CHARACTERIZING EVOLUTION OF MATHEMATICAL MODELS

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Formulating a mathematical model is a dynamic process, and attending to changes in students' models as they engage in modeling tasks is critical for informing pedagogical interventions. In this report, we coordinate constructs from literature on mathematical modeling, quantitative and covariational reasoning, and semiotics to characterize changes in mathematical models. We illustrate the application of these constructs using data from an undergraduate solving a modeling task.

Keywords: Modeling, Using representations

Mathematical modeling remains an important skill for K–20 students (Bliss et al., 2016; CCSSM, 2010), and relatively little is known about how to prepare teachers to interact with a student-modeling task dynamic. This is partly because the field has yet to bridge task-centered analyses common to research with teacher-centered "tips" on what to notice in students' work, because it is not clear what to direct teachers to focus on as students model. A cognitive view of modeling (Kaiser, 2017) can be leveraged to provide just such a link. A detailed exploration of how a mathematical model evolves with an eye toward identifying pivotal moments in students' reasoning that could be used for teacher education and teacher intervention has yet to be undertaken. In particular, it would be useful to teachers and teacher educators to know when, how, and with what level of scaffolding to intervene in a student's modeling process, since the best results from teaching with a modeling approach are obtained when students work out their own solutions (Kaiser, 2017). To make progress in this area, it is first necessary to address the methodological problem of articulating criteria for determining whether, and the extent to which, a student's model has changed. In this methodological paper, we will use micro-analytic techniques to document the changes that a student introduces to his model and network a set of theoretical constructs for explaining those changes.

Theoretical Perspectives

The target construct in this exploratory work is *model evolution*, specifically, revisions that a student might make to his/her model during mathematical modeling. A priori, there are several related theories and attendant constructs that might serve to articulate and trace changes to a model (elaborated below) that offer partial explanations of how individuals' models evolve. The Networking Theories Group (2014) proposed strategies like combining, coordinating, integrating locally, and synthesizing, for conducting parallel analysis of empirical phenomenon. Combining and coordinating involve generating "deeper insights into an empirical phenomenon" (p. 120). Integrating and synthesizing involve development of new theory by building on a small number of already-stable theoretical approaches. Here, we foreground our efforts coordinating theories of symbolic forms, multiple representations, and quantitative reasoning. We use one undergraduate's work on a modeling task to facilitate the theory-building. The end result is a method for using observable indicators to trace the evolution of a model. We then apply the method to identify and characterize significant conceptual hurdles experienced by the student, along semiotic and cognitive (Greca & Moreira, 2001) dimensions.

The mathematical modelling process is often conceptualized as an iterative cycle. One approach to studying individuals' modeling activity is to examine what are termed *modeling competencies* (Kaiser, 2017; Maaß, 2006). This approach is cognitive in nature and addresses how students come to understand a real problem and choose mathematical representations for it as they work on challenging tasks, which encourage accounting for various constraints. These competencies are *not* meant to suggest student abilities, but rather describe *phases* of modeling in a way that opens the process to observation and analysis. Phases include formulating a problem to solve (identifying aspects or characteristics that need to be modeled), systematizing (selecting relevant entities and relationships, identifying variables, making assumptions, or estimating parameters), mathematizing (representing entities and relationships in mathematical notation), analyzing (using mathematical techniques to arrive at mathematical conclusions), validating (evaluating the model and establishing its scope), and communicating (sharing conclusions obtained from its use) (Blomhöj & Jensen, 2003; Blum & Leiß, 2007). Student decisions made during each phase contribute to the dynamic evolution of the model. The systematizing and mathematizing phases can be viewed as model construction whereas validating and verifying can be viewed as a reflective monitoring process (Czocher, 2018). Empirically, the phases do not proceed linearly (Czocher, 2016) and the process draws on a complex interplay of mathematical, nonmathematical, and perhaps even scholastic knowledge (Stillman, 2000). Throughout these phases, the model can be refined, modified, or entirely rejected (and replaced) as it evolves to meet the modeler's problem-solving needs. In the next sections, we lay out additional relevant theories that offer insight into ways models could change.

Greca and Moreira (2001) argued that comprehension of a topic in physics is tantamount to being able to predict phenomena without needing to reference mathematical formalism. They distinguish among physical models, mathematical models, and mental models, while also maintaining that an integration of all three is necessary for building understanding. In their elaboration, "a *physical theory* is a representational system in which two sets of signs coexist: the mathematical signs and the linguistic ones" (p. 107). Physical theories are not direct presentations of observations of phenomena or objects; instead, statements of physical theories are about simplified and idealized physical systems, which they term *physical models*. The role of mathematics, then, is to formalize the theory as statements without semantic content. They characterize the *mathematical model* as a "deductively articulated axiomatic system, which can express the statements of the theory in terms of equations" (p. 108), but also acknowledge that the term may also extend to the mathematical theory the syntactic structure is derived from. *Mental models*, then, are internal and idiosyncratic representations of phenomena, which contrast physical and mathematical models which are socially mediated. Finally, they posit "families" of distinct mental models that serve as explanations for phenomena. For example, explaining a physical phenomenon like motion as interactions of forces rather than as a consequence of a linear, causal agent. Greca and Moreira's (2001) demarcation and explanation of interaction among mental, physical, and mathematical models is compatible with a competency view of mathematical modeling, even extending from physics education to modeling other phenomena of interest. We note some changes in vocabulary. We use *mathematical representation* to refer to an outward expression of an individual's mathematical model (Greca and Moreira's "mathematical model"). *Mathematical model* refers to the attendant conceptual system. Empirically, only the mathematical representations are observable, though changes in these may indicate shifts in students' perceptions of the relevant mathematical or physical concepts.

We take two further positions from mathematics education. First, we hold that mathematical models can be expressed in other conventional forms, besides equations (e.g., tables, graphs, words, etc. (Hitt & S., 2014)). Second, the modeling cycles that are common descriptions of students' mathematical modeling activity tend to conceptualize the modeling process as deriving mathematical expressions and variables from the physical problem. In contrast, Greca and Moreira (2001)asserted that variables in equations have meanings only *after* the they are interpreted through a physical model. From our own experience, we suggest that there may not be a general rule regarding whether the mathematical model or the physical model come first when addressing the kinds of modeling problems that are found in educational research. However, the point is tangent to two other useful theories: Sherin's (2001) elaboration of how individuals use mathematical models as templates to adapt to the problem at hand and Thompson's (2011) theory of quantitative reasoning.

A priori, mathematizing a situation would involve generating mathematical models and assigning semantic meanings drawn from physical models of physical theories. Identifying quantities and describing how they vary is just such a link. Thompson's (2011) theory of quantitative reasoning offers relevant insights. First, Thompson asserts that quantities are mental constructs, not characteristics of objects in the world. It immediately follows that a *quantification process* is carried out by an individual in order to conceive of quantities and that the process is non-trivial. Quantification is taken to mean "the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bilinear, or multi-linear) with its unit" (p. 37). One can then also imagine an object, an attribute, and a measure in such a way that the value of the measure takes on different values at different moments. Observing a phenomenon and conceptualizing that there are quantities and that they can vary (or may be constant) is foundational to formulating physical models and articulating physical theories. These mental acts may become quite familiar or nearly automatic if one has much experience in the context. For example, quantities like distance and velocity may be more readily available for high school students than torque, electrical current, or GDP.

Quantitative reasoning entails conceiving of quantities and relationships among quantities. Thus, deriving a mathematical model entails quantitative reasoning. However, conceiving of covarying quantities is a non-trivial mental act. Thompson (2011) explains that covariational reasoning involves conceiving of invariant relationships among quantities whose values may vary independently. The difficulty lies in imagining how a situation can change, the quantities conceived from it can change, but that a relationship among them stays the same. In deterministic language, mathematical modeling entails discovering the invariant relationships that govern the quantities involved. Coordinating quantities and attending to relationships among quantities, variant or invariant, is *covariational reasoning* (Carlson, et al., 2002). It involves identifying ways to combine quantities through operations and trace their changes, rates of changes, and intensities of changes whether they are directly measurable or not (e.g., Johnson, 2015). Relationships can be identified through observation, a priori reasoning, or through knowledge of principles rooted in physical theory. When these relationships are expressed externally in mathematical notation, they become the mathematical representation of a physical model. The mathematical representation brings with it the relevant mathematical model (mathematical concepts, objects, and structures) and the physical meanings of its constituents (attributes, measurements, quantities). Thus, the theory of covariation in conjunction with theory of quantification elaborates an important aspect of how physical models are formalized into mathematical models.

At its core, a mathematical model presents a system of signs used to stand in for a physical system. Naturally, a theory of modeling should attend to semiotic processes that imbue meaning to the signs and to how (perhaps multiple) systems of signs are coordinated. Following Kehle and Lester's (2003) application of Piercian semiotics to mathematical modeling, we view as a process of unification among a sign, a referent (the object the sign stands for), and an interpretant. The interpretant has a dual role; it is the individual's reaction to the sign and object and simultaneously defines the sign/object pairing through the individual's reaction. Generally, interpretants can be actions, emotions, thoughts, or ideas. An interpretant is the fundamental unit of inference between an object and a sign and it is subjective and idiosyncratic.

Sherin's (2001) theory of symbolic forms, which explains how meaning is read from equations, can be construed an extended example of semiosis. A symbolic form consists of a template and a conceptual schema (the idea to be expressed in the equation). For example, _ + _ = _ expresses a "parts-of-a-whole" relationship. The blanks can be filled with a single symbol or a group of symbols representing quantities or combinations of quantities (perhaps related via other symbolic forms). Familiarity with symbolic forms helps individuals "know" to use certain operators (e.g., + or ×) and to know where to place the symbols of quantities in an equation.

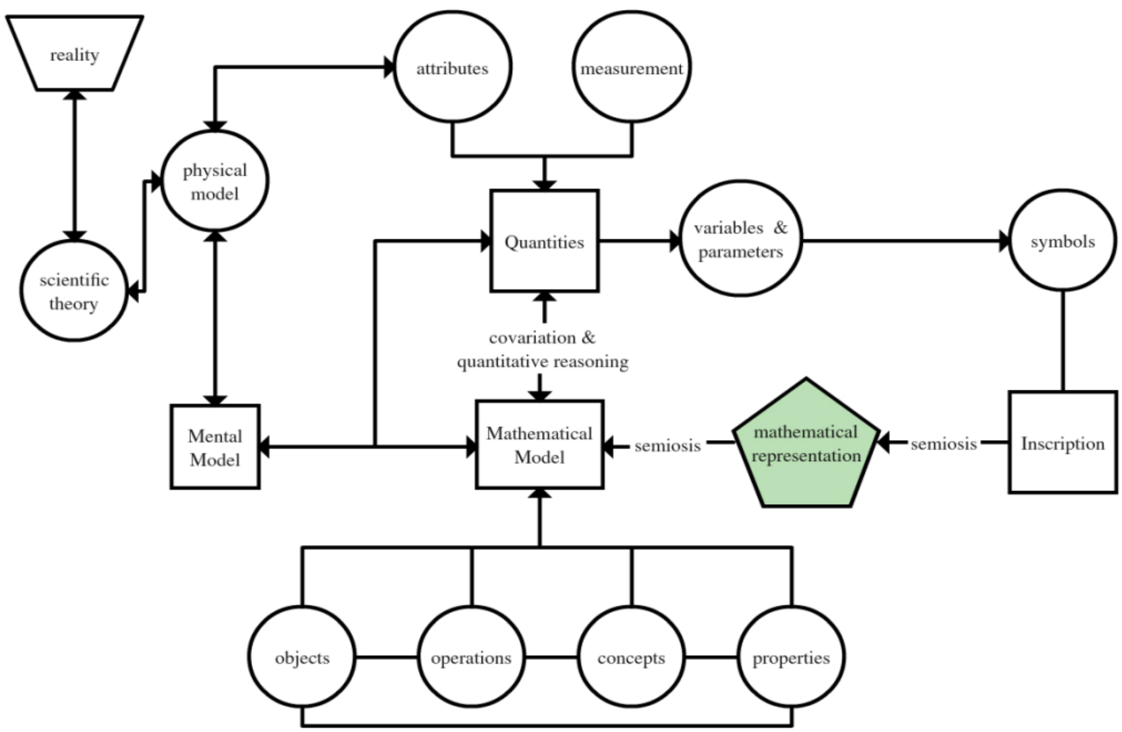

Figure 11: Networking of Major Theoretical Constructs (rectangles), Components (circles), and Mental Processes (arrows) to Hypothesize Sites for Change in How Physical Meaning is Projected onto Mathematical Symbols.

These theories operate at different grain sizes, were developed among different populations, and have different scopes. They are also asymmetrical in terms robustness. However, each layer of theory suggests a dimension of evolution and offers nuance to describe changes in a model at varying grain sizes. The networked theories are shown in Figure 11.We suggest that attending to the aforementioned constructs may produce a more comprehensive and nuanced account of how a student's mathematical model evolves.

Methods

To illustrate the coordination of constructs outlined above, we draw on data generated from a larger project examining instances of students' validating activity as they solved modeling, application, and word problems. Participants ranged from 6th grade to undergraduates. Here, we focus on Merik, an engineering major who had completed Calculus III (vectors) at a large southern university, and his work on The Monkey Problem. Merik's work was purposefully selected to explore the plausibility of the networked theory and articulate criteria for identifying changes in a model because (a) he was exceptionally verbal, (b) he often explained his own reasoning without prompting, and (c) he used multiple approaches to solve the problem.

> ***The Monkey Problem***: *A wildlife veterinarian is trying to hit a monkey on the tree with a tranquillizing dart. The monkey and the veterinarian can change their positions. Create scenarios where the veterinarian aims the tranquilizing dart to shoot the monkey. (And later: create models to represent the situation).*

The networked theory predicts that evolution of Merik's model might proceed along external (symbolic) or internal (meaning-making) dimensions. Thus we conducted three parallel analyses that could highlight opportunities to observe the representation or its meaning changing. The first coded Merik's work according to a modeling competencies framework (Blum & Leiß, 2007; Czocher, 2016) to document the phases of modeling, specifically *validating* activity since it is hypothesized to precipitate changes to the model. The second documented the various representations Merik used to capture and express his reasoning mathematically. The third sought evidence of shifts in meanings of the representations.

To analyze representations, we acknowledged that "what we ultimately observe are the external components (representations), but these cannot be disengaged from the conceptual systems" (Lesh & Doerr, 2003, p. 213). We introduce the term *inscription* to mean writing without implying anything about the inscription serving as a sign. We use the term *representation* to imply that the inscription serves as a sign for Merik. These theoretical commitments necessitated that methodologically we search for overt changes to inscriptions as indicators of changes to the model and separately search for evidence of changes to the cognitively-generated meanings for those inscriptions. To trace changes to representations, we first documented changes to inscriptions by attending to their spatial and temporal organization on Merik's paper. In the first case, we judged his model to have changed if either (1) the system of signs comprising by the representation changed (i.e., the type of representation changed, introducing a symbolic equation after working with a graph) or (2) a new inscription was created in a different location on the page. Each new mathematical representation was called a *parent*. We judged Merik's model to have changed through considering whether Merik's attention switched to another parent and whether there were substantive changes to the parent. To identify substantive changes, we considered (a) whether there was evidence to infer that information or meaning was distributed to the representation or removed from it, (b) whether Merik modified an inscription, or (c) whether Merik modified an inscription in a way suggestive of transporting meaning to or from another parent. We considered these alterations to a parent representation as a *child*. Children amending the same parent were called *siblings*. We recorded the *first time* an inscription was introduced and briefly annotated the nature of the change.

For example, Merik's first inscription appeared at time 0:51 (**Figure 12**). We labeled his *tree representation* as Parent 1 and this first version as Child 1A. At time 2:10, Merik added inscriptions x, θ, y producing Child 1B. Merik's talk indicated that these symbols were signs standing for the quantities *distance between veterinarian to the tree*, *distance from monkey to the ground*, and *angle formed by the veterinarian aiming his dart gun at the monkey*, respectively. His talk about 1A yielded evidence of both an implicit physical theory (kinematics) and an implicit mathematical model (right triangle geometry). The modification of the inscription from 1A to 1B signals introduction of quantities. At 2:38, Merik inscribed a new parent (#2), capturing his ideas that the dart, due to gravity, would not travel along a linear path but rather a parabolic one. The change, combined with his talk, and in addition to introducing gravity explicitly as a quantity, indicated that his mental model and physical model had shifted, necessitating a new mathematical model and mathematical representation. The symbolic form used in Representation 2A was _ - _ . It combined velocity and gravity. Notably absent are an explicit reference to time as a variable (though implicitly it was present in his talk as a quantity) and dimensional analysis. At time 4:25, Merik returned to Representation 1C, substituting in specific numbers for x, y, and h (with h standing for "hypotenuse"). We infer these symbols were parameters for Merik that could be fixed momentarily and changed from scenario to scenario.

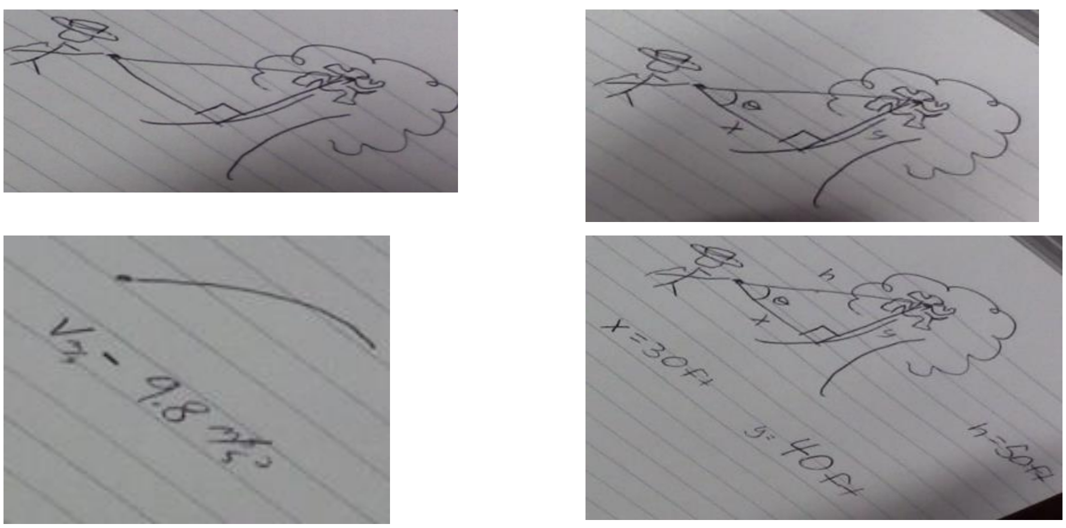

Figure 12: Merik's Representations
Clockwise from top left: Representations 1A, 1B, 1C, 2A

To help us visualize the evolution of Merik's representations, we plotted the parent and child representations over time by recording each time it could be inferred that Merik's attention was on a given sibling (Figure 13). The parent and sibling tracking system allowed us to resolve methodological issues distinguishing representations and models from one another. We next sought shifts in mental, physical, and mathematical models, and quantitative reasoning.

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Results

Seventy-four total explicit switches in attention and modifications were identified belonging to 11 distinct parent representations (comprising 22 temporally distinct siblings). The introduction of new parent representations signaled either that Merik was refining existing ways of thinking or to introducing new approaches (see Figure 13 for evolution over time). Of the 31 instances of validating identified using techniques from Czocher (2016), only 10 co-occurred with a shift in (any aspect of) the model, suggesting that most adjustments were not consequences of validating. We share three examples of model evolution in detail.

One

At time 9:45, Merik introduced Representation 4A: $f(x) = Ax^2 + Bx + C$, a standard form quadratic equation because he asserted that it would trace the motion of the dart. At his time, he had already introduced x as distance between the veterinarian and the base of the tree holding the monkey. However, he treated the equation as a symbolic form combining quantities for initial position (C) and gravity (A), evidenced by his substitutions (representation was "$= -10x^2 + Bx + 0$"). He set $f(x) = 0$ and solved for B obtaining $B = \frac{94}{3}$ and finally representation 4C, $f(x) = \frac{94}{3}x - 10x^2$. His attention shifted back to Representation 1 and he created 1D by inscribing a parabolic arc to connect the veterinarian and the monkey, leaving the hypotenuse of the triangle in place. Throughout this excerpt we infer that Merik was attempting to describe a spatial parabola, yet the substitutions in his symbolic form suggested a temporal component he did not explicitly attend to. By this we mean that his physical model included a trace of the path of the dart which can be described as a parabola. However, his mental model, which helped him to interpret the meaning of the symbolic form included a temporal component to which he did not explicitly attend. Thus, from our perspective, the meanings ascribed to the representations did not align; however, Merik initially indicated no misalignment due to his certainty that the situation could be represented with a quadratic form. It was not until 15:34 that Merik acknowledged that time would play a role in the quadratic equation, noting that "So yeah, um, x would be time, which means I couldn't use, yeah plugging in these [gestures to x=30 and y=40 in representation 1C] did nothing cause this equation is based on time." This analysis foregrounds the importance of explicit quantitative referents for variables when symbolic forms are used to generate mathematical models and that covariational reasoning was largely absent.

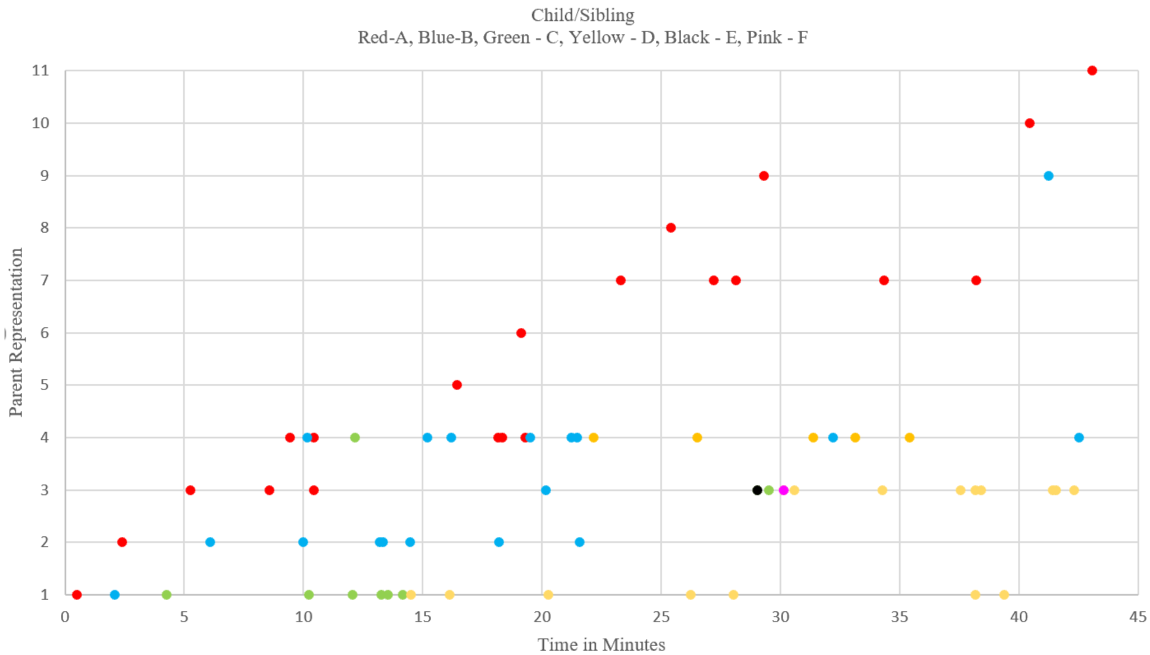

Figure 13: Evolution of Merik's Representations

Two

At 16:45, Merik introduced an analogical problem situation. He described trying to swim across a river with a current. He explained that the situation "applies the same with gravity because of the flow of the river going this way is just your rate of gravity working downwards, but your motion is that way." Introduction of a sketch of the river (Figure 14) is indicative of abductive reasoning (a type of semiotic inference). The shift leverages a different physical model to support his mental model of how the motion of the dart could be modeled, but still using kinematics as a physical theory.

Three

At 27:18, Merik summarized a difficulty that had emerged in treating the path of the dart as a parabola: he could not use right-triangle trigonometry to determine the launch angle. He acknowledged that it shouldn't matter that the path was curved. The interviewer intervened with the intention of suggesting an additional mathematical model: angle between tangent vectors. She asked "Can you think of a way that you might be able to find an angle measure between two curves? Have you ever studied anything like that?" She asked him to draw to curves in an xy-plane. He said, "I guess a straight line can be a curve," and produced representation 3E (**Figure 14**). Merik realized, from a mathematical representation that did not overtly have meaning connected to the monkey problem that he could use tangent lines to curves (via derivatives) to find the angle subtended by the two curves in question. He concluded, "that's the direction…at that exact instant which means that's where it's aimed. So if you do it from the point that leaves the barrel that's the way…". Later, at 29:30, he introduced representation 9A, a version of the law of cosines, to express θ. He also confirmed that θ would be a function of x, progressively attaching more information (in the forms of quantities, relations, and dependencies) to his representation. This analysis demonstrates that a shift in mathematical model co-occurred with a static physical theory (kinematics) and static mental mod

el (projectile follows an arc).

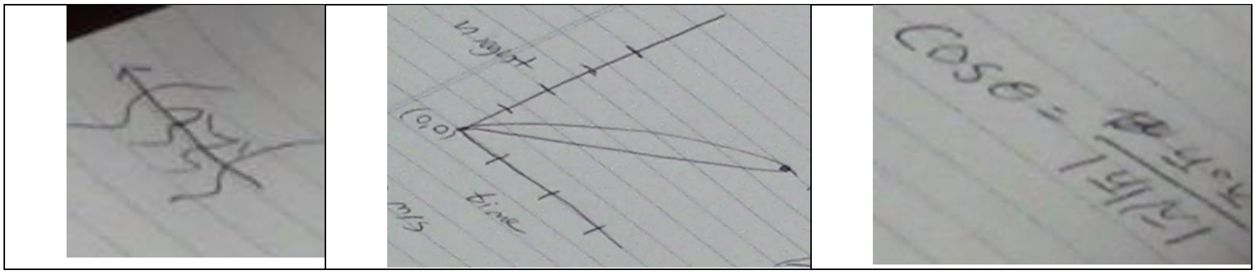

Figure 14: Representations 5A (river), 3E (coordinate axes), and 9A (law of cosines)

Our analysis exemplifies the complexity and richness of mathematical modeling when accounting for the attendant cognitive acts. We have networked theories to hypothesize and then demonstrated empirically several dimensions along which a model can change, including the adding or taking away from the inscriptions themselves, shifting attention among representations, introducing or removing quantities (attributes and how they are measured), semiotic processes (ascribing or shifting meaning of an inscription), mathematical models (including concepts, objects, operations, or procedures), physical models, and mental models. Thus, we demonstrated the plausibility of integrating the networked theory to explain the evolution of a mathematical model by tying together theories of modeling, quantitative reasoning, covariational reasoning, and semiotics.

Shifts in the ever-evolving model are important to identify because they co-occur with the critical competencies of mathematical modeling and are junctures where new ideas can be introduced and developed. We hold that sensemaking and monitoring are ongoing throughout mathematical modeling, noting that tracing representations is only partially predictive of shifts in the model. It will be important to also trace shifts evident in the students' speech that do not co-occur with changes in inscriptions. Though the Monkey Problem did not call for it (Merik worked solely within kinematics), we can easily imagine student-task interactions where a shift in physical theory is called for. We further identified at least two sites for facilitator intervention: one that was used successfully (supporting Merik in finding a mathematical model that would allow him to completely determine an angle) and one that was missed by the interviewer (the mis-alignment between a spatial and temporal parabola). Finally, we posit that the richness of structures is created and maintained through quantitative reasoning. We recommend that future research explore a parallel analysis that would trace quantification as the evolutionary thread rather than representations.

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