EXPLORING STUDENTS' MENTAL MODELS IN LINEAR ALGEBRA AND ANALYTIC GEOMETRY: OBSTACLES FOR UNDERSTANDING BASIC CONCEPTS

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In this paper, we discuss the relevance of 'Grundvorstellungen' (GVs), a didactical category to analyze students' mental models in comparison to the intended mathematical meanings in the context of Linear Algebra and Analytic Geometry. Diagnostic tasks were used to reveal students' conceptual understanding in this field of expertise. In particular, an open item format was chosen to elicit students' individual GVs and to explore how they use them while working on mathematical tasks. 30 students from upper secondary school participated in our study; data was collected by a paper-and-pencil test. The results show that elaborated representations of GVs foster students' understanding of mathematics and facilitate the process of finding problem solving strategies.

INTRODUCTION

Research on students' understanding of mathematical content is huge and varies with respect to constructs and categories employed for analyzing different facets. Some authors elaborate on procedural aspects of knowledge construction and underline the role of abstraction when students delve into mathematics (cf. Dreyfus, 2012). Other research investigates the role of mental models that students build up and to which degree these adequately reflect the mathematical properties of a specific concept (cf. Fischbein 1989; Vinner & Tall, 1981). While introducing the term *concept image*, Tall and Vinner (1981) explicitly accentuate the individual understanding that students develop when trying to make sense of the mathematics they encounter in the classroom.

In German didactics tradition, the construct of *Grundvorstellungen*, abbreviated here as GV, serves as essential tool to capture both normative and intuitive interpretations of mathematics. Vom Hofe, Kleine, Blum and Pekrun (2005) emphasize that the value of the construct lies in interpreting GVs as "elements of connection or as objects of transition between the world of mathematics and the individual world of thinking" (p. 2). In our study we are interested in gaining insight into upper secondary students' GVs in the field of Linear Algebra and Analytic Geometry and how those influence students' performance. In order to reveal what students really know and understand diagnostic tasks were employed.

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THEORETICAL BACKGROUND

Exploring the role of intuition for the learning of mathematics has a long tradition in PME research. One essential starting point for subsequent research was provided, for instance, by the seminal work of Fischbein (1989) who differentiates algorithmic, intuitive and formal knowledge. In particular, he stresses:

> To think by manipulating pure symbols which obey only formal constraints is practically impossible. Consequently, we produce models which confer some behavioral, practical, unifying meaning, to this symbols. (p. 9)

These kind of students' models of mathematical concepts and procedures and how their individually constructed knowledge conflicts with the mathematically intended one have been studied in depth. One promising approach lies in analyzing students' *concept images* in relation to the intended *concept definitions*. Here, Tall and Vinner (1981) use the term *concept image* "to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). When working on mathematical tasks, students base their decisions on the *concept image*. To that effect, Vinner (1994) could show how obstacles in calculus occurred since students retain, for instance, a restricted *concept image* of a tangent developed earlier. This *concept image* of a tangent to a circle provokes difficulties in students' learning of calculus when confronted with the analytical definition of a tangent.

Concept images help to identify prototypes that students apply inappropriately in specific situations so that obstacles occur due to incorrect generalization of constructs. Considering GVs, the construct provides a broader scope to analyze students' sense-making and occurring hindrances (cf. Prediger, 2008). Vom Hofe et al. (2005) point out that three significant aspects characterize the process of building up GVs during mathematical concept acquisition:

- constitution of meaning of mathematical concepts based on familiar contexts and experiences,
- generation of generalized mental representations of the concept which make operative thinking (in the Piagetian sense) possible,
- ability to apply a concept to reality by recognizing the respective structure in real life contexts or by modeling a real life situation with the aid of the mathematical structure. (p. 2)

Prediger (2008) uses an explorative item format to access multiple facets of students' GVs when dealing with multiplication of fractions. From the mathematical viewpoint, the following GVs can be activated in the given situation:

- repeated addition, repeated adjoining (temporal-successive interpretation)
- part of interpretation
- scaling up and down
- multiplicative comparison
- area of rectangle. (p. 10)

When comparing to students' GVs, Prediger (2008) showed "that the individual models for multiplication were more heterogeneous and more distant from the mathematically intended models than for addition" (p. 10). Most students still applied the model for multiplication of natural numbers (repeated addition) which cannot be transferred to the multiplication of fractions. Here, analyzing student performances reveals inadequate GVs that occur as implicitly learned rules in another context.

Our study aims at exploring students' GVs in Linear Algebra and Analytic Geometry, a school topic which introduces a great variety of constructs and concepts. Classroom activities in this area are characterized by a dominance of algorithmic procedures and the use of schemata to arrive at solutions (Tietze, Klika & Wolpers, 2000). As a result, such treatment does often not allow students to develop a deep understanding of the mathematical concepts at hand (Malle, 2005).

In particular, we draw on the work by Wittmann (2003) who distinguishes the following three GVs to capture the interplay between Geometry and Algebra:

- *Algebraization*: Students use algebraic expressions to structure a presented (geometric) situation (parametrization, vectorization), and place geometrical objects in the coordinate system.
- *Geometrization*: Students translate algebraic equations into a geometric object to use for further interpretation.
- *Structural Generalization*: Students attend to overriding structural features, and they are involved in abstraction and generalization to bring together concepts on a meta-level.

Mostly teaching of Linear Algebra and Analytic Geometry is restricted to paying attention to developing GVs in *Algebraization* or *Geometrization* (Wittmann, 2003). However, to attain comprehensive understanding that pretends insular knowledge the development of GVs in *Structural Generalization* is decisive (Tietze, Klika & Wolpers, 2000).

RESEARCH QUESTIONS

The research at hand is part of a larger study to survey significance and construction of diagnostic tasks as an instrument to understand students' difficulties with main concepts of Linear Algebra and Analytic Geometry in school (cf. Schueler, 2013). With respect to the theoretical background we pay attention to students GVs on *Algebraization* (GV_A), *Geometrization* (GV_G), and *Structural Generalization* (GV_{SG}). In particular, we pursue the following research questions:

- Do students have preferred GVs (GV_A, GV_G or GV_{SG}) in the field of Linear Algebra and Analytic Geometry?
- How do students deal with mathematical tasks that entail interconnections of different GVs?

METHODOLOGY

Qualitative methods are used for exploring students' GVs while working on specific tasks. During a period of five weeks we observed corresponding lessons and analyzed the teaching material in order to construct a set of diagnostic mathematical tasks implying key aspects of Linear Algebra and Analytic Geometry in school. Data was collected by a one-hour paper-and-pencil test composed of seven diagnostic tasks. In this paper we focus on three tasks to highlight different facets of GVs.

The sample consists of 30 students that range in age from sixteen to eighteen. Among them, 18 female and 12 male students who attend grade 12 of a German high school. In addition to the test we collected some information about students' general performance level in mathematics and their self-assessment compared to the average of the class; these results are not presented in this paper.

RESULTS AND DISCUSSION

For the sake of brevity the presentation of results is limited to exemplary findings which illustrate students' solutions against the background of the three basic GVs discussed in the theory section. In addition, we enrich our presentation by discussing essential mathematical aspects and by reporting typical obstacles.

Task 1

a) Explain with your own words the concept 'vector'.
b) Describe situations of application in which it is essential to use vector algebra.

Introducing vectors in school is based on at least two different approaches, i.e. vectors are considered as equivalence classes of arrows or as n-tuples. In an equivalence class of arrows a vector is defined as an infinite set of arrows with same length, same orientation and same direction. The n-tuple model is based on abstract understanding of a vector as an ordered list of elements.

In task a) we intend to reveal students' prevalent GVs. In addition, task 1a) emphasizes what relevance students' attach to the use of vectors in applications. Table 1 summarizes the answers given by students.

equivalence class of arrows	n-tuple	incorrect answer	no answer
54%	17%	23%	6%

Table 1: Students' answers to problem 1a).

Having observed the lessons, we can confirm that both aspects of the vector concept were introduced in class. However, 54% of students rely on the geometric understanding of an equivalence class of arrows (GV_G) while only 17% of them consider the n-tuple concept (GV_A). Thus, the majority of students are able to define a vector as an equivalence class of arrows. Reviewing relevant teaching material we assume that the preference for a geometric association (GV_G) results from the fact that the majority of the lesson material deals with geometric problems.

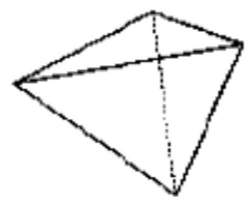
The answers given to task 1b) underline this aspect as 87% of students use vector algebra in geometric situations for example by considering the routes of airplanes (GV_G).

In sum, 29 % of students are not able to define the vector concept correctly. About 92% of the incorrect answers result from a deficient geometric interpretation of a vector as single arrow, placed at a concrete position in a three-dimensional coordinate system. Only 12% of students use both GVs to describe the vector concept. The combination of the geometric and the algebraic definition of a vector requires focusing on general mathematical characteristics common to both approaches. These thoughts refer to aspects of structural generalization (GV_{SG}) and present an elaborated understanding of the concept of vectors.

Task 2

The geometric figure is called a regular tetrahedron. It consists of four equilateral triangles.

Draw a figure to illustrate a convenient way to place the tetrahedron in a Cartesian coordinate system. Describe the position of the tetrahedron as accurately as possible.



In task 2 the students were asked to give a possible parameterization of a tetrahedron. This task demands students to activate different facets of GV_A. In the first place, the task strongly refers to GV_A in terms of using algebraic expressions to describe and structure a presented geometric figure. In addition, a correct solution requires the understanding of typical characteristics of a tetrahedron like equal edge length. That is, task 2 furthermore addresses key aspects of studying global features of a geometric figure.

In order to find a solution to this problem the students need to choose a convenient way of placing the Cartesian coordinate system and its point of origin and of translating the geometric characteristics of a tetrahedron into algebraic expressions. Table 2 demonstrates the distribution of the students' answers to task 2.

correct answer	incorrect answer	no answer
47%	33%	20%

Table 2: Students' answers to problem 2.

47% of the students are able to give an adequate visualization of the tetrahedron. However, it is notable that the correct solutions differ with respect to placing the point of origin. The majority of students identify one surface of the tetrahedron with the x1-x2-plane as shown in Figure 1. Figure 2 shows an example of an alternative way that students chose to locate the Cartesian coordinate system.

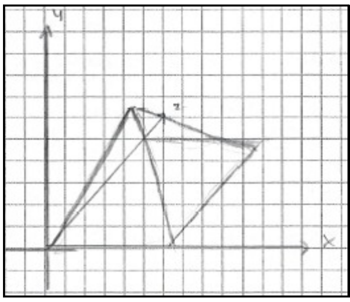


Figure 1

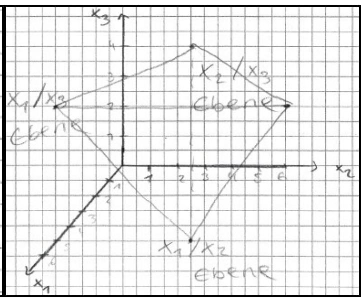


Figure 2

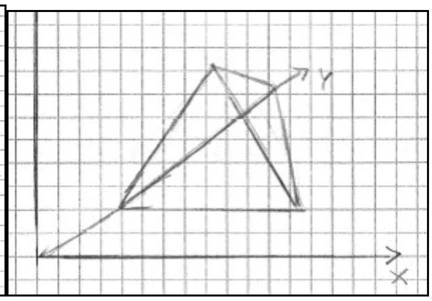


Figure 3

However, 53% of the students are not able to give a correct answer. Analyzing the incorrect answers leads to two major problems. First, we observed that students face difficulties when identifying geometric characteristics of the tetrahedron. On the one hand some students disregard the aspect of equilateral triangles and on the other hand they misinterpret the tetrahedron as a square pyramid as shown in Figure 3. The second difficulty lies in choosing a position of the tetrahedron in the Cartesian coordinate system that facilitates algebraic parameterization. In sum, applying GV_A which capture the process of algebraization, is problematic due to lacking understanding of some basic geometrical features.

Task 3

a) *Describe the position of the planes (i) or (ii) in a Cartesian coordinate system. Draw a figure which illustrates the position of the plane.*

(i) $x_1 = 4$

(ii) $x_1 - x_3 = 0$

b) *Give a possible equation of a plane which lies vertical to the $x_1 - x_3$-plane. Explain your choice.*

In-depth understanding of geometric objects in Linear Algebra and Analytic Geometry manifests itself in the ability to switch between a geometric characterization of an object and the corresponding algebraic expression. The ability of combining effectively these different representations is part of GV_{SG}. In task 3 a) the students were asked to give an adequate geometric description of a plane which is presented in coordinate form, whereas subtask b) deals with this problem vice-versa. Table 3 sums up students' answers.

3 a)	correct answer	incorrect answer	no answer
	35%	35%	30%
3 b)	correct answer	incorrect answer	no answer
	44%	40%	16%

Table 3: Students' answers to problem 3a) and 3b).

Our findings show that 35% of the students answer task 3a) correctly, and 44% of them are able to give an adequate solution to task 3b). The relation between correct and incorrect answers for both tasks is hardly different indicating that both GVs (GV_A and GV_G) are equally assessable for students. However, considering the number of students that are not able to provide an answer at all, it appears that algebraization allows more students to approach the mathematical content. From explanations that students wrote to task 3a), we can gather that the missing answers are due to deficient understanding of the coordinate form of a plane (cf. Schüler, 2013). Several students brought forward the argument that the expression $x_1 = 4$ does not describe a plane but a single point in the coordinate system. This argumentation reveals a typical obstacle, i.e. students interpret the missing of a coordinate signifies it to be zero (cf. Wittmann, 2003).

CONCLUSION

The presented problems stress in manifold ways the relevance of GVs in learning Linear Algebra and Analytic Geometry. Considering our exemplary findings we are able to underline the function of GVs as *hinges* which facilitate the transition from students' individual understanding of situations described in tasks to the respective mathematical models.

Regarding research question one and two our findings show that students neither have a preference for GV_A nor GV_G. However, the tasks would allow combing both GVs as required in the category GV_{SG}. Given that GV_{SG} are essential for developing a deep understanding, teaching would profit from using contexts that encourage structural generalization.

Reviewing teaching material and schoolbooks traditionally used in the majority of German high schools shows that the preference of daily practice is to emphasize either GV_A or GV_G, i.e. dealing with characteristics of geometric figures is almost limited to finding an algebraic expression. This proceeding leads to the phenomenon that students learn solution strategies by heart and try to memorize how to fit them to tasks without activating a deeper mathematical understanding (cf. Tietze, Klika & Wolpers, 2000). Such behavior could be seen as well in students' task performance in our study.

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