ON TEACHING ACTIONS IN MATHEMATICAL PROBLEM-SOLVING CONTEXTS

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In this research report, the question of the role of the teacher in problem-solving contexts is addressed, particularly in relation to the development of mathematical concepts. Data extracts from a problem-solving session are used to draw out three sorts of teaching actions that aim to push forward the mathematics in the classroom. These teaching actions are discussed in light of verbatim extracts and of available theoretical constructs from the research literature.

Keywords: Mathematics, Problem-solving, Teaching Actions, Analytical Geometry

Introduction

The literature abounds with work conducted on mathematical problem-solving from a variety of perspectives, illustrating the beneficial outcomes for students of a sustained practice of solving problems in classrooms: on the meaning given to mathematical concepts and their relevance for everyday life; on the development of critical, logical and autonomous thinking; on the active engagement in doing mathematics; on the development of positive relationships with mathematics, and so on (Boaler, 1998; Borasi, 1992, 1996; English & Gainsburg, 2015). Through this literature, one dimension that appears in need of further research concerns the role of the teacher in problem-solving contexts. As Stein, Boaler and Silver (2004) explain, problem-solving work is mostly focused on students – who are plunged into authentic mathematical practices as are mathematicians – hence the problem-solving literature says little about teachers, their role, and classroom events or actions in which teachers interact. This leads to wonder about *What is the role of the teacher in problem-solving contexts?* In parallel, another salient issue that remains little studied, as English and Gainsburg (2015) illustrate, concerns the development of content in problem-solving contexts. There is a need to investigate the interplay of mathematical concept development and problem-solving endeavors in the classroom, and *How is problem-solving used as a powerful means to develop mathematical concepts*? It is these two issues that orient this research report, namely teachers' actions in problem-solving contexts that aim at covering content and developing concepts, that is, at pushing forward mathematics in the classroom. As Jaworski (e.g., 2011) often noted, we have numerous theories about students' learning and mathematical activity, but the same is not true for teaching and the teacher's role. There is thus an important need to conduct studies that aim to develop conceptualisations of the teacher's roles *in the action* of teaching in problem-solving contexts: this research report aims to contribute to reflections and conceptualizations on these teacher's roles.

In order to do this, an analysis is conducted on an extract taken from a session in a Grade-10 classroom. Once aspects of the research are grounded both theoretically and methodologically, the extract is presented and then analyzed in relation to three kinds of teaching actions that contributed to concept development – that is, that pushed forward the mathematics of the classroom – using theoretical concepts and verbatim extract as supporting illustrations to clarify the nature of these teaching actions.

Theoretical Grounding – Problem Solving as Engaging in a Dynamic Process

Grounded in the enactivist theory of cognition for conceptualizing problem-solving environments as non-linear endeavors (Proulx & Simmt, 2016), the research is also strongly inspired by the work of Borasi (1992) and Lampert (1990) on inquiry and problem-solving, who aim to place students in authentic problem-solving situations. In their work, mathematical problem-solving is conceived of as a process that does not follow a pre-specified thread of events – analogous to the development of mathematics itself as a discipline – where numerous questions and ideas arise amid problem-solving endeavors, these often becoming central issues that can redirect the inquiry being undertaken (see also Cobb et al., 1994).

Remillard and Kaye Geist (2003) termed these "emergent" events as openings in the curriculum, where occasions offer themselves to inquiry and (can) redirect the flow of classroom events; something akin to Van Zoest et al. (2015) notion of building on ideas unfolding in the classroom. Borasi (1992, p. 202) addressed these matters in terms of flexibility, where authentic mathematical problem-solving spaces are conceived for tackling unanticipated events:

> The open-endedness that characterizes inquiry requires extreme flexibility in terms of curriculum content and choices. A teacher will often need to deviate from the original lesson plan in order to follow a new lead, pursue valuable questions raised by the students, or let the class fully engage in a debate stimulated by difference in opinion or different solutions.

Mathematics teaching is thus here conceived as a dynamic process that emerges amid interactions between teacher and students. As Curcio and Artzt (2004) assert, teachers are themselves engaged in problem-solving when they teach through problem-solving, deploying an expertise in action, while teaching, in reaction to the events of the classroom. Teachers navigate through the "material" of the classroom, act with what happens, and attempt to push the mathematics of the classroom forward through analyzing and synthesizing on-the-spot the mathematical ideas shared and produced. It is in this sense that, in Bednarz and Proulx (2009), teachers' actions are conceived along three interconnected dimensions. *First*, following work on mathematics teachers' practices (e.g., Roditi, 2005), teaching is conceived of as an event that happens in the action of, in relation to, the task in which the teacher is engaged: *teaching is enacted in action*. *Second*, along a situated cognition perspective (Lave, 1988), teaching is conceived of as situated, deployed in relation to a specific context. As Roditi (2005) insists, teaching actions are not independent of students' learning or the classroom environment, a context that plays an immense role in the kinds of actions deployed in the act of teaching: *teaching is a situated practice*. *Third*, aligned with Mason and Spence's (1999) knowing-to act in the moment, teaching is conceived of as a practice adapted in real time to events of the classroom. Teachers constantly need to adapt their responses to the dynamics of the classroom as situations often drift from the planned script: *teaching is a practice deployed in the moment*, while it is enacted. Along these dimensions, teachers are seen as continually reflecting on possibilities, offering and inventing new avenues and representations in relation to students, thinking of additional explanations to clarify or resituate the tasks offered, choosing to emphasize some aspects and not others, knowing that an explanation or a representation may eventually benefit students' understanding, and so on. Teaching practice can thus hardly be considered as a preestablished practice designed in advance for reacting to situations, but rather an expertise enacted in context, in action, as a knowing-to act adapted to situations and deployed on-the-spot in (inter)relation to classroom events. It is along this theoretical perspective that teachers' actions for pushing forward the mathematics of the classroom, are considered.

Methodological Considerations

This research report is part of a wider research program focused on studying the teaching of mathematics through problem-solving in elementary and secondary classrooms. We collaborate with groups of teachers who regularly invite us into their classrooms to experiment various kinds of problem-solving approaches and to interact, assess and reflect with us on the teaching that goes on in these sessions. Because it inserts itself in regular classrooms, the research does not want to be disruptive and follows the teachers' teaching plans, with the tasks given in class to students being chosen by and with teachers (often coming from their teaching materials and workbooks). The problem-solving sessions usually follow the same trend, starting with a task presented to students, written on the board or handed on paper, where students are given relative amounts of time to address it. After this, in a plenary manner, students are asked to share their strategies/solutions and thoughts with the group, while making sure that these are clearly explained and justified for other students to understand and ask additional questions if necessary. Students are also invited to interact between each other in relation to the ideas shared, to question or challenge them, add to them, etc., thus aiming to create a community of inquiry (Borasi, 1992; Lampert, 1990). These various interactions in turn often provoke new inquiries, where students can be asked to explore new issues or additional questions (Cobb et al., 1994).

Data-collection focuses on classroom discussions and interactions, as well as traces left on the board, all chronologically recorded as field notes by a research assistant (RA) or videotaped. Data analysis is carried out in two phases. The *first phase* consists of on-the-spot meetings (PI, RA, and teachers) to discuss teaching events that occurred during the sessions that stood out and deserved attention (in this case, on teaching actions that pushed the mathematics forward). These meetings offer a first level of analysis, which also affords interpretations of teaching events from the teachers' perspectives and permits refinement of, and adds to, the observational notes. This first level of analysis revealed salient issues about three explicit teaching actions that succeeded in pushing forward the mathematics of the classroom, that is, actions that enabled mathematical concept development: validation practices, reformulation practices, and summarizing practices. This three-pronged orientation toward teaching actions was used to orient the subsequent data analysis. This *second phase* consisted of attending to the data in relation to Desgagné's (1998) notion of available constructs from the mathematics education literature, which could give these teaching actions deeper theoretical meaning: here, e.g., Forman and Ansell's (2001) revoicing, Shimuru's (2004) *yamada*, Lampert's (1990) establishment of a community of validation.

Data Extract from the Problem-Solving Session

The extract is taken from a session in a Grade-10 classroom of about 30 students, who were working on analytical geometry in relation to distances (points, midpoints, lines, etc.) and had been initiated to usual algebraic formulas. This extract was chosen for its capacity to illustrate patterns of teacher and student interactions that were common to almost every session conducted/experimented in classrooms. For this precise session, the teacher wished to experiment with tasks along a mental computation context (following our work on mental computation, e.g., Proulx, 2014), with the intention to see how students would engage in it. One task given to students was "*Find the distance between (0,0) and (4,3) in the plane*" (given orally, with points drawn on a Cartesian plane on the front board), who had 15 seconds to answer without recourse to paper and pencil or any other material. When time was up, students were invited to share and justify their solutions to the group. The following is a synthesis of the strategies engaged in and the discussions, questions, and explorations that ensued.

The first strategy referred to applying the usual distance formula (D=$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$), leading to 5 as a distance. A second strategy suggested drawing a triangle in the plane, with sides 3 and 4, for then finding the hypotenuse by using Pythagoras (Figure 1a).

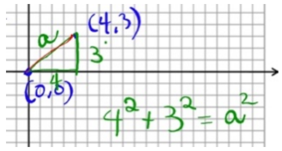


Figure 1a – Drawing the right triangle

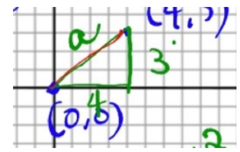


Figure 1b – Close-up on the triangle

Another student then suggested a third strategy: coming to the board to trace a red segment to count directly on it from (0,0) to (4,3) as in Figure 1b. Starting from (0,0), she counted "the number of points" to arrive at (4,3), counting the number of whole-number coordinate points from (0,0) to (4,3). While doing this at the board, she suddenly stopped and mentioned that her red segment did not go through the points she envisaged, which made the counting difficult. The teacher then traced another segment going through square diagonals linking two separate points, which could enable counting the number of (whole-number) coordinate points from one point to the next (giving 4 as a distance, Figure 2). The student agreed that for this case, it would work.

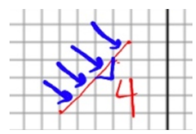
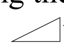


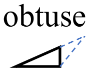
Figure 2 – Line drawn through square diagonals

The teacher then asked if the measure obtained with square diagonal lengths was identical to that obtained with the side of the square (drawing  on the board).

> One student asserted that both lengths were not identical, because the diagonal of the square was not of the same length as the square's side. Another explained that both lengths were different, because the hypotenuse is always the longer side in a triangle. Finally, a student claimed that the diagonal was longer, because it faces the wider angle.

The teacher then asked if that last assertion about facing the wider angle was always true, and if so why (drawing on the board a random right triangle ).

> One student, pointing at the triangle, stated that it was indeed the case in this drawn triangle. Another student explained that, in a triangle, the bigger the angle the longer the opposite side, mentioning that if the side-hypotenuse had been longer, the opposite angle would have been wider. And, because the sum of the (measures of the) angles in a triangle is 180°, then the 90° angle is always the wider one, the other 90° being shared between the remaining two angles.

Using the drawing of the triangle, the teacher simulated the variation of the right angle toward an obtuse one and traced the resulting side obtained, showing how it would become longer (drawing  on the board). He then moved it toward producing an acute angle, asking students if their "theory" about opposite side of the angle worked for any angle, like acute ones.

> One student asserted that it works for isosceles triangles, with equal sides facing equal angles, and another mentioned that it is the same for the equilateral triangle, since it is "everywhere the same" with same angles and same side lengths.

The teacher explained that these ideas about the diagonals being longer than the side underlined the fact that this initial strategy amounted to counting diagonals, that is, the number of diagonals of a unit square. And, that this offered a different sort of measure for the (same) distance between the two points: one in terms of units and one in terms of diagonals. A student added that if one knows the value of the diagonal, then one could find the number of unit squares for the diagonal-segment by multiplying by that factor.

One student offered a fourth strategy to find the distance, suggesting to use the sine law with angles of 45°. The teacher asked the student how he knew that both angles were 45° in the triangle. As skepticism grew in the classroom, the teacher suggested that students inquire, in small groups or individually, if the triangle's angle were 45° or not, and to be able to convince others. After 5-6 minutes of exploration, students were invited to share their findings.

> One student explained that on her exam checklist there is an isosceles right triangle with 45° angles. Thus with this triangle of side length of 4 and 3, one cannot directly assert that it is 45° because it is not an isosceles triangle as its sides are not of equal measure. Another student illustrated at the board that if one "completes" the initial triangle into a rectangle ( , see Figure 3a), then the hypotenuses of both triangles are the rectangle's diagonal which cuts it in two equal parts and thus cuts its angle in two equal 45° parts.

As the teacher highlighted that the two arguments were opposed, one student replied not in agreement with the last argument, drawing on the board a random rectangle with its diagonal (Figure 3b), and asserting that in this rectangle *it was not certain* that the angle was divided into two equal parts. Another student added that because the sides of the triangle were not identical (of 3 and 4), then the diagonal would not necessarily cut the 90° angle in two equal parts of 45°.

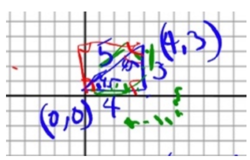


Figure 3a – The "completed" rectangle

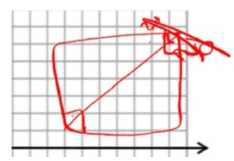


Figure 3b – The "counter" rectangle

The teacher highlighted that this last argument reused aspects of the precedent "theory" that the longer side faces the wider angle in the triangle. Hence, following this, a longer side needed to face a wider angle. Then a counter-example was offered to the group.

> The student who made reference to the checklist asserted that it happens in their exams that right triangles don't have 45° angles, for example, one with 32° and 58°; coming to the board to draw it (Figure 4). She completed her drawing to create a rectangle, explaining that the diagonal cuts as well this rectangle in two parts, but that the angles obtained are not of 45°.

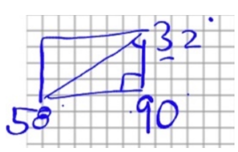


Figure 4 – The triangle counter-example with angles of 32° and 58°, and the rectangle

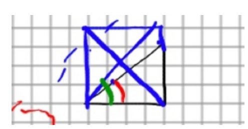


Figure 5 – Comparing triangles within a square

The teacher asserted that this offered a counter-example, with a type of right triangle frequently met that did not have angles of 45°. One student added that because all sides were different, then their associated angles would be different, the longer side needed to face a wider angle, leading

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to different angles. The teacher then highlighted the work of one student who drew a square in his notebook to assess the 45º situation. Drawing a triangle of sides 3-4-5, he extended the cathetus of 3 toward one of 4 to create a 4x4 square. Then, because in the previous unit-square the angles were of 45º, in this 4x4 they were 45º as well (Figure 5). Comparing hypotenuses of both triangles, it illustrated that in the initial 3-4-5 right triangle the angle is smaller than the right triangle of side 4 and 4. All this led students to appear to agree with the fact that the angle was not 45º, ending the explorations (and leading to offer another task to be solved by the students).

Pushing Forward the Mathematics: Validating Practices

One kind of teaching actions enacted in the session are validation practices. These, in problem-solving contexts, are related to the consideration of the classroom as a mathematical community of validation as Boaler (1998), Borasi (1992) and Lampert (1990) call it. In this community, members are encouraged to generate ideas, questions, and problems, to solve them, to share their understanding, to negotiate meaning, to develop explanations and justifications to support their solutions, to question others' solutions, and so on. As Lampert (1990) explains it is the teacher's role to makes sure that students' justifications are adequate, that arguments are clearly stated, that solutions are expressed in an intelligible manner and are accessible; essential conditions for the mathematical community to take shape and flourish. In sum, it is the teacher's role to create and sustain this mathematical community. As a way of example, when Max mentioned that the triangle's angle was 45º for using the sine law, the following interaction led to a request for validation from the classroom community:

Teacher: And, how do you know that the angle is 45º? [some students appeared opposed]
Max: Because it is 45º for both angles [some students agree, others express disagreement]
Teacher: Ok, so maybe 45º maybe not. We are not sure. But we need to arrive at something, we need to agree, we need to know if it is or not 45º. So, I will ask you to take a couple of minutes, alone or in small groups, to see if it is or not 45º, and to be able to explain it and even be able to convince others of it. You can use all you have, workbooks, textbooks, notes, whatever. After that, we will share your findings. Ok, go! [students start to work]

For Cobb et al. (1994) and Lampert (1990), a community of validation develops, analyzes, questions, and argues for what is or not mathematically acceptable. Hence the mathematics produced within the classroom community is validated by the community itself (in which the teacher takes part). The teacher's requirement for and establishment of these validation practices aim to give status to the mathematical productions of the classroom, to make official the ideas shared by making them accessible and reusable in the future as they are justified and argued for. These validation practices contribute to the advancement of mathematical ideas in the classroom.

Pushing Forward the Mathematics: Reformulating Practices

Another sort of teaching actions relates to the reformulation of students' ideas. While students share their strategies, as Cobb and Yackel (1998) underline, the teacher can re-describe students' reasoning and reformulate it in different terms: one in which they would not necessarily have used but that are aligned with the meaning they are making. As an example, when Sandra explained her strategy of counting the points on the red segment, the teacher not only asked her to clarify what she did, but also re-explained and reformulated the underlying idea.

Sandra: Well, I just counted the points.
Teacher: What do you mean exactly by this?
Sandra: (coming to the board) Well, if you draw a line from here to here, it gives 1, 2, 3…

Teacher: Ah, OK, you trace a segment from one point to the other, so from (0,0) to (4,3).
Sandra: Yes. But, finally, this thing does not work as I thought.
Teacher: Ok, it is difficult to count directly on the plane. So, Sandra, you wanted to count the number of coordinate points that your segment crosses from (0,0) to (4,3). And it's difficult because it does not cross them directly. But it could work! If you take this segment and link these two other points (Figure 2). Then, here, it goes though the square diagonals and we can directly count the points crossed on the plane.

The teacher here resumed and reformulated Sandra's explanations, not to correct them (even if some adjustments could be made), but mostly to clarify them, deepening them, drawing out its key elements for all to see. This is aligned with what Forman and Ansell (2001) call *revoicing*:

> there is a greater tendency for students to provide the explanations […] and for the teacher to repeat, expand, recast, or translate students' explanations for the speaker and the rest of the class. The teacher revoices students' contributions to the conversation so as to articulate presupposed information, emphasize particular aspects of the explanation. (p. 119)

These reformulating practices are a way for the teacher to insert him or herself into students' explanations and work with the mathematical ideas produced in the moment. It is also a way to make ideas accessible to the classroom community for engaging in subsequent validation practices, since some ideas are not always mathematically adequate (like numerous arguments made around the 45° angle). These are then also clarified and reformulated for students to understand and validate as a community. By making the mathematical ideas accessible to all and highlighting them, the teacher is making the mathematics of the classroom advance.

Pushing Forward the Mathematics: Summarizing Practices

A third sort of teaching action is about summarizing practices. At different moments in the classroom, when ideas are shared in the problem-solving process, the teacher underlines explicitly some ideas produced that have important mathematical potential and to which students need to pay attention: those that will be useful or reused in the future. Hence in problem-solving contexts, the teacher has an important role to play in underlying the important mathematical ideas produced and making sure that these are clear and accessible, as Stein et al. (2004) mention. These summarizing practices can happen at varying moments during a problem-solving session, and the teacher at any moment can opt to underline, validate, put forth, thus summarize, the mathematical ideas shared for leaving traces about the mathematical productions of the classroom. For example, in relation to the query about the square diagonal, after arguments and ideas were shared, the teacher summed up the ideas and came back on Sandra's strategy, extending it to the calculation of distances:

Teacher: Let's go back to the square diagonal. It is longer than the side. In Sandra's strategy, we count the number of diagonals, the ones from one point to the other, from (0,0) to (4,3). We talked about points crossing. We would here have 4 diagonals, which is also a possible measure of the distance between (0,0) and (4,3). We would thus have two measures: one of 5 in terms of square-sides and one of 4 in terms of square-diagonals. Same distance, measured in two different ways, hence offering two different measures.

When it is done at the end of the inquiry, as Shimizu (2004) mentions, this summing up is an occasion to conclude by highlighting the mathematical ideas worked on during that inquiry, what is called in Japanese the *yamada* of the activity. For example, the strategy offered for concluding the 45° exploration was a way of summing up most ideas shared in class and of settling the issue.

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Summarizing practices in problem-solving contexts are essential acts that contributes to leaving mathematical traces of what was covered in the classroom, enabling students to pay attention to important mathematical concepts, hence pushing the mathematics forward.

Final Remarks

These three teaching actions succeeded in pushing the mathematics forward in the classroom, highlighting the important role of the teacher in the evolution of the classroom and how mathematics is tackled and develops in it. The analysis of these teaching actions offers an initial way, albeit preliminary, to address the questions that triggered this research report, namely, about the possible teacher actions that aim to contribute to the development of mathematical concepts in problem-solving contexts. Much more needs to be studied in terms of the outcomes of these practices, for example, on student's personal concept-development and how they operationalize these in various situations. But, at this point, this initial analysis offers promise for better understanding and conceptualization of teachers' roles in problem-solving contexts.

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