CONNECTIONS AND SIMULTANEITY: ANALYSING SOUTH AFRICAN G3 PART-PART-WHOLE TEACHING

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In this paper analysis of Grade 3 mathematics teaching in South Africa shows evidence of associations between teaching and learning outcomes in an adapted learning study. The intervention dealt with partitioning and part-part-whole relations, taking a structural approach within tasks and representations. Our analysis of this teaching emphasizes simultaneity of examples, and connections within and across examples and representations. This analysis indicated differences in enactment of a jointly planned lesson that related to different patterns of learning outcomes between the three classes. Episodes of teaching containing work with representations marked by connections and simultaneity closed gaps in learning outcomes seen in the pre-test.

INTRODUCTION

Difficulties with linking teaching and learning in any direct way have been noted in the literature. Complexity relates to the need to take prior understandings into account, making it hard to directly compare the efficacy of teaching. However, writing also notes the importance of teaching for the possibilities of learning, and acutely so in contexts of disadvantage.

In this paper, we share a micro-analysis of videotaped Grade 3 mathematics teaching in South Africa that shows evidence of associations between teaching and learning outcomes in an adapted learning study intervention (Lo & Pong, 2005). The intervention dealt with part-part-whole relations, taking a structural approach and introducing structural representations – both new to participating teachers and students. Data were collected on students' prior understandings of this topic. Our analysis of teachers' work with part-part-whole representations emphasizes simultaneity of examples and connections within and across examples and representations. This analysis indicated differences in enactment of a jointly planned lesson that related to different trajectories of performance for three classes. Further, this focus suggested that teaching episodes marked by connections and simultaneity could close gaps in pre-test performance.

We begin with an overview of literature on part-part-whole structures and representations within additive relations, noting that operational conceptions are more prevalent in South African curricula.

THE PART-PART-WHOLE RELATIONS AND REPRESENTATIONS

Additive relations is an area with a variety of nomenclatures for problem types, useful representations, and solution strategies. An area of contention relates to directionality

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in the teaching and learning of additive relations. A significant body of work advocates counting as the fundamental base for addition and subtraction (Carpenter, Fennema, Franke, Levi, & Empson, 1999). Addition and subtraction, in this view, are built on an operational approach. Standing counter is a more structural approach in which addition/subtraction is viewed fundamentally as a relation between parts and wholes (Schmittau, 2003).

Parallel to this discussion are representational options that push more in either operational or structural directions. The empty number line representation (a) advocated in the RME literature (Beshuizen, 1999) tends to align with more operational conceptions, while variations of part-part-whole representations (b) push towards structural relations (Figure 1):

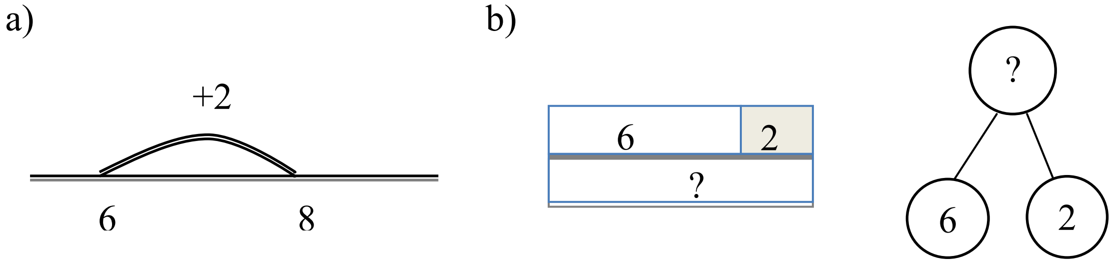


Figure 1: Part-part-whole representations.

Structural orientations to additive relations, in task and representation terms, were taken in this study. Systematicity, equivalence, commutativity, completeness and inverse relations can be dealt with in the context of part-part-whole problems. These ideas require connection between partition examples and help to build generality into specific working (Mason & Johnston-Wilder, 2004).

THEORETICAL FRAME

Variation theory (VT) forms the theoretical base for our analysis. VT argues the need for variation in the midst of invariance, as a condition for learning (Marton & Pang, 2006), necessitating a focus on *what* is simultaneously available and whether, and if so, how, connections between examples are drawn. Schmittau (2003) recognizes part-part-whole relations as the central invariant feature of all additive relation problems – with examples and representations linked to this general theme. Representations can remain invariant across examples, emphasizing their *general* usefulness. Alternately, invariant examples allow for introduction of new representational pathways, providing openings for connections between representations and expanding representation spaces.

RESEARCH DESIGN

Learning studies share common features with Japanese lesson study. As in lesson studies, the teachers were involved in the development, teaching and retrospective analysis of lessons. The broader study involved two sub-study cycles during 2013, each of three weeks' duration, with the three Grade 3 teachers/classes in one suburban school in Johannesburg. In this paper we analyse results from the videorecorded first lesson together with learner performance on two worksheets in the first study. Analysis of student pre-test performance indicated differences between the classes in prior understandings of part-part-whole relations, but performance profiles shifted on the worksheets set after sections of teaching – summarized in Table 1:

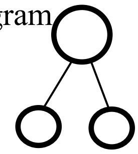
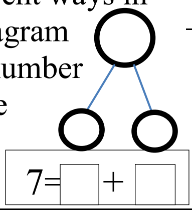
RESULTS PRESTEST				
Task 1: Split 9 marbles in two boxes	No of correct partitions	Class 3.1 (n=40)	Class 3.2 (n=39)	Class 3.3 (n=44)
	10	0%	0%	0%
	4-9	12%	46%	23%
	0-3	88%	54%	77%
Task 2: Split the number 12 in as many different ways as you can 12 = __ + __	No of correct partitions	Class 3.1 (n=40)	Class 3.2 (n=39)	Class 3.3 (n=44)
	10	10%	8%	27%
	5-9	48%	41%	55%
	0-4	42%	51%	18%
RESULTS WORKSHEET 1				
Task: Split number 7 in different ways in a triad diagram 	No of correct partitions	Class 3.1 (n=43)	Class 3.2 (n=43)	Class 3.3 (n=46)
	8	33%	63%	37%
	5-7	44%	18,5%	50%
	0-4	23%	18,5%	13%
RESULTS WORKSHEET 2				
Task: Split number 7 in different ways in triad diagram and in number sentence 	No of correct partitions	Class 3.1 (n=43)	Class 3.2 (n=44)	Class 3.3 (n=46)
	8	25%	50%	66%
	5-7	46%	36%	30%
	0-4	29%	14%	4%

Table 1: Pre-test, worksheet 1 and worksheet 2 results

Pre-test results indicated that while class 3:2 were stronger on the marble splitting activity, class 3:3 were stronger on producing abstract number partitions of 12. Worksheets 1 and 2 followed segments of teaching that are analysed in this paper. Worksheet 1 results showed class 3:2 performing better than the other two classes, in spite of lower performance in abstract number partitioning in the pre-test. In contrast, worksheet 2 data showed class 3:3 outperforming 3:2. Class 3:1 performed weakly throughout.

Our analysis of teaching explored what produced these shifts in performance. In the teaching sections preceding worksheets 1 and 2 we saw differences in the three teachers' work with examples and representations. Salient features of contrast related to which examples were elicited, whether examples were simultaneously visible, and how they were represented and connected within and across examples, and episodes.

FIRST SECTION OF TEACHING

In the planning meeting the teachers had agreed that in the first section they would introduce the idea of splitting a 'whole' into two 'parts'. The triad diagram – a new representation – was to be introduced within the activity of splitting 7 monkeys between two trees (Cobb, Boufi, McClain & Whitenack, 1997). As the descriptive summaries indicate, Teacher 3:1 did not adhere to this plan

Teacher 3:1

Reporting subsequently that she thought 7 would be too easy, teacher 3:1 worked with whole values of 26 and 10 in this section. The first episode consisted of 16 'separate' offers of partitions, 5 of these incorrect. For the first five correct examples, the split offered was represented in a triad diagram. These five triad partitions were then transferred to a table with split values verbally replayed. No gestures or actions emphasized either the connection between representations or the part-part-whole relationship. Thus, the table and triad representations were visible simultaneously but we described the connections between them as 'weak'.

In the next episode a concrete situation with ten monkeys and two trees was visible on the board. Physical splitting actions and table representations of the parts were produced with simultaneous visibility of four partitions in the table, but with each partition produced 'separately' with all monkeys returned to trees after each partition. No explicit connection was made verbally or gesturally by the teacher in support. Her instruction for worksheet 1 was to work on partitions of 30, rather than 7 – the planned whole value.

Teacher 3:2

The teacher introduced the concrete situation visually and orally, and asked students to split the monkeys in different ways. Eight unique partitions of 7 were offered, with some partitions produced by moving monkeys from one partition arrangement in the two trees to another. As the students physically split monkeys between the trees, the teacher verbally 're-played' their actions in numerical terms and subsequently wrote all the different partitions in a table on the board. Across all eight examples offered, the teacher coherently connected students' physical split results to verbal and tabular representations. This coherence between representations, with tabular representation added after the first three examples making all examples visible simultaneously – marked 'strong' connection.

In episode 2 the teacher returned to monkeys to be split between the two trees. She introduced the triad diagram and verbally related it to the concrete situation. Gesturing supported verbal connections between ‘monkeys in trees’ and ‘parts’ in the triad model. Three different numerical partitions were produced by learners, without physical actions of moving monkeys. The teacher rubbed out the numbers in the triad when she moved to the next example. The same situation used in episode 1 was thus linked to a ‘new’ representation, providing an expanded representation space and a pathway to it from a situation that was familiar.

Teacher 3:3

Teacher 3:3 dealt with just one example of splitting number 7. The representation space included, simultaneously, a verbal description of the visible concrete monkeys/trees situation, physical splitting actions with the whole and part values resulting from this action then transferred into a triad diagram. Gestures and verbal descriptions maintained ‘strong connections’ between the concrete situation, actions and triad diagram. We note gestural and verbal representations repeatedly as research continues to note their salience within mathematics teaching (Alibali et al., 2014). The teacher opened activity to individual working at this point to produce more examples of a split. The inclusion of only one example of splitting 7, in VT terms, provides limited possibility for students to discern other partitions of 7 or to see the invariance of representations across examples. Table 2 summarizes the teaching preceding worksheet 1, including the number and simultaneity of examples, the whole number, the representation space and the nature of connections.

Teacher	No of examples Whole number	Representations	Connections	Simultaneity of examples
3.1	16 (26 –whole)	V Ta Tr	Weak	Different partitions visible
3.1	3 (10 –whole)	V C A Ta	Weak	Different partitions visible
3.2	8 (7 –whole)	V C A Ta	Strong	All partitions visible
3.2	4 (7 –whole)	V C Tr	Strong	Rubs out examples
3.3	1 (7 –whole)	V C A Tr	Strong	One example

Table 2: Section 1 teaching: V = verbal, C = concrete situation, A= physical action, Ta = table and Tr = triad

Teachers 3:1 and 3:2 both present multiple examples of splits of the given number, in contrast to teacher 3:3. In teacher 3:2’s work, the table and triad representations record the outcomes of physical splitting of monkeys between the two trees, whereas in teacher 3:1’s first episode, the split is enacted on abstract numbers. Further, while no attention was given to systematic production of different splits in any of these teaching episodes, teacher 3:2 did produce a complete set of splits of 7 in her Episode 1, with checks in her questioning (‘Is there still another way?’). In class 3:1, the use of 26 is an unwieldy choice for producing completeness, but the production of only four splits of 10 in Episode 2 suggests lack of focus on this aspect anyway.

Verbal descriptions and gesturing connecting between the concrete situation, splitting action and triad representations were consistently present in teacher 3:2 and 3:3's lessons, in contrast to teacher 3:1's lesson. Simultaneous presence of whole and parts could be seen in all three classes, but in class 3:1 the whole faded as the teacher started talking about 'pair of numbers'.

This analysis confirms that multiple examples of split are more useful pedagogically than a single example from the perspective of learner performance, seen in the contrasts in performance on worksheet 1 between classes 3:2 and 3:3. But careful selection of examples and strong and consistent connections between representations are also critical within teachers' handling of sequences of examples.

SECOND SECTION OF TEACHING

The teachers had agreed to continue with partitions of 7, expanding the splitting activity to a number sentence representation, with worksheet 2 following, prior to a final missing part problem task (completed in two classes only and therefore omitted from current analysis). Teacher 3:1 and 3:2 both handled one episode, while teacher 3:3 handled two episodes before worksheet 2. Teacher 3:1 used 9 as the whole instead of 7 and introduced a missing part problem before worksheet 2.

Teacher 3:1

Rather than linking Section 1 representations to number sentences Teacher 3.1 used whole value 9 and dealt with three examples as missing part addition problems. In the first example the concrete situation, triad and number sentence were simultaneously visible. Strong connections were maintained between the teacher´s talk and moves from concrete situation to triad and symbolic form, in contrast to the other two examples where connections became weaker. In these subsequent examples, a triad diagram was presented in one example, and a concrete situation and number sentence in the third example, without connection to the triad. Therefore connections between representations were less consistent. Further, worksheet 2 with whole value 7, was disconnected from the teaching.

Teacher 3:2

Teacher 3:2 returned to the concrete situation and the triad diagram. With the seven monkeys/two trees visible on the board, the teacher verbally linked the concrete situation to the triad, and then transferred the triad partition to a number sentence. Across the four examples dealt with, the different representations were simultaneously present but with more sporadic verbal reference to the concrete situation, but with monkeys/trees remaining visible. Across all four examples verbal descriptions and gesturing connected representations and maintained visibility of the part-part-whole relationship, again marking 'strong connection'. Her rubbing out each example of splitting 7 resulted in a lack of simultaneous representation of instances, and therefore no possibility for linking examples.

Teacher 3:3

Worksheet 1 was followed by students splitting 7 monkeys again using concrete situation, physical action, triad and table. Initially, splitting was demonstrated with physical actions leading to results presented in triad form, but physical actions were dropped in the next three examples with direct moves to triad representations. The next four instances were presented in a table. Across this episode the teacher´s verbal descriptions and gestures connected different partitions and representations. The teacher handled eight partitions of 7, all shown on the board simultaneously. While not all representations were visible for all the partitions, there were always multiple examples using the same representation and multiple representations presented simultaneously across examples, strongly connected through talk and gesture, and additionally, possibilities to discern completeness in the example space.

In the next episode the teacher referred to monkeys while using one partition from the triad to link to a number sentence representation. In this example she connected the 'whole' and 'parts' from the triad to monkeys in trees and transformed the partition to a number sentence with coherent verbal description and gestures. Connections were therefore, again, strong. Table 3 overviews the teaching preceding worksheet 2.

Teacher	No of examples Whole number	Representations	Connections	Simultaneity of examples
3:1	3 (9 –whole)	V C A Tr N	Mostly weak	Fleeting appearance of representations
3:2	4 (7 –whole)	V C Tr N	Strong	Rubs out examples
3:3	8 (7 –whole)	V C A Ta Tr	Strong	All partitions visible
3:3	1 (7 –whole)	V C Tr N	Strong	One example

Table 3: Section 2 teaching: V = verbal, C = concrete situation, A= physical action, Ta = table, Tr = triad and N = number sentence

These descriptions indicate overlap relating to simultaneous presence of parts and whole. Teacher 3:3 produced a complete set of splits of 7 in the second episode, as teacher 3:2 did in the first episode. By leaving the eight splits on the board teacher 3:3 provides opportunity to discern the complete set of partitions of 7, in contrast with teacher 3:2 who rubs out split examples as she proceeds in this section. In class 3:1 and 3:3, some representations were not used across all the presented examples. In class 3:3 though, sporadic representations were strongly connected to each other within examples, compared to fleeting representations connected in more limited ways in class 3:1. Teacher 3:3's episode 2 included only one example, but this example was linked to the previous concrete situation using a split from the triad diagram to provide an expanded representation space. Thus, there were differences in the extent to which teacher talk connected between representations within and across examples. In class 3:2 and 3:3 verbal descriptions and gesturing supporting connections between representations were consistently present, compared to class 3:1. Contrasts related to invariance of the whole value across all episodes in class 3:2 and 3:3, while teacher 3:1 varied the whole several times.

CONCLUDING COMMENTS

The comparatively strong attainment of Class 3:2 on Worksheet 1 and Class 3:3 on Worksheet 2 points to strongly connected representation spaces and simultaneity of examples contributing directly to improved understandings. The 'newness' of the triad representation and the structural approach has, in all likelihood, made it more possible for us to see sharper distinctions in shifting performance patterns between the three classes than would be possible on a more familiar topic where prior understandings would figure. While acknowledging this, these findings point to significant possibilities for progressing learning through attention to simultaneity in the example space, and strong connections between representations and across example spaces and representations.

References

Alibali, M. W., Nathan, M., Wolfgram, M. S., Breckinridge Church, R., Jacobs, S. A., Johnson Martinez, C., & Knuth, E. J. (2014). How teachers link ideas in mathematics instruction using speech and gesture: A corpus analysis. *Cognition and Instruction*, *32*(1), 65-100.

Beshuizen, M. (1999). The empty number line as a new model. In I. Thompson (Ed.), *Issues in teaching and numeracy in primary schools* (pp. 157-168). Buckingham: Open University Press.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction.* Portsmouth, NH: Heinemann.

Cobb, P., Boufi, A., McClain, K., & Whitenack, J. (1997). Reflective discourse and collective reflection. *Journal for Research in Mathematics Education*, *28*(3), 258-277.

Lo, M. L., & Pong, W. Y. (2005). Catering for individual differences: Building on variation. In M. L. Lo, W. Y. Pong, & C. P. M. Pakey (Eds.), *For each and everyone: Catering for individual differences through learning studies* (pp. 9-26). Hong Kong: Hong Kong University Press.

Marton, F., & Pang, M. F. (2006). On some necessary conditions for learning. *The Journal of the Learning Sciences*, *15*(2), 193-220.

Mason, J., & Johnston-Wilder, S. (2004). *Designing and using mathematical tasks*. Hertfordshire, UK: Tarquin Press.

Schmittau, J. (2003). Cultural-historical theory and mathematics education. In A. Kozulin, B. Gindis, V. S. Ageyev, & S. S. Miller (Eds.), *Vygotsky´s educational theory in cultural context* (pp. 225-245). Cambridge: Cambridge University Press.