DEVELOPING AND USING DEFINITIONS FOR PRISMS AND PYRAMIDS

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Teaching geometry courses for preservice elementary teachers, we observed that difficulty with classifying shapes (and, in particular, composite 3-D shapes) persists even after work with simple shapes to support the writing of accurate and unambiguous definitions. We conducted a self-study of our teaching of 3-D concepts to uncover the concept images of pyramid and prism that emerge. We sought to understand the nature of those observed difficulties. We found that using both simple and composite shapes in classification activity exposed more nuanced and complex concept imagery than simple shapes alone. Opportunities to articulate assumptions create a space for all learners to make the language more precise and to create concept definitions that are more resilient.

Keywords: Geometry and Geometrical and Spatial Thinking, Instructional Activities and Practices, Teacher Education-Preservice

Background

Human perceptions of the physical world are primarily made up of 3-D shapes. Many curricula around the world provide opportunities to identify and name three-dimensional solids in early grades (Sinclair, Cirillo, & de Villiers, 2017). However, only a few studies on learners' definitions and conceptions of 3-D solids were discussed in the review conducted by Sinclair, Cirillo, and de Villiers (2017). Bozkurt and Koc (2012) reported that many of the first-year Turkish pre-service elementary teachers (PSETs) in their study found it difficult to define prism; 60% of them either could not provide a definition for prism or could not go beyond stating the fact that it was a term for 3-D shape. Another study identified a variety of concept images that Turkish PSETs hold about the base of 3-D shapes. Many were limited and/or contradictory in nature (Horzum & Ertekin, 2017). Tanguay and Grenier (2010) found that preservice secondary teachers had difficulty defining and describing regular polyhedra, which hindered their later attempts to develop a proof for the existence of only five possible regular polyhedra.

In our geometry courses for PSETs, we have observed the difficulty that our students have in classifying shapes, and, in particular, composite 3-D shapes. This difficulty persists even after significant work with simple pyramids and prisms to support the writing of accurate and unambiguous definitions. Initially, we speculated that this difficulty was related to our PSETs' ability to write and apply formal definitions, but wondered if it was also related to unarticulated concept imagery. The challenge was to create opportunities to articulate problematic concept images and expose the hidden contradictions that make classification difficult. Our analyses have uncovered layers of complexity in PSETs' conceptions of prism and pyramid.

We will provide findings related to two research questions:

1. What concept images of pyramid and prism emerged from in-class activities that focused on defining and classifying 3-D solids?
2. What is the nature of PSETs' difficulty in using established concept definitions and images to classify composite 3-D shapes as pyramids, prisms, or neither?

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Theoretical Framework

In our work to help PSETs understand prisms, pyramids, and related concepts, we strive to create opportunities for PSETs to experience cognitive disequilibrium (Piaget, 1985). This is the moment when there is an imbalance between prior knowledge (schema) and experiences that cannot (yet) be explained by it. The process of engaging students in the act of defining is one of iteration and revision; we move back and forth between examining concrete shapes built from wooden models or other commercially made building materials, and writing and revising emerging concept definitions. The activity described later in this paper, *Prism, Pyramid or Neither?* is one of our attempts to perturb the equilibrium of our students in the hopes that they are able to articulate deeply held concept imagery about these shapes and to demonstrate how resilient their conceptions have become.

We used Tall and Vinner's (1981) framework of concept definition and concept image to frame our PSETs' experiences with classification activity. Tall and Vinner describe a *concept image* as "the total cognitive structure that is associated with the concept which includes all the mental pictures and associated properties and processes" (p. 152). These authors distinguish this from a *concept definition* or "a form of words used to specify that concept" (p. 152). For example, one of our students described a pyramid as having "a tippy or pointed top, a base opposite to the top, and triangles around the top point." Individual concept definitions may be different from the "formal concept definition," which is a definition accepted by the mathematical community.

Data Collection and Analysis

This study followed the *Self-study of Teacher Education Practices* as we undertook action research to systematically study our own practices and to make our knowledge and beliefs, along with the dilemmas, decisions, and reflections, explicit (Vanassche & Kelchtermans, 2015). Self-study research makes it possible to share what we have learned from our practices so that it can be examined and transformed by other teacher educators (Bullough & Pinnegar 2001). We adopted an "inquiry as a stance" approach and acknowledged that self-study is an ongoing and complex process (Cochran-Smith, 2003).

We collected data in a geometry course required for all PSETs in a Midwest university including lesson plans and observation notes of about 100 minutes of lessons, lesson stories written by assigned students spanning the first two lessons of the semester, as well as written work from 58 students from three different sections taught by the same instructor on classifying composite polyhedrons.

We will present three stories, built from our data. First, we will use collected data to describe two episodes of classroom instruction related to classifying polyhedra and creating concept definitions for prisms and pyramids. Data collected at the classroom level were analyzed in order to examine existing and emergent concept imagery as the class worked to construct concept definitions for prism and pyramid. Once these definitions had been constructed, we used the quantitative results of the *Prism, Pyramid or Neither?* assignment to determine areas of both success and struggle for individuals involving classifying polyhedra using those classroom-constructed concept definitions. In this activity, we showed PSETs composite polyhedra built by composing pyramids and prisms in different ways. We asked them to identify each as a prism, pyramid, or neither. We were able to interpret their written justifications and identify specific concept images that interfered with classification activity.

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Findings and Discussion

Emergent Images and Definitions

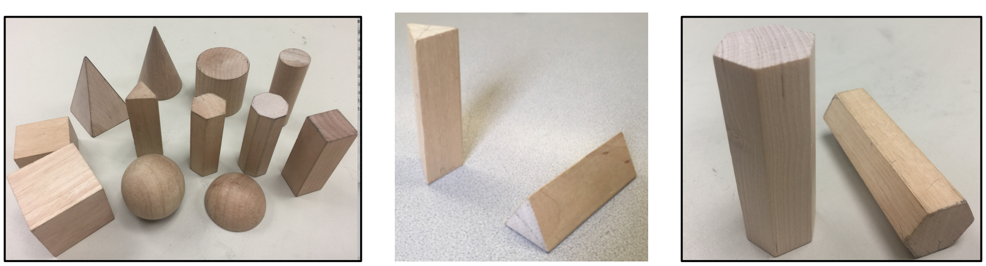
In this section, we discuss two classroom episodes. These episodes are amalgams of data from three separate teachings of the same content. They are written to represent the depth and breadth of conversations that occurred, even if each varied in minor ways from the others.

Episode 1. Working in small groups of 3 to 4, preservice elementary teachers were asked to come up with different ways to categorize a set of 15 wooden 3-D shapes that included both polyhedra (e.g., triangular prism, rectangular pyramid) and non-polyhedra (e.g., cylinder, sphere) as seen in Figure 1a. They were asked to record their thoughts on the question, "How are items in a category like one another and how are they different from other shapes?" A variety of categories was proposed and discussed during the follow-up whole-class discussion.

Many issues emerged during this part of the lesson that gave rise to the need for more precise definitions. For example, students had different meanings for the word *face*. Some considered *flatness* as part of their definitions of *face*; thus, a hemisphere would have only one face. Others argued that a hemisphere had two faces—a flat one and a curved one. The word *side* was also problematic. Some used the word *side* to refer to the *faces* of a prism, while others used that word to refer to the *edges* of a prism.

These discussions led to a classification scheme that separated polyhedra from non-polyhedra, with polyhedra being 3-D shapes that had only straight edges and flat faces. The instructor then assigned students to learn more about polyhedra, prisms, and pyramids by visiting the interactive 3-D shapes by Annenberg Learner (https://www.learner.org/interactives/geometry/3d.html).

Episode 2. To begin the second lesson, PSETs shared what they had learned from the website about the definitions as well as their current thoughts on the similarities and differences between pyramids and prisms. Initially, many students had a limited conception of *base*, similar to their Turkish counterparts (Horzum & Ertekin, 2017). The instructor helped students to transcend orientation-dependent conceptions of *base* by drawing attention to identical triangular and hexagonal prisms placed in different orientations (Figure 1b, 1c).



a b c

Figure 1. Classifying wooden models.

Our definition for the base of a prism as a special type of paired, opposite, congruent, and parallel faces required more clarification. When exposed to the hexagonal prisms in Figure 1c, PSETs encountered multiple pairings that fit this description. Introducing the idea of congruent *cross-sections* encouraged the distinction between base and lateral faces (non-base face). Similar discussions explored the idea of *apex* in pyramids as a special kind of vertex where all the edges from the base connected. Finally, the PSETs arrived at the class definitions: prisms are polyhedra with two congruent, parallel bases and all lateral faces are rectangles; and pyramids

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are polyhedra with a base, an apex opposite to the base, and all lateral faces triangles.

To further solidify the concept definitions and images, the instructors passed out to each group two composite shapes that were made up of two prisms, two pyramids, or one of each and asked the students to name the two shapes that made up the composite shape, describe the way they were connected, and decide whether the composite shape was a prism, pyramid, or neither.

This proved to be a challenging task for many students. For example, one group of students was not sure if the composite shape in Figure 2 was a pyramid. They could identify that this composite shape was made by connecting the base of a square pyramid with a lateral face from a hexagonal prism. They thought it could be named a pyramid because there were an apex and a base, but they admitted it didn't quite look like a typical pyramid. The instructor reminded students to justify their decision using the definition of pyramid that the class had agreed upon. Finally, students examined the shape further and noticed that some of the lateral faces weren't triangles, thus confirming that this composite shape was not a pyramid.

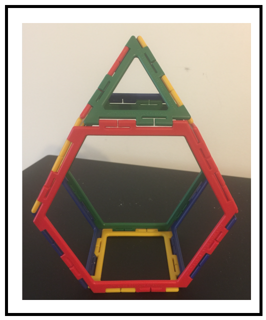

Figure 2. Is this composite shape a pyramid?

Classification Difficulty

We were also able to document our PSETs' difficulties in using established concept definitions and images to classify composite 3-D shapes. After the class activities described above, PSETs were asked to complete an online quiz. The online quiz had two components. The first component asked them to determine if each of 12 composite shapes was a prism, pyramid, or neither. The second part asked them to "write a careful justification to explain whether a composite shape shown is a prism, pyramid, or neither" for three composite shapes selected at random from a subset of 4 of the original 12 shapes.

Because of space limitations, we included here results from only six items: two from the items with the highest percentage of correct responses and four from the items of lowest percentage to provide insights into PSETs' overall performance. We use bold print to indicate correct responses. While the majority of PSETs were able to correctly identify that Figure 3a was a prism and 3b a pyramid, the other shapes were not so easily classified. The quantitative data alone are evidence of the difficulty our students faced in applying definitions to classify 3-D shapes. However, the justifications for these classifications (provided in the next section) illustrate strongly held concept images that might help explain the nature of these difficulties. Below, we describe several concept definitions and images strongly held by some PSETs when classifying composite shapes. We drew support from both the analysis of the quantitative data summarized above and the analysis of the written work.

Difficulties associated with pyramid. Despite the class discussion tending to the classification of the composite shape in Figure 2, about a quarter of PSETs still classified the shape in Figure 3c as a pyramid because it had an apex, a base, and triangular lateral faces. For

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example, Dakota (mistaking the hexagon for an octagon) wrote, “The composite shape is a pyramid because it has an octagonal prism with another octagon pyramid on top. The definition of a pyramid includes a polygonal base—an octagon, and lateral faces that are triangles that run into one vertex.” She and many PSETs still ignored the fact that some lateral faces on Figure 3c were not triangles; thus, it could not be a pyramid.

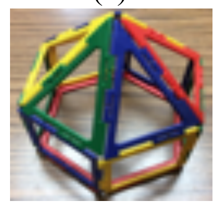
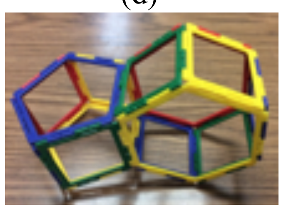
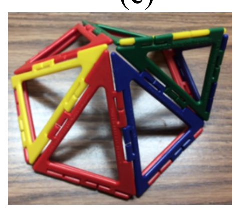
	(a) 	(b) 	(c) 
Prism	**94.83%**	5.17%	6.90%
Pyramid	0.00%	**84.48%**	25.86%
Neither	5.17%	10.34%	**67.26%**
	(d) 	(e) 	(f) 
Prism	32.76%	3.45%	**41.38%**
Pyramid	1.72%	48.48%	6.90%
Neither	**65.52%**	**48.48%**	51.72%

Figure 3. Classification of composite shapes by PSETs (N = 58).

Shape 3e was another difficult one to classify. It was made up of a hexagonal pyramid and a triangular prism connecting through their lateral triangular faces. Half of the PSETs judged it to be a pyramid and half of them judged it to be neither. Our analyses of the written justifications revealed that the class definition of pyramid as “polyhedra with a base, an apex opposite to the base, and all lateral faces are triangles” might not be explicit enough to help PSETs to correctly classify Figure 3e. For example, Ryann wrote, “Pyramid. Because it has one face that is a polygon and all the other lateral faces are triangles they also come to a point.” Ryann’s concept definition of pyramid was previously sufficient to classify all of the wood blocks in Figure 1a, but it was not sufficient to tackle the complexity of this composite figure.

Some PSETs did recognize that there seemed to be odd faces disconnected from the apex that were not bases. This was a moment of disequilibrium (Piaget, 1985) that proved important. Some decided to modify their concept definition to accommodate the new type of face. Katelyn wrote,

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> The composite shape is a pyramid because even though it has an extension on one of the lateral faces, all lateral faces are triangles and both bases are polygons. It is a combination of a triangular pyramid and a hexagonal pyramid. It fits the definition of a pyramid, which is a polyhedron where the base is a polygon and all lateral faces are triangles. It breaks no rules and meets all the requirements to be considered a pyramid.

Our analysis uncovered additional concept images and definitions of pyramid that were useful in helping the PSETs to make the correct classification. For example, some PSETs decided that Figure 3e was not a pyramid because "it had a side stick out." Alex stated that for a shape to be classified as a prism or pyramid, "The shape would have to be able to lay on any one of its sides." Jasmin said this is not a pyramid because it is "capable of rocking back and forth." The concept image of a pyramid that could lay stable on each of its many faces was a strong one. Others had included in their concept definitions of pyramid the requirement of having only one base. As Anjou reasons, Figure 3e isn't a pyramid because it has two bases:

> Our definition for pyramid is—a polyhedron where the lateral faces are triangles, has one base which is a polygon. Now. Although all the lateral faces meet at a common point, this shape has more than one base so it has more than one base so it can't be a pyramid either.

Difficulties associated with prism. About 51.72% of the students declared that Figure 3f was neither a prism nor a pyramid. Most justified eliminating pyramid as a choice due to the lack of an apex. However, the fixed orientation impacted PSET's ability to see it as a prism. As Jessica says, "This shape is not a prism because all of the lateral faces are not rectangles, some are triangles and also this shape does not have two congruent bases."

Just like Jessica, many PSETs had the concept image of a base as a face on which the whole 3-D shape sits. So if a student assumes the square as the base, the octagons become lateral faces and it is impossible to find another congruent square parallel to that square base. The idea that shapes retain their form as they are rotating in space (or, for 2-D, on a plane) is a critical conceptual understanding that students need to develop in making sense of both 2-D and 3-D shapes. Some were able to overlook the orientation and identified this shape as a prism, as Jamal wrote, "This composite shape is a prism because it has two congruent bases when I flip the shapes sideways (yellow shapes at the bottom) and the rest of the lateral faces are rectangles which fits in the definition of a prism."

What is it about the shown orientation that renders it unrecognizable? The answer is to return to concept imagery around the term *base*. When classifying simple polyhedra in class, students always oriented their prisms so that they were resting on one of the bases. By doing so, it became a habit to define *base* as a face on which the whole 3-D shape "sits." The singular case is also described as "the bottom" (as in a basement) and the pair of bases as "the bottom and top" of a prism. Jessica explains: "This shape is not a prism because all of the lateral faces are not rectangles, some are triangles and also this shape does not have two congruent bases."

The case of overgeneralization. As seen in Figure 3a and 3b, the majority of the PSETs were able to recognize the composite shape in 3a as a 9-gonal prism made by connecting a hexagonal prism and a pentagonal prism at a congruent lateral face. Also, they recognized the shape in 3b as a pyramid made by connecting two triangular pyramids at a congruent lateral face. Unfortunately, that led some students to believe that the overall shape of a 3-D composite is a prism if it is made up of two prisms, and is a pyramid if it is made up of two pyramids, and is neither if it is made up of a prism and a pyramid.

The statement is obviously false but the challenge is to recognize what parts of this statement

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contain some truth. It is true that composing a prism and a pyramid will always result in a shape that is neither a prism nor a pyramid. Figure 3c is one example. In the composition process, the figure cannot retain the property of having two congruent parallel bases (thus, not a prism) nor can it cannot retain the property that all lateral faces are triangles (thus, not a pyramid).

The first two parts of the generalization, however, are not always true. In Figure 3e we see a counterexample of a composite shape, made up of two pyramids, that is not a pyramid. The two shapes were connected via a lateral face such that their bases did not create a composed pyramid; their bases are not in the same plane as in Figure 3b. Another counterexample is shown in Figure 3d, which was composed of the same two prisms in Figure 3a by matching the same lateral faces, but twisted so that their original bases would not lie on the same plane.

The composite polyhedra were selected for this activity because they had the potential to bring a variety of concept imagery to the surface related to the terminology used to define prisms and pyramids. However, this last discussion indicates that this activity has the potential to generate even more false generalizations if we do not recognize the complexity of this topic.

Conclusions and Implications

We have chosen to closely examine our instruction related to prisms and pyramids partly because of the difficulties we observed PSETs having with classification activity related to both 3-D and 2-D shapes. However, we were surprised at the depth to which we were able to take our analysis, indicating that composing, decomposing, and classifying 3-D shapes is far more complex than we previously thought. The power of self-study is to uncover assumptions, and we feel that the methodology was successful in that regard.

From this experience, we find that it's not enough to use simple ready-made solids such as wooden blocks when exploring 3-D concepts with PSETs. While they are sufficient to sort polyhedra from non-polyhedra, they lack the complexity necessary to lead to a deep discussion about related concepts (e.g., base, lateral face, edge, side, and apex) to make the properties of prisms and pyramids clear. Using simple polyhedra allows for ambiguity and assumption.

Composite shapes and complex polyhedra (including platonic solids) have the power to cause disequilibrium (Piaget, 1985) and perturb the concept imagery that PSETs take for granted. The concept images that PSETs have of prisms and pyramids as well as related concepts are myriad and rich, but often go unarticulated. Opportunities to articulate assumptions create a space to make the language more precise and to create concept definitions that are more resilient.

Supporting students in their examination and classification of polyhedrons has long-term implications at all levels. The act of composing and decomposing are central to the development of measurement concepts (Feikes, Schwingendorf, & Gregg, 2008). A robust understanding of prism and pyramid is important to the future study of measurement concepts such as volume and surface area. Many concept images are formed throughout the teaching episodes presented here. Specifically, using cross sections to make distinctions between bases and lateral faces on prisms has great promise when it comes time to develop formulas for volume.

Furthermore, teaching the act of defining rather than a memorized definition (de Villiers, 1998) creates a space to challenge and refine concept imagery that conflicts with more formal concept definitions. This, in turn, supports a more robust understanding of the concepts we are trying to define. Providing a myriad of activities that help students assimilate increasingly complex shapes into their schema for polyhedra, prisms, and pyramids challenges assumptions and opens up opportunities for nuance and precision in the way we are all able to collectively negotiate meaning and shared understanding.

Both the theoretical frameworks by Piaget and Tall and Vinner have supported the design

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and analyses of numerous mathematics education research projects in the last 40 years. In this study, we found them also to be helpful in illuminating our quest for understanding the nature of difficulties behind 3-D classification. One area for future research is to continue to explore the use of their constructs to understand the nature of student difficulty with other challenging topics related to 3-D solids, such as surface area and volume, for both PSETs and K-12 learners. Another study could focus more on the general conceptions of definition held by PSETs and the impact of curricular moves on those conceptions. Leikin and Zazkis (2010), looking across multiple research studies, argued that teachers' concept images and their understanding of the notion of definition influence the ways in which teachers introduce mathematical content to their students. Working in conjunction with methods instruction, it would be important to extend the study into a field experience where PSETs were tasked with designing and/or implementing lessons on 3-D shapes with elementary students.

References

Bozkurt, A., & Koc, Y. (2012). Investigating first year elementary mathematics teacher education students' knowledge of prism. *Educational Sciences: Theory and Practice*, *12*(4), 2949–2952.

Bullough, R. V., Jr., & Pinnegar, S. (2001). Guidelines for quality in autobiographical forms of self-study research. *Educational Researcher, 30*(3), 13–21.

Cochran-Smith, M. (2003). Learning and unlearning: The education of teacher educators. *Teaching and Teacher Education, 19*, 5–28.

de Villiers, M. (1998). To teach definitions in geometry or to teach to define? In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 248–255). Stellenbosch, South Africa: University of Stellenbosch.

Feikes, D., Schwingendorf, K., & Gregg, J. (2008). *Connecting mathematics for elementary teachers*. Boston: Addison-Wesley Longman.

Horzum, T., & Ertekin, E. (2017). Prospective mathematics teachers' understanding of the base concept. *International Journal of Mathematical Education in Science and Technology*, *49*(2), 176–199.

Leikin, R., & Zazkis, R. (2010). On the content-dependence of prospective teachers' knowledge: A case of exemplifying definitions. *International Journal of Mathematical Education in Science and Technology, 41*(4), 451–466.

Piaget, J. (1985). *The equilibration of cognitive structures: The central problem of intellectual development*. Chicago, IL: University of Chicago Press.

Sinclair, N., Cirillo, M., & de Villiers, M. (2017). The learning and teaching of geometry. In J. Cai (Ed.), *Compendium in research in mathematics education* (pp. 457–489). Reston, VA: National Council of Teachers of Mathematics.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151–169.

Tanguay, D., & Grenier, D. (2010). Experimentation and proof in a solid geometry teaching situation. *For the learning of mathematics*, *30*(3), 36-42.

Vanassche, E., & Kelchtermans, G. (2015). The state of the art in self-study of teacher education practices: A systematic literature review. *Journal of Curriculum Studies, 47*(4), 1–21.

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