PRESERVICE SECONDARY MATHEMATICS TEACHERS' PERCEPTIONS OF PROOF IN THE SECONDARY MATHEMATICS CLASSROOM

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Despite the recognized importance of mathematical proof in secondary education, there is a limited but growing body of literature indicating how preservice secondary mathematics teachers (PSMTs) view proof and the teaching of proof. The purpose of this survey research was to investigate how PSMTs in Australia and the United States perceive of proof in the context of secondary mathematics teaching and learning. PSMTs were able to outline various mathematical and pedagogical aspects of proof, including: purposes, characteristics, reasons for teaching, and imposed constraints. In addition, PSMTs attended to differing, though overlapping, features of proof when asked to determine the extent to which proposed arguments constituted proofs or to decide which arguments they might present to students.

Keywords: Reasoning and Proof, Teacher Education-Preservice, Mathematical Knowledge for Teaching, High School Education

Introduction

The importance of proof in the discipline of mathematics, and subsequently in the teaching and learning of mathematics, is recognized in the literature worldwide (e.g., Hanna & de Villiers, 2008; Stylianou, Blanton, & Knuth, 2009), and is echoed in recent policy documents and national curriculum (e.g., ACARA, 2017; Common Core State Standards Initiative, 2010). This importance largely rests on the notion that engaging in proof helps students reason about mathematical ideas as they critique arguments or construct their own logically sound explanations or justifications. In fact, both the Australian and United States mathematical proficiencies and practices reference the importance of the reasoning abilities of students, with emphases placed on constructing, justifying, and communicating arguments. However, research has revealed that secondary mathematics teachers often hold a limited view on the purpose of proof instruction and its appropriateness for all students (Bergqvist, 2005; Knuth, 2002). Specifically, teachers often relegate proof to verifying formulas in high school geometry, neglecting the explanatory role proof can play in the learning of mathematics at all levels (Hanna, 2000; Knuth, 2002). Moreover, teachers often focus on the structure rather than substance of a proof and have difficulty evaluating proofs presented pictorially (Dickerson & Doerr, 2014; Tsamir, Tirosh, Dreyfus, Barkai, & Tabach, 2009).

A recent study by Dickerson and Doerr (2014) suggested that teachers' ability to accept and promote less formal modes of proof representation (e.g., visual & concrete models) develops over time, with more veteran teachers less likely to impose strict standards for mathematical language and format. While both experienced and veteran teachers in this study appreciated the explanatory role of proof (Hanna, 2000), espousing proof as a vehicle to build student understanding of specific mathematics content and generalized thinking strategies, they differed considerably in terms of the value they afforded to explicit logic, detail, and precise language in writing proofs (Dickerson & Doerr, 2014). Views of proof as a formalistic mechanism are magnified for prospective teachers (Boyle, Bleiler, Yee, & Ko, 2015; Varghese, 2009) whose most recent experiences with proof are often in the context of upper-level mathematics courses. For example, in a study by Varghese (2009), secondary level student teachers expressed uneasiness with respect to teaching proof, with many indicating that proof should only be introduced to students planning to study advanced mathematics. Without a more complete understanding of the nature of proof and its role in learning mathematics, it is unlikely that beginning teachers will be equipped to enact proof instruction for all students as envisioned by mathematics education policies.

Purpose of the Study

The chief aim of our study was to investigate how preservice secondary mathematics teachers (PSMTs) in Australia and the United States perceive the importance and purpose of proof in the context of secondary mathematics teaching and learning. The overarching research question guiding this study was: *What are preservice, secondary mathematics teachers' conceptions of proof and proof teaching in a secondary classroom context?*

Theoretical Perspective

Our study is grounded in a situative perspective that takes into account what, when, and how mathematical knowledge is required in various practices such as teaching (Adler & Davis, 2006; Putnam & Borko, 2000). Specific to our study, Knuth (2002a, 2002b) demonstrated how teachers' conceptions of proof in the discipline of mathematics were sometimes at odds with their views on the role of proof in mathematics teaching and learning. This perspective also underlies the various frameworks that have been developed to delineate mathematical knowledge that would support the work of teaching proof (e.g. Lesseig, 2016; Steele & Rogers, 2012). For instance, Lesseig's (2016) Mathematical Knowledge for Teaching Proof (MKT for Proof) framework describes Common and Specialized Content Knowledge related to constructing and understanding proof, explicit knowledge of proof components, and the functions of proof that would support teachers' classroom work with proof. This particular framework provided a lens through which to analyze our data and grounded our subsequent interpretations of PSMTs' conceptions of proof and proof evaluations.

Methods

Participants and Contexts

The purpose of this study was to investigate PSMTs' conceptions of proof in the context of secondary mathematics teaching and learning. Data for this paper come from the results of a survey distributed to students enrolled in secondary mathematics teacher preparation programs across 6 different universities in the United States and Australia. Twenty-two PSMTs completed the survey, with exactly half of the participants from each country. The 11 PSMTs from Australia were all enrolled at the same university whereas the 11 PSMTs from the United States were split amongst five different universities. Half of the participants were in their first year of their preparation program, and the other half of the participants were in their second to third year of the program. All PSMTs had taken at least 3 college-level mathematics courses, with the majority of PSMTs (13) having taken 10 or more college-level mathematics courses. Exactly half had taken a course focusing on proof.

Data Collection

The survey was created in Qualtrics and was completed electronically. The three-part survey was qualitative in nature, with the majority of questions being open-ended.

Part I. The first part of the survey focused on PSMTs' conceptions of the nature and role of proof in mathematics and in teaching mathematics. This section included four open-ended questions: (1) What purpose(s) does proof serve in mathematics? (2) What makes an argument a

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proof? (3) If proofs are to be taught to students, what are your reasons for teaching proof? and (4) What will be the constraints, if any, on teaching proofs?

Part II. The second part of the survey focused on what secondary preservice teachers attend to when evaluating whether or not an argument is a proof. For five different statements, PSMTs evaluated between one and three student-generated arguments, decided whether or not each argument constituted a proof, and provided an explanation to support their decision. The five statements were drawn from the content areas of geometry, algebra, elementary number theory, and infinite geometric series. Student-generated arguments varied in clarity, generality, and approach. The survey purposefully included both symbolic and visual representations as well as deductive and empirical modes of argumentation.

Part III. The third part of the survey focused on what PSMTs attend to when deciding what kinds of arguments are most helpful for a group of students working on mathematical ideas underlying each of the five statements from Part II.. PSMTs were prompted to provide an explanation to support their selection of argument(s) they would share with students.

Analysis

Part I Coding. Once all survey responses were collected, the first two authors separately read through the Part I responses provided by PSMTs from their respective countries. Initial codes were drawn from the literature. For example, the codes VERIFY, EXPLAIN, SYSTEMATIZE, DISCOVER, and COMMUNICATE, adapted from de Villiers (1990), provided an initial lens to analyze questions 1 and 3. In addition, codes for question 2 such as LOGIC, THEOREM, and GENERAL were determined *a priori*, as they captured essential proof understandings one would expect from those familiar with mathematics, including future teachers (Lesseig, 2016). After the first pass of coding, researchers met virtually to discuss other themes that arose throughout the analysis and refined codes to incorporate these additional themes as well as to remove themes that were not applicable to certain questions. On the second pass of Part I, the first two authors analyzed the data from both countries using the agreed-upon codes. Once completed, they met virtually to discuss similarities and differences in their analyses and came to consensus on codes for each PSMT response to each question. The inter-rater reliability (IRR) was 85% for Part I and was calculated as the number of PSMT responses for which there was initial agreement on one or more codes (as more than one code could be used per response), divided by the total number of PSMT responses. A total of 60 responses were coded for Part I.

Parts II and III Coding. The codes used for the open-ended responses in Parts II and III were discussed and decided upon after researchers had read through all PSMT responses from their respective country. The first two authors then separately coded responses from both countries. After separately coding all data, the researchers discussed responses that did not initially have full agreement and came to consensus. The IRR was calculated at 83% for Part II, and 86% for Part III. There was a total of 134 responses coded for Part II, and a total of 50 responses coded for Part III.

Results

Findings for each of the three parts of the survey are presented below. For each of the survey parts, a table of researcher-generated codes has been provided to indicate the five most common responses offered by each cohort (see Tables 1-6). Overall, responses from both Australian and US cohorts appeared strikingly similar with little variation.

Part I – Conceptions of Proof and Proof Teaching

What purpose(s) does proof serve in mathematics? Participants indicated that proof establishes an axiomatic system to formalize mathematical knowledge, provides verification that a mathematical statement is true (or false), and helps to explain why a statement is true or makes sense. In this way, the notion of proof acting as a 'failsafe' during the teaching and learning of mathematics underpinned many statements regarding the purpose of proof. This response from Grant (Aus) describes the role proof plays in verification and systemization, "(proof serves) to establish fact, it provides the means where fundamental truths can be established which then provide the foundations for further understanding to be built."

Table 1: The purpose of proof in mathematics (PSMTs' conceptions)

Code	Code Description	PSMT (AUS)	PSMT (USA)	PSMT (Total)
SYSTEM	To establish an axiomatic system, referring to the formalization of mathematical knowledge	4	5	9
VERIFY	To provide verification that a statement is true or false	5	2	7
EXPLAIN	To explain why a statement is true or makes sense	2	4	6
BUILD-U	To build or deepen understanding of the mathematical concepts underlying the proof	1	2	3
COMM	To communicate mathematics	0	2	2

What makes an argument a proof? For the second survey question, participant responses aligned closely with the criteria identified as *Essential Proof Understandings* in Lesseig's (2016) MKT for Proof framework. Those criteria stipulate that a theorem has no exceptions, a proof must be general, a proof is based on previously established truths, and the validity of a proof depends on its logic structure. The most frequently elicited responses were that a proof is based on accepted statements, follows a logical structure, and removes any doubt about the veracity (i.e. truth or falsehood) of the statement.

Table 2: What makes an argument a proof? (PSMTs' conceptions)

Code	Code Description	PSMT (AUS)	PSMT (USA)	PSMT (Total)
THEOREM	It is based on accepted statements or theorems	2	7	9
LOGIC	It follows a logical structure	3	3	6
INFALLIBLE	It removes any doubt about the truth or falsehood of the statement	2	3	5
GENERAL	It proves the statement in general by covering all cases within the domain	2	2	4
VERIFY	It provides verification that a statement is true or false	2	0	2

If proofs are to be taught to students, what are your reasons for teaching proof? According to the proffered responses for survey question three, participants feel that proof should be taught to students to impart a variety of mathematical skills (e.g. logical reasoning), to build an understanding of the mathematical concepts underlying the proof, and to increase student agency. Walter's (US) response illustrates ways in which PSMTs' saw proof as valuable in promoting reasoning and argumentation skills that were useful beyond mathematics as well as a vehicle for both building and assessing understanding:

> The ability to convey reasoning and to justify in writing is a vital skill in the real world. In any job, you have to persuade others (often through proving that your way is right). Proofs also press for deep understanding and makes students' thinking visible to the teacher.

Table 3: Reasons for teaching proofs to students (PSMTs' conceptions)

Code	Code Description	PSMT (AUS)	PSMT (USA)	PSMT (Total)
T-SKILLS	To teach skills in logical reasoning, argumentation and problem solving	5	5	10
BUILD-U	To build or deepen understanding of the mathematical concepts underlying the proof	3	3	6
S-AGENCY	To increase or build student agency	2	4	6
DISC	To discover or explore an idea or create new knowledge	0	3	3
ASSESS-U	To assess student understanding	0	2	2

What will be the constraints, if any, on teaching proofs? Most respondents tended to assert that proof is a difficult topic, skill, or body of knowledge to teach. PSMTs' responses reflected their perceptions of the complexity of proof by way of questioning students' prior knowledge or ability to deal with the abstract nature of proof. PSMTs identified additional resource constraints such as a lack of time, curricular emphasis, or adequate preparation for teaching proof.

Table 4: The constraints on teaching proofs (PSMTs' conceptions)

Code	Code Description	PSMT (AUS)	PSMT (USA)	PSMT (Total)
S-KNOW	Students may not have the requisite mathematical content knowledge or skills for proof	3	4	7
TIME	Takes too much time to teach properly	2	4	6
CURRIC	There is not enough clear direction in standards or curriculum	3	3	6
ABSTRACT	Proof is too abstract for this age group	0	3	3
T-KNOW	Teachers may not have the knowledge and skills necessary to teach proof	1	2	3

Part II - Proof Evaluation

The purpose of Part II in the survey was to determine the extent to which PSMTs felt that proposed arguments constituted a proof, and ascertain the features of proof to which PSMTs attended. Participants attended to a variety of features in describing why they felt a proposed argument constituted a proof or identifying what they felt was missing in the argument. The most common rationales provided were in relation to whether (or not) the arguments proved the statement in general (i.e. they covered all cases within the domain), were based on accepted statements or theorems, and followed a logical structure, aligning with the top five characteristics of proof outlined in Part I of the survey. To a lesser degree, participants argued (in the

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affirmative or the negative) that the proofs were error-free and were easy to follow or understand. Table 5 below displays the counts for these top five characteristics evident in responses from US and Australian participants. Counts for two additional characteristics frequently cited in Part III are included for comparison.

Table 5: Characteristics PSMTs attend to when evaluating a proof.

	General	Logic	Theorem	Correct	Clear	Representatio	Accessible
AUS	23	20	9	15	8	4	0
US	26	17	16	9	10	6	0
TOTAL	49	37	25	24	18	10	0
	25.1%	19%	12.8%	12.4%	9.2%	5.1%	0%

*Percentages in Tables 5 and 6 are calculated based on total number of codes applied across responses. There were a total of 195 codes applied to the proof evaluation questions (Table 5) and 90 total codes in response to why an argument would be helpful (Table 6). Also, the last two columns have been included to facilitate comparison with the five most popular responses in Table 6.

Part III - Identification of proof features

Our intent in Part III of the survey was to identify features of proof that PSTs attend to when evaluating arguments or deciding which arguments to present in a classroom. In an identical manner to Part II, we assigned codes to responses regardless of whether PSTs referenced a particular feature in a positive or negative manner.

When deciding which arguments they might present to a group of students, PSMTs not only valued arguments that were correct and general (characteristics mentioned in Part II), but also considered pedagogical features such as whether the proof was accessible to students and how different representations might support student learning. As shown in Table 6, the mode of representation and clarity of the argument were the two most commonly mentioned characteristics across all four questions (20% & 18.9% respectively), yet when compared with findings from Part II, logic and theorem were not commonly espoused features. Several PSMTs attended to how accessible the argument might be for different levels of students.

Table 6: Helpful characteristics of proofs for students (PSMTs' perceptions)

	Representation	Clear	Correct	Accessible	General	Logic	Theorem
AUS	7	9	4	4	7	4	1
US	11	8	5	5	0	0	1
TOTAL	18	17	9	9	7	4	2
	20%	18.9%	10%	10%	7.8%	4.4%	2.2%

Discussion and Conclusion

Overall survey responses reveal that many PSMTs still hold rather formalistic views of proof. This is perhaps not surprising given participants' current status in teacher education programs. Participants' most recent experiences with proof have been in the context of taking pure mathematics courses wherein opportunities to consider proof from a teacher perspective are uncommon. However, we claim that to carry out the work of mathematics teaching wherein reasoning and proof remains central, PSMTs need to consider pedagogical aspects of proof. Specifically, our findings highlight the need to enhance PSMTs' views of the role of proof and extend their understanding and acceptance of other forms of proof that may be more accessible to secondary students.

PSMTs from both Australia and the US cited systematization and verification as the primary roles of proof. Proof was most commonly described as a mechanism for establishing truths in mathematics and demonstrating that a result is true beyond a reasonable doubt. Previous researchers (e.g., Knuth, 2002; Varghese, 2009) have discussed how this dominant view has the potential to limit teachers' perspectives on when and how proof should be utilized in secondary mathematics. However, in our case we see this not necessarily as a deficit, but as a common starting point. Indeed, a number of PSMTs also acknowledged the explanatory role of proof (Hanna, 2000) and recognized proving as a venue for students to both build and communicate understanding. Coupling verification with explanation, and explicitly discussing these complementary roles of proof in preservice teacher education (Bleiler-Baxter & Pair, 2017) has the potential to expand PSMTs' view of proof and their ability to integrate proof more consistently into their future teaching.

In light of our theoretical perspective, we noted that while many of PSMTs' responses were consistent across survey items, others appeared somewhat inconsistent. For example, when asked to identify necessary components of a proof, PSMTs stated in Part I that proofs should be based on established facts or theorems and follow a logical progression, and subsequently included statements about the logical structure or use of theorems in their rationales for the acceptance (or not) of the student-generated arguments as proof. On the contrary, despite the fact that only 20% of PSMTs explicitly mentioned that a proof must be general, generality was the characteristic attended to most often when PSMTs were asked to evaluate the student-generated arguments from Part II. Indeed, in their evaluations, PSMTs demonstrated a robust understanding of proof from a mathematics perspective, and made explicit statements that examples do not constitute proof. This understanding is particularly notable in light of prior research documenting students and preservice teachers' acceptance of empirical arguments as proof (Harel & Sowder, 2007; Simon & Blume, 1996). In addition, the characteristics PSMTs attended to when evaluating arguments in Part II did not necessarily map directly onto characteristics PSMTs used when deciding which proofs they might present to students in Part III, wherein the use of multiple representations that might be accessible to a range of students was valued. Differences across survey sections highlight the importance of considering PSMTs' actions across multiple settings. From a research perspective, these findings suggest that measures of PSMTs' proof conceptions should be situated in tasks of teaching (Steele & Rogers, 2012).

Finally, we note that merely knowing that a proof must be general was not necessarily sufficient, as PSMTs had different interpretations of what constituted generality. This finding was most evident in their assessment of two "non-traditional" arguments included in the survey, one a visual argument, the other a generic example proof (Karunakaran, Freeburn, Konuk, & Arbaugh, 2014; Mason & Pimm, 1984) for a claim about divisibility. Given their accessibility to students, visual arguments and generic example proofs have the potential to 'bridge the gap' from empirical toward more deductive modes of argumentation (Karunakaran et al., 2014). Thus, we contend that increasing PSMTs' proficiency with constructing and assessing generic example proofs and visual arguments is an important step toward enhancing the role of proof in secondary mathematics classrooms. It is worth noting that the invitation to participate in this study, which resulted in a convenience sample from participants from multiple universities in the US and one university in Australia, is a limitation in this study. Although results from students across universities and countries were similar, the different requirements for the programs at the different universities could have affected PSMTs' perspectives. A future direction could involve disaggregating the data by type of program in which the students were enrolled.

There is widespread agreement that reasoning and proof should be an integral part of *all* students' mathematical experiences across content areas and throughout the grades (Stylianou et al., 2009). Meeting this goal requires that PSMTs develop a nuanced understanding of the various purposes and characteristics of proof and what might be an acceptable proof given a particular purpose. This includes the ability to distinguish among a range of valid and invalid arguments presented visually, verbally, or symbolically, as well as the ability to determine which of those arguments might be accessible to students. Moreover, PSMTs need opportunities and experiences within their teacher preparation programs that allow them to confront (and reconcile) their views of proof from mathematical and pedagogical perspectives.

References

Adler, J., & Davis, Z. (2006). Opening another black box: Researching mathematics for teaching in mathematics teacher education. *Journal for Research in Mathematics Education*, *37*, 270-296.

Australian Curriculum and Reporting Authority (ACARA) (2017). *The Australian curriculum: Mathematics.* Retrieved from https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/key-ideas/

Bergqvist, T. (2005). How students verify conjectures: Teachers' expectations. *Journal of Mathematics Teacher Education*, *8*, 171-191.

Bleiler-Baxter, S.K., & Pair, J.D. (2017). Engaging students in roles of proof. *The Journal of Mathematical Behavior*, *47*, 16-34.

Boyle, J. D., Bleiler, S. K., Yee, S. P., & Ko, Y.Y. (2015). Transforming perceptions of proof: A four-part instructional sequence. *Mathematics Teacher Educator*, *4*(1), 32-70.

Common Core State Standards Initiative. (2010). Common Core State Standards for Mathematics. Washington DC: National Governors Association for Best Practices and the Council of Chief State School Officers.

de Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras*, *24*, 17-24.

Dickerson, D. S., & Doerr, H. M. (2014). High school mathematics teachers' perspectives on the purposes of mathematical proof in school mathematics. *Mathematics Education Research Journal*, *26*(4), 711-733.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics*, *44*, 5-23.

Hanna, G., & de Villiers, M. (2008). ICMI Study 19: Proof and proving in mathematics education. *ZDM: International Journal on Mathematics Education*, *40*(2), 329-336.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. in F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning,* National Council of Teachers of Mathematics (pp. 1-60).

Karunakaran, S., Freeburn, B., Konuk, N., & Arbaugh, F. (2014). Improving preservice secondary mathematics teachers' capability with generic example proofs. *Mathematics Teacher Educator*, *2*(2), 158-170.

Knuth, E. J. (2002a). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education*, 33(5), 379-405.

Knuth, E. J. (2002b). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education*, *5*(1), 61-88.

Lesseig, K. (2016). Investigating mathematical knowledge for teaching proof in professional development. *International Journal of Research in Education and Science*, *2*(2), 253-270

Mason, J., & Pimm, D. (1984) Generic examples: seeing the general in the particular. *Educational Studies in Mathematics, 15*(3), 277-289.

Putnam, R.T., & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning? *Educational Researcher*, *29*(1), 4-15.

Simon, M.A., & Blume, G.W. (1996). Justification in the mathematics classroom: A study of prospective elementary teachers. *The Journal of Mathematical Behavior*, *15*, 3-31.

Steele, M.D., & Rogers, K.C. (2012). Relationships between mathematical knowledge for teaching and teaching practice: The case of proof. *Journal of Mathematics Teacher Education*, *15*(2), 159-180.

Stylianou, D.A., Blanton, M.L., & Knuth, E.J. (2009). *Teaching and learning proof across the grades: A K−16 perspective*. New York, NY: Routledge.

Tsamir, P., Tirosh, D., Dreyfus, T., Barkai, R., & Tabach, M. (2009). Should proof be minimal? Ms T's evaluation of secondary school students' proofs. *Journal of Mathematical Behavior, 28*, 58–67.

Varghese, T. (2009). *Secondary-level student teachers' conceptions of mathematical proof.* Issues in the undergraduate mathematics preparation of school teachers: The Journal, Volume 1: Content Knowledge.

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.