crowley K. 1994. Constraints on learning in nonprivileged domains. *Cogn. Psychol.* 27:194–226

Siegler RS, Duncan GJ, Davis-Kean PE, Duckworth K, Claessens A, et al. 2012. Early predictors of high school mathematics achievement. *Psychol. Sci.* 23:691–97

Siegler RS, Jenkins EA. 1989. *How Children Discover New Strategies*. Hillsdale, NJ: Erlbaum

Siegler RS, Lortie-Forgues H. 2015. Conceptual knowledge of fraction arithmetic. *J. Educ. Psychol.* 107:909–18

Siegler RS, Pyke AA. 2013. Developmental and individual differences in understanding of fractions. *Dev. Psychol.* 49:1994–2004

Siegler RS, Shrager J. 1984. Strategy choices in addition and subtraction: how do children know what to do? In *The Origins of Cognitive Skills*, ed. C Sophian, pp. 229–93. Hillsdale, NJ: Erlbaum

Siegler RS, Thompson CA, Schneider M. 2011. An integrated theory of whole number and fractions development. *Cogn. Psychol.* 62:273–96

Stanovich KE, West RF. 2000. Individual differences in reasoning: implications for the rationality debate? *Behav. Brain Sci.* 23:645–65

Tian J, Braithwaite DW, Siegler RS. in press. Distributions of textbook problems predict student learning: data from decimal arithmetic. *J. Educ. Psychol.*

Torbeyns J, Schneider M, Xin Z, Siegler RS. 2015. Bridging the gap: fraction understanding is central to mathematics achievement in students from three different continents. *Learn. Instr*. 37:5–13

Tunç-Pekkan Z. 2015. An analysis of elementary school children's fractional knowledge depicted with circle, rectangle, and number line representations. *Educ. Stud. Math.* 89:419–41

University of Chicago School Mathematics Project. 2015a. *Everyday Mathematics Assessment Handbook* (4th ed.). Columbus, OH: McGraw- Hill Education

University of Chicago School Mathematics Project. 2015b. *Everyday Mathematics Student Math Journal* (4th ed., Vols. 1 and 2). Columbus, OH: McGraw-Hill Education

University of Chicago School Mathematics Project. 2015c. *Everyday Mathematics Student Reference Book* (4th ed.). Columbus, OH: McGraw-Hill Education

Valverde GA, Bianchi LJ, Wolfe RG, Schmidt WH, Houang RT. 2002. *According to the Book: Using TIMSS to Investigate the Translation of Policy into Practice through the World of Textbooks*. New York: Springer Science+Business Media

Vamvakoussi X, Vosniadou S. 2010. How many decimals are there between two fractions? aspects of secondary school students' understanding of rational numbers and their notation. *Cogn. Instr.* 28:181–209

Ward JM, Mazzocco MM, Bock AM, Prokes NA. 2017. Are content and structural features of counting books aligned with research on numeracy development? *Early Child. Res. Q.* 39:47–63