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From "You have to have three numbers and a plus sign" to "It's the exact same thing": K–1 students learn to think relationally about equations

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ABSTRACT

This research shares progressions in thinking about equations and the equal sign observed in ten students who took part in an early algebra classroom intervention across Kindergarten and first grade. We report on data from task-based interviews conducted prior to the intervention and at the conclusion of each school year that elicited students' interpretations of the equal sign and equations of various forms. We found at the beginning of the intervention that most students viewed the equal sign as an operational symbol and did not accept many equations forms as valid. By the end of first grade, almost all students described the symbol as indicating the equivalence of two amounts and were much more successful interpreting and working with equations in a variety of forms. The progressions we observed align with those of other researchers and provide evidence that very young students can learn to reason flexibly about equations.

1. Introduction

Algebra is a critical "gatekeeper" in mathematics education in the United States. Indeed, research has shown that completion of Algebra II correlates with access to and graduation from college as well as future earnings (National Mathematics Advisory Panel, 2008). Unfortunately, traditional routes to algebra, in which an arithmetic-focused K–5 mathematics education is followed by a disconnected treatment of algebra in the secondary grades, have led to high failure rates in school algebra that have marginalized large numbers of students and kept them from college, career, and economic opportunities (Kaput, 1998; Moses & Cobb, 2001; Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999).

In response, scholars now advocate that students have long-term algebra experiences beginning in the elementary grades (hereinafter, "early algebra"). The call for early algebra is *not* a call for "algebra early" (Carraher, Schliemann, & Schwartz, 2008). That is, the goal is not to push secondary school algebra down to the earlier grades but rather to build young students' informal intuitions about structure and relationships into formalized ways of thinking mathematically (Kaput, Carraher, & Blanton, 2008). This work often

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builds on students' informal intuitions about arithmetic (e.g., Carraher, Schliemann, Brizuela, & Earnest, 2006; Russell, Schifter, & Bastable, 2011).

Our program of research, Project LEAP (Learning through an Early Algebra Progression) explores young students' algebraic thinking and the instruction, tasks, and tools that support such thinking. In this work, we developed and tested a Grades 3–5 early algebra intervention framed by core algebraic ideas and thinking practices (see Blanton, Brizuela et al., 2018 for an elaboration). Our goal was to produce a series of lessons and a set of assessments from which we could better understand how curriculum and instruction in early algebra impacts students' algebraic thinking and document progressions in the development of their algebraic thinking. We found that students who experienced the intervention outperformed their control counterparts on measures of algebra understanding and used more algebraic strategies in problem solving (Blanton et al., 2015; Blanton, Isler-Baykal et al., 2019; Blanton, Stroud et al., 2019).

In our current work, we are drawing from research by Blanton and colleagues (Blanton, Brizuela, Gardiner, & Sawrey, 2017; Blanton, Brizuela et al., 2018) and extending our Grades 3–5 program of research to develop and test a Grades K–2 early algebra intervention with the goal of establishing a research-vetted Grades K–5 program of early algebra education. In the work described here, we focus on students' understandings of mathematical equivalence—in particular, the meaning and use of the equal sign—and explore how students' responses to tasks addressing their interpretations of equations and the equal sign shifted over the course of two school years.

2. Students' understandings of mathematical equivalence

Mathematical equivalence is a relationship between two quantities or expressions that are equal or interchangeable (Kieran, 1981). This relationship is denoted by the equal sign. The concept of the equivalence relationship denoted by the equal sign is widely acknowledged as foundational to algebraic thinking (Baroody & Ginsburg, 1983; Carpenter, Franke, & Levi, 2003; Kieran, 1981; National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State

Construct Map for Knowledge of the Equal Sign as Indicator of Mathematical Equality

Level	Description	Core equation structure(s)
Level 4: Comparative Relational	Successfully solve and evaluate equations by comparing the expressions on the two sides of the equal sign, including using compensatory strategies and recognizing transformations maintain equality. Consistently generate a relational interpretation of the equal sign.	Equations that can be most efficiently solved by applying simplifying transformations: For example, without adding 67 + 86, can you tell if the number sentence "67 + 86 = 68 + 85" is true or false?
Level 3: Basic Relational	Successfully solve, evaluate, and encode equation structures with operations on both sides of the equal sign. Recognize relational definition of the equal sign as correct.	Operations on both sides: $a + b = c + d$ $a + b - c = d + e$
Level 2: Flexible Operational	Successfully solve, evaluate, and encode atypical equation structures that remain compatible with an operational view of the equal sign.	Operations on right: $c = a + b$ No operations: $a = a$
Level 1: Rigid Operational	Only successful with equations with an operations-equals-answer structure, including solving, evaluating, and encoding equations with this structure. Define the equal sign operationally.	Operations on left: $a + b = c$ (including when blank is before the equal sign)

Fig. 1. Construct map for knowledge of the equal sign as indicator of mathematical equality. Reproduced from Matthews, Rittle-Johnson, McEldoon, & Taylor (2012). Measure for Measure: What Combining Diverse Measures Reveals About Children's Understanding of the Equal Sign as an Indicator of Mathematical Equality. *Journal for Research in Mathematics Education, 43*(3), p. 320.

School Officers., 2010). Indeed, understanding of the equal sign has been found to be predictive of early algebraic competence in the elementary grades (Matthews & Fuchs, 2020) and critical to success solving algebraic equations in the middle grades (Knuth, Stephens, McNeil, & Alibali, 2006).

A complete understanding of the equal sign is often referred to as a *relational* view of the symbol. A relational view of the equal sign entails understanding that the symbol indicates the "sameness" or interchangeability of the two sides of an equation (Carpenter et al., 2003; Jones, Inglis, Gilmore, & Dowens, 2012). Unfortunately, decades of research have demonstrated that elementary and middle school students consistently struggle to develop such a relational view and instead often demonstrate an *operational* view (Behr, Erlwanger, & Nichols, 1980; Matthews, Rittle-Johnson, McEldoon, & Taylor, 2012; McNeil & Alibali, 2005; McNeil, Hornburg, Brletic-Shipley, & Matthews, 2019).

An operational view of the equal sign treats the symbol not as an indication of an equivalence relationship but rather as a signal to compute or "put the answer" (Carpenter et al., 2003). Such a view manifests itself in students' solutions to missing value equations, their evaluation of true-false equations, their abilities to accurately encode equations, the definitions they propose for the equal sign, and—in later grades—their algebraic equation solving success. For example, students with an operational view of the equal sign often solve missing value equations such as $8 + 4 = \_\_ + 5$ by writing 12 or 17 in the blank (Carpenter et al., 2003); they have difficulty accepting equations such as $5 = 5$ (because they lack an operation) or $7 = 3 + 4$ (because they are "backwards") (Falkner, Levi, & Carpenter, 1999); they do not appropriately attend to the equal sign when encoding equations; they often define the equal sign in an "operational" manner, stating that the symbol means "the answer is" or "the total," and they are less successful solving algebraic equations (Knuth et al., 2006; Knuth, Alibali, Hattikudur, McNeil, & Stephens, 2008; McNeil & Alibali, 2005).

Matthews et al. (2012); see also Rittle-Johnson, Matthews, Taylor, & McEldoon, 2010) assessed Grades 2–6 students' knowledge of mathematical equivalence using missing value equations, true-false equations, equation encoding, and definition tasks. Tasks were presented using a variety of equation structures (equations of the form $a + b = c$, $a = a$, $c = a + b$, and $a + b = c + d$). They found that the structure of the equation had a large influence on performance, with equations with operations on both sides proving the most difficult, while item type did not. Generating a relational definition of the equal sign was especially difficult for students, even more so than solving or evaluating equations with operations on both sides. These findings led to the development of a four-level construct map describing a progression of understanding from rigid operational to comparative relational views. Students with a rigid operational view define the equal sign operationally and have difficulty with any equations not in $a + b = c$ form. Students with a comparative relational view define the equal sign relationally and evaluate equations as whole entities, using compensation strategies to solve and evaluate equations (see Fig. 1 for the entire progression).

Blanton, Otalora Sevilla et al. (2018) conducted a Kindergarten classroom teaching experiment that included 4 lessons addressing students' understandings of the equal sign. Like Matthews et al. (2012), they found levels of thinking about mathematical equivalence in Kindergarten students that they characterized as operational, emergent relational, and relational. Like Matthews et al. (2012), they described students at an operational level as those who defined the equal sign operationally and were only successful writing, interpreting, and solving equations of the form $a + b = c$. Students at the emergent relational level were those who still viewed the equal sign's role as an operator yet began to accept equation forms other than $a + b = c$. This acceptance, however, still often included referring to equations of the form $c = a + b$ as "backwards" and equations of the form $a = a$ as containing a hidden "plus zero." Finally, students at the relational level offered relational definitions of the equal sign and were successful interpreting and solving missing value equations of the form $a + b = c + d$.

In the study presented in this paper, we built on the work of Blanton, Otalora Sevilla et al. (2018) to develop a longitudinal classroom intervention focused in part on the development of a relational view of the equal sign. In what follows, we describe this intervention and address the question *How do students' understandings of the equal sign and interpretations of various equation forms shift over the Kindergarten and first-grade years in response to an early algebra intervention?* We discuss our observations in terms of Matthews et al.'s (2012) and Blanton, Otalora Sevilla et al.'s (2018) frameworks.

3. Methods

The data we report here are part of a larger longitudinal study in which, building on our work in Grades 3–5, we developed, tested, and refined an early algebra intervention to engage Grades K00–2 students in early algebraic thinking. We focus here on a subset of data collected during the first two years of the intervention.[1] In Year 1, the intervention took place in two Kindergarten classrooms in the Northeastern U.S. In Year 2, these two classrooms of Kindergarten students were enrolled in three first-grade classrooms (along with other students who did not experience the Kindergarten intervention), and the intervention continued in two of these three classrooms. The school district in which the participating school is located serves 10 % students of color and 16 % students categorized as low SES.

3.1. Participants

Twenty students (10 from each of the two participating Kindergarten classrooms) were interviewed before and after the early algebra intervention in Kindergarten. The students were selected by their classroom teachers to represent a diverse academic range and

[1] Several lessons and interviews scheduled for the end of Grade 2 (Spring 2020) were not conducted due to school closures implemented in response to Covid-19.

included students who the teacher thought would feel comfortable talking with a researcher. In first grade, 11 of these students were placed in the two intervention classrooms while the other nine students were placed in the classroom that did not receive the intervention. Ten of the 11 students in the intervention classrooms participated in the mathematical equivalence portion of the end-of-first-grade interview. In the results, we share data from these 10 students who experienced the intervention and completed the interview tasks that are the focus of this paper across the two years.

3.2. Early algebra intervention

Eighteen lessons of approximately 30 min each were taught in each of Kindergarten and first grade. Kindergarten lessons were taught during the second semester of the school year (Spring 2018) while first grade lessons were taught throughout the school year (Fall 2018–Spring 2019).[2] Lessons typically took place once per week, depending on school schedules, and were taught by a member of our research team—a former elementary school teacher who participated in the lesson development process—during students' regular mathematics instruction time. The lessons were designed to engage students in the early algebra concepts and practices developed in our previous Grades 3–5 work, including mathematical equivalence, generalizations about arithmetic properties and the properties of even and odd numbers, the use of variable, and functional thinking (see Blanton, Stroud et al., 2019 and Fonger et al., 2018 for details regarding the development of the curricular framework used in the previous and current studies). Each lesson began with a "Jump-start" to review previous concepts or prompt students to think about new concepts to be addressed in the lesson. The main body of the lessons included small-group (often partner) investigations around specific tasks. The lessons concluded with whole-group discussions of students' findings and a "Review and Discuss" question that served as a formative assessment of students' thinking.

Seven lessons in Kindergarten (Lessons 7–11 and 13–14 of 18) and six lessons in first grade (Lessons 4–9 of 18) were primarily focused on engaging students in thinking about mathematical equivalence and equations, and this concept was often revisited in the Jumpstarts of later lessons. Many of these lessons included work with balance scales. Balance scales and other balance models are often used to teach elementary and middle grades students about mathematical equivalence (e.g., Alibali, 1999; Fyfe, McNeil, & Borjas, 2015; Linchevski & Herscovics, 1996). Rationales for the use of such a model include the analogy that can be made between balancing a scale and balancing an equation (e.g., Warren & Cooper, 2005), the model's grounding in physical experience (e.g., Alibali, 1999; Araya et al., 2010) and the link such a model provides between concrete experience and abstract representations (e.g., Fyfe et al., 2015). In a study conducted with a different population of Grades K–2 students, we found that students' work with balance scales supported their development of a relational view of the equal sign (Stephens et al., 2020).

In Kindergarten, we started our equivalence lessons by having students compare the weights of unspecified quantities of identical items with a pan balance, using this as a context to develop the notion of balance and introduce the equal sign and equations. For example, given two opaque bags containing an unknown number of equally weighted manipulatives, students were asked to use the scale to determine if the quantities in the bags were equal or not equal and then after checking the contents of the bags wrote equations such as $5 = 5$ to represent the relationship between equal quantities. Subsequent lessons examined simple true-false and missing value equations, often using pan balances and number balances (see Fig. 2) to explore the relational meaning of the equal sign. The number balance was introduced by having students place one weight on the left side of the scale and then think about where they would need to place a weight on the right side to make the scale balance. The resulting balanced scale was represented with an equation (e.g., $4 = 4$ or $7 = 7$). This progressed to placing one weight on the left side of the scale and then asking students if they could place *two* weights on the right side to make the scale balance. The resulting balanced scale was again represented with an equation (e.g., $5 = 4 + 1$ or $3 = 2 + 1$). These activities were designed to intentionally challenge students to think about equations in forms other than $a + b = c$.

First-grade students built on their Kindergarten experiences by working with more challenging true-false and missing value equations of the form $a + b = c + d$ (e.g., $4 + 1 = 5 + 2$). Throughout these lessons, students were encouraged to think about the equal sign as a symbol indicating that the numbers or expressions on either side of the equation were the same or the same amount.

When not participating in our early algebra intervention (i.e., outside of the 18 lessons), students were taught mathematics by their regular classroom teachers. Teachers used *enVisionmath* and *GO Math!* curricula as resources during their regular mathematics instruction.

3.3. Data collection

The ten students who are the focus of the data shared here participated in individual task-based interviews prior to and approximately two weeks following the conclusion of the Kindergarten intervention and prior to and approximately two weeks following the conclusion of the first-grade intervention. One member of our research team conducted all interviews. We focus here only on the two Kindergarten interviews and the end-of-first-grade interview. These interviews addressed a wide range of early algebra content. The particular tasks that are the focus of this paper are those that engaged students in thinking about mathematical equivalence, including describing the meaning of the equal sign, giving an example of its use, identifying equations from non-equations, evaluating equations as true or false, and solving open number sentences (see Table 1 for specific interview tasks). Note that Kindergarten students were asked "Which of the following are equations?" because they were less familiar at the time of the pre-interview with the idea of evaluating equations as true or false or solving equations. The focus was thus on students' interpretations of various forms of (true)

[2] This schedule reflects the timing of our research cycle and not a commentary on when lessons should be taught. We believe Kindergarten lessons could be started during the first semester and taught throughout the year as they were in first grade.

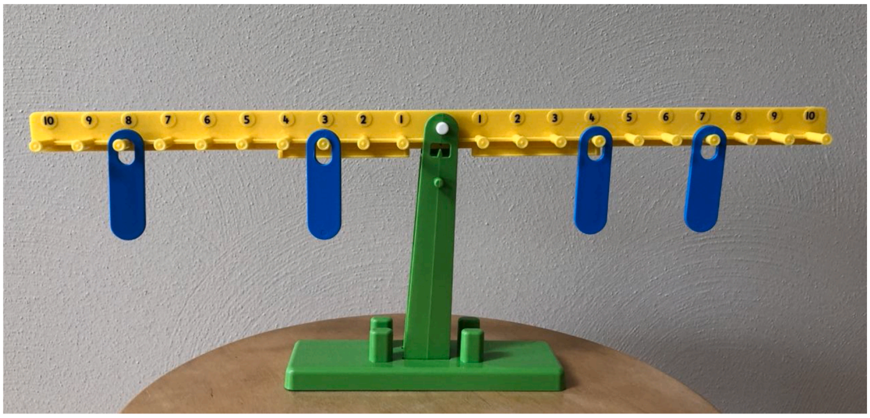

Fig. 2. Number balance representing the equation $8 + 3 = 4 + 7$.

equations and an investigation of which forms were accepted.

For the questions explicitly about the equal sign, students were shown the symbol "=" alone on a piece of paper and were orally asked the prompts shown in Table 1. They were encouraged to write their examples of how the symbol could be used on paper. For questions asking students to identify equations, evaluate equations as true or false, and solve open number sentences, students were shown the equations on paper and were asked to orally respond to each task one at a time. Paper was provided for students to use at their discretion. All tools used during the instructional intervention (e.g., cubes, balance scales) were also available for students to access during the interview, though they were not required to use them. The interviews lasted about 30 min overall and were videotaped for subsequent analysis.

3.4. Data analysis

Codes used to analyze students' responses to interview tasks were initially developed by drawing from the literature on students' understandings of mathematical equivalence (e.g., Carpenter et al., 2003; Matthews et al., 2012) as well as our previous work (described in Blanton, Stroud et al., 2019) and then refined based on common student responses observed during initial viewing of the interviews.

Student interpretations of the equal sign's meaning were characterized as *relational*, *operational*, or *other*. Responses identified as relational were those that suggested an understanding that the symbol indicates an equivalence or "sameness" of the quantities or expressions on either side of an equation. Operational responses were those that suggested an interpretation of the equal sign as a signal to compute or produce an answer. Responses that were vague and difficult to interpret were placed in the Other category. Student responses to equation identification, true-false, and open equation items were coded for correctness as well as explanation or strategy. Often these explanations or strategies shed further light on students' interpretations of the equal sign as a relational or operational symbol.

All interviews were reviewed and coded by two coders. When disagreements arose, coders met to review and refine the coding

Table 1
Mathematical equivalence interview tasks.

Kindergarten	Grade 1
Have you seen this symbol (=) before?	*Have you seen this symbol (=) before?*
• *What is it?*	• *What is it?*
• *What do you think this symbol means?*	• *What do you think this symbol means?*
• *Can you give me an example of how you think this symbol can be used?*	• *Can you give me an example of how you think this symbol can be used?*
Which of the following are equations?	*Is the equation true or false?*
$2 = 2$	$8 = 8$
$4 + 1 = 5$	$3 + 2 = 9$
$4 < 7$	$7 = 3 + 4$
$3 + 8$	$4 + 5 = 9 + 3$
$5 = 2 + 3$	$3 + 6 = 2 + 7$
$10 > 5$	
$4 + 2 = 3 + 3$	*Find the missing value*
	$5 = \_\_$
	$4 = b + 3$
	$5 + 4 = \_\_ + 3$
	$6 + 2 = \_\_ + 6$

scheme until agreement was reached. Other members of the research team were occasionally included in these discussions as needed until a stable coding scheme was in place.

4. Results

Table 2 shows how the ten participating students (labeled A–J) described the equal sign, the forms of equations they produced to show how the equal sign could be used, and which legitimate equation forms they accepted (in Kindergarten) or correctly evaluated as true or false (in first grade).

4.1. Equal sign interpretations and equation production

As shown in Table 2, prior to the intervention, only one of the ten students described the equal sign as a relational symbol ("It means the same" [J]). Five students described the symbol operationally (e.g., "What it makes" [A] and "It's like the end of the math problem" [D]) while two described it in ways that were too vague to categorize (e.g., "It means is" [B], which might be interpreted as "is *the same as*" [relationally] or "is *the answer*" [operationally]) and two simply stated they did not know. When asked to provide an example of how the equal sign could be used, nine students wrote an equation of the form $a + b = c$ while one student [J] wrote " $= 5$."

Students progressed in terms of their descriptions of the equal sign and the examples they produced over the course of the intervention. At the end of Kindergarten, six of the ten students described the equal sign relationally (e.g., "It means on both sides it's the same" [A] and "It means the numbers are equal" [G]) while four described it operationally (e.g., "It means the answer for the two

Table 2
Student responses to equal sign definition, example equation, and equation interpretation tasks across time.

Student	Pre-Kindergarten Interview			Post-Kindergarten Interview			Post-Grade 1 Interview		
	Equal sign description	Form of equation produced	Form of equations accepted	Equal sign description	Form of equation produced	Form of equations accepted	Equal sign description	Form of equation produced	Forms of equations evaluated correctly
A	Operational	$a+b=c$	$a+b=c$ $c=a+b$	Relational	$a+b=c$	$a+b=c$ $c=a+b$ $a=a$ $a+b=c+d$	Relational	$a=a$	$a+b=c$ $c=a+b$ $a=a$ $a+b=c+d$
B	Other	$a+b=c$	$a+b=c$	Operational	$a+b=c$	$a+b=c$ $c=a+b$ $a=a$ $a+b=c+d$	Operational	$a+b=c$ $a=a$	$a+b=c$ $c=a+b$ $a=a$ $a+b=c+d$
C	Don't know	$a+b=c$	$a+b=c$	Relational	$a+b=c$	$a+b=c$ $c=a+b$ $a=a$ $a+b=c+d$	Relational	$c=a+b$	$a+b=c$ $c=a+b$ $a=a$ $a+b=c+d$
D	Operational	$a+b=c$	$a+b=c$ $c=a+b$ $a+b=c+d$	Operational	$a+b=c$	$a+b=c$ $c=a+b$ $a=a$	Operational and Relational	$a+b=c$ $a=a$	$a+b=c$ $c=a+b$ $a=a$ $a+b=c+d$
E	Other	$a+b=c$	$a+b=c$	Relational	$a+b=c$	$a+b=c$ $c=a+b$ $a=a$	Relational	$a=a$	$a+b=c$ $c=a+b$ $a=a$ $a+b=c+d$
F	Don't know	$a+b=c$	$a+b=c$	Relational	$a=a$	$a+b=c$ $c=a+b$ $a=a$ $a+b=c+d$	Relational	$a=a$	$a+b=c$ $c=a+b$ $a=a$ $a+b=c+d$
G	Operational	$a+b=c$	$a+b=c$ $c=a+b$	Relational	$a+b=c$	$a+b=c$ $c=a+b$ $a=a$	Relational	$a+b=c$	$a+b=c$ $c=a+b$ $a=a$ $a+b=c+d$
H	Operational	$a+b=c$	$a+b=c$	Operational	$a=a$	$a+b=c$ $c=a+b$ $a=a$ $a+b=c+d$	Relational	$a=a$	$a+b=c$ $c=a+b$ $a=a$ $a+b=c+d$
I	Operational	$a+b=c$	$a+b=c$ $c=a+b$	Operational	$a+b=c$	$a+b=c$ $c=a+b$ $a=a$	Relational	$a=a$	$a+b=c$ $c=a+b$ $a=a$ $a+b=c+d$
J	Relational	Other	$a+b=c$	Relational	$a+b=c$	$a+b=c$ $c=a+b$ $a=a$ $a+b=c+d$	Relational	$a=a$	$a+b=c$ $c=a+b$ $a=a$ $a+b=c+d$

numbers" [I]). When asked to provide an example of the equal sign's use, eight of the ten students again offered an example of the form $a + b = c$ while two offered examples of the form $a = a$. At the end of first grade, eight of the ten students described the equal sign exclusively relationally, one student (B) described it operationally (though this student still gave an example of the equal sign's use by writing an equation of the form $a = a$), and one student (D) described it both operationally and relationally. This student explained, "[The equal sign] has two purposes. Like, 1 plus 1 equals 2. And then another one, like 5 is equal to 5. Like, 5 is the same number as 5." The examples they provided of the equal sign's use were more varied than in Kindergarten, with only one student offering (only) an example of the form $a + b = c$, one offering a $c = a + b$ example, six offering (only) $a = a$ examples, and two students offering examples of both $a + b = c$ and $a = a$ forms. Despite working with equations of the form $a + b = c + d$ during the first-grade intervention, no students offered such an equation as an example of how the equal sign could be used.

4.2. Equation interpretation

The third columns at each time point in Table 2 show the legitimate equation forms that students correctly accepted as equations (in Kindergarten) or correctly identified as true or false (in first grade). We detail these results here, along with the justifications that students provided for their responses. Recall that at the end of first grade, students were also asked to solve four missing value equations (as shown in Table 1). We share those results here as well.

4.2.1. Equations of the form $a + b = c$

All ten students identified $4 + 1 = 5$ as an equation before and after the Kindergarten intervention and identified $3 + 2 = 9$ as false at the end of first grade. Prior to the intervention, eight of the ten students justified this choice by referring to the elements of the equation or the order in which they appeared, or both. For example, "It has a plus sign, an equal sign, and three numbers," (C) and "There's two numbers first, and after the equal sign there's what it equals" (F). At the end of Kindergarten, seven students (A, B, E, F, H, I, J) focused solely on the existence of the equal sign rather than on the operation symbols or the order of the elements. One student (A) even explained that "equations don't need plus signs, but they do need equal signs." At the end of first grade, all students stated that $3 + 2 = 9$ was false, with eight (A, B, E, F, G, H, I, J) pointing to the incorrect addition and two (C, D) using the number balance to justify that the amounts on either side are not the same.

4.2.2. Equations of the form $c = a + b$

As shown in Table 2, four Kindergarteners identified $5 = 2 + 3$ as an equation prior to the intervention while six did not. Those who did not accept this as an equation referred to the statement as "backwards" (B, J), "upside down" (H), or needing to be "turned around" (F). One student (C) explained, "the plus sign needs to be where the equal sign is" while another (E) similarly commented, "the plus sign has to come first." Of the four students who did accept this statement as an equation, three (A, D, I) still referred to the equation as backwards (e.g., "It's backwards, but it's still an equation because it has the two signs [referring to the equal sign and addition sign] and the three numbers" [A]), indicating a preference for the $a + b = c$ form. At the end of Kindergarten, all students accepted this as an equation and none of them referred to it as backwards. At the end of first grade, all students identified $7 = 3 + 4$ as a true equation. Five (B, F, G, H, I) of these students simply confirmed the computation (e.g., "[True], because 3 plus 4 equals 7" [B]) while three (C, D, J) invoked the idea of balance and used the number balance to show that the statement was true. Two students (A, E) did not elaborate on how they knew the equation was true. Also, at the end of first grade, all ten students correctly identified 1 as the value of b in the equation $4 = b + 3$.

4.2.3. Equations of the form $a = a$

When asked to identify all statements on their papers that were equations, none of the ten Kindergarteners identified $2 = 2$ prior to the intervention. Their explanations revealed a preference for equations of the more familiar $a + b = c$ form (e.g., "There is no plus in the middle" [B], and "You have to have three numbers and a plus sign" [G]). At the end of the Kindergarten year, all ten students identified $2 = 2$ as an equation and none of them indicated that they felt an operation was missing, focusing instead on the existence of the equal sign. At the end of first grade, all students identified $8 = 8$ as true and justified this by indicating the "sameness" of the two sides of the equation (e.g., "Eight is the same as eight" [A, F, I]). Also, at the end of first grade, all ten students identified 5 as the missing value in $5 = \_\_$.

4.2.4. Equations of the form $a + b = c + d$

When asked to identify all statements on their papers that were equations prior to the intervention, only one (D) of the ten Kindergarteners identified $4 + 2 = 3 + 3$. This student was rather unsure, however, stating, "I'm just gonna circle it. I don't really know. It might be when I get to first grade, second grade, or fourth grade." The other Kindergarteners objected to the form of the equation, indicating, for example, "It has an extra number" (A), "There's two pluses" (C) and "They just mixed it around. It's crazy" (H). At the end of Kindergarten, six of the ten students identified this statement as an equation, generally stating that the statement was an equation because it had an equal sign. Of those who did not identify the equation as such, their focus remained on the form (e.g., "The equal sign needs to be at the end, not in the middle" [G] and "It's too long" [I]), once again indicating preference for the $a + b = c$ form. At the end of first grade, all ten students correctly identified $4 + 5 = 9 + 3$ as false and $3 + 6 = 2 + 7$ as true. Given $4 + 5 = 9 + 3$, nine of the students justified this choice by computing and stating the amounts on both sides of the equal sign were not the same while one (I) gave a vague statement that was difficult to interpret. The equation $3 + 6 = 2 + 7$ invoked more holistic explanations focused on the idea of balance, with two students (A, F) modeling the equation on a number balance and three students (B, H, I) using a compensation strategy, stating, for example, "[True], because they took away 1 from 3 and put it to 2. The 6, if you took away 1 from the 7 it would be the same as that" [I].

First grade students were further asked to solve the missing value items $5 + 4 = \_\_ + 3$ and $6 + 2 = \_\_ + 6$. Nine of the ten students solved both items correctly while one student (D) solved both incorrectly. Given $5 + 4 = \_\_ + 3$, six students (C, E, F, G, H, J) used computational approaches and two (A, I) used the number balance to determine the missing value. One student (B) used a compensation strategy to solve this equation, explaining "if you take 1 away [from the 6 written in the blank], it would be 4 and 5…I'm taking 1 away [from the 6] and adding 1 to the other [i.e., the 3], so it would be the same thing." All nine students who completed $6 + 2 = \_\_ + 6$ correctly noticed, at least informally, the Commutative Property of Addition (e.g., "You flip it through" [B], "It's the exact same thing" [H]). The student who solved both open equations incorrectly demonstrated operational thinking, adding all given numbers ("$5 + 4 + 3 = 12$" and "$6 + 2 = 8$ and then $8 + 6$ equals something else"). Interestingly, this is the student who, earlier in the same interview, described the equal sign both operationally and relationally.

5. Discussion

Our findings regarding students' growing understandings of mathematical equivalence over the course of a two-year early algebra intervention support Blanton, Otalora Sevilla et al.'s (2018) findings that—provided supportive instruction—even very young students are capable of developing relational views of the equal sign and the ability to work with varied equation forms. The ways in which students shifted over the two-year period in terms of the way they explained the equal sign's meaning, the equation forms they accepted and correctly interpreted as true or false, and the equation forms they were able to successfully solve are consistent with Blanton, Otalora Sevilla et al.'s (2018) and Matthews et al.'s (2012) progressions as well. Although we report here on results from only ten students, our data coupled with the work of others point to the possibilities of engaging even very young students in developing understandings of mathematical equivalence and offer the opportunity to examine shifts and contradictions within individual students as well as the group of ten students and generate questions for future research.

5.1. Shifts in students' thinking over time

Not surprisingly, we found that prior to the intervention nine of the ten students produced an equation of the form $a + b = c$ when asked to provide an example of how the equal sign could be used. When asked whether a set of equations were, in fact, equations, four of these nine students additionally accepted an equation of the form $c = a + b$ at this same time point. As both Blanton, Otalora Sevilla et al. (2018) and Matthews et al. (2012) note, this equation form is often the next accepted form after the familiar $a + b = c$ because it can be viewed through an operational lens. Indeed, three of the four Kindergarteners who accepted this equation prior to the intervention referred to it as "backwards."

After seven lessons in Kindergarten focused on mathematical equivalence with particular attention paid to the concept of balance, all ten students accepted equations of the forms $c = a + b$ (without referring to the equation as backwards) and $a = a$ as equations, and six of the ten accepted an equation of the form $a + b = c + d$. This progression of equation acceptance likewise fits the progressions outlined by Blanton, Otalora Sevilla et al. (2018) and Matthews et al. (2012), who found that equations with operations on both sides are typically the last equation form with which students are successful. By the end of first grade, after another six lessons focused on mathematical equivalence, all ten students correctly evaluated as true or false equations of all four types and were successful solving open equations of the forms $c = a + b$, $a = a$, and $a + b = c + d$.

As suggested by Matthews et al.'s (2012) construct map, we found that assigning a relational meaning to the equal sign took longer for students to achieve than evaluating as true or false or solving equations of even the most complex form (i.e., $a + b = c + d$). Only one student described the equal sign relationally prior to the intervention, six of ten did so at the end of Kindergarten, and nine of ten did so at the end of first grade (although one of these students [D] also gave an operational definition). We also observed stability on this measure such that once a student described the equal sign as a relational symbol, they continued to do so without "regressing" to an operational description.

While it was not our goal in Kindergarten or first grade to push students to use compensation (rather than computational) strategies to evaluate or solve equations, we did see some glimpses of compensatory thinking or strategies that focused on equation structure rather than computation at the end of first grade. For example, the equation $3 + 6 = 2 + 7$ led three first graders to use a compensation strategy and, when asked to solve missing value equations, one first grader used a compensation strategy given $5 + 4 = \_\_ + 3$. The more "obvious" $6 + 2 = \_\_ + 2$ led almost all first graders to at least informally note the underlying structure in this equation. Such thinking is important as it indicates a more structural, holistic view of equations that considers relationships across the equals sign as opposed to a view that considers each side of an equation separately as a computation to perform.

5.2. Why teach such young students about the equal sign?

While our findings suggest that young students *can* develop a relational view of the equal sign and fluency with a wide range of equation types, some may ask, why should they? Formal algebra is many years away, and elementary school is generally viewed as the time to focus on arithmetic and computational fluency. Our response is that a) engaging students in early algebraic thinking around mathematical equivalence does not require sacrificing traditional goals of elementary school mathematics and b) *when* we engage students in this work actually does matter.

First, research shows that students' computational fluency is deepened through tasks that promote a relational view of the equal sign. The students in our study were engaged in computation when determining if equations such as $5 = 2 + 3$ or $9 + 6 = 4 + 11$ were true or false and when solving open equations such as $12 = \_\_ + 8$. There is no reason students need to work with equations solely of the form $a + b = c$ when engaging in computational practice. McNeil, Fyfe, Petersen, Dunwiddie, and Brletic-Shipley (2011), for example, found that seven and eight-year-old children whose computational practice activities included equations such as $\_\_ = 9 + 4$ and $\_\_ = 2 + 2$ gained just as much arithmetic fluency and developed stronger conceptions of mathematical equivalence than students whose practice consisted solely of equations such as $9 + 4 = \_\_$ and $2 + 2 = \_\_$.

Second, waiting too long to engage students in thinking relationally about the equal sign has consequences. Students who experience years of work with equations exclusively in the form $a + b = c$—which Powell's (2012) analysis of elementary school mathematics textbooks suggests is the case for many—develop rather entrenched views of the equal sign as an operational symbol that are difficult to eradicate. In the first study of the association between age and performance on equivalence problems, McNeil (2007) found that seven-year-old children outperformed nine-year-old children on equivalence tasks (e.g., $9 + 4 + 3 = 9 + \_\_$) whether or not a lesson focusing on the equal sign as a symbol indicating "sameness" preceded the tasks. While many theories of learning and development attribute children's difficulties to something that they *lack*, McNeil proposed a "change resistance" theory to explain this finding, suggesting instead that the difficulty is due to the knowledge students *have*. Specifically, McNeil proposed that the repeated focus on arithmetic procedures common to the early school years lead children to form internal representations of operational patterns in long-term memory that interfere with their interpretations of external problems.

While McNeil (2007) found improvement in students' performance solving equivalence tasks from ages 9–11 (suggesting a U-shaped pattern), older students still continue to struggle with mathematical equivalence. Even middle school students often endorse operational meanings of the equal sign (Knuth et al., 2006; McNeil & Alibali, 2005). This has consequences when students are asked to engage in the more formal algebra of the middle grades, with students holding an operational view having more difficulty than those holding a relational view when asked to solve algebraic equations with variables (Alibali, Knuth, Hattikudur, McNeil, & Stephens, 2007; Knuth et al., 2006). Even in the middle grades, *when* this understanding of mathematical equivalence develops also matters. Alibali et al. found that middle school students who acquired a relational understanding of the equal sign closer to the beginning of sixth grade were more successful solving equivalent equations problems on a written assessment at the end of eighth grade than students who acquired a relational understanding at a later grade.

5.3. Dual interpretations of the equal sign

McNeil et al. (2006) found that the interpretations middle school students provided when asked to describe the equal sign's meaning were influenced by the context in which the equal sign was presented. Students were more apt to provide operational interpretations when presented with equations of the form $a + b = c$ than of the form $c = a + b$ or $a = a$. Equations with operations on both sides were the best at eliciting relational interpretations of the equal sign.

In this study, students were only asked to explain the equal sign's meaning when presented with the symbol "=" alone. However, the notion that students might hold multiple meanings of the symbol at the same time is supported by the fact that at the end of Kindergarten, all four students who provided an operational interpretation of the equal sign when asked to explain its meaning also accepted equations that did not conform to this interpretation. One of these students even gave an example of an equation of the form $a = a$ when asked for an example of how the equal sign could be used. At the end of first grade, the student who gave only an operational interpretation of the equal sign likewise gave an equation example of the form $a = a$ and also correctly interpreted equations of all four forms. Evidence of a dual interpretation of the equal sign was best illustrated by a student who explicitly provided both operational and relational interpretations of the equal sign with example equations to match. An interesting question for future research would be to investigate the impact of these dual interpretations on students' success on algebraic tasks. While evidence suggests that holding a relational interpretation benefits students, does simultaneously "holding on" to an operational interpretation negatively impact students' success?

In sum, we argue that our findings—that even very young students *can* develop a relational view of the equal sign—coupled with previous findings that spending instructional time developing this conception promotes computational fluency, that later success with traditional algebra tasks is tied to equal sign understanding, and that *when* a relational view is developed matters, points to the possibilities and benefits of engaging students early and often with relational thinking.

6. Conclusion

Our study lends support to the consensus among some researchers (e.g., Carpenter et al., 2003; McNeil et al., 2011) that developmental readiness is not the driving force behind students' tendencies to hold operational rather than relational views of the equal sign. Rather, we found that instruction involving nonstandard equations focused on the notion of balance and including frequent discussions about the meaning of the equal sign was well within the grasp of Kindergarten and first grade students, and that this instruction impacted their understanding in important ways. Specifically, it helped them shift towards a relational view of the equal sign and, relatedly, work flexibly with multiple equation forms—developing understandings that are foundational to success in arithmetic *and* algebra.

Author statement

Stephens, Veltri Torres, Sung, and Strachota collaborated on the conceptualization of this paper, the development of coding schemes, and the coding of the data. Blanton and Murphy Gardiner developed the early algebra lessons implemented and collected the interview data. Murphy Gardiner served as the classroom teacher in the described study. Stephens, Murphy Gardiner, Blanton, Stroud, and Knuth conceptualized and secured the funding for the overarching project. Stephens wrote the first draft of the manuscript and all authors offered feedback and approved the final submission.

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