STATIC AND EMERGENT THINKING IN SPATIAL AND QUANTITATIVE COORDINATE SYSTEMS

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The body of research examining students' graphing understandings across STEM fields indicates students are not developing productive meanings for graphs. We conjecture such failings may, in part, be explainable by features of students' use of coordinate systems and graphing activity that are under examined. In this theoretical report, we present a conceptual analysis of different ways students may reason about graphs and coordinate systems. Specifically, we describe two different uses of coordinate systems—spatial and quantitative—students might leverage and two ways of reasoning—static and emergent—students might engage in as they construct or interpret graphs. We characterize how a student may engage in each kind of reasoning in each use of coordinate system. We intend this paper to serve as a theoretical lens for future empirical studies examining students' developing graphing understandings.

Keywords: Cognition, Learning Theory, Spatial Thinking

There is a need for U.S. students to enter STEM fields, with mathematics often serving as a "gatekeeper" for student success in these fields (Crisp, Nora, & Taggart, 2009). As such, it is important for students to have experiences in their K–12 schooling that are attentive to the needs of potential STEM coursework and careers. A common way STEM fields communicate information is through graphical representations. Paoletti, Rahman, Vishnubhotla, Seventko, and Basu (2016) analyzed the graphs depicted in commonly used STEM textbooks and practitioner journals, finding the most common uses of graphs were to mathematize a spatial situation or phenomena (e.g., Figure 1, left) or to represent two covarying quantities (e.g., Figure 1, center). In contrast, the researchers noted most graphs in popular precalculus and calculus textbooks were used to represent two decontextualized quantities (e.g., Figure 1, right). Hence, there are discrepancies between the graphs students experience in their math classes and are expected to interpret in other STEM fields.

Researchers have identified persistent difficulties students experience as they engage in constructing and interpreting graphs in their mathematics (e.g., Leinhardt, Zaslavsky, & Stein, 1990) and science courses (e.g., Potgieter, Harding, & Engelbrecht, 2008). For instance, researchers have identified a range of difficulties including (a) drawing graphs by connecting points without considering what happens between points (Yavuz, 2010); (b) treating graphs as literal representations of a situation (e.g., interpreting a time-speed graph of a biker as the biker's traveled path) (Clement, 1989); and (c) attending to one quantity while ignoring other quantities (i.e. reasoning variationally) (Leinhardt et al., 1990). Collectively, this research indicates instructional approaches have not provided students sustained opportunities to develop meaningful ways of understanding and interpreting graphs.

Dewey (1910) stated, "vagueness disguises the unconscious mixing together of different meanings, and facilitates the substitution of one meaning for another, and covers up the failure to have any precise meaning" (p. 130). In this report, we attempt to clarify a vagueness, in Dewey's terms, regarding how students may think about graphs and coordinate systems. We hypothesize supporting students becoming explicitly aware of the subsequent distinctions may alleviate some difficulties students experience as they engage in graphing activity. In this theoretical report, we

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present a conceptual analysis of different ways students may reason about coordinate systems and graphs. Specifically, we describe two different uses of coordinate systems—spatial and quantitative (Lee & Hardison, 2016)—and two ways students might reason as they construct or interpret graphs within these coordinate systems—static and emergent thinking (Moore & Thompson, 2015). For each use of coordinate system, we characterize how a student may engage in each kind of graphical reasoning, creating a framework that will be useful for future studies.

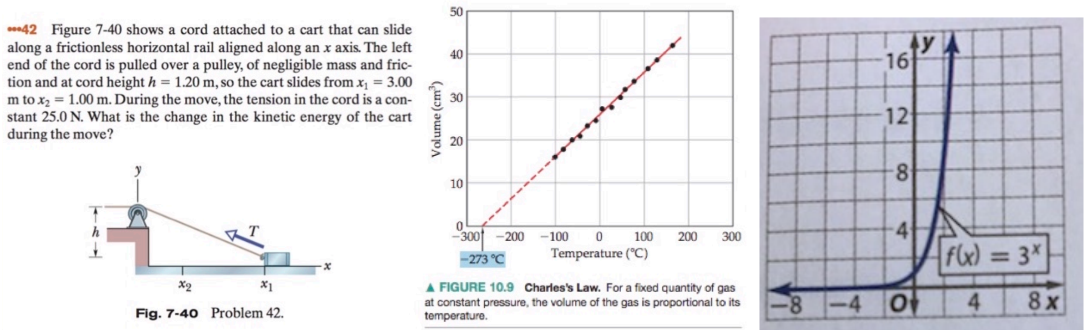

Figure 1. Examples of using coordinate systems to (left) mathematize space (Halliday, Resnick, & Walker, 2011), (center) represent two covarying quantities (Brown et al., 2012), and (right) represent two decontextualized quantities (Glencoe, 2014).

Conceptual Analysis

In this theoretical report, we present a conceptual analysis of ways students may reason about graphs and coordinate systems; we intend for this conceptual analysis to inform future empirical research. Thompson (2008) characterized several uses of conceptual analyses; we leverage two of these uses, namely "describing ways of knowing that might be propitious for students' mathematical learning," and "describing ways of knowing that might be deleterious to students' understanding of important ideas and in describing ways of knowing that might be problematic in specific situations" (p. 46). We present our first-order models (Steffe & Olive, 2010) of how students may reason about graphs and underlying coordinate systems. These models are based on research examining students' graphing understandings (e.g., Moore & Thompson, 2015), research examining students' construction and use of coordinate systems (e.g., Lee, 2017), principles of quantitative reasoning (Thompson, 2011), and our experiences working with students. We point the reader to examples from extant literature in which we infer students are engaging in reasoning compatible with our conceptual analysis.

Bridging Two Theoretical Frameworks

In this report, we examine how a distinction between two uses of coordinate systems and a distinction between two ways of thinking about graphs (i.e., traces within coordinate systems) can create four different ways students may construct or reason about graphical representations within coordinate systems. Below, we describe two frameworks: one for students' understandings of coordinate systems and one for students' thinking about graphs.

Two Uses of Coordinate Systems

Lee and Hardison (2016) found curricular materials often give students rules for "generating" a Cartesian plane and plotting points within it, and these materials rarely address students' conceptions of coordinate systems. They (Lee, 2016; Lee & Hardison, 2016; 2017) have distinguished between two uses of coordinate systems in students' thinking: spatial coordination and quantitative coordination.

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Spatial coordination refers to an individual using a coordinate system to represent or mathematize a space or physical phenomena. This involves establishing spatial frames of reference (Levinson, 2003; Rock, 1992), such as a reference point or orienting vectors, to locate objects within the space or physical phenomena (e.g., Figure 1, left). In this case, an individual can produce quantities by measuring attributes of the space/situation using the established frames of reference and thus coordinate such measurements to locate objects in the space or situation (e.g., a map).

Quantitative coordination refers to an individual coordinating sets of quantities by obtaining a geometrical representation of the product of measure spaces. In this case, the quantities being coordinated are already established and abstracted from the space/situation and superimposed onto some new space. This use of coordinate system, as a result of coordinating quantitative frames of reference (Joshua, Musgrave, Hatfield & Thompson, 2015), allows the individuals to coordinate quantities and construct graphs representing relationships between these quantities (e.g., Figure 1, middle). These graphs are not projections of physical objects or phenomena onto the same space containing the original objects or phenomena.

Static and Emergent Shape Thinking

Moore and Thompson (2015, under review) differentiated between students' static and emergent shape thinking. A student's *static shape thinking* entails his thinking of a displayed graph as a shape (i.e. graph-as-wire) that can possibly be manipulated (e.g., translated, rotated). In such a case, properties of the perceptual shape and the shape itself are the focus of the student's thinking. For instance, a student emphasizing the properties of a shape may argue that a straight line moving up from left-to-right unquestionably represents a linear function with a positive slope even if positive x-values are represented to the left of the y-axis in the Cartesian coordinate system or if the line is represented in the polar coordinate system.

Moore and Thompson also noted static shape thinking can take the form of students' making iconic or thematic associations (e.g., Clement, 1989; Leinhardt et al., 1990). A student reasons iconically when he incorporates visual features of an event in a graph (e.g., drawing a graph resembling a hill because a biker is traversing a hill). A student reasons thematically when he incorporates aspects of a phenomena in his graph that are unnecessary from the researcher's perspective (e.g., an object traveling at a varying speed necessarily implying a curved graph).

In contrast to static shape thinking, Moore and Thompson characterized *emergent thinking* as conceiving a displayed graph simultaneously in terms of "what is made (a trace) and how it is made (covariation)" (2015, p. 785). Critical to such a conception is a student's construction of a point on a graph as a multiplicative object; when using the term multiplicative object, Thompson and colleagues draw on Piaget's notion of "and" as a multiplicative operator. Specifically, Thompson, Hatfield, Yoon, Joshua, and Byerley (2017) noted, "A person forms a multiplicative object from two quantities when she mentally unites their attributes to make a new attribute that is, simultaneously, one and the other" (p. 98). Hence, when reasoning emergently, a student understands a point as simultaneously representing two quantities and imagines a graph being created by the trace of the point as the quantities vary.

A student with sophisticated emergent thinking may, at times, appear to treat a graph as a static shape (e.g., 'shifting' the graph in a direction) for various reasons whilst being able to unpack the static thinking in terms of the graph's emergence. Although such 'shifting' activity could indicate static thinking, if the student is able to unpack his new graph in terms of representing an emergent trace constituted by the new distances, such reasoning would constitute emergent, rather than static, thinking in the quantitative coordinate system.

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Static or Emergent Thinking in Spatial or Quantitative Coordinate Systems

In combining the frameworks elaborated above, we differentiate between static and emergent thinking, which characterizes a student's reasoning when producing or interpreting a graph (i.e., a trace within a coordinate system), and quantitative and spatial coordinate systems, which characterizes a student's understanding of the coordinate system potentially containing a graph. We provide an example of each kind of reasoning using a billiard context at the Infinity Pool Hall. Additionally, we provide a description of student reasoning for each case. We reiterate these are first-order models of how students may reason; we point the reader to empirical examples where appropriate.

Table 1: Four ways of reasoning about graphs within coordinate systems.

		Ways of Reasoning About a Graph (Moore & Thompson, 2015)	
		Emergent Reasoning	Static Reasoning
Uses of Coordinate Systems (Lee & Hardison, 2016)	Spatial Coordination	Case A: Emergent thinking within a spatial coordinate system	Case B: Static thinking within a spatial coordinate system
	Quantitative Coordination	Case C: Emergent thinking within a quantitative coordinate system.	Case D: Static thinking within a quantitative coordinate system

Case A: Emergent thinking in a spatial coordinate system (Emergent, Spatial)

To imagine a student reasoning emergently in a spatial coordinate system, consider a student seeing the red 3-ball moving from the left wall to the top middle pocket in a straight line (Figure 2, top). If asked to describe the location of the red ball throughout its journey toward the middle pocket, the student may establish a spatial frame of reference consisting of a horizontal axis and a vertical axis through which the student can gauge the horizontal and vertical locations of the ball. Using this spatial frame of reference, the student can describe the ball's movement in terms of varying horizontal and vertical distances within a spatial coordinate system. By conceiving of the ball's location as simultaneously composed of vertical and horizontal components, the student conceives of the ball's location as a multiplicative object and therefore can reason emergently about the ball's path in terms of its horizontal and vertical components. Figure 2 (bottom) depicts instances of a student's potential emergent imagery when coordinating the ball's trajectory in relation to its horizontal and vertical components.

There are several features critical to a student reasoning emergently in a spatial coordinate system. First the student must conceive of an object or phenomena happening in a spatial system and imagine the object or phenomena as producing a trace in this space. The student then overlays a coordinate system onto the spatial system in order to explicitly coordinate and/or represent how the object or phenomena is producing the imagined trace in terms of the quantities represented in the coordinate system. Hence, the student must conceive of the object or phenomena as representable by a multiplicative object, which she can decompose as simultaneously representing the orienting quantities in the spatial coordinate system. The student can then explicitly coordinate how the orienting quantities are changing as the object moves or phenomena occurs to mathematize the situation. For an empirical example of a student reasoning emergently in a spatial coordinate system, see Lee (2016).

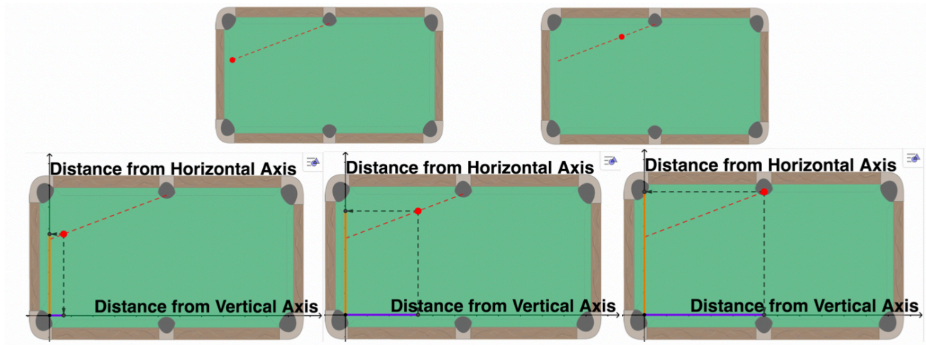

Figure 2. (top) Two instantiations of the red 3-ball traveling from the wall of the table to the pocket and (bottom) three instantiations of how a student may reason about the ball's path as an emergent trace by coordinating its vertical and horizontal components.

Case B: Static thinking in a spatial coordinate system (Static, Spatial)

To imagine a student reasoning statically in a spatial coordinate system, consider the logo on an Infinity Pool Hall table (Figure 3, left). A student sees the logo, which he interprets as composed of two circles tangent at a point (i.e., shapes), and is tasked with describing the shape of this logo mathematically. To do this, the student defines a coordinate system by establishing a spatial frame of reference through which he could describe the location and shape of the logo. This includes choosing a reference point and defining orienting quantities. After constructing the spatial coordinate system, the student may mathematically describe the shapes in the logo within the coordinate system using known equations. For example, in Figure 3 (middle) a student decides to use Cartesian coordinates with the origin at the intersection of the circles, and describes each circle using memorized rules related to the general form of a circle, $(x - h)^2 + (y - k)^2 = r^2$. Alternatively (Figure 3, right), a student might define a polar coordinate system as in Figure 3c, with the pole at the intersection of the circles and use the recalled formula $r = a\cos\theta$.

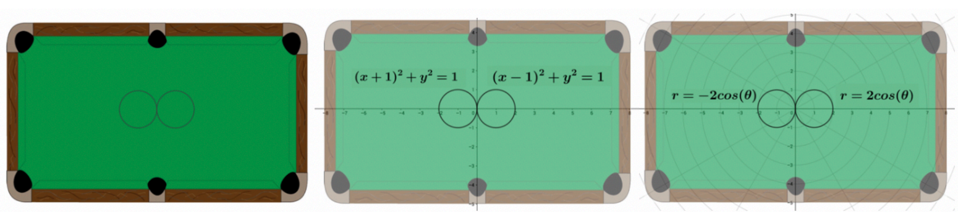

Figure 3. (left) A pool table with logo, and two possible ways to spatially orient the pool table and mathematize the circles in the logo in (middle) Cartesian and (right) polar coordinates.

As exemplified in the above examples, static thinking in a spatial coordinate system entails (a) conceiving of a static shape or object to be located or described mathematically in a situation and (b) establishing a spatial frame of reference through which one can situate, coordinatize, and mathematize the shape. For an example of a teacher using coordinate systems in ways compatible with this description in their classrooms, see Disher (1995).

In this example, we emphasize a student using memorized rules to mathematize the conceived shape, rather than reasoning about an emergent trace. We note that a student *could* demonstrate an understanding of analytic rules as representing an emergent trace of covarying quantities; in such a case, the student would not necessarily be reasoning statically when representing the 'shape' via an analytic rule if the shape's emergence is implicit in his understanding.

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Case C: Emergent thinking in a quantitative coordinate system (Emergent, Quantitative)

To imagine a student thinking emergently in a quantitative coordinate system, consider a student seeing the red 3-ball's path, now with the yellow 1-ball and blue 2-ball on the table (Figure 4, we positioned the balls and trajectory to mimic the cities and path of the car in Saldanha and Thompson's (1998) Car Problem). The student may be asked to create a graph representing the red 3-ball's distance from the yellow 1-ball and blue 2-ball as it moves to the pocket. After conceiving the quantities and how they vary, the student may construct a Cartesian coordinate system with the horizontal and vertical axes representing the red 3-ball's distances from the blue 2-ball and yellow 1-ball, respectively. Having constructed a quantitative coordinate system, the student may construct an emergent trace within this coordinate system to represent the relationship between the 3-ball's distance from the other balls (e.g., Figure 4a–d).

We highlight several critical features in this example. First a student needs to conceive of two quantities covarying and intend to represent the relationship between the quantities. The student must then disembed (Steffe & Olive, 2010) the two quantities from the context and insert them onto two number lines to construct a quantitative coordinate system with the intention of simultaneously representing the two quantities in a product space. Each quantity would then be represented by one of the orienting quantities in the coordinate system (see Moore, Paoletti, and Musgrave (2013) for examples in Cartesian and polar coordinate systems). The student understands a point on the emergent trace in this quantitative coordinate system as a multiplicative object simultaneously representing the two quantities.

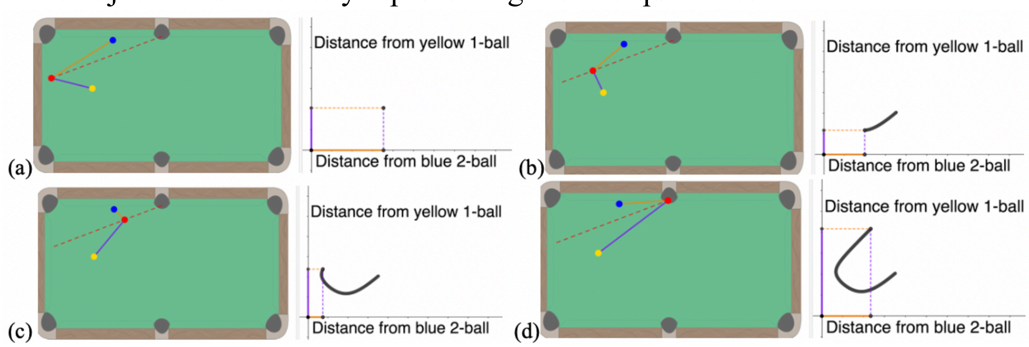

Figure 4. Four instantiations of how an individual may reason about and represent the red 3-ball's distance from the blue 1-ball and yellow 2-ball as an emergent trace.

Case D: Static thinking in a quantitative coordinate system (Static, Quantitative)

We modify the same situation described in Case C to characterize how a student may reason statically while using a quantitative coordinate system. Recall, static thinking may entail graphically representing features of the situation. For instance, Figure 5 (center) shows a student duplicating the 3-ball's path as a representation of the ball's distance from the other two balls, and Figure 5 (right) shows a student using the balls' and pocket's relative locations as points on his graph. In both cases, we highlight that although the coordinate system is meant to represent (from the researcher's perspective) the 3-ball's distance from the 1-ball and 2-ball, the student may not explicitly use the coordinate system in this way. One indication that a student is interpreting the coordinate system quantitatively is if the student interprets her constructed graph by describing the 3-ball's relative distance from the other two balls. For instance, in Figure 5 (center), the student may describe that the 3-ball's distance from the 1-ball increases at a constant rate with respect to its distance from the 2-ball (see Paoletti, 2015 for an empirical example).

Reasoning statically in a quantitative coordinate system can be unproductive in several ways: (a) treating a graph in a quantitative coordinate system as a shape which can be moved around the coordinate system without considering how the translation relates the underlying relationship between quantities, (b) engaging in an iconic translation while interpreting or creating a graph in a quantitative coordinate system, or (c) engaging in a thematic association while interpreting or creating a graph in a quantitative coordinate system. In each case, in order for the system to be a quantitative coordinate system *from the student's perspective*, the student needs to make some indication that he is representing the quantities defined on the coordinate axes.

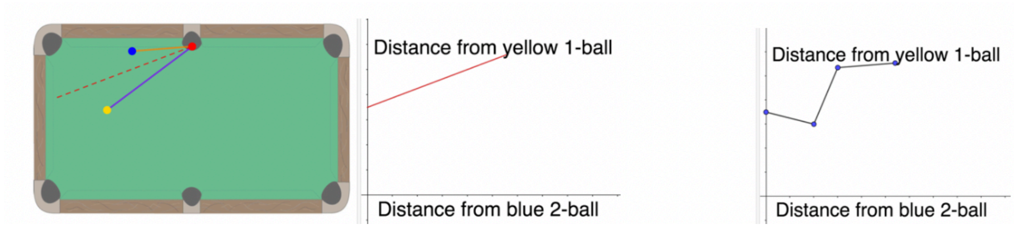

Figure 5. Two examples of static thinking in a quantitative coordinate system.

Concluding Remarks and Areas for Future Research

In this report, we have presented a conceptual analysis that distinguishes meanings students may hold as they engage in graphing activity within coordinate systems. These distinctions can provide insights into some difficulties students experience as reported in extant literature. For instance, students representing the path of a biker (Clement, 1989) may be indicative, not of a misconception, but of the students leveraging a spatial coordinate system to mathematize a situation. In this case, and others, misconceptions identified in the literature may be partially explainable by students' reasoning about coordinate systems and graphs in ways inconsistent with what the researcher (or teacher) intended.

If we intend for curriculum designers, teachers, and students to maintain and convey particular meanings for graphs within coordinate systems, mathematics education researchers must be explicit about these meanings. We intend for the hypothetical models we elaborated, which explicitly address students' meanings for both coordinate systems and graphs, to serve as a resource for future research into how students may develop understandings of spatial and quantitative coordinate systems, as well as graphs in these systems. Further, the similarity between the two uses of coordinate systems discussed here and the uses of coordinate systems observed by Paoletti et al. (2016) in STEM resources (e.g., Figure 1, left, middle) underscores the importance of students explicitly understanding and using coordinate systems for each purpose. Hence, there is a need to develop and test curricular materials that support students in explicitly understanding the differences between spatial and quantitative coordinate systems, as such differences are relevant for their potential future STEM studies and careers.

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