The metaphor of transition for introducing learners to new sets of numbers

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Is the natural number 7 rational? Is it complex? We argue that the answers to these questions relate to the ways numbers are taught. Commonly, a new kind of numbers is presented as an expansion of a previously familiar kind of numbers, which results in a nested image of the relations between number sets. In this article, we introduce an alternative approach, in which one transitions between different numerical domains, some subsets of which are isomorphic.

Is the natural number 7 rational? Is it complex? Based on our experience with raising such questions to many students and teachers, we speculate that most members of the MERGA community will answer affirmatively. This might relate to a common way of teaching, where a new kind of numbers is presented as an *expansion* of a previously familiar kind, resulting in a nested image of number sets (see Figure 1). In this short theoretical discussion, we introduce an alternative perspective, in which one *transitions* between different numerical sets, some subsets of which are isomorphic.

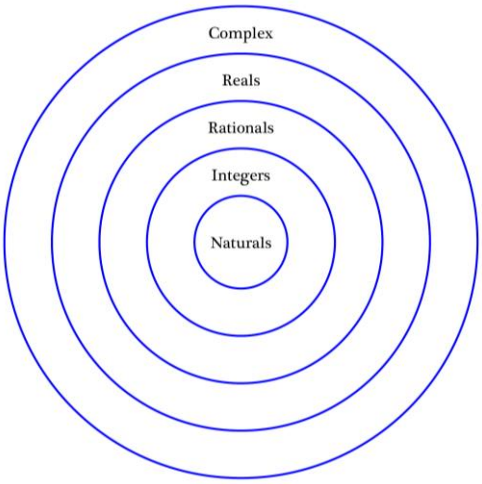

Figure 1. Nested image of number sets.

The Metaphor of Expansion

Many scholars argue that mathematics emerges from communication, which is replete with ubiquitous and often transparent metaphors (e.g., Lakoff & Núñez, 2000). Drawing on experiences that are expected to be common to the communicating actors, metaphors can open the door even to the most abstract mathematical ideas (e.g., Barton, 2008; Sfard, 2008). This feature turns metaphors into a powerful didactical tool that becomes handy when new

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numbers are introduced and when they are related to those numbers with which learners are already familiar.

In instructional settings, new kinds of numbers are often "grown" from an *expansion* of the concept of number: novel elements are introduced to a familiar number set yielding its expansion. For instance, González-Martín et al. (2013) maintain that,

> the learning of different sets of numbers can be seen as a progressive extension of the initial perception of numbers through the algebraic structure of nested number sets, from the primitive notion of counting, to the ideas of comparing, measuring and solving equations (p. 230)

At least three reasons can be offered for the didactical appeal of expanding learners' concept of number:

- Different number sets share many familiar number-symbols, words, related concepts, and properties (e.g., commutativity, associativity, identity). This allows teachers to develop new numbers out of the ones that students are already familiar with.
- The expansion epitomizes mathematics as a highly connected and coherent body of structural relationships. Given that numbers accompany students' learning all the way from kindergarten to university, every encounter with new numbers turns into an opportunity to perpetuate this image.
- This perspective aligns well with a common narrative, in which new numbers are positioned as a patch that resolves issues and inadequacies with numbers of the "old" kind. Naturals do not allow subtracting a larger number from a smaller one, hence the integers. Not all divisions of two natural numbers result in a natural number, hence the rationals. While bearing some resemblance to the development of numbers throughout mathematical history (e.g., Kline, 1972), an expansion of the familiar presents a sensible rationale for introducing new numbers.

As with any metaphor taken literally, expansion comes with its issues. For instance, it draws attention to the introduced add-ons, while glossing over the changes that they impose on the familiar structure. This might at least partially explain why students often assume that their previously held truths about numbers remain intact. At the elementary-school level, well-documented examples concern the notions of successor and density that children "carry over" from natural to rational numbers. For instance, pupils can claim that 2.4 is the next number after 2.3 and that 7.5 is the only number between 7.4 and 7.6 (e.g., Vamvakoussi & Vosniadou, 2010). Similar phenomena occur in a more advanced context. Kontorovich (2018a) showed that many tertiary students continue referring to complex numbers with a zero imaginary part as positive and negative. In fact, some of his participants even became irritated with the questionnaire specifying the number set for each question and lamented "Why do you always mention whether it's $\mathbb{R}$ or $\mathbb{C}$? 2 is positive no matter where!".

In research and practice, the exemplified ways of thinking are often stigmatized as products of students' "bias", "naivety", and "overgeneralization". However, we suggest that the metaphor of expansion may play a role in the robustness of these ways of thinking. Indeed, it seems more reasonable to expect expansion to enrich familiar concepts rather than transform them beyond recognition. Of course, a diligent teacher will emphasize the ways in which new numbers are different from the "old" ones. Yet, it is still not easy to keep track of what changes and what remains valid after the expansion. For instance, NCTM standards (2000) prescribe understanding complex numbers as solutions to quadratic equations that do not have real-number roots. Students are usually introduced to the quadratic formula in the system of real numbers. Accordingly, it seems to be taken for granted that the quadratic formula remains intact even after renouncing square-rooting negatives – one of the most prominent taboos of reals.

The Metaphor of Transition

The issues that we described in relation to the expansion metaphor appear serious enough to consider whether it is the only way to introduce new numbers. The alternative that we bring to the fore is the metaphor of *transition*. Within it, learners are not asked to mobilize familiar numbers to engender new ones but encouraged to depart from one numeric set to arrive at another. Transitions take place between distinct domains, situating the differences between them as an expected norm rather than an anomaly. For travellers, an appreciation of transition implies that the destination is foreign, and its mysteries are waiting to be discovered. It also means that the luggage carried from the port of departure should be selected carefully since not everything will continue to be useful. Overall, for the sake of a positive experience, transitioning students had better be attentive and alert to the rules and customs of the foreign terrain, as these are likely be different from the familiar. This is not to say that similarities between the new and the old will not be recognized. Such instances would be a pleasant surprise, enabling to leverage previously gained knowledge and experiences in new circumstances.

The transition metaphor may be viable for introducing new kinds of numbers. Specifically, it may offer a cohesive frame to attune learners' mindsets to the encounter with new number-names, symbols, and operations; to enhance their readiness to adjust and make sense of new number rules; and to explain why some familiar mathematical truths should be lost in transition. Transition also provides room to grow insights and appreciations of the familiar kind of numbers from the newly developed perspective.

To illustrate the metaphor of transition, let us consider an example where a somewhat extremal attempt is made to disconnect between real and complex numbers. Imagine a teacher who welcomes students to a new mathematical domain consisting of dots residing on a plane with one special dot O. "What can be done with them?", students ask. "Well, there is one operation we can do, let's call it "tāpiritanga" and "tāpiria" as its process." Then, the teacher shows how tāpiritanga of the dots z_1 and z_2 yields another dot z_3 via a so-called parallelogram law (see Figure 2). Through a guided investigation, students can find out that "tāpiritanga" is commutative (i.e., z_1 tāpiria z_2 is the same as z_2 tāpiria z_1), associative (i.e., z_1 tāpiria z_2 and then tāpiria z_3 is the same as z_2 tāpiria z_3 and then tāpiria z_1), and tāpiria of O to any dot leaves this dot intact. To impede students from carrying over "old" meanings of the concept, the teacher refrains from referring to dots as numbers. Instead, the teacher invites students to consider whether numerical domains with which students were familiar until now and the new world of dots have something in common. To support this process of discerning similarities, the teacher can reveal that "tāpiritanga" is "addition" in Māori (see Zazkis et al., 2021 for more illustrations of this sort).

We acknowledge that teaching with the metaphor of transition in mind is likely to come with issues. Supporting students in establishing productive relations between different kinds of numbers is probably among the first issues to emerge. Teaching experiments are needed to show what these issues can look like and how they can be handled. What we wonder about is whether students who transitioned between numerical sets will adhere to the abovementioned ways of thinking as students for whom the concept of number was expanded. Another point to consider is how the rules of new numbers can be harnessed to make students re-appreciate numbers of the familiar kind. For instance, will the students in our example enjoy the fact that a "flat" version of the parallelogram law works as the addition of reals on a number line?

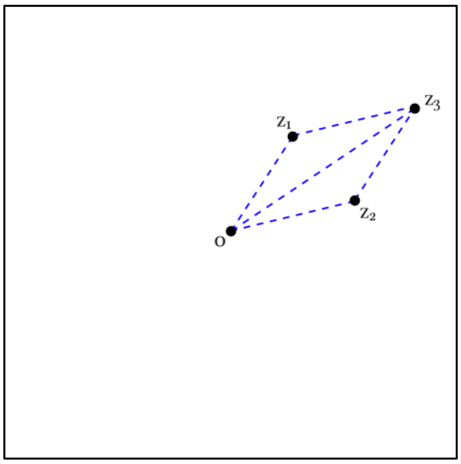

Figure 2. z_3 as a result of the 'tāpiritanga'-operation between z_1 and z_2.

Images Underpinning the Relations Between Number Sets

Herein we draw on the notion of subset to illuminate the mathematical grounds for the metaphors of expansion and transition. To recall, the set A is called a subset of the set B if every element in A is also an element in B. The expansion metaphor draws on the nested relationship among number sets, commonly visualized as presented in Figure 1: natural numbers are a subset of integers, which are a subset of rationals, which are a subset of reals, which in turn is a subset of complex numbers. To be explicit, we consider the subset relation of numbers as a mathematical stance rather than a deductively derivable result. Within this perspective, recognizing 7 as an element of natural numbers warrants its being an integer, rational, real, and complex number.

This recognition may become easier or harder depending on how numbers are represented. For instance, when numbers appear as dots, the dot entitled "7" remains fixed when the natural number line extends to the negative direction to become the integer line. The "7"-dot stays in place when the dotted line becomes dense with rationals and reals, and even when it expands to become the Argand plane. The situation is different when symbolic representation starts playing a more significant role, especially when different kinds of numbers are defined through symbols. For instance, complex numbers are often characterized by a real and an imaginary part. Then, $7 + 0i$ and 7 become different representations of the same mathematical object. In this sense, one could argue that $7 + 0i$ *is* 7, in more or less the same way that "seven" in English is "whitu" in Māori. This is opposed to a common students' claim that "the addition of zero i has no impact".

The transition metaphor draws on an image in which different number sets are isomorphic to some subset of each other. To recall, two sets are isomorphic if there exists a bijection between their elements that preserves a binary relationship, for instance addition and multiplication. Figure 3 depicts this relation with an example of real and complex numbers. From this standpoint, the natural 7 is different from the integer 7 (or $+7$), rational 7 (or $\frac{7}{1}$), and from the complex 7 (or $7 + 0i$). Yet, these numbers could be considered equivalent, if one wishes to identify them as such. Similarly, the relationship between natural and rational numbers is captured by considering naturals as isomorphic to a subset that, mathematically speaking, is perfectly embedded in rationals.

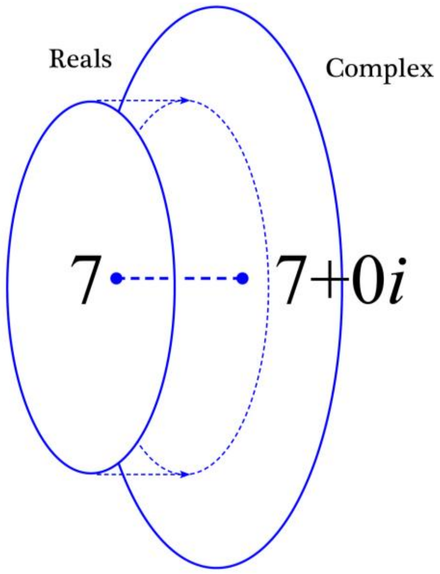

Figure 3. Isomorphic image of real and complex numbers.

An isomorphic image can help in resolving what may appear as an issue within the nested view on numbers. Zazkis (1998) discussed an incident, where her pre-service classroom was divided around the quotient in the division 12 by 5: some of the students argued for 2 with a whole-number quotient in mind, while others advocated for 2.4, implicitly assuming rational-number division. In a similar vein, Kontorovich (2018b) reported on a student who struggled to cope with the fact that $\sqrt{9}$ was 3 when approached as the (real) square root function, but the application of De Moivre formula on the complex 9 entailed 3 and -3. In both cases, the difference of the results is an issue within the nested number image but not necessarily with the isomorphic view. Through the latter lens, identically appearing words and symbols can be interpreted rather differently in different number sets.

Specific images of the relation between number sets underpin mathematical software. In MAPLE, the command $isprime$ tests for whether the input is a prime number. Working with an older version of MAPLE, we witnessed that it outputted "true" for $isprime(7)$ but "false" for $isprime\left(\frac{14}{2}\right)$, $isprime(7.0)$ and $isprime(3.5 \times 2)$. This was because the programmers intended for $isprime$ to operate with integer arguments. In MAPLE, the result of division was considered a rational number, and a rational $\frac{14}{2}$, and similarly 7.0 and 3.5×2, were not identified with an integer 7. Such programming may appear infelicitous to those adhering to the nested image: if all the four inputs point at the same number, how come that their outputs are not the same?! The devotees of the isomorphic image may be more accommodating since for them all these "7"s are different numbers a priori. Yet, we do appreciate that the current version of MAPLE explicates that the input of $isprime$ must be an integer.

Concluding Remark

We started with a question whether the natural 7 is also rational and complex, and suggested that the answer depends on the metaphoric lens through which one considers relations between number sets. We hope that the members of the MERGA community will share our curiosity in the metaphor of transition as a refreshing alternative to the hegemonic metaphor of expansion. The nested and isomorphic images underpinning the metaphors may appear conflicting, but we consider them as complementary viewpoints – one from “above” and one from “aside” – on the same mathematical structure (see Figure 4). Furthermore, we believe that, for the learning of mathematics, it is useful for students and teachers to be able to flexibly switch between the two images.

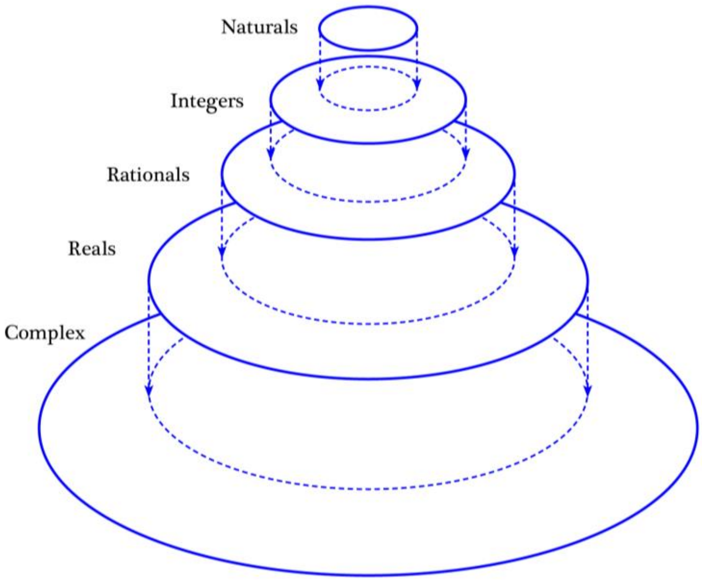

Figure 4. Visualization of relations between number sets.

Note. This paper is an amended version of Kontorovich et al. (2021).

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