GENERALIZATION OF AN INVARIANT RELATIONSHIP BETWEEN TWO "QUANTITIES"

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In this report, we present an analysis of two prospective secondary mathematics teachers' generalizing actions in quantitative contexts. Specifically, we draw from a teaching experiment to report how Lydia and Emma engaged in different generalizing processes for the same task. Based on these differences, we found Lydia's generalizing actions (i.e., coordinating quantities) to be a more productive generalization than Emma's (i.e., slopes of tangent lines). The former was extendable to a new case, instance, or situation, whereas the latter was constrained to a specific representational system.

Keywords: Cognition; Algebra and Algebraic Thinking; Design Experiment

Several researchers have illustrated that teachers' covariational reasoning is critical to supporting their students in understanding major pre-calculus and calculus ideas (Ellis, 2007a; Confrey & Smith, 1995; Thompson, 1994; Thompson, 2011). Moreover, Ellis (2007a) reported that quantitative reasoning is an integral part of students constructing productive and global generalizations. In this paper, we characterized two pre-service teachers' (PSTs') generalizing actions during a teaching experiment focused on graphing and modeling covariational relationships. We give specific attention to how two PSTs' generalizing actions differed when reasoning about the rate of change between the distance a rider has traveled (i.e., arc length) around a Ferris wheel and the rider's vertical distance from the horizontal diameter of the Ferris wheel (i.e., height). More specifically, this paper reports (a) the generalizing actions of two PSTs, one which coordinated quantities, and one which focused primarily on steepness of tangent lines and (b) implications of those generalization actions on their later activities.

Theoretical Framework

We investigated PSTs' generalizing actions about relationships between quantities in dynamic situations. We use *quantity* to refer to a conceptual entity an individual constructs as a measurable attribute of an object (Thompson, 2011). Relatedly, Ellis, Tillema, Lockwood, and Moore (2017) introduced a generalization framework (Table 1) involving three major forms of students' generalizing—relating, forming, and extending—by building on Ellis' (2007b) taxonomy of generalizations, in which students' generalizing actions were viewed as different than students' final statements of generalization. In this paper, we are using this framework to illustrate two PSTs' generalizing actions by situating "generalization" within the perspective of the learner and with sensitivity to their reasoning about quantities. In other words, we do not take mathematical correctness as a necessary criterion of generalization, instead focusing on the process of how individuals identify their own patterns/similarities determined across cases.

Ellis et al.'s (2017) framework has two major forms of generalizing—intra-contextual forms and inter-contextual forms—in which students' generalizing actions occur within one context, task, or situation in the former and across situations, contexts, or tasks in the latter. Intra-contextual forms of generalizing include students searching for and identifying similar elements, patterns, and relationships across cases, numbers, or figures in order to form a similarity or regularity and extend it to new cases, instances, situations, or scenarios. Inter-contextual forms of generalizing include students forming similar relationships across contexts, problems, or situations and transfer. In this study, we illustrate several intra- and inter-contextual generalizations and their relationships to forms of abstraction presented in the work of two PSTs.

Table1: Ellis et al. (2017) generalization framework

Intra-contextual forms of generalizing		Inter-contextual
Forming	Extending	Relating
Relating Objects • Operative • Figurative • Activity Search for Similarity or Regularity Identify a Regularity • Extracted • Projected Isolate Constancy	Continuing Operating • Near • Projection Transforming • Constructing a Quantity • Recursive Embedding Removing Particulars	Relating Situations • Connecting Back • Analogy Invention Relating Ideas or Strategies (Transfer)

Methods

The data we present and analyze is from a teaching experiment (Steffe & Thompson, 2000) conducted over the course of a spring semester at a large public university in the southeastern U.S. with two PSTs. Our goal in the teaching experiment was to investigate the PSTs' ways of thinking and create models of their mathematics. Specifically, we explored how PSTs conceived of situations quantitatively and represented particular quantitative relationships under particular coordinate system constraints. In this paper, we focus on Lydia and Emma, who, at the time of the study, were two junior undergraduate students enrolled in both content and pedagogy courses. They had completed at least two additional courses beyond a traditional calculus sequence with at least a C as their final grade.

Lydia and Emma participated in 12-videotaped sessions (interview and teaching sessions) and we digitized their written work. Each session was approximately 1-2 hours in length. The second author served as the teacher-researcher (TR), and at least two project members observed each teaching session. We analyzed PSTs observable and audible behaviors (e.g., talk, gestures, and task responses) in details by relying on the generative and axial methods (Corbin & Strauss, 2008) combined with conceptual analysis techniques (Thompson, 2008). We drew on Ellis et al.'s (2017) analytic framework for categorizing students' generalizing actions and reflection generalizations in order to identify forms of generalization and abstraction.

Analysis and Findings

In this section, we analyze Lydia and Emma's activities during two tasks—Taking a Ride Task and Circle Task—from the teaching experiment to illustrate their generalizing actions and implications of those generalizing actions. For the Ferris wheel task (Figure 1a), we draw attention to how Lydia and Emma used the same verbal statement to describe their identified regularity, but these verbal statements signified different reasoning and generalizing actions. Then, for the Circle Task (Figure 1b), we show implications of those generalizing actions for Emma and how those generalizing actions influence Emma's activity in a later session.

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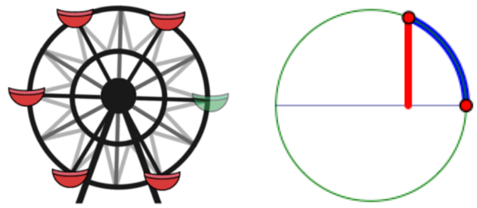


(a) (b)

Figure 1. (a) Ferris Wheel and (b) simplified version used during the Circle Task.

Taking a Ride Task

The Taking a Ride Task occurred during the first session of the teaching experiments. The PSTs watched an animation of a Ferris Wheel rider that was indicated by a green bucket (see Figure 1a) who travels at a constant speed counterclockwise starting from the 3 o'clock position (Desmos, 2014). Then, we asked them to describe how the height of the rider above the horizontal diameter of the Ferris wheel changes in relation to arc length it has traveled. Both Lydia and Emma described the height as increasing less and less for the first quarter of a rotation. However, their generalizing actions entailed marked differences, which we will illustrate below.

Lydia's generalizing actions. In the Taking a Ride Task, Lydia, first, *identified a regularity* in directional changes of height with respect to the arc length it traveled (i.e., height is increasing or decreasing as the arc length increases). In other words, we inferred from her activity she identified a predictable stable feature of the riders' height from horizontal diameter in relation to arc length it traveled for each quarter turn. We note that, for the second quarter turn, she continued her generalizing actions with a *minor accommodation*. That is, she did not fully extend her identified regularity of directional changes of height in relation to arc length to the next case (i.e., second quarter turn); she conceived that the height began to decrease as the arc length continued to increase.

The TR then asked Lydia if she could provide more information—in addition to her observation that the arc length increases and the height increases during that first quarter—about how the height and arc length change in relation to each other. Lydia used the spokes of the Ferris wheel (i.e., each of the black bars connecting the center of the wheel to its edge) to partition the Ferris wheel into equal arc lengths. This partitioning activity was to become an important tool for Lydia to *establish a way operating* with respect to identifying how height changes in relation to arc length.

(a) (b) (c)

Figure 2. Lydia engaging the Ferris wheel task.

After Lydia used the spokes to create equal partitions in arc lengths (see the green dots in Figure 2a), she drew associated heights (see the green bars in Figure 2a). Then, with support from the TR's questioning, Lydia constructed successive amounts of change in height corresponding to successive equal changes in arc length (see Figure 2b, circled in blue). That is, Lydia's partitioning activity led her to *construct a quantity*, namely change in height. The TR next asked Lydia if she noticed anything to describe the relationship between height and arc length in the first quarter of trip. The following excerpt demonstrates her response.

Lydia: Um, the distance from A to B and then B to – the distance from – Okay, so this is A, B, and C [*labels as shown in Figure 2c*], from B to C the distance is less than from A to B.
TR: So what distance is less?
Lydia: Um, from point B to C [*motions along arc*], um, that distance [*indicates circled vertical distance from Figure 2c*] from – is less than the – from A to B [*unclear if pointing along arc or along vertical distance*], I feel like.
TR: Like when you're saying distance, what do you mean? Do you mean like –
Lydia: Vertical distance.
TR: Vertical distance, okay, gotcha.
Lydia: Yes. And then also if we were to say at the top this is D [*labels pi/2 on circle d*], then from C to D [*motions along arc then vertically down to the center*] is less than B to C [*unclear motion*].

We inferred from her activity that she was *establishing a way of operating* that involved the construction of a new quantity (i.e., change in height) and associated partitioning activity. With that quantity constructed, she was then able to *search* for regularity or pattern in that quantity's variation (e.g., decreasing increases). After conceiving that change in height decreased along with those equal partitioning in arc length as shown in Figure 2, Lydia concluded that "as the arc length is increasing... [the] vertical distance from the center is increasing ... but the value that we're increasing by is decreasing." We interpreted this to indicate that Lydia had *identified the regularity* in how height changes in relation to arc length in the first quadrant. To say more, we inferred that she *extracted* this regularity because she identified a common feature of a change in one quantity (i.e., height is increasing less and less) across multiple cases relying on a change in another quantity (i.e., with respect to equal increase in arc length). We did not have evidence that she could generalize her regularity to any size of equal partitions (i.e., a *projected* regularity).

Emma's generalizing actions. In the Taking a Ride task, Emma began similarly to Lydia. She initially drew several consecutive vertical line segments (see red line segments in Figure 3) and discussed the directional variation of these segments with respect to traveling along the arc. Thus, Emma also *identified a regularity* in directional changes in the height of the rider (i.e., the height is increasing or decreasing). She specifically explained, "you're going to be increasing your heights or your distances from the center until you get to 3 pi over 2, and then, you're going to go back to decreasing your heights." As such, she made a *minor accommodation* to her identified regularity in the directional change in height with respect to the arc length for the first quarter rotation in order to extend it to other quarter turns.

Noticing that Emma used the phrase "slowly increasing" to describe the height of the rider above the horizontal diameter, the TR asked Emma to explain more. Emma drew a blue line segment from the 12 o'clock to the 3 o'clock position (see Figure 3) and responded, "there isn't like a linear trend that the lines [*the vertical red segments in the Figure 3 representing the different heights of the rider along the ride*] follow…. Like the lines [*referring to the vertical red* *segments in Figure 3*] don't increase at the same rate." We inferred that she used an *analogy invention* by creating a new situation (i.e., a linear trend by drawing a linear line) with the purpose of comparing and contrasting the two situations in order to show the increase in height was not linear. Although her associating the current situation with an analogous situation did not provide an explanation of why the height increased "slowly" over the given interval, she determined the increases in height were not constant or "linear".

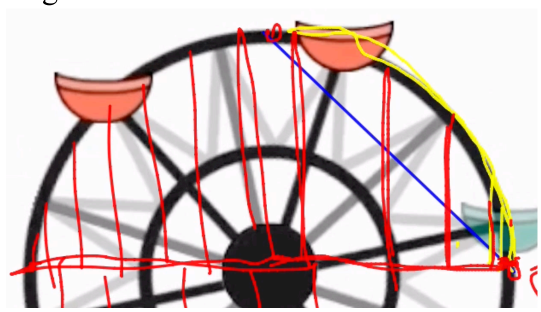


Figure 3. Emma engaging the Ferris wheel task.

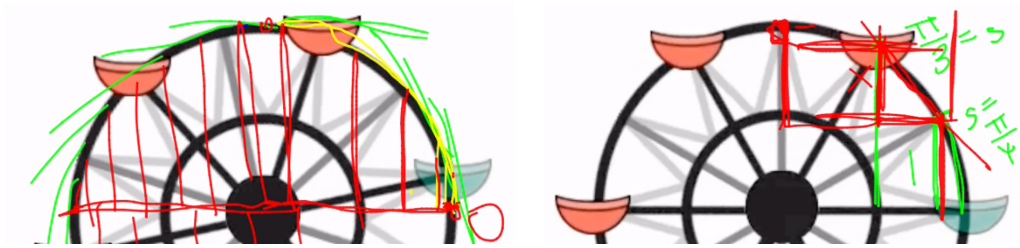
Continuing, Emma explained, "The lines increase … more steeply, and then, they slowly increase less and less [*makes a curved shape with her hand*] as you reach that inflection point at the top [referring to the point at the 12 o'clock position]." For the second quarter, she said, "They [*i.e., the heights*] start decreasing at a slower rate and then begin to decrease at a faster rate as you reach pi." It is important to note that, in contrast to Lydia, we did not have evidence that Emma coordinated the quantity of arc length beyond imagined movement in order to investigate how the increase or decrease in height changes. Although she imagined the movement at the beginning as height increasing steeply, her overall reaction to the first quarter in height was "increasing less and less", which was akin to Lydia's statement. The underlying generalizing activity of this statement, however, was quite different than that of Lydia's, which we illustrate below.

The TR returned Emma's attention to the first quarter turn and asked her to discuss how she saw the height increasing at a steeper and then slower rate. She referenced "the concavity" of the Ferris wheel in the first quarter, saying, "Well, you can see that the concavity of the – of like, I guess the arc length, the line that follows the circle…". An explanation for Emma's actions is that they were influenced by a prior context or information (i.e., *transfer*) such as presented relationships between the concavity of a curve and classifications of rates of change (e.g., concave down means increasing or decreasing at a decreasing rate).

As Emma continued, she transitioned to comparing the steepness of tangent lines that she drew along the Ferris wheel, starting at the 3 o'clock position and ending at the 9 o'clock position, as seen in Figure 4a. She said,

> …like, if these are the tangent lines showing the slopes and then you start to hit a slope that's more horizontal [*moves to the top of the circle and draws tangent lines*] as you reach your maximum point where you start decreasing … more steeply as you get back down, so like you – like these are horizontal [*motioning to the top of the wheel*]; these are vertical [*motioning to the left side of the wheel*]. So, you can see the steepness of your increasing or decreasing….

Here, Emma compared the steepness of tangent lines, concluding the tangent lines were becoming horizontal or less steep in the first quarter turn and they were getting vertical or steeper in the second quarter turn. She related these features to the height increasing at a decreasing or increasing rate. Similar to her reference to concavity, an explanation of her actions is that they were influenced by prior knowledge (i.e., *transfer*) relating the slope of lines tangent to a function graph to classifications of rate. Thus, by means of the generalizing action of the *transfer*, Emma *identified a regularity* in the steepness of tangent lines as the rider rotated around the Ferris wheel. We inferred that Emma began to identify regularity by *extracting* a few cases (i.e., multiple tangent lines that were getting steeper/horizontal, see Figure 4a), and then she immediately described a predictable feature of the entire interval, which was a *projected* identification of a regularity.



(a) (b)

Figure 4. (a) Emma draws the tangent lines; (b) Emma identifies the amount of change in height.

It is important to note that when we asked Emma to talk about height increasing slowly (or at a decreasing or increasing rate), she persistently described tangent lines or some other idea vaguely connected to the tangent lines (e.g., ambiguous values of derivative functions as the values of the slope of tangent lines). In fact, when the TR guided her in identifying amount of changes in height in relation to each equal arc length increments along the Ferris wheel as shown in Figure 4b, she neither spontaneously noted that changes in height were decreasing, nor did she relate the changes in height to her slope of tangent lines. She instead reverted back to discussing the steepness of tangent lines and associating those features with increasing or decreasing "rates" of height. We inferred that her image of tangent lines and their steepness has little to do with explicitly coordinating changes in height with systematic changes in arc length.

The Circle Task

In the Circle Task, which occurred during a later teaching experiment session, PSTs watched an animation of a point moving along the circle and without any physical context like an amusement park ride (see Figure 1b). The TR asked them to graph the relationship between the height of the point above the horizontal diameter of the circle (i.e., the red segment in Figure 1b) and the arc length (i.e., the blue arc in Figure 1b). In this section, we only provide Emma's data, because she engaged in generalizing actions including both coordinating quantities (i.e., Lydia above) and the steepness of the tangent lines (i.e., Emma above) in order to show how the former generalization enabled her to accommodate to new situations; however, the latter generalization did not. In addition, due to space constraints and because her activity was consistent with that described above, we will not describe Emma's generalizing actions in a great detail. Therefore, we will only provide implications of those two different generalizing actions as it relates to Emma considering a new situation.

Emma graphed the relationship between quantities on a Cartesian coordinate system in which the vertical axis was labeled "blue" and the horizontal axis was labeled "red" (Figure 5a). She then described how the red segment changed in relation to the blue arc by coordinating quantities. She equally partitioned the arc length on the horizontal axes and determined the corresponding changes in the red segment and indicated those changes by orange line segments (Figure 5a). She explained, "… over the same arc lengths, my heights are, um, increasing but at a

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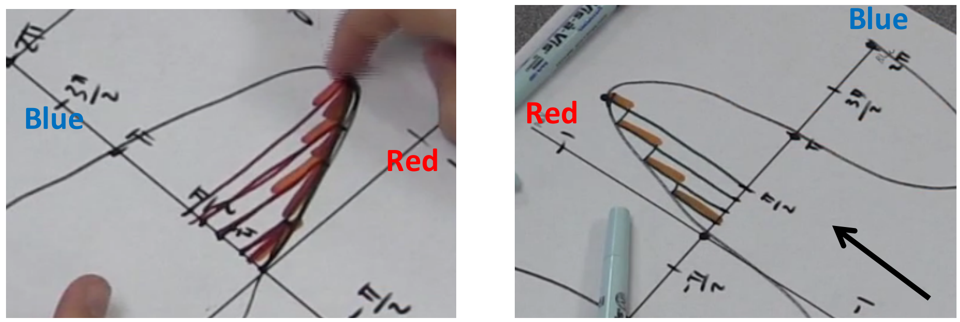
decreasing rate, so like, the rate that they're increasing is less and less each time." Verbally, this is nearly the same statement that she used in the Taking a Ride task in order to refer to how the height changes, but in this case it stemmed from her coordinating quantities and their changes.



(a) (b)

Figure 5. Emma's graphs in two differently-oriented Cartesian coordinate system.

Next, we asked Emma to graph the same relationship in a differently-oriented coordinate system (i.e., the vertical axis labeled as red and horizontal axis labeled as blue). She graphed the relationship as shown in Figure 5b. Importantly, she identified and illustrated (Figure 5b) that the amount of change in red was decreasing with sensitivity to the alternative orientation of the coordinate axes. Because Emma had briefly mentioned comparing tangent lines at the onset of the problem, we raised this idea in order to determine if she connected it to her partitioning activity. Referring to the graph in Figure 5b, she explained, "it's increasing at a decreasing rate, so like, the slope follows this change." She further explained that the slopes became more horizontal and she estimated the slope of the tangent lines in each increment by estimating the vertical change. In this case, her generalizing activity relative to the steepness of the tangent lines was compatible with her generalizing activity relative to coordinating quantities.

Turning to Figure 5a, Emma conceived the slopes of the tangent lines to indicate something different than that from coordinating quantities. She claimed the red quantity "is increasing at a decreasing rate", when considering the orange line segments (Figure 5a). She also claimed, "Slope is increasing at an increasing rate", when considering slopes of the tangent lines (e.g., becoming steeper and estimated the slope of the tangent lines). The TR drew her attention to these differing claims, which did not perturb Emma. Rather, these conclusions existed independent of each other. She transformed her coordinating quantities to take into account a different coordinate orientation, but her image of slope and associated generalizations were specific to a particular orientation. We do not mean to imply a deficiency in Emma's reasoning, but rather point out that due to its roots in "steepness", her generalizations regarding tangent lines and "rate" did not take into account different orientations (e.g., dy/dx versus dx/dy).

Discussion

In summary, after both reasoning about directional change in height, Lydia and Emma used similar verbal statements to describe their identified regularity in the relationship between height and arc length. Lydia's initial generalization was connected to coordinating the quantities of height, arc length, and their changes. Emma's initial generalization, along with her scheme and mental actions, primarily involved associating the steepness of tangent lines along the circle with the "rate" of height. We did not have evidence that the quantity of arc length and its variations were central to Emma's reasoning. Thus, we can claim that Emma and Lydia made similar

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generalized statements, but the objects that were associated and what regularity they identified were different.

Byerley and Thompson (2017) reported teachers holding different meanings for a slope, and illustrated that some of these meanings are useful in limited circumstances. For instance, they identified slope as an index of steepness as such a meaning, which correspond to Emma's case. Specifically, in Emma's case, her generalization slope as an index of steepness in association with "rate" did not enable her to conceive invariance among graphs in different coordinate orientations. With respect to Figure 5a, her meaning led her to conclude "increasing at an increasing rate." With respect to Figure 5b, her meaning led her to conclude "increasing at a decreasing rate." On the other hand, when coordinating quantities, Emma was able to conceive invariance in "rate" among the two graphs; both represented red increasing at a decreasing rate with respect to blue. Ellis (2007a) argued that generalizations rooted in coordinating quantities are more powerful, and we interpret Emma's activity as such in that her generalization rooted in quantities was more flexible than that rooted in the perceptual steepness of a tangent line.

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