PERSPECTIVES ON THE NATURE OF MATHEMATICS AND RESEARCH

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When people address early mathematics education, commonly they write or reference policies, standards, "scope and sequences" and curriculum, or documents on instructional strategies. These are important; however, we believe that the core consideration should be the nature of mathematics and the development of mathematics *in children*.

An 80-year-old incident illustrates what we mean. A mother attending parents' night asked Fawcett (1938), "How is Willie doing in mathematics?" Fawcett replied: "Madam, you ask the wrong question. You should ask, 'How is mathematics doing in Willie?'"

This is what we mean by the "nature of mathematics and the development of mathematics in children": The mathematics that *does well in Willie* and all other children. We develop this position by describing *learning trajectories* and our theoretical framework for them, *Hierarchic Interactionalism*.

Learning Trajectories: Construct and Theory

Learning trajectories are a device whose purpose is to support the research-grounded development of a curriculum or other unit of instruction, as well as to conduct rigorous research in learning and teaching. The term "curriculum" stems from the Latin word for race course, referring to the course of experiences through which children grow. Thus, the notion of a path, or trajectory, has always been central to curriculum development and study. Simon stated that a "hypothetical learning trajectory" included "the learning goal, the learning activities, and the thinking and learning in which the students might engage" (Simon, 1995, p. 133). Building on Simon's definition, emphasizing a cognitive science perspective and a base of empirical research, "we conceptualize learning trajectories as descriptions of children's thinking and learning in a specific mathematical domain, and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain" (Clements & Sarama, 2004, p. 83).

The name "learning trajectory" reflects its roots in a constructivist perspective. That is, although the name emphasizes learning over teaching, both these definitions clearly involve teaching and instructional tasks. Some appropriations of the learning trajectory construct emphasize only the "developmental progressions." Although studying either psychological developmental progressions or instructional sequences separately can be valid research goals, and studies of each can and should inform mathematics education, we believe the power and uniqueness of the learning trajectories construct stems from the inextricable interconnection between these all three components, goal, developmental progression, and correlated instructional tasks.

Our learning trajectories base *goals* on both the expertise of mathematicians and research on students' thinking about and learning of mathematics (Clements, Sarama, & DiBiase, 2004; Fuson, 2004; National Governor's Association Center for Best Practices & Council of Chief State School Officers, 2010; Sarama & Clements, 2009). This results in goals that are organized into the "big" or "focal" ideas of mathematics: overarching clusters and concepts and skills that

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are mathematically central and coherent, consistent with students' (often intuitive) thinking, and generative of future learning. Our goals also include productive dispositions, including, curiosity, imagination, inventiveness, risk-taking, creativity, and persistence (National Research Council, 2001). With that in mind, we turn to the question of *how* children think about and learn mathematics.

Research is reviewed to determine if there is a natural developmental progression (at least for a given age range of students in a particular culture) identified in theoretically- and empirically-grounded models of children's thinking, learning, and development (Carpenter & Moser, 1984; Griffin & Case, 1997). That is, researchers build a cognitive model of students' learning that is sufficiently explicit to describe the processes involved in the construction of the mathematical goal across several qualitatively distinct structural levels of increasing sophistication, complexity, abstraction, power, and generality.

The issue of what is meant by a *natural* developmental progression is sure to arise. We believe the research supports a synthesis of aspects of previous theoretical frameworks that we call *Hierarchic Interactionalism* (for a full explication, see Sarama & Clements, 2009). The term indicates the influence and interaction of global and local (domain specific) cognitive levels and the interactions of innate competencies, internal resources, and experience (e.g., cultural tools and teaching). Mathematical ideas are represented intuitively, then with language, then metacognitively, with the last indicating that the child possesses an understanding of the topic and can access and operate on those understandings. The tenets of Hierarchic Interactionalism therefore lay the foundation for the creation of both the developmental progression and instructional tasks of research-based learning trajectories.

1. *Developmental progression.* Most content knowledge is acquired along developmental progressions of levels of thinking. These progressions play a special role in children's cognition and learning because they are particularly consistent with children's intuitive knowledge and patterns of thinking and learning at various levels of development,
2. *Domain specific progression.* These developmental progressions often are most propitiously characterized within a specific mathematical domain or topic. Children's knowledge, that is, the objects and actions they have developed in that domain, are the *main determinant of the thinking within each progression, although hierarchic interactions occur at multiple* levels within and between topics, as well as with general cognitive processes (e.g., executive, or metacognitive processes, potentialities for general reasoning and learning-to-learn skills, and some other domain general developmental processes). See Figure 1 for an illustration.
3. *Hierarchic development.* Development is less about the emergence of entirely new processes and products and more an interactive interplay among specific existing components of knowledge and processes. Also, each level builds hierarchically on the concepts and processes of the previous levels. The learning process is more often incremental and gradually integrative than intermittent and tumultuous. A critical mass of ideas from each level must be constructed before thinking characteristic of the subsequent level becomes ascendant in the child's thinking and behavior. Successful application leads to the increasing use of a particular level. However, under conditions of increased task complexity, stress, or failure this probability level decreases and an earlier level serves as a fallback position.
4. *Co-mutual development of concepts and skills.* Concepts constrain procedures, and concepts and skills develop in constant interaction.

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5. *Initial bootstraps.* Children have important, but often inchoate, premathematical and general cognitive competencies and predispositions at birth or soon thereafter that support and constrain, but do not absolutely direct, subsequent development of mathematics knowledge.
6. *Different developmental courses.* Different developmental courses are possible within those constraints, depending on individual, environmental, and social confluences.
7. *Progressive hierarchization*. Within and across developmental progressions, children gradually make connections between various mathematically-relevant concepts and procedures, weaving ever more robust understandings that are hierarchical in that they employ generalizations while maintaining differentiations.
8. *Consistency of developmental progressions and instruction.* Instruction based on learning consistent with natural developmental progressions is more effective, efficient, and generative for the child than learning that does not follow these paths.
9. *Learning trajectories.* A particularly fruitful instructional approach is based on hypothetical learning trajectories. Curriculum developers design instructional tasks that include external objects and actions that mirror the hypothesized mathematical activity of children as closely as possible. These tasks are sequenced, with each corresponding to a level of the developmental progressions, to complete the hypothesized learning trajectory. *Specific learning trajectories are the main bridge that connects the "grand theory" of hierarchic interactionalism to particular theories and educational practice.*
10. *Instantiation of hypothetical learning trajectories.* Hypothetical learning trajectories must be interpreted by teachers and are only realized through the social interaction of teachers and children around instructional tasks.

For example, consider one *goal* regarded as important in all standards documents: young children should learn to be competent in whole number, including meaningful verbal and object counting and the application of counting to solve a variety of arithmetic problem types. The *developmental progressions* for each of these learning trajectories are sampled in the left column of Figure 1. The second column provides an example of children's behavior and thinking for each level. The third column presents an example of an instructional task designed to catalyze that level of thinking.

In summary, learning trajectories describe the goals of learning, the developmental progression through which children pass, and the learning activities in which students might engage. The *source* of the developmental progressions— the thinking and learning processes of children at various levels—are extensive research reviews and empirical work that cannot be presented here due to space constraints. Also beyond the scope of this chapter are the complex, cognitive actions-on-objects that underlie the LTs (see Sarama & Clements, 2009). Here we will provide one illustration of both cognitive actions-on-objects and how different trajectories grow not in isolation, but interactively.

Consider learning a critical competence—counting on, used especially at the *Counting Strategies* level in Figure 1b. Children need to develop competencies from three trajectories: counting (Fig. 1a), subitizing (not shown, but see Clements & Sarama, 2009; Sarama & Clements, 2009), and the addition and subtraction trajectory (Fig. 1b) to learn to count on meaningfully. From the counting trajectory, they learn to count forward from any number. Then they learn to understand explicitly and apply the idea that each number in the counting sequence includes *the number before, hierarchically*. That is, 5 includes 4, which includes 3, and so forth. From the subitizing trajectory they quickly learn to recognize the number of—not just visual

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sets, but also *rhythmic patterns.* From the addition and subtraction trajectory, children learn to interpret situations mathematically, such as interpreting a real-world problem as a "part-part-whole" situation. They also learn to use counting to determine what is missing. The creative combination of these developments allows them to solve *meaningfully* problems such as, "You have three green candies and six orange candies. How many candies do you have in all?" by counting on. They understand that these numbers are two parts and that they need to find the whole. They also understand that the order of numbers does not matter in addition. They know, in practice, that the sum is the number that results by, starting at the first number and counting on a number of iterations, equal to the second number. They can use counting to solve this, starting by saying "siiiiix…" because they understand that word can stand for the counting acts from 1 to 6 (because 6 includes 5…). They know *how many more* to count because they use the subitized "rhythm of three" "Du de **Du" ("Doo – Day – Doo")** "seven (du…), eight (day…), nine (*du*)—*nine!*"

Consider Justin, who participated in the successful scale-up of the learning-trajectories-based Building Blocks curriculum (Clements, Sarama, Spitler, Lange, & Wolfe, 2011). At pretest, he operated at the Reciter level of counting, as he verbally counted correctly but when counting toy bananas, broke one-to-one correspondence as he counted a space between the bananas. He did not solve any arithmetic problems. After 7 months moving through the learning trajectories for counting, subitizing, and the counting-based addition and subtraction trajectories (among others), Jason showed remarkable growth. He counted up to 30 randomly-arranged objects accurately and could verbally count up or down from any number in that range. In arithmetic, he solved a variety of problems. For "…you have 3 candies and I gave you 2 more; how many do you have?" Justin put out 3 fingers, then 2 more, and then said, "Five. I was just counting but no words" (i.e., he didn't count out loud). Later, , shown 6 blocks, which were then covered with a cloth, and 4 secretly removed, leaving 2, he said "Two. There were six." "So, how many am I hiding?" Justin quickly counted the two and then counted, pointing to the table and said, "Four." These solutions suggest he was now operating at the Counting Strategies level of arithmetic.

a. Counting

Developmental Progression	Example Behavior	Instructional Tasks
Reciter Verbally counts with separate words, not necessarily in the correct order.	Count for me. "one, two, three, four, six, seven."	Provide repeated, frequent experience with the counting sequence in varied contexts.
Corresponder Keeps one-to-one correspondence between counting words and objects (one word for each object), at least for small groups of objects placed in a line.	Counts: ☆ ☆ ☆ ☆ "1, 2, 3, 4" But answers the question, "How many?" by re-counting the objects or naming any number word.	*Kitchen Counter* Students click on objects one at a time while the numbers from one to ten are counted aloud. 

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Counter (Small Numbers) Accurately counts objects in a line to 5 and answers the “how many” question with the last number counted.	Can you count these? ☆ ☆ ☆ ☆ “1, 2, 3, 4… four!”	*How Many?* Tell students you have placed as many cubes (3, hidden) in your hand as you can hold. Ask them to count with you to see how many. Take out one at a time as you say the number word (so, when they say “two” they *see* two). Repeat the last counting number, “three,” gesturing in a circular motion to all the cubes, and say “That’s how many there are in all.”
Counter and Producer (10+) Counts and counts out objects accurately to 10, then beyond.	Counts a scattered group of 19 chips, keeping track by moving each one as they are counted.	*Road Race* Board game.
Counter from N (N+1, N-1) Counts verbally and with objects from numbers other than 1 (but does not yet keep track of the *number* of counts).	Asked to “count from 5 to 8,” counts: “5, 6, 7, 8!” Determines numbers just after or just before immediately.	*One more!.* Have the children count two objects. Add one and ask, “How many now?” Have children count on to answer. Add another and so on, until they have counted to ten.

b. Arithmetic

Developmental Progression	Example Behavior	Instructional Tasks
Small Number +/- Finds sums for joining problems up to 3 + 2 by counting-all with objects.	Asked, “You have 2 balls and get 1 more. How many in all?” counts out 2, then counts out 1 more, then counts all 3: “1, 2, 3, 3!.”	*Finger Word Problems* Tell children to solve simple addition problems with their fingers.
Find Result +/- Finds sums *by direct modeling, counting-all, with objects.*	Asked, “You have 2 red balls and 3 blue balls. How many in all?” counts out 2 red, then counts out 3 blue, then counts all	*Places Scenes (Addition)—Part-part-whole, whole unknown problems.* Children play with toy on a background scene and combine groups.

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	5.	
Counting Strategies +/- Finds sums for joining (you had 8 apples and get 3 more…) and part-part-whole (6 girls and 5 boys…) problems with finger patterns and/or by counting on.	Counting-on. "How much is 4 and 3 more?" "Fourrrrr…five, six, seven [uses rhythmic or finger pattern to keep track]. Seven!"	*How Many Now?* Have the children count objects as you place them in a box. Ask, "How many are in the box now?" Add one, repeating the question, then check the children's responses by counting all the objects. Repeat, checking occasionally.

Figure 1. Selected Levels/Descriptions from the Learning Trajectories for Counting and counting-based Arithmetic (these and other figures adapted from Clements & Sarama, 2014)

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