MENTAL MATHEMATICS AND OPERATIONS ON FUNCTIONS

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This study is part of a larger research program aimed at studying mental mathematics with objects other than numbers. It concerns operations on functions in a graphical environment with Grade-11 students. Grounded in the enactivist theory of cognition, particularly in problem-posing, the study aims to characterize students' mathematical activity in this mental mathematics environment. The data analysis offers understandings of strategies that students brought forth: algebraic/parametric, graphical/geometric, numerical/graphical. These are discussed in relation to implications for research on solving processes and potential for studying functions.

To highlight the relevance and importance of teaching mental calculations, Thompson (1999) raises the following points: (1) most calculations in adult life are done mentally; (2) mental work develops insights into number system/number sense; (3) mental work develops problem-solving skills; (4) mental work promotes success in later written calculations. These aspects stress the non-local character of doing mental mathematics with numbers where the skills being developed extend to wider mathematical abilities and understandings. Indeed, diverse studies show the significant effect of mental mathematics practices with numbers on students' problem solving skills (Butlen & Peizard, 1992; Schoen & Zweng, 1986), on the development of their number sense (Murphy, 2004; Heirdsfield & Cooper, 2004), on their paper-and-pencil skills (Butlen & Peizard, ibid.) and on their estimation strategies (Schoen & Zweng, ibid.). For Butlen and Peizard (ibid.), the practice of mental calculations can enable students to develop new ways of doing mathematics and solving arithmetic problems that the traditional paper-and-pencil context rarely affords, because it is often focused on techniques that are in themselves efficient and do not require other actions. Overall and across contexts, it is thus generally agreed that practicing mental mathematics *with numbers* enriches students' learning and mathematical written work about calculations and numbers. This being so, as Rezat (2011) explains, most if not all studies on mental mathematics focus exclusively on numbers/arithmetic. However, mathematics taught in schools involves more than numbers, which rouses interest in knowing what mental mathematics with objects other than numbers might contribute to students' mathematical activity. In this study, issues of functions, mainly operations on functions in graphs, are investigated. This paper reports on the strategies brought forth by Grade-11 students.

THEORETICAL GROUNDING OF THE STUDY: AN ENACTIVIST FRAME

Recent work on mental mathematics points to the need for better understanding and conceptualizing of how students develop mental strategies. Researchers have begun to

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critique the notion that students choose from a toolbox of predetermined strategies to solve mental mathematics problems. E.g. Threlfall (2002) insists on the organic emergence and contingency of strategies in relation to the tasks and the solver (what he or she understands, prefers, knows, has experienced with these tasks, is confident with; see also Butlen & Peizard, 1992). This view on emergence is also discussed by Murphy (2004), who outlines perspectives that conceptualize mental strategies as flexible emergent responses adapted and linked to specific contexts and situations. Because the enactivist theory of cognition (c.f. Maturana & Varela, 1992; Varela, Thompson & Rosch, 1991) has been concerned in mathematics education with issues of emergence, adaptation, and contingency of learners' mathematical activity, it offers a way to contribute to conceptualizations about students' meaning-making and mathematical strategies. In particular, the distinction made between problem-posing and problem-solving offers ways to address questions about the emergence and characterization of strategies.

For Varela (1996), problem-solving implies that problems are already in the world, independent of us, waiting to be solved. Varela explains, on the contrary, that we specify the problems that we encounter through the meanings we make of the world in which we live, leading us to recognize things in specific ways. We do not choose problems that are out there in the world independent of our actions. Rather, we bring problems forth: "The most important ability of all living cognition is precisely, to a large extent, to pose the relevant questions that emerge at each moment of our life. They are not predefined but enacted, we bring them forth against a background." (p. 91). The problems that we encounter, the questions that we ask, are as much a part of us as they are a part of our environment: they emerge from our interaction with/in it. The problems we solve are relevant for us as we allow them to be problems.

If one adheres to this perspective, one cannot assume, as René de Cotret (1999) explains, that instructional properties are present in the tasks presented and that these *causally determine* solvers' reactions. As Simmt (2000) explains, it is not tasks that are given to students, but mainly prompts that are taken up by students who themselves create tasks with. Prompts become tasks when students engage with them, when, as Varela would say, they pose problems. Students *make* the "wording" or the "prompt" a multiplication task, a ratio task, a function task, an algebra task, and so forth. Nonetheless, each prompt is designed following specific intentions in specific ways, which can play a role in how solvers pose problems (e.g. one does not react to two square-root functions in the same way as one does with two linear functions). In sum, each prompt can be seen to have what Gibson (1979) refers to as affordances:

> The affordances of the environment are what it offers the animal, what it provides or furnishes […] I mean by it something that refers to both the environment and the animal in a way that no existing term does. It implies the complementarity of the animal and the environment […]. If a terrestrial surface is nearly horizontal (instead of slanted), nearly flat (instead of convex or concave), and sufficiently extended (relative to the size of the animal) and if its substance is rigid (relative to the weight of the animal), then the surface

> affords support […]. Note that the four properties listed – horizontal, flat, extended, and rigid – would be physical properties of a surface if they were measured with the scales and standard units used in physics. *As an affordance of support for a species of animal, however, they have to be measured relative to the animal. They are unique for that animal. They are not just abstract physical properties*. (p. 127, emphasis added)

These affordances for Maturana and Varela (1992) play the role of triggers in relation to the solver's posing. Hence reactions to a prompt do not reside in either the solver or the prompt: they emerge from the solver's interaction with the prompt, through posing the task. Strategies are thus triggered by the prompt's affordances, but determined by the solver's experiences, where issues explored in a prompt are those that resonate with and emerge from the student, as Threlfall (2002) explains:

> As a result of this interaction between noticing and knowledge each solution 'method' is in a sense unique to that case, and is invented in the context of the particular calculation – although clearly influenced by experience. It is not learned as a general approach and then applied to particular cases. […] The 'strategy' […] is not decided, it emerges. (p. 42)

This emergent/adapted perspective offers a specific way of talking about solving problems, avoiding ideas of possession (acquisition of, choice of, of *having* things, etc.) in favor of issues about emergence, flux, movement, interactions, relations, actions, and so forth. It is this perspective that orients this research.

METHODOLOGICAL ISSUES, DATA COLLECTION AND ANALYSIS

One intention of the research program is to study the nature of the mathematical activity that students brought forth when working on mental mathematics. This is probed through (multiple) case studies conducted in educational contexts designed for the study (classroom settings/activities). This reported study is one of these case studies, taking place in two Grade-11 classrooms. Classroom activities/tasks were designed with the teacher (covering two 75-minutes sessions for each group), in which students had to operate mentally on functions in a graphical environment, that is, they had to solve without paper-and-pencil or any other computational/material aids. For example, using a whiteboard, a typical prompt consisted of showing two functions in the same graph and ask students to add or subtract them (see Figure 1).

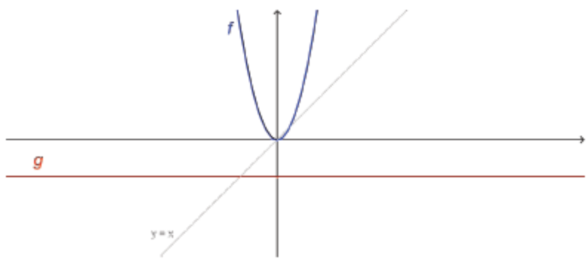

Figure 1: Example of a graphical prompt on operations on functions [$f(x) \pm g(x)$].

The activities were conducted by the regular teacher and had the following structure: (1) a graph is shown on the board and instructions are given orally; (2) students have 20 seconds to think about their solutions; (3) at the teacher's signal, students have 10 seconds to write their answer (on a sheet of paper showing a blank Cartesian graph) and then hold it up to show the teacher; (4) the teacher asks various students to show/explain their answers to others. Six thematic blocks, each composed of 6-10 prompts, were organized. The 1st block introduced students to the ideas, where both the graphs and the algebraic expressions of the functions were offered (prompts consisted of a combination of linear and constant functions). For the 2nd block, graphs of two functions (sometimes three) were given without their algebraic representation, and students had to add them mentally (functions varied from a combination of constant with linear, quadratic, square root, constant, rational, and step functions). In the 3rd block, still on the same graph, students were given the representation of one function and the result of an operation and were instructed to find the function that had been added to or subtracted from the first to obtain the resulting function (functions varied from a combination of two linear, two square-root, or a combination of a constant with a linear or square root functions, see Figure 3). The 4th block was similar to the second, but focused on subtractions. The 5th block differed in that only algebraic expressions of functions were given. These algebraic expressions could not be "directly" computed, like $f(x)=|x|$ or $f(x)=[x]$ with $g(x)=x$ or $g(x)=x^2$. The 6th block focused on symmetry, where students had to add two (linear, quadratic, by parts) functions that looked symmetrical in the graph (see e.g. Figure 2 and 4).

Data collection focused on students' strategies recorded in note form by the PI and a research-assistant, for each of the four sessions. To analyze the data, repeated interpretative readings of the field notes about the various strategies that emerged were conducted, and combined with the existing literature on functions to enrich the analysis. These repeated interpretative readings underlined three strategies, which are reported below: algebraic/parametric, graphical/geometric, graphical/numerical.

FINDINGS – ON STRATEGIES BROUGHT FORTH

Strategy 1. Algebraic/Parametric

Even when prompts were proposed in a graphical context without algebraic expressions, many students engaged in algebraic-related solving. Students referred to what Duval (1988) calls significant units for "reading" the graphical representation of a linear function and offered an interpretation in relation to the algebraic expression. That is, students brought forth parameters from the algebraic expression (the *a* and *b* of the linear function $f(x)=ax+b$) to make sense of the graphs and add them. However, because the resulting function had to be expressed graphically, they explained their answer and strategy algebraically by blending aspects of graphical information. For example, in the following addition prompt (see Figure 2), where neither function had an algebraic expression attached, many students explained that "BOTH FUNCTIONS LOOKED SYMMETRICAL, SO THE 'a' PARAMETER OF EACH LINE WOULD CANCEL OUT, AS WELL AS THE 'b' AND THUS GIVE x=0" (quotations in are taken from students' words and translated from French to English).

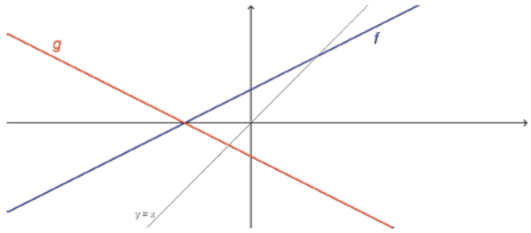

Figure 2: Addition of function graphical prompt.

In prompts where e.g. a linear function *f* would be added to a constant function, even if no algebraic expression was attached to the functions, students would say that the "a" parameter of the function *f* does not change when added with a constant function that "DOES NOT HAVE AN 'a' PARAMETER, SO THE FUNCTION'S STEEPNESS STAYS THE SAME AND ONLY THE 'b' CHANGES" giving a function parallel to *f* with a *y*-intercept at "b" instead of at 0. Thus students generated algebraic information from the graphs of the functions in order to operate and develop their solutions. They were able to draw out an algebraic context, to pose it as an algebraic task, and to solve with/in that context. Even if no algebraic expression was attached to the functions, students illustrated affordances of the prompt for them, showing that there were potential algebraic pathways in them and for them (of course, students' algebraic prominence or preference when working with functions is not new, see e.g. Vinner's, 1989, "algebraic bias"). They thus posed the prompt as an algebraic problem, solving it in relation to algebraic aspects generated for the functions.

Strategy 2. Graphical/Geometric

When facing a function that was not linear (e.g. quadratic, square root, rational, hyperbolic), students generated particular ways of working with slope and parallelism. They assigned a constantly changing rate of change/slope to some nonlinear functions with which they were dealing (students used the expressions *slope* and *rate of change* interchangeably, hence the "/"). E.g. with the addition of a quadratic and a constant function (see Figure 1), students explained that the rate of change of the quadratic function was not affected by the addition of a constant function, because a constant function "DID NOT HAVE A VARIATION" and thus the slope of the quadratic function: "WILL CONTINUE TO VARY IN A CONSTANT WAY". When students said *constant*, they meant that its appearance was not affected. Thus the resulting function of their addition would have the "SAME RATE OF CHANGE AS THE QUADRATIC FUNCTION" but would simply be "TRANSLATED DOWN" in the graph because the constant function was "NEGATIVE". Although it is not clear what exactly students meant by this "CONSTANTLY CHANGING" rate of change/slope for nonlinear functions (especially e.g. when they were dealing with $f(x)=1/x$), many of them brought forth a language that enabled them to solve their problem (and talk about it) and not worry about the variation in the function. As one student said about the square-root function, "ITS RATE OF CHANGE IS LEFT UNTOUCHED WHEN I ADD THE CONSTANT FUNCTION, SINCE IT HAS NO VARIATION".

In cases where students faced more than one nonlinear function, the above constantly changing rate of change strategy appeared insufficient, as they began analyzing functions in terms of "parallelism". For example, in Figure 3 where the function g is to be found, some students expressed that "EACH FUNCTION WAS PARALLEL TO THE OTHER" and that *g* had to be a constant function "FOR THE CURVE TO BE TRANSLATED DOWN" and that it was "NEGATIVE FOR BRINGING THE CURVE LOWER".

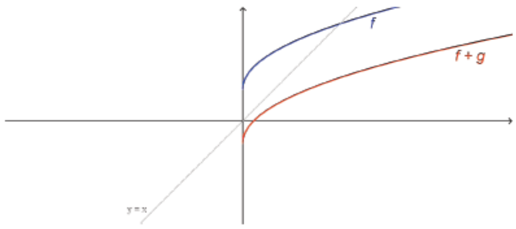

Figure 3: A prompt for which the parallelism strategy was used.

Again, this vocabulary and idea of parallelism (which can be mathematically questioned) emerged as a way of making sense without going into details about the fluctuation in image for each function. Somehow students defined these meanings through their use, in their emergent use for solving their problems. Theirs was a strategy well tailored/generated for their problem, which in turn made their problem about that strategy. To some extent, students offered a geometrical interpretation of rate of change/slope as a property not of the function, but of the curve present on the graph. They were talking about a geometric rate of change/slope, something reminiscent of Zaslavsky, Hagit and Leron's (2002) concept of slope seen as a geometric concept rather than slope seen through the lens of analytical geometry. Through their geometrical rate of change, students brought forth the nonlinearity of nonlinear functions and developed ways of engaging with/in it. By posing the prompt in geometrical terms, they generated a graphical/geometric strategy to solve it.

Strategy 3. Graphical/Numerical

Students brought forth specific points in the graphs of functions (related to Even's (1998) pointwise approach). In sum, the prompts were posed as numerical or pointwise tasks by students. Through those points, they generated exact and approximate answers (Kahane, 2003), which they combined to find the resulting function. In Figure 4 e.g. students had to find the function resulting from the addition of *f* and *g*. In this case, they would bring forth specific points: (1) where *f* cross the *x*-axis (*x*-intercept); (2) where both *f* and *g* intersect; (3) where *f* and *g* cross the *y*-axis (*y*-intercept); (4) where *g* cross the *x*-axis (*x*-intercept). For case (1), the operation is an exact calculation as the addition of the image for *f* (which is of length 0) with the one for *g* results in an image for *f*+*g* that is the same as that for *g* (it has the same image for *g* to which 0 was added). For case (2), the operation is an approximate calculation, as both images at *f* and *g* are the same, so the resulting image is double the value of the intersection point; but a precise location is impossible without knowing the exact location of the intersecting point in terms of precise length. For case (3), the same approximate calculation applies, as both images are added. For case (4), an exact answer is obtained, as in case (1). In doing this, students mingle both exact and approximate calculations to find points for the resulting function.

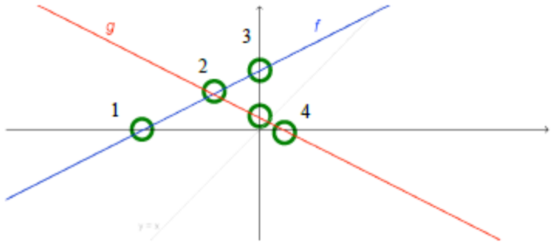

Figure 4: An addition of function prompt for which points were outlined

Students generated precise and approximate points to determine the resulting function. In so doing, they were no longer in an algebraic context, but in a blend of numerical and graphical contexts, generating numbers/coordinates that had meaning for them in the graph. E.g. when they referred to the *x*-intercept, they did not attempt to find its meaning in the algebraic expression (see Moschkovich, 1999), but worked in the graphical context to gain information for computing the resulting function. The same is true for the *y*-intercept, not treated as parameter *b*, but as a point in the graph. Their posing was numerical or pointwise, making the task about points.

DISCUSSION OF FINDINGS AND FINAL REMARKS

These strategies enacted on the spot as emergent reactions tailored to their problems offer illustrations of students' mathematical activity in this mental mathematics environment. Through their entry into the prompts, students posed their problems, making emerge affordances of the problems, that is algebraic, geometric, procedural, and so forth. Thus an algebraic posing of the functions produced an algebraic strategy; a graphical posing produced a graphical strategy; a numerical/pointwise posing produced a numerical pointwise strategy. These affordances are to be seen relative to students and the prompts, as affordances for *those* students interacting with these prompts: they do not exist in themselves, but are brought forth in the interaction with the prompt when posing the task and making them emerge.

Three main lessons can be learned from this analysis. First, it shows how students illustrated significant meaning-making capacities, as they were fluent in linking algebraic (symbolic expression), numerical (coordinate values in *x* or *y*) and graphical aspects of functions. This seems to contrast with what we know from other studies, as students are frequently reported as experiencing difficulties of many kinds when linking graphs of functions with other representations (see e.g. Even, 1998; Hitt, 1998; Moschkovich, 1999). Second, even if more research is obviously needed, this fluency underlines the potential of these mental mathematics activities for studying functions, as it occasioned numerous (and even alternative) ways of conceiving and operating on functions, e.g. algebraic, graphic, and numerical. Third, and possibly most important, the creative and adapted nature of these approaches, seen through problem-posing, underlines the importance of being attentive to students' mathematical activity when they are solving (their) problems. It shows how sensitive we ought to be, following

Threlfall (e.g. 2002), not to constrain students' mathematical doings into specific frames of expected solutions or reducing them to already known categories of solving: it offers a window onto students' mathematical activity that allows us to embrace its creative character and adaptive nature when students are solving (their) problems.

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