REFLECTED ABSTRACTION AS A MECHANISM FOR DEVELOPING PEDAGOGICAL CONTENT KNOWLEDGE

Michael Tallman
Oklahoma State University
michael.tallman@okstate.edu

John Weaver
Oklahoma State University
john.weaver@okstate.edu

Tallman (2015) argued that pedagogical content knowledge is a form of content knowledge with particular characteristics that endow it with pedagogical utility, and he conjectured that an essential characteristic of a teacher's content knowledge is her conscious awareness of the mental actions and conceptual operations that comprise her mathematical schemes. We report the results of a study that explored this conjecture by examining the pedagogical implications of engaging a pre-service secondary teacher in a mathematical intervention to support her construction of a powerful scheme for constant rate of change and to engender her conscious awareness of the mental processes that constitute this scheme.

Keywords: Pedagogical content knowledge; Reflecting abstraction; Mathematical knowledge for teaching; Pre-service teacher education

Researchers have long been concerned with conceptualizing the knowledge base that enables teachers to effectively support students' conceptual mathematical learning. Shulman's (1986, 1987) notion of *pedagogical content knowledge* (PCK), which he defined as a "special amalgam of content and pedagogy," represents a significant advance to this end (1987, p. 8). While researchers since Shulman have strived in various ways to elaborate the PCK construct, there is an aspect of Shulman's initial conceptualization that is almost never abandoned: the notion that PCK is a combination of content knowledge and pedagogical knowledge (Depaepe, Verschaffel, & Kelchtermans, 2013). Although many researchers follow Shulman in defining PCK as an amalgam of knowledge of content and knowledge of pedagogy (e.g., Ball, Thames, & Phelps, 2008; Fennema & Franke, 1992), the relationship between these distinct types of professional knowledge, as well as the specifics of their synthesis, remains elusive.

To investigate the relationship between teachers' content knowledge and their instructional actions, Tallman (2015, in submission) compared an experienced secondary mathematics teacher's knowledge of sine and cosine functions with the knowledge he enacted in the context of teaching. Through this comparison, Tallman identified the influences that affected the nature and quality of the subject matter knowledge the teacher leveraged in his lesson planning and instruction. Tallman's analysis revealed that the inconsistencies between the teacher's personal and enacted mathematical knowledge resulted from him possessing weak connections and multiple schemes for particular concepts related to trigonometric functions, as well as from his unawareness of the mental actions and conceptual operations that comprised these often powerful but uncoordinated cognitive schemes. Tallman concluded from these results that PCK is a form of content knowledge with particular characteristics that endow it with pedagogical utility; he argued that the pedagogical character of PCK derives from the specific ways in which a teacher's content knowledge informs her enactment of effective pedagogies. Additionally, Tallman conjectured that an essential characteristic of a teacher's content knowledge is the extent to which she is consciously aware of the mental actions and conceptual operations that comprise her own mathematical schemes.

In this paper, we report the results of a study that explored this conjecture by examining the pedagogical implications of engaging a pre-service secondary teacher in a mathematical intervention to support her construction of a powerful scheme for constant rate of change and to engender her conscious awareness of the mental processes that characterize this scheme. We leveraged Piaget's (2001) notion of reflected abstraction to stimulate such conscious awareness.

Theoretical Background

Piaget proposed abstraction as the mechanism of scheme construction and refinement and distinguished five varieties: *empirical*, *pseudo-empirical*, *reflecting*, *reflected*, and *meta-reflection* (Piaget, 2001). We discuss only *reflecting* and *reflected* abstraction because of their unique role in the construction and refinement of mathematical schemes and because these two types of abstraction served as design principles and analytical constructs in the present study. Reflecting abstraction involves the subject's reconstruction on a higher cognitive level of the coordination of actions from a lower level, and results in the development of *logico-mathematical knowledge*, or schemes at the level of *operative thought*. Reflecting abstraction is thus an abstraction of actions and occurs in three phases: (1) the *differentiation* of a sequence of actions from the effect of employing them, (2) the *projection* of the differentiated action sequence from the level of activity to the level of representation, or the reflected level, and (3) the *reorganization* that occurs on the level of representation of the projected actions (Piaget, 2001). A subject must differentiate (dissociate) actions from their effects before she can construct an internalized representation of them, what Piaget called projecting actions to the level of mental representation (i.e., the "reflected level"). Additionally, the subject must coordinate the actions that produced the effect before she can project and represent them on this higher cognitive level. Once a subject differentiates actions from their effect and coordinates them, she is prepared to project these coordinated actions to the reflected level where they are organized into cognitive structures, or schemes.

Reflected abstraction involves operating on the internalized actions that result from prior reflecting abstractions, which results in a coherence of actions and operations accompanied by conscious awareness. It is the act of deliberately operating on the actions and operations that result from prior reflecting abstractions that engenders such awareness. To consciously operate on actions at the level of representation suggests that one has symbolized coordinated actions at this higher level. Reflected abstraction thus relies on what Piaget called the *semiotic function*, or the subject's capacity to construct mental symbols to represent aspects of her experience. As a result of the conscious awareness of internalized actions that occurs as a byproduct of reflected abstraction, the subject's ability to purposefully assimilate new experiences to the reflected level provides evidence that she has engaged in reflected abstraction. Additionally, performing operations on the symbols the subject constructs to represent coordinated actions at the level of representation results in increasingly organized cognitive structures. Reflected abstraction is therefore the means by which systems of organized actions at the level of representation become progressively coherent and refined.

Conceptual Analysis of Rate of Change

As we mentioned above, we engaged a pre-service secondary teacher in a mathematical intervention to support her construction of a particular scheme for rate of change. In this section, we briefly describe the meanings we designed the intervention to support.

A mature rate of change scheme relies upon productive conceptualizations of ratio, rate, and continuous variation. A *ratio* is a multiplicative comparison of the measures of two constant (non-varying) quantities while a *rate* defines a proportional relationship between varying quantities' measures (Thompson & Thompson, 1992). Constructing a rate therefore involves

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images of smooth continuous variation (Thompson & Carlson, 2017), as well as the expectation that as two quantities covary, multiplicative comparisons of their measures remain invariant.

A rate is a reflectively abstracted constant ratio (Thompson & Thompson, 1992, p. 7), which means that constructing a rate involves internalizing the coordinated actions involved in multiplicatively comparing particular values of covarying quantities. An individual has done so if on the basis of her conceptualization of a particular quantitative situation she anticipates that subsequent multiplicative comparisons of the covarying quantities' measures will yield the same numerical value.

Rate of change is a quantification of two covarying quantities and results from a multiplicative comparison of changes in covarying quantities' measures. The rate of change is constant if changes in the quantities' measures are proportional. Conceptualizing changes in quantities' measures as quantities themselves involves a quantitative operation with attention to a point of reference (Joshua et al., 2015). Conceptualizing rate of change requires students to construct changes in quantities' values as quantities themselves and then to abstract an invariant multiplicative relationship between changes in the quantities' values.

Methods

This study's experimental methods proceeded in four phases. First, we video recorded all nine lessons from a pre-service secondary teacher's instruction of rate of change and slope in a 7th grade class during her student teaching semester. We then asked the participant, Samantha, to use the data analysis software Studiocode (Studiocode Version 5.8.4, Sportstec, Ltd., 2015) to identify segments of her first two lessons that exemplify high quality instruction as well as segments that indicate room for improvement. She wrote brief justifications for each selection. Samantha performed this analysis of her teaching videos independently and met with the research team after she had finished to answer clarifying questions and to provide additional rationale for her written responses. We then conducted a teaching experiment (Steffe & Thompson, 2000) during the third phase of the study to construct a model of Samantha's scheme for constant rate of change and to characterize the evolution of this scheme as she engaged in particular instructional experiences designed to promote reflecting and reflected abstractions. The teaching experiment consisted of eight teaching episodes that each lasted between 60 and 90 minutes. During the fourth phase of the study, we asked Samantha to analyze five teaching videos (including the two she analyzed prior to participating in the teaching experiment) using the codes "High Quality Instruction" and "Room for Improvement" and to provide written rationale for her selections. We again interviewed Samantha to probe her justifications for her selections and to ask focused questions about the segments of her instruction she identified as exemplars of high- or low-quality teaching. Our focus in the fourth phase of the study was to determine whether the criteria by which Samantha evaluated the quality of her instruction had changed, and to identify the role played by her content knowledge in making such evaluations.

Results

Due to space limitations, we discuss only our analysis of the data relevant to Samantha's first lesson on rate of change.

Samantha's Instruction of Rate of Change

Samantha began her teaching of rate of change by asking students to describe what they think of when they hear the word "rate" and to provide examples of rates. She then defined a rate as "a ratio of two quantities with different units" and described a unit rate as "a rate with a denominator of one." After defining these terms, Samantha asked students to interpret the

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meaning of a car traveling at a constant speed of 65 miles per hour. She validated a student's response, "You're traveling 65 miles in an hour" and then restated for the rest of the class the student's interpretation: "Every 65 miles that I travel, that's an hour of time; or if I travel for an hour, I've gone 65 miles." Samantha then asked students to "find the unit rate in miles per hour" if a car travels 155 miles in two hours. After students divided 155 by two, Samantha explained that the resulting 77.5 represents the number of miles the car travels each hour.

Samantha's definition of rate as a "ratio of two quantities with different units" supported students' understanding of constant rate as the change in some quantity's measure that corresponds to a one-unit increase in another quantity's measure. Her discussion of the meaning of a vehicle traveling at a constant rate of 65 miles per hour is a case in point. She encouraged a meaning based on a coordination of additive changes, instead of a proportional correspondence between continuously varying changes in accumulated distance and accumulated time.
Toward the end of the lesson, Samantha asked her students to compute the unit rate provided the values in Table 1. Her discussion of computing this unit rate supported a conception of rate as an additive comparison grounded in images of discrete variation. In response to the task, a student proposed dividing the weight of two books (6 lbs.) by the number of books—a reasonable strategy considering that just prior to this task the class had computed the unit rate of a car traveling 155 miles in two hours by dividing the car's distance traveled (155 miles) by two hours. Another student recognized that the number of pounds per book was evident in the third column of the table (3 pounds corresponds to one book). Samantha noticed that the two students were focusing on comparing the *values* in the table, rather than determining the *change* in the number of pounds that corresponds to adding one additional book. She subsequently directed students' attention to these changes.

Table 1: Find the unit rate.

Books	0	1	2	3
Pounds	0	3	6	9

Samantha's Pre-Intervention Video Analysis

After Samantha concluded her instruction of rate of change, but prior to the teaching experiment, we asked her to watch the videos of her first two lessons and to identify moments that exemplify high quality instruction as well as moments that suggest room for improvement. She categorized her opening discussion—in which she asked students to describe what they think of when they hear the word "rate"—as indicating room for improvement. In her written justification Samantha explained,

> I like that I ask students what they think of when they hear the word 'rate.' However, I think there is room for improvement because I tend to talk to one student (whoever answers the question) instead of addressing the whole class with the student's answer.

Samantha identified her explanation of the meaning of 65 miles per hour as representing high quality instruction and rationalized her selection as follows:

> I think this portion went really well because we really isolated what a unit rate is. I used speed limits because I knew it was something the students could understand. I liked that I asked the students what '65 miles per hour' means. When a student explained it using the same words found in the question, I asked if another student would repeat what he said but say it in their own words. I thought this was an effective way to get more than just one

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> student involved and to check that others understood the concept. Additionally, I liked that I had students give several examples. This helped me to see that they knew what a unit rate was and it kept them engaged.

We notice that Samantha prioritized students' engagement and participation as important criteria for evaluating the quality of her teaching. This focus on students and their activity was reflected in five of the six written justifications she provided for her selections. Samantha's appraisal of her teaching was based primarily on the extent to which she sustained students' interest and elicited their contributions. None of her written responses reflected a critical evaluation of the meaning of rate of change her instruction supported.

Teaching Experiment

We leveraged Piaget's notion of reflecting and reflected abstraction to design an instructional sequence that would enable Samantha to construct a scheme for rate of change that reflects the meaning we articulate in our conceptual analysis above. This instructional sequence unfolded over nine teaching episodes. On many occasions throughout the teaching experiment, Samantha repeatedly and convincingly demonstrated her understanding of constant rate of change as an invariant multiplicative relationship between changes in the measures of covarying quantities.

To engender reflected abstractions—that is, to support Samantha's awareness of the mental actions and conceptual operations that comprised her rate of change scheme—we provided Samantha with her written work to particular tasks in the instructional sequence and prompted her to compare the thinking required to correctly solve select pairs of tasks. We asked her to articulate her comparison in writing and then to respond to our clarifying questions about what she had written. Although the tasks in the instructional sequence varied substantially in terms of the information provided and the quantity Samantha was asked to compute or represent, all tasks in the sequence could be solved by leveraging a meaning for constant rate of change as a proportional correspondence between changes in covarying quantities' measures. After having recognized that similar or identical thinking is required to solve all 12 pairs of tasks from the instructional sequence we asked her to consider, we prompted Samantha to summarize in writing the meaning for rate of change that might enable a student to reason productively about all tasks in the instructional sequence. We include her response in Figure 1.

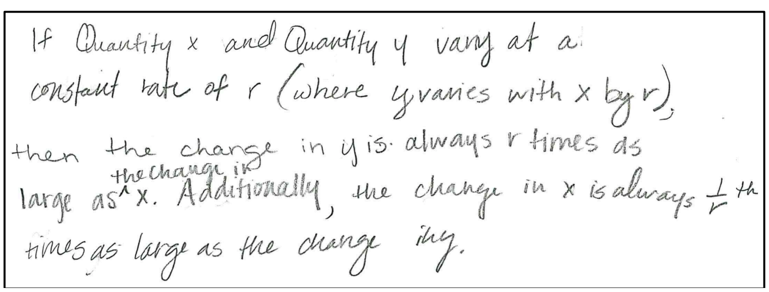
If Quantity x and Quantity y vary at a constant rate of r (where y varies with x by r), then the change in y is always r times as large as the change in x. Additionally, the change in x is always $\frac{1}{r}$ th times as large as the change in y.

Figure 1. Samantha's meaning for constant rate of change.

At the conclusion of the teaching experiment, Samantha was able to demonstrate her conception of constant rate of change as a proportional relationship between changes in covarying quantities' measures and to articulate the mental imagery, actions, and operations that comprised her meaning for constant rate of change. We were interested in assessing the implications of Samantha's sophisticated rate of change scheme and her awareness of its contents for the criteria she leveraged to evaluate her teaching and to propose alternative instructional actions.

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Samantha's Post-Intervention Video Analysis

Just as she had done in her pre-teaching experiment video analysis, Samantha identified the opening discussion of the first lesson as indicating room for improvement. In this discussion, Samantha invited students to describe what they think of when they hear the word "rate" and to suggest some examples of rates. Prior to the teaching experiment, Samantha critiqued this episode of her first lesson based on her lack of success in getting more students involved in the discussion. The following is Samantha's justification for identifying this opening discussion as "room for improvement" after having participated in the teaching experiment:

> I think there is room for improvement here because I think there is a better way to introduce rates. I feel like asking students what they think of when they hear a word is a fine way to get them talking and engaged but it doesn't direct the conversation to the mathematical meaning of a rate. Instead, this question sort of just starts a guessing game between myself and the students rather than engaging in mathematical discourse. … I could've started the conversation off by posing a question like, 'Clara runs 7 miles per hour. How far will she run in 2 hours?'

We were interested in why Samantha suggested posing the question, "Clara runs 7 miles per hour. How far will she run in 2 hours?" so we asked her write a transcript of a hypothetical alternative to the opening discussion from the first lesson and to explain why she prefers this alternative. After having asked students to determine how far Clara will run in two hours, Samantha's transcript indicated her asking them to determine how far Clara will run in three hours, half an hour, and 4.3 hours. She then prompted students to generalize a (multiplicative) relationship between Clara's distance traveled (in miles) and the number of hours Clara had spent running. The following is Samantha's explanation for why she expected this alternate introduction to be more effective than how she actually began her first lesson:

> I think this interaction reflects an ideal unfolding of the conversation because we start with a basic example that is easy to understand (i.e. Suppose Clara runs 7 miles per hour. How far will she run in 2 hours?) This part gives students an easy starting point to use. Then I ask some follow up questions, like how far would she go in half an hour, another good benchmark. I move on to give an example that needs a little more explanation so that we can start thinking about how to break up the time (4.3 hours) and figure out what the respective number of miles is. It's not as obvious as 4 hours and I could have even chose something like 0.59 hours instead. … I think that opening with these examples fosters a conversation of how to find rate of change and what it means for those quantities to vary directly. The end of the conversation opens up many routes; we could talk about finding time and then add some initial values and dig into changes in quantities.

Samantha's alternate introduction to the first lesson reflects both her image of what it means to understand constant rate of change as well as her expected trajectory through which students must progress to construct this understanding. Rather than proposing instructional actions that simply elicit students' participation, Samantha's transcript and accompanying rationale demonstrate that her criteria for effective teaching had shifted to incorporate attention to the mathematical meanings her instruction supported.

The transformation in the criteria by which Samantha evaluated the quality of her teaching was evident in her analysis of other episodes from the first lesson. Regarding her definition of rate as a "ratio of two quantities with different units," Samantha commented:

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> I'm not really a huge fan of doing a ton of definitions so I don't know why I did this. I don't think I would use this definition (the book definition) for rate again. Although it connects rates to ratios, it doesn't really give any meaning to rates. One could recite this definition without actually knowing how to do any problem that involves rates.

Samantha's appraisal of her discussion of what it means to say that a car travels at a constant speed of 65 miles per hour similarly reflected her attention to the understanding of rate she promoted:

> I think we are getting closer to what I would really like conversations about rate to look like. This is what I would call the first stage of understanding rates of change. The students seem to have an understanding that 65 miles per hour means that for every hour traveled, one goes 65 miles. I think this a great place to start conversations about rates of change. Maybe some other contexts could get the conversation moving further so we could generalize the idea by talking about changes in quantities.

Additionally, Samantha expressed contrition for asking students to determine the unit rate provided the values in Table 1. Specifically, she acknowledged that the context did not support students' conception of rate of change as conveying a relationship between *changes* in quantities' measures:

> I see what I am getting at here. I am trying to get at the idea of changes in pounds for changes in books and how that is a rate. Honestly, the examples I chose aren't the best for trying to make students think hard about changes in quantities. The problems are almost trivial, clearly every book is three pounds. Once students see that, it's hard to get them to talk about changes in quantities because the answer to the question has already been found and can be explained simply.

Samantha critically evaluated the mathematical meanings her instruction supported in 14 of the17 justifications she provided for the segments of her lessons that exemplify high quality instruction or indicate room for improvement. In Samantha's initial analysis of her teaching videos, none of her written responses suggested that she was attending to the understanding of rate of change her instruction promoted. This shift in the criteria by which Samantha evaluated her teaching quality did not escape her attention. When asked what she initially considered when evaluating her instruction, Samantha explained, "I think it was student engagement, using something the students care about or would be interested in, asking good questions, getting them to repeat back." Samantha acknowledged the importance of attending to issues of classroom management and students' participation, but then suggested that instruction viewed only through this lens does not enable one determine whether students are being provided with opportunities to construct productive ways of understanding. Having come to this conclusion, Samantha described the additional criterion that she considered after having engaged in the teaching experiment to assess the quality of her teaching:

> As we went over rate and what it means to be a rate together, I started thinking more about what am I doing to promote understanding. … Going back I think cared a lot more about classroom management versus when I went back and looked at these I acutely cared about, ok wait, is this promoting understanding. … I cared more about what they knew.

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Discussion

Our findings expose a common assumption that underlies much of the literature on mathematical knowledge for teaching as well as mathematics teacher preparation and professional development programs: Mathematical knowledge for teaching is comprised of a variety of distinct but related knowledge “types” that coalesce in the practice of teaching. Mathematics teacher education programs based on this assumption focus on supporting teachers in developing these categories of knowledge without seriously attending to how knowledge in one domain informs or derives from knowledge in another. The central findings of our study suggest that teachers’ image of effective teaching—and the pedagogies they enact to ensure that their practice conforms to this image—are influenced by having constructed powerful mathematical schemes and by having achieved an awareness of the mental actions and conceptual operations that comprise these schemes. The results of our study therefore support Tallman’s (2015, in submission) hypothesis that engaging pre-service teachers in mathematical experiences designed to engender reflecting and reflected abstractions has positive implications for their image of effective teaching and the pedagogical actions they enact (or envision enacting) to achieve their instructional goals.

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