FLEXIBLE USE AND UNDERSTANDING OF PLACE VALUE VIA TRADITIONAL AND DIGITAL TOOLS

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Place value is a key concept for numbers that is introduced in early mathematics. It is necessary to have a flexible understanding of place value for efficient arithmetic strategies and success in written algorithmic arithmetic. In our research we explore typical mistakes and misconceptions that occur with 3rd graders in German elementary school and investigate the various underlying actions when manipulating numbers in a place value chart. We also report on a follow-up quantitative study that compares real and virtual manipulatives for place value and their effect on learning of place value.

INTRODUCTION

As demonstrated in the PME 36 plenary by Mariotti (2012), it is helpful to analyse artefacts from a semiotic perspective for their semiotic potential. The Theory of Semiotic Mediation (TSM) can be used to analyze teaching experiments at all school levels and with artefacts of any type. This latter is particularly important when we want to find out whether and how new artefacts, in particular digital ones, can improve the teaching and learning process. In our research about place value and flexible interpretation of place value charts we realized that instead of focusing on sign productions it was more helpful to look at the individual *actions* that can be observed when children work with a place value chart. Using Artefact-Centric-Activity-Theory (ACAT, Ladel & Kortenkamp 2013) we designed a digital artefact as an alternative to traditional paper-and-pencil place value charts. This paper will report about the necessity of teaching place value, will analyze typical tasks found in textbooks, and show how these tasks correspond (or not) to certain actions with either artefact, the traditional and the electronic one. Finally, it will elaborate on a pre-study that helped to design an experiment that is currently conducted with 3rd graders (N > 300) in a quantitative study that shall contribute to the research for designing learning environments for place value that has been started by Hiebert and Wearne (1992), who conclude:

> The data reported here suggest that understanding, as measured by the place-value tasks, does not translate directly into procedures but that it does interact with procedures to yield increased flexibility and power. However, this interaction is influenced by the in-structional environment and, in this case, flourished more when instruction attempted to facilitate students' understanding rather than procedural proficiency. (p. 121)

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PLACE VALUE AND NUMBER SYSTEMS

The decimal number system (in fact, any positional system) is a powerful tool for writing mathematics and doing arithmetic. Any number[1] can be written using only 10 different digits *in a unique way using a finite number of places*. It is easily possible to compare, add, subtract and –a bit harder– multiply or divide two numbers when we are given the numbers of bundles in the *maximal bundling*.

The underlying idea of the number system carries the proof for this non-obvious fact: By *creating bundles* of ten objects, bundles, bundles of bundles, etc. *repeatedly* until there are less than 10 objects of any same bundle size available we end up with a unique maximal bundling of objects. Creating bundles of objects is a basic task in many exercises in early mathematics learning, as it constitutes the basic operation for working with larger numbers.

Another key concept besides bundling is the *part-whole-concept* that concerns the fact that each number can be partitioned into smaller parts that add up to whole. While this is trivial for most of us, it is still an important fact that is not obvious to all children. It is used in almost all further arithmetic work. Creating bundles and the part-whole-concept are connected: By replacing 10 single objects with a bundle of ten objects we do not change the whole quantity, as we replace a part with another part of same size.

While any partition of a quantity into parts is feasible, some are more useful than others. If we are working with bundles, any partition that is created by (repeated) bundling, counting all bundles of the same size in one part, is called a *decimal partition*. More formally, if $n = a_0 \cdot 10^0 + a_1 \cdot 10^1 + \ldots + a_k 10^k$ then the summands form a decimal partition of n. The unique representation of a number results in a decimal partition created by less than 10 bundles of each bundle size and is called the *standard (decimal) partition* of n. In German schools, a different notation for decimal partitions is used: Instead of powers of ten a single or double letter is used. E (Einer/Ones) is used for 10^0, Z (Zehner/Tens) for 10^1, H (Hunderter/Hundreds) for 10^2, T (Tausender/Thousands) for 10^3, ZT (Zehntausender/Ten thousands) for 10^4, HT, M, ZM, …. A typical notation used for the number 324 while introducing place value is 3H 2Z 4E (standard partition), but also 32Z 4E, 24E 3H, or even 324E (nonstandard partitions). We follow the naming convention of Ross (1989), who also claims that "Understanding place value requires an elaboration of the student's emerging understanding of a part-whole concept." (p. 47)

Due to the structure of *decimal partitions* we can easily find a correspondence between tokens in a place value chart and such a decimal partition. Any placement of tokens in a place value chart corresponds to the decimal partition that has exactly as many bundles of a given size as there are tokens in the table cell for that size.

[1] We consider positive integers, though most of the following is applicable to decimal fractions, too.

Nonstandard partitions are important for flexible arithmetic. Dividing 320164 by 4 is much easier if the dividend is interpreted as 32 Tenthousands + 16 Tens + 4 Ones, just to give one example. They become even more important in written arithmetic, as we start with single digits and carryovers that directly lead to nonstandard partitions.

Abstraction levels

Sayers & Barber (2014) discuss teaching of place value to 5-6 year with a particular emphasis on the teacher and the manipulatives used in the classroom, and they conclude "In sum, place value is difficult to understand and to teach." (ibid., p. 34)

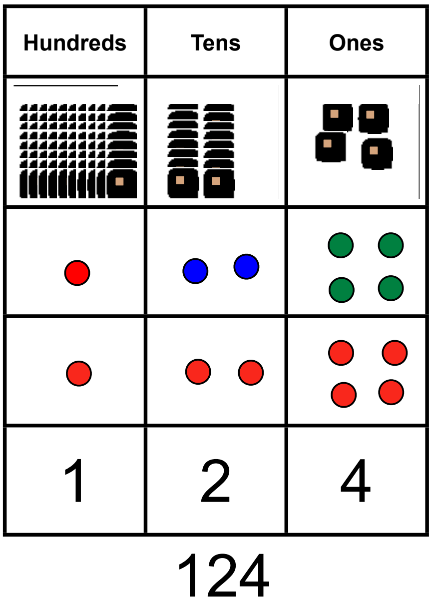

Figure 1: Abstraction 7–11

This difficulty is documented in the literature. Gerster & Walter (1973) describe eleven levels of abstraction from bundling up to standard notation of numbers. Levels 1–6 are just for creating bundles and exchanging between them. Only levels 7–11 are concerned with place value: (7) sort objects into a place value chart, (8) replace objects by iconic representations, for example coloured tokens, (9) replace icons by undistinguishable tokens, (10) replace tokens by a digit representing their number, (11) omit the chart and write the number as a sequence of digits.

While it is neither clear that all these abstractions have to be followed one by one, nor at all, they influence teaching in primary school. According to Grevsmühl (1995) it is important to exchange and replace not only objects that are clearly distinguishable by their volume, shape or colour, but special care should be taken to exchange tokens of the same kind differing only by their position in a place value chart.

Typical activities when teaching place value

An analysis of seven German textbook series for primary schools showed that we can categorize place value related demonstrations and exercises into

- Bundling activities as a preparatory task,
- Grouping and sorting of bundles and placing them in charts,
- Exercises based on the transfer between different representations of numbers (tokens or digits in a chart, the standard number representation, or spoken number words),
- Exercises that use a place value chart as a tool, and
- Exercises where number representations (either in a chart or in standard notation) are changed.

Our main focus lies on the last two categories, as they involve nonstandard partitions as well as standard partitions. The third category, representation transfer, relies on students being able to create standard partitions from nonstandard partitions in a place value chart using repeated bundling to find a maximal bundling. This includes questions for the minimum number of tokens needed to represent a number.

The tool use of place value charts occurred mainly when the algorithms for written addition and subtraction were introduced. For addition, the carryover is easily explained using the bundling process in a chart, and for subtraction, depending on the strategy used, we can demonstrate how to unbundle for borrowing.

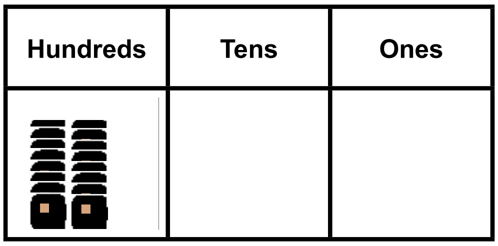

Figure 2: What number is this?

Some questions ask for the consequence of the movement of one or several tokens from one place to the other. It is important to note that the place value chart as a tool is now used differently: When we are placing objects that still carry their bundle size as an attribute, for example by being a base block of 100 or 10, by having a different colour, or by being an iconic representation of a bundle, we must not place these in the wrong cell of the chart, because this violates the rules and leads to a conflicting situation. A 10-stick place in the tens' place represents *one ten*, while the same stick in the hundreds' place could represent either *one hundred, ten hundreds*, or again *one ten* – or, even better: none of these, as it is illegal to place 10-sticks there (Fig. 2). On abstraction level 9 –having the same tokens for counting any bundle size– it is no longer visible that this is illegal, and it leads to exercises like the following: "Paul places 324 with 9 tokens in a place value chart. He moves one token from the tens' place to the hundreds' place – what number does he have now?" While this exercise makes sense for children that already understand place value, it can be confusing for children who just learned (or are about to learn) that moving a token between different places changes their value and is illegal unless the token is replaced by its corresponding unbundled bundle.

We conclude with the statement that there are both *value-preserving* and *value-changing* operations taking place in typical exercises. For us, it is important to support *value-preserving* operations, as these are the underlying mechanism for creating standard partitions from nonstandard partitions and also to create many new decimal partitions from one representation of a number in the place value chart.

THEORETICAL FRAMEWORK: ACAT

Our theoretical framework has been introduced in detail in Ladel & Kortenkamp (2013) and we recall it here only briefly and for better understanding of our study. In ACAT (Artefact-Centric Activity Theory), based on the work of Engeström (1987) and Leont'ev (1978) special attention is given to a mediating artefact between a subject (here: the student) and an object (here: the notion of place value), and the internalization and externalization processes occurring along this line of interaction. ACAT allows for the derivation of rules for the design of the artefact.

For our research, we used an interactive place value chart[2] that works as an App on iOS devices. Its major design decision has been that moving a token with the finger should be *value-preserving*. In order to achieve that effect, the app is automatically unbundling tokens of a higher value when moving them to a lower place. When the student is moving tokens to a higher place this is only possible when there are enough further tokens of the same bundling size that can be bundled with the moving token. If so, tokens are automatically bundled and replaced with a single token. The standard configuration of the App does not colour the tokens in order to support the abstraction process described earlier, but it is possible to switch on an automatic recolouring as well ("Montessori mode").

RESULTS

The App and the theoretical framework leading to its design have been demonstrated at PME 2013. We report on experimentation with it that has been carried out by us in laboratory situations and the classroom.

Interview Pre-study

In our pre-study we combined several exercises into a guided interview. The children were interviewed by one of the authors following the various questions and tasks that had to be carried out. The interactive place value chart App was introduced during the interview by the interviewer (I). We give some exemplary results.

In the first set of questions children had to compare nonstandard decimal partitions in the typical notation (e.g., 32Z 4E, see above). After deciding which one is larger, or whether they are equal, they were asked for a justification of their answer. We could identify the following types of mistakes: (a) Numbers were created just by omitting the letters denoting the bundle size – 14E 2Z becomes 142; (b) Only the largest bundle is considered, such that 5Z 3E (=53) is considered to be larger than 4Z 15E (=55); (c) The bundle size letters are ignored completely (5Z 3E becomes 5 and 3); (d) Only the largest number of a certain bundle type is used to decide which number is larger. The answers showed that not all children have understood the notation that is used on a daily basis in schools for decimal partitions.

The next item was to ask children how many tokens they need to represent 35 in a (two-column) place value chart. Next they were asked to show one representation. The children had red and blue tokens and strips of 10 blue tokens. One of the interviews highlighted the problem of mixing abstraction levels mentioned before. Here is part of the transcript after a student (S) placed three red tokens in the tens' place and five blue tokens in the ones' place:

I: Is it necessary to use blue tokens there (*points to the ones' place*) and red tokens there (*points to the tens' place*)?

S: No, not really. I just did it that way. […]

[2] Available from https://itunes.apple.com/app/id568750442.

I: Is there another way to put the tokens?

S: Yes, 30 tokens here (*points to the tens' place*) and 5 tokens there, as there are.

I: (*points to the tens' place*) Does it matter whether there are 3 or 30 there?

S: It does not, not really.

I: Can you explain that?

S: Actually it is the same. The 3 is there (*points to the tens' place*), the 5 is there (*points to the ones' place*) and when I put 30 there then there is a 0 instead of a 5 there.

The student shows that he is able to abstract from the actual colour of the tokens, but he uses an interpretation of nonstandard partitions that will interfere with activities that require bundling and unbundling. 30 tokens in the tens' place are not the same as 3 tokens in the tens' place.

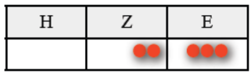

Figure 3

The last item was concerned with the action of moving a token from one place to the other. Children were asked what number is shown in the place value chart of Fig. 3. After answering the question they had to tell what they think will happen when one token is moved from the tens' place (Z) to the ones' place (E).

Of course, there are two possible answers to that question, depending on whether the token is unbundled or not. Our textbook analysis showed that both behaviours are used currently: In a "what if one of the Z-tokens is used as a E-token" scenario students should answer that the value of the number is decreased by 9. Questions of that type are also contained in the national comparison VERA-3 (see Stanat et al. 2012). On the other hand, using the place value chart as a tool for the introduction and explanation of written arithmetic the behaviour must be value-preserving, such that we end up with one Z-token and thirteen E-tokens.

Unsurprisingly, the students preferred the behaviour of traditional place value charts in their answers. Immediately after that they were given the interactive place value chart and were asked to move a token. Here is a transcript excerpt of another student.

I: Let's try it here. Here is the 23. Move a token from the tens to the ones!

S: (*moves a token*). Ooooh. Ey Caramba.

I: Ey Caramba. What is happening?

S: They become many.

I: Yes. Look here, the numbers are shown above. Can you try again and see? […]

S: I move (S. *moves a ten token to the ones*)

I: What number is it now?

S: 13 Ones.

I: Yes, and there is a ten (*points to the ten token*)

S: Strange.

I: What happened?

S: (*thinks*) These are 10 single ones! Now, these (points to tokens).

I: Why?

S: Because that wouldn't work. See, if you did on token, and then, if you do not have ten single ones, then it is only one. If you have ten single ones, then it is a ten. (S *moves one token from the ones to the tens. Automatically, nine other tokens are bundled into it and the ten tokens are replaced by one token.*)

S: Oops, what happened there?

I: Yes, what happened there?

S: The others moved over. Because the one, you know, if it moves to the tens, it would be eleven only and then nine come as well and 9 plus 1 is ten, so its two tens again, twenty, so its 23 again.

After being surprised by the magic behaviour of the App the student is able to explain this behaviour in detail. It seems to support the necessary flexible representation of numbers in the place value chart.

Quantitative Study

Based on the above we designed a study which shall reveal whether our digital artefact that interprets the action of moving a token from one place to the other differently, that is, preserving the value of the number, can improve the fluency of the students when creating nonstandard partitions. Also, we measure how the fluency –either acquired using the digital or virtual artefact, or already existing before– in creating nonstandard partitions in a place value chart influences the ability to transfer between written nonstandard partitions and the numbers represented by these.

Currently, we run the experiment with over 300 students in grade 3. The students are assigned by random to one of two groups; in group A each student has access to an iPad with our place value chart software, in group B students work with paper and pencil only. Each student has to work without further support on a test with three parts. Part I and III contain questions of the type "Which is larger – 22E 5Z or 22Z 5E?" or "Which is larger – 1H 12Z 5E or 1H 3Z 5E?" where two decimal partitions have to be compared, and questions where a (nonstandard or standard) decimal partition should be written in standard notation. The questions are designed such that students without a proper understanding of decimal place value are likely to fail.

Part II of the test consists of two activities. The students are asked to interpret numbers given by tokens in a place value chart as a number and to write them down. Next, students are asked to place tokens in a chart in order to represent a given number. For each number they are encouraged to find distinct representations.

The data will be interpreted using statistical implicative analysis (SIA, Gras et al. 2008) with Boolean variables that further differentiate between the ability to interpret non-standard representations and only standard representations. Our first data set (N=37) supports our hypotheses that the use of the iPad lets students create more nonstandard partitions (1.58 vs. 1.06 on average) than in the non-technology group. Also, the iPad group performed about 7.9% better on average in part III than in part I, while the other group performed 7.6% *worse*. However, without further data, to be collected in late January 2014, it is impossible to discuss this further, but we hope to answer the question whether an activity-theoretic driven design of an electronic place value chart can support a flexible understanding of place value and whether this leads to better performance in place value tasks.

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