PROSPECTIVE TEACHERS' CONCEPTIONS ABOUT A PLANE ANGLE AND THE CONTEXT DEPENDENCY OF THE CONCEPTIONS

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In our study, we looked for an answer to the research question "How prospective teachers comprehend the concept of a plane angle and what kind of variation there is between the conceptions? The study shows that the prospective teachers interpret the concept "plane angle" which in principle is familiar to everybody in many different ways. In this paper, we categorise eight interpretations of a different type, appearing in the future teachers' ideas. We also show some examples of the fact that individuals can use different concept images of a plane angle when performing the tasks of different types even in the same test.

INTRODUCTION

The concept of a plane angle has been in the course of centuries a concept which even the mathematical science community has found hard to define and hard to approach from one single point of view (Matos, 1990; Keiser, 2004). The following three modes of definition have been the ones most frequently applied as the definition of an angle at different times. An angle is defined either (1) as a rotation by which one of two intersecting straight lines is made to merge into the other or (2) as a region defined by two half lines starting from the same point or (3) as the common region defined by two intersecting half planes (Mitchelmore & White, 2000). In fact, the Latin word angulus means literally "a little bending." According to these definition alternatives, an angle can be understood to be either a measurable quantity, a geometric construction or a plane region. Keiser's studies (Keiser, 2004; Keiser et al., 2003), in particular, show that these different interpretations of the concept of an angle reflect the differences discovered in didactic research in the interpretations of individuals to this concept rather well. Our study intends to acquire information on the ways in which prospective class teachers and prospective subject teachers of mathematics grasp and make sense of the concept of a plane angle. We used two different task types in our study. As Tall and Vinner (1981) and after that many others have noted our understanding about concepts is based both to the more holistic and visual concept image and to the more formal concept definition. In our study, we tried to get information of both of these modes of understanding regarding the concept of a plane angle. The study revealed that in different contexts the prospective teachers seem to base their decisions on different concept images although tasks focus on the same concept.

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RESEARCH QUESTIONS

The main research question of our study is *how prospective teachers comprehend the concept "plane angle" and what kind of variation there is between the conceptions?*

Primarily, we were here interested in the variation of the conceptions between individuals. However, as a secondary task we also wanted to look as whether student's answers to the definition task and to the point selection task both reflected similar conception of a plane angle or did the answers as well show that individuals may apply different concept images to the same concepts when performing different processes.

METHOD

The way in which an individual grasps a mathematical concept can be examined either by observing how the individual uses the concept spontaneously in speech and action without actually being aware of the observation, or by planning a test situation in which the informant is asked to do something that reveals as much as possible of the way in which this individual interprets the meaning of the concept. In our earlier case study (Joutsenlahti & Silfverberg, 2007) we used the former method of collecting research data from schoolchildren, whereas in Silfverberg and Joutsenlahti (2007) and in the present study the latter method was used for the purpose of examining teacher students' interpretations of an angle.

An analysis of the definitions given by the informants can be done in several ways revealing different aspects the understanding about the concept image and concept definition (Tall & Vinner, 1981). For instance, we can focus on checking (1) how correctly a definition defines the concept in comparison with the normative interpretation; (2) how adequately the form of a given definition meets the formal criteria set for a mathematical definition (Hershkowitz, 1990; Leikin & Winicki-Landman, 2000a, 2000b; de Villiers, 1995); (3) how well a definition given by an informant corresponds with the concept form which this informant seems to have on the basis of the situations in which the concept was actually applied (Tall & Vinner, 1981; Vinner & Dreyfus, 1989; Vinner, 1991); (4) what kind of linguistic form the informant uses in providing a definition (Barnbrook, 2002).

The research data was collected by a questionnaire handed out to 191 Finnish university students. Hundred (100) of them were pursuing studies to become subject teachers in mathematics and science for grades 7 through 12, and 91 to become class teachers for grades 1 through 6. In item 1 of the questionnaire we just asked the prospective teachers to define the concept of a plane angle. In item 2 (Figure 1) the students were asked to choose from a given set of points the ones, which they thought to belong to the given angle. In item 3 (Figure 2) the students picked from the given set of points the ones, which they thought belonged to the angle of the triangle added into the same figure that was used in item 2. Items 2 and 3 were developed from the testing method presented by Hershkowitz et al. (1987). The item where it was asked students to write a definition for a plane angle was given as a first item on a questionnaire but it

was on the same paper as the other two items so the answerer could answer the questions in any order and easily correct the definition after answering more concrete items 2 and 3 if she/he felt it necessary.

Which of the points

A, B, C, D, E, F, G, H, and I belong to the angle α ?

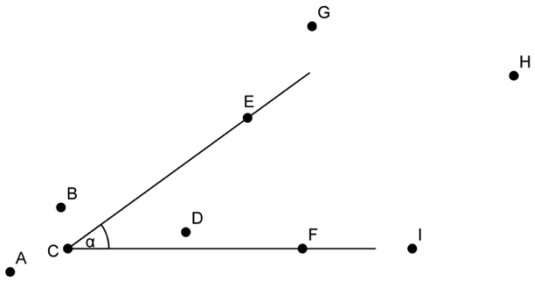


Figure 1: Item 2 of the questionnaire

Which of the points A, B, C, D, E, F, G, H, and I belong to the angle α in a triangle CFE ?

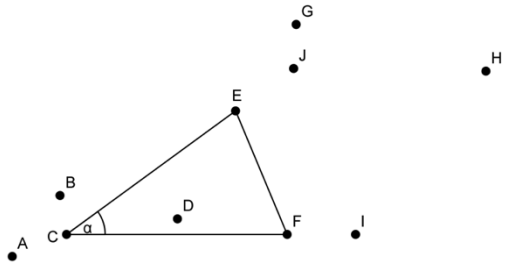


Figure 2: Item 3 of the questionnaire

RESULTS

Based on the earlier research literature we could expect that especially three particular classes of interpretations to the plane angle would be found in our data, namely an angle interpreted as (1) an amount of turning about a point between two lines; (2) a shape, formed by two lines or rays diverging from a common point (the vertex); (3) one of the two regions into which the two sides of the angle (rays) split the plane.

In the following, we call them as turning interpretation (TI), line interpretation (LI) and region interpretation (RI). However, our data revealed that we have to both broaden and specify the range of these three possibilities how the concept of a plane angle can be interpreted. First there were few student teachers, who interpreted an angle to be limited only to a 'corner' of an angle (corner interpretation CI) consisting only the vertex of an angle or/and its 'close surrounding'. There were also different interpretations of LI and RI depending if the angle was considered including only those elements visible in the actual drawn picture of the angle or if the angle was considered continuing endlessly in a direction it specifies. Finally, we classified interpretations into eight categories, namely to TI and CI and to six categories shown in Figure 3.

Point selection task (Item 2)

We will begin by presenting first a summary of student's answers to the point selection task (item 2). In the whole data (n =191) there were two prospective teachers who thought that only the vertex C from the given points belonged to the angle α reflecting the interpretation CI and two prospective teachers who didn't select any of the points to the angle α probably because they interpreted a plane angle in item 1 to be a measure (TI) and not a kind of geometrical line or region construction. Table 1 presents the sub-categorisation we applied to LI and RI in our study. Correspondingly Table 2 shows how the students' answers were distributed into these sub-categories.

Point selection task (Item 3)

When the angle α was placed as an angle of the triangle it did not essentially seem to affect the fact whether an angle was addressed according to the line interpretation or the area interpretation. In the whole data, 63.8 % of the student teachers chose the points in item 2 according to the line interpretation and 27.2 % according to the area interpretation. In the item 3, the corresponding percentages were 64.9 % and 32.5 %. However, as it was considered an angle of the triangle student teachers chose generally only the inner points of the triangle or the points belonging to the sides of the triangle. When 67.5 % of the students chose in the item 2 also the points of the angle belonging outside the part of the drawn angle, in item 3 only 11.5 % of the students did the same.

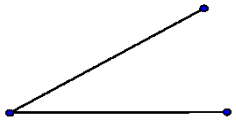
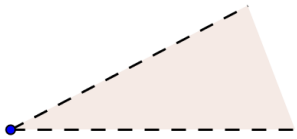
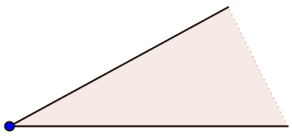
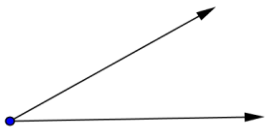
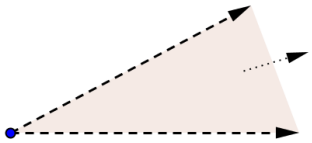
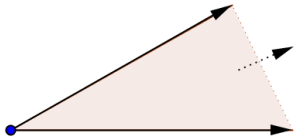
The type of interpretation	The line construction of two line segments or two rays	The region construction with boundaries excluded or	included
Bounded by the drawing (visible)	 CEF	 D	 CDEF
Continuing to the infinity (imaginary)	 CEFI	 DH	 CDFHI

Table 1: Classification of the line and region interpretations to the concept of a plane angle. The combinations of capital letters next to the pictures refer to the corresponding selection of the points in Figure 1.

The type of interpretation: Bounded or infinite/line or region construction	The line construction of two line segments or two rays		The region construction with boundaries excluded		The region construction with boundaries included		
	Class teachers (n=91)	Subject teachers (n=100)	Class teachers (n=91)	Subject teachers (n=100)	Class teachers (n=91)	Subject teachers (n=100)	Total
Bounded by the drawing (visible)	23	33	0	0	3	1	60
	25.3 %	33.0 %	0.0 %	0.0 %	3.3 %	1.0 %	31.4 %
Continuing to the infinity (imaginary)	32	34	1	0	21	26	114
	35.2 %	34.0 %	1.1 %	0.0 %	23.1 %	26.0 %	59.7
Total	55	67	1	0	24	27	174
	60.4 %	67.0 %	1.1 %	0.0 %	26.4 %	27.0 %	91.1 %

Table 2: Distribution of students' responses to item 2 (n=191).

The definition task (Item 1)

The writing of the definition to the concept of the plane angle proved to be a difficult task to many prospective students and especially to the prospective class teachers. Some students left the item 1 totally unanswered. Because of this, we restrict the examination here to the data concerning only prospective subject teachers (n=80). About half of the respondents had attempts to write the description in the form of the definition, such as "An angle is formed by…", "When two lines intersect, …", An angle is a relation between.." etc. Because the answers to this item were so vague we do not here give precise numbers of the occurrence of different types of definitions. Instead of that, we present some general observations from to what kind of ideas and concept images of the angle the attempts of defining an angle concept seemed to be based.

In the point selection task there were very few such students who restricted the angle concept so that only the vertex C would belong to the angle. However, in the defining task (item 1) remarkable many students seemed to have a kind of vertex or "sharp point interpretation" of a plane angle as the following examples show "Point of convergence formed by two line segments", The common point of two lines which forms an acute or an obtuse angle", "An angle is an acute or an obtuse point in a solid" etc. An interesting observation as well was that roughly estimated every fourth of the respondents who gave a written definition in item 1 described it so that it did not seem to base on the same concept image as the point selections in item 2 would reflect. In the following, we will present five examples of the inconsistencies between the definitions students' wrote as an answer to item 1 and the selection of points they made in item 2.

Example 1. A student teacher wrote a definition "Two straight lines intersect each other. An angle stays between the lines". However, in item 2 the student considered that the points C, E and F only belonged to the angle α and did not choose the points D, H and I. The selection seems to base on the concept image "a combination of line segments intersecting each other" instead of that what was written in the definition.

Example 2. Another student teacher defined an angle as "the region between two straight lines which have a common point" but chose only the points C, D, E and F which corresponds the interpretation "the finite region bounded by two line segments including the boundaries".

Example 3. The definition given by a student teacher was "An angle consists of two line segments and of their common intersection or of the common point from which the sector opens". However, she chose only the points C, E, F and I seeming to be applying a concept image "combination of two rays starting from the same point".

Example 4. The definition which a student teacher wrote was "An angle consists of two line segments and of their common intersection point from which the sector opens". In item 2, he chose the points: C, E, F and I seeming to apply more like a conception "Angle is formed by two rays starting from the same point"

Example 5. After writing the definition "Two straight lines meet each other at one single point" the prospective teacher chose in item 2 the points D and H reflecting a concept image "An angle is one of the infinite regions between straight lines without boundaries".

One possible explanation for the fact that the concept images applied in items 1 and 2 do not correspond to each other can be the fact that respondents use the concept straight line when they actually mean the concept line segment or vice versa. But these linguistic inaccuracies do not explain all the incompatibilities of the concept images as one can see also from the examples above. It is important to notice from the point of view of the theory that the concept images which some individuals used seemed to be at least in some extent dependent also on the context where they was taken in use.

DISCUSSION

To summarise the results of our study revealed fairly clearly that prospective teachers interpret the concept of an angle by several ways. Some respondents interpreted an angle as a line consisting of two line segments, some consisting of two rays, and some as a region defined by these elements. On the other hand, interpretations differed as to whether an angle continues outside the part shown in the drawing in the direction determined by the angle, or not. The results of our examination showed that even the adults who have completed their years of mathematics studies at lower and upper secondary school – and many of whom have also pursued the studies of mathematics at university level – still cherish various notions (beliefs) on such basic concepts of elementary mathematics as an angle, and these different notions and beliefs can remain very much alive although we use concepts in our mutual discussions regularly.

Possibly, this fact can partly be explained by school mathematics learning practices. The nature of the exercises typical of school mathematics, like calculate …, draw …, classify …, define the magnitude of … etc., seem to allow communication on the issues to be examined as well as the completion of the exercises even though the basic concepts are understood in ways that are fundamentally different.

In our view, mathematical concept formation could be enhanced by deliberately drawing attention to the differences of the interpretations learners may have even of the basic concepts of mathematics and by critically debating and negotiating the various ways of interpretation in line with the socio-constructivist learning theory. Our research also showed fairly clearly that few Finnish prospective teachers were not at all able to put their idea of the concept definition of the concept plane angle into words. It seems that neither school nor university studies ensure that students are familiar with the idea of mathematical definition or with the requirements of the form and formulation of such a definition. It seems highly likely that the significance of learning to formulate a definition – i.e. being forced to analyse the content and meaning of a concept and to search for an explicit and easily comprehensible way of expressing this meaning – is not held in adequate esteem on any level of school education.

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