MATHEMATICAL CERTAINTIES IN HISTORY AND DISTANCE EDUCATION

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An historical case is presented in which extra-mathematical certainties lead to invalid mathematics reasonings, and this is compared to a similar case that arose in the area of virtual education. A theoretical-methodological instrument is proposed for analysis of certainties. The article suggests the need for teachers to be aware that certainties of mathematics facts are not always based on mathematics understandings.

BACKGROUND AND OBJECTIVES OF THE PAPER

In Euclid's *Elements*, the author supported his theory of the parallels in the Fifth Postulate; there he established that two lines that are not equally inclined in relation to a third line will always have to intersect. Said proposal engages a behavior in the infinite (Kline & Helier, 2012), hence throughout history mathematicians resorted to different means to convince themselves of their truth—states Lovachevski (1974, pg. 2). Saccheri, for instance, decided to establish that truth by resorting to a double reduction to the absurd: denying the existence (no parallel to *l* crosses *P*), and denying unicity (more than one line crosses P). Denial of the existence produced a contradiction. From the second possibility, Saccheri deduced theorems that, albeit contradiction-free, seemed odd to him. This was sufficient for him to reject the second possibility, from which he derived the veracity of the Fifth Postulate as the sole possible option. According to Kline & Helier (2012, pg. 508), "when Saccheri concluded that the Fifth Postulate was the necessary consequence of the others, he was only able to show that when a person intends to establish something of which s/he is already convinced, s/he will be satisfied even if his/her demonstration has nothing to do with the facts."

Another attempt to demonstrate the parallels postulate arises in Legendre. In 1800, he published, according to descriptions by Lovachevski, that the sum of the angles of a triangle cannot be greater than 180°. He moreover argued that said sum could not be less than 180°. From his analysis, Lobachevski deducted that Legendre's reasons were incorrect and that "the biases in favor of the position accepted by all had probably induced him at each step to precipitate his conclusions or add what was still not legitimate to admit in the new hypothesis" (1974, p. 3).

In a critical reading of history, Lovachevski questioned the absence of logical rigor of the demonstrations of the Fifth Postulate; he objected to the ontology and idealistic epistemology that was the foundation of those attempts, by suggesting that "the concepts themselves did not encompass the truth that he wanted to demonstrate" (1974, pg.1) and by raising an empirical route as the alternative proof, by way of

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astronomical observations. With an open spirit, he built hyperbolic geometry, admitting with it "the existence of Geometry in a broader sense than what Euclid has presented" (1974, p. 1).

This passage through history illustrates how biases—taken on by Saccheri or Legendre- can disturb mathematics reasoning, and how certainty and convincement of mathematics facts can be strongly tied to extra-mathematical sources, such as ontological or epistemological commitments. The subsequent text contains arguments based on empirical evidence derived from a case study (Mariana), that that historical phenomenon associated with convincement and certainty also arises in mathematics instruction processes. The regularity of that phenomenon in such dissimilar arenas suggests, in one way or another, its generality, and raises the need for teachers to have knowledge of it and consider it in their didactic practices.

Research on certainty and convincement has been directed toward the professional arena of mathematics, such as that of the teaching of the subject. For the mathematician, convincement and certainty are drivers that boost its activity in the stages of heuristic development, and a guide for certifying its findings during proof processes (Tymoczko, 1986). The mathematics education community has carried out different studies that implicitly use the point of departure that, like what happens with mathematics, certainty is also important in building mathematics knowledge in the classroom. Some of those works have been recreated in extra-class environments and have focused either on the students (e.g., in Balacheff, 2000) or on the teachers (e.g., in Harel & Sowder, 2007); others, developed in classroom environments with intervention, have basically focused on the students (e.g., in Krummheuer, 1995). Unlike any of the foregoing, in this work the point of departure is an historical phenomenon associated with the building of certainties so as to take it as an epistemological laboratory that enables explaining the presence of the very phenomenon in current training environments using a virtual forum. This raises the challenge of having theoretical elements and analytical instruments that make it possible to distinguish states of certainty (with respect to the statements of mathematics contents that arise there) experienced by the students enrolled and that they express in writing. Below, the authors of this paper propose the instrument of analysis that has been developed for that purpose.

PROPOSAL OF AN INSTRUMENT TO DISTINGUISH EPISTEMIC STATES OF CERTAINTY AND OF PRESUMPTION OR DOUBT

This research deems that, associated with their assertions of mathematics content, subjects can experience internal states of certainty (when they associate the highest degree of probability to what they believe in) or of presumption (when they associate lower degrees of probability to what they believe in). Such states are known as "epistemic states" in Rigo (2013).

In the design of the theoretical-methodological instrument proposed below, there is a convergence of perspectives from different disciplines, namely: from philosophy

(Wittgenstein), psychology (Bloom, Hastings & Madaus) and sociology (Abelson). Of particular relevance to this study was the contribution of linguistics works, such as those of Hyland (1998), which made it possible to resort to analysis of the meta-discourse of the participants in the virtual forum so as to reveal the communicative intentions (many of which are unconscious) that they project through their writings.

The authors of this research consider that a person (who takes part in a virtual forum) experiences a degree of certainty, or of presumption or doubt, in a mathematics statement when one or more of the criteria that appear in Table 1 are met. Said criteria are sufficient, albeit not necessary.

Elements of speech	The person resorts to language emphasizers that can reveal a greater degree of commitment to the truth of what he is saying; for instance, when the person uses the indicative mode of verbs (e.g., I have).
Action	The subject carries out actions that are consistent with his discourse.
Familiarity	The person resorts to forms of sustentation based on familiarity (result of repetition, memorization and customs).
Cognitive formulation	The person resorts to forms of justification based on mathematics reasons.
Determination	The person spontaneously and determinedly expresses his adherence to the veracity of a mathematics statement, indicating some degree of determination. That degree may be higher when the subject maintains a belief, in spite of having the collective against him. He may even make efforts to convince others of the truth of his position.
Interest	The participations of a person who shows interest concerning a specific mathematics fact in a virtual forum are:
	-*Systematic*. That is to say, the subject answers all questions addressed to him in the most detailed manner possible.
	-*Informative*. His assertions, procedures and/or results are sufficiently informative.
	-*Clear and precise*.
Consistency	The person's varying interventions show consistency.

Table 1: Theoretical-methodological instrument for distinguishing states of certainty.

METHODOLOGICAL ASPECTS

The qualitative research reported here focuses on an interpretative-type case study (Denzin & Lincoln, 1994). The empirical study was carried out in the Diploma Program on Fundamental Themes of Algebra, the purpose of which was to strengthen the training of people who provide advice on algebra topics to adults in the process of obtaining their secondary school certificates. The teaching activities are carried out remotely by using the Moodle platform, through which the students receive support, are assessed and given feedback by a tutor. The episode analyzed here pertains to Module IV. It was selected due to the fact that the advisors tended to use sustentation in their responses. The episodes begin with the tutor asking the students to complete a task and they end with the agreement of the students on the solution to the task. For this report, the participations of three students were chosen given that those students appear to have experienced very different epistemic states when faced with the task proposed, despite the fact that none of the three answered the task correctly.

EPISODE: "THE MILLION DOLLAR PROBLEM"

The episode dealt with resolution of the following problem: You will get one million dollars if you can find a two digit number that simultaneously meets the following conditions: a) If you add to the first digit of the number we seek, a figure that is twice the second figure, the result is 5; b) If you add four times the second figure to double the first digit of the number we seek, the result is 7. The students were expected to conclude that no number could meet the problem's conditions, and that they would see that this was the case when they charted the equations in a graph that would produce two parallel lines.

1st Fragment. Mariana's first intervention. Presence of certainties

Mariana started with the following participation:

1.1	x+2y=5; 2x+4y=7
1.2-1.6	... Since the equations do not contain an equal unknown, the substitution method is applied … to eliminate one unknown, which leaves us with 2x+4y=10. After that step, you can do the operation.
1.7	2x+4y=10 <u>2x-4y=7</u> 4x+0=17
1.8-1.9.	We separate the terms and solve for "x". [So]... x=4.25
1.10-1.11	Obtaining the value, we substitute in one of the 2 equations: 2(4.25)+4y=7
1.12-1.13	We do the operation, separate the terms and solve for "y". 8.5+4y=7; 4y=7-8.5; 4y=1.5; y=0.375
1.14-1.15	We prove. First equation: 4.25+2(0.375)=5; 5=5. Second equation: 2x-4y=7; 2(4.25)-4(0.375)=7; 8.5-1.5=7; 7=7

During her resolution, Mariana used different equation systems. The first came from the translation from common to algebraic language (1.1); then she obtained an equivalent equation (at 1.5), and after that, at 1.7, she obtained a modified equation, by changing a sign (of the term 4y from the second equation). To obtain the value of x, she used the first equation at 1.7, and to obtain the value of y, she began with the second equation in 1.1 and ended with the second equation in 1.7. To prove the operation, she used the first equation from 1.1 and the second from 1.7.

In her resolution, Mariana liberally applied the rules of algebra, by capriciously changing the signs of the terms of the equations and by indistinctly using the equations that appear in those systems and combining them in an *ad hoc* manner, as they suited her purposes. It would seem that this responded to a specific objective, namely: to obtain values for literals *x* and *y*, an objective that may possibly have been derived from an interpretation of the literal only as an unknown, excluding the variable's other uses.

During this process, it would seem that Mariana experienced high degrees of presumption and even of certainty. Amongst other reasons, this is because of her *determination* to be the first to submit her answers and procedures to the judgment of the group; her use of *emphasizers*, specifically due to the indicative mode of the verbs (at 1.2 or 1.14); because her *actions* were the result of the procedures that she was announcing, for example when she announced that the substitution method was to be applied (1.2), all of her subsequent actions were aimed at trying to apply rules that she believed belonged to that method; because she sustained her assertions in *schemes* based on *familiarity* (such as the addition and subtraction method), at 1.7, or what she called the 'substitution method', at 1.2. She also demonstrated her certainty by showing *interest* in resolving the problem, by explaining her solution in a detailed manner, answering all of the questions in the problem, resolving the system raised without the tutor requesting it, and by presenting her resolution clearly.

2nd Fragment. José's questioning

José expressed the following to refute Mariana's answer:

2.1-2.2 Hello Mariana. You really surprised me. But you've [missed] a small detail.

2.9-2.11 2x-4y=7. In that step, you changed the sign (it should be ***+4***y or multiply by -1, but the entire equation), that's no longer the original equation. What do you think?

José realized that Mariana had not correctly applied the rules of algebra (changing the sign in the system at 1.7), and that that had consequences ("that's no longer the original equation", 2.9), and he informed her of it, waiting for her reaction.

3rd Fragment. Mariana's reply. Explicitation of reasons and ontological commitments, and strengthening certainty

Below is Mariana's reply

3.1-3.4 You are indeed completely right [José], the entire equation is affected. But the purpose of the system of equations is to arrive at the result by eliminating one of the unknowns. If I affect my entire equation, I would be left with 3 and I would not have an unknown to solve.

3.5-3.10 x+2y=5; 2x+4y=7. In this case, since the equations do not have one same unknown, the substitution method is applied where one of the two equations is multiplied by a number that serves to eliminate one unknown. 2(x+2y)=2(5), leaving us with 2x+4y=10. All is well so far.

3.11-3.12 After that step, you can do the operation
2x+4y=10
<u>-2x-4y=-7</u>
40+0=3

3.13-3.16 Once the value has been obtained, we substitute in one of the two equations: 2(3)+4y=7. We do the operation, and solve for "y"... y=0.25.

3.17-3.18 We prove it. First equation: x+2y=5; 3+2(0.25)=5 and I don't get 5.
Second equation: 2x-4y=7; 2(3)-4(0.25)=7; 6-1=5; nor do I get the 7.

3.19-3.20 So I only affect 4y, in order to not affect the whole equation, and much less my result. You may not see it as correct, but it is [correct] for me because the objective is to find the correct value.

3.22 Let's prove it. First equation: x+2y=5; 4.25+2(0.375)=5; 5=5.

At the beginning of the fragment (from 3.1 to 3.4), Mariana told José that he was right. Yet she subordinated those reasons to what she thought should be obtained from a system of equations: "to arrive at the result". This was probably because she believed that absurdities would be derived from her classmate's answer (such as "I would not have an unknown to solve" and "I would be left with 3", possibly referring to 3.12). At a second point (from 3.4 to 3.18), she followed José's suggestion, perhaps with the idea of 'mathematically showing him his error' by letting a contradiction arise from his suggestion: "I don't get 5" and "nor do I get the 7", without realizing that the mistake did not come from the resolution, but from the arbitrary nature of her manipulation of algebraic language (e.g., by assuming at 3.12-3.13 that x=3 or using the system of equations that best suited her ends). At the third point (3.19-3.20), she once again sustained the advisability of her method, once again subjecting it to the obtainment of her objectives: "to find the correct value" (3.20), and at the fourth (3.22) she proved its validity without realizing that she needed to substitute the values in the two equations at 1.1 and not just in the equation that best suited her interests.

In Mariana's second intervention, she very likely strengthened her epistemic states of certainty by being able to make her objectives and arguments explicit, and 'demonstrating' her classmate's error and the validity of her principles and her method, all of which she did with *determination* and with a *consistent attitude*. Her certainty can also be inferred from the use of emphasizers (not just due to the assertiveness of her language, but also due to the use of the indicating mode in "I [don't] get", at 3.18, "it is" at 3.2 or "much less" at 3.19). Her *interest* can moreover be seen in her reiteration of her resolution, clarification of her points of view, and public refute of her classmate despite her understanding that he was right, to a certain extent.

4th Fragment. Jeimy's participation. Doubt

4.2-4.4 I think I have a problem too. I'm trying to do the second exercise and cannot find the value for x or for y. My equations are: x+2y=5; 2x+4y=7

4.5 Then I change the sign in the first equation.

4.6-4.8 -x-2y=-5
<u>2x+4y=7</u>
x+2y=2; x=2-2y. Substituting in the first equation (2-2y)+2y=5 [so] 2=5

4.9 And I don't get any value ?????????????? What's going on? Help! ...

By rigorously applying the rules of algebra, Jeimy arrived at an absurdity that made her doubt her work. Without presupposing anything, she simply detected it and asked for help.

MAJOR FINDINGS

The case of Mariana is interesting. Although she shows her knowledge of some of the rules of algebra (see 3.5 to 3.12), her ontological commitments concerning the characteristics that must be possessed by mathematics tasks and, particularly by systems of equations –of having an unknown to solve and a precise and numerical solution that can be found- they appear to represent an obstacle that prevent her from fully applying those rules.

Mariana, like Saccheri or Legendre, was faithful to her ontological principles (or biases) and, just like them, those commitments and certainties lead her to "admit demonstrations that had nothing to do with the facts" and "they lead her to precipitate her conclusions or add things that were not legitimate" (see pg. 1 of this text).

Jeimy, like Mariana, faced a problem that jeopardized her beliefs (of the existence of a numerical and sole solution to all mathematical tasks) and her algebraic knowledge. But while Marianna obstinately held on, with not a trace of doubt, to an ideal of the mathematics object, subjecting the rules of algebra to those ontological commitments, Jeimy preferred to maintain her logical rigor—like Lovachevski did, taking due distance- by scrupulously following the rules of algebra. Unlike Mariana, Jeimy allowed herself to doubt the results obtained—like Lovachevski did—recognize her lack of knowledge and ask for help—a metacognitive openness that placed her in a position to learn.

An important didactic consideration stems from the analysis presented. And it has to do with the help that can be given to Mariana. José's participation reveals that it was not enough to demonstrate her algebraic errors because in one way or another she was already aware of them. What Mariana appears not to have realized, and perhaps she would need some help with this, is that her beliefs and ontological commitments (that she probably took as unquestionable and unmovable truths) lead her to lose logical rigor in application of algebraic rules and, in the final instance, represented an obstacle to moving forward in her learnings.

This texts show that certainty of mathematics facts can have deep roots in extra-mathematical considerations, such as ontological commitments, and that certainty is not always or necessarily tied to mathematical comprehension. Given the information presented here, it is important that teachers and their professors become aware of the phenomenon because it has significant consequences in the learnings of students.

References

Balacheff, N. (2000). *Procesos de prueba en los alumnos de matemáticas* [Proof processes among mathematics students]. Bogotá: Universidad de los Andes.

Denzin, N. K., & Lincoln, Y. S. (1994). Introduction. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 1-18). California: Sage.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspective on the learning and teaching of proof. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 805-842). Charlotte, NC: NCTM.

Hyland, K. (1998). Persuasion and context: The pragmatics of academic metadiscourse. *Journal of Pragmatics*, *30*, 437-455.

Kline, M., & Helier, R. (2012). *Matemáticas para los estudiantes de humanidades*. México: Fondo de Cultura Económica.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229-270). Hillsdale, NJ: Lawrence Erlbaum Associates.

Lombardo-Radice, L. (1974). Lobacevskij, matemático-filosofo. In N. I. Lobacevskij (Ed.), *Nuovi Princípi della Geometria, con una teoria completa delle parallele* (pp. 13-54). (S. Ursini, Trans.). Turin: Universale Scientifica Boringhieri.

Rigo, M. (2013). Epistemic schemes and epistemic states: A study of mathematics convincement in elementary school classes. *Educational Studies in Mathematics*, *84*(1), 71-91.

Tymoczko, T. (1986). The four-color problem and its philosophical significance. In T. Tymoczko (Ed.), *New directions in the philosophy of mathematics* (pp. 243-266). Boston: Birkhäuser.