A THEORETICAL FRAMEWORK FOR THE FUNCTION OF GENERALIZATION IN LEARNING MATHEMATICS

Toru Hayata, Masataka Koyama

Graduate School of Education, Hiroshima University, Japan

The purpose of this study is to clarify for what students do generalize something in learning mathematics. In this study, we make a distinction between generalization and extension, and focus on the function of generalization in terms of its meaning, purpose, and usefulness. Through reviewing literature on generalization and philosophical considerations, six functions with their examples are identified; variablization, purification, unification, discovery, association, and socialization. We propose a new theoretical framework for the function of generalization in learning mathematics, suggesting that the framework has possibility of a principle of didactics for teachers and a guideline in forming mental habit for students.

INTRODUCTION

In mathematics classrooms, we evaluate more students' mathematics activities based on mathematical knowledge than their static mathematical knowledge. Students are expected to be improved as the result of their activity. We call such improvement by the term of *learning*. In learning mathematics, generalization is one of most important mathematics activities. Generalization is to extending the range of reasoning and/or communication from the particular (concrete something) to the general (abstract something). In that sense, generalization is essential to mathematics. In our daily life, however, knowledge about the particular is enough for most of our purpose, and such knowledge sometimes may be more useful than knowledge about the general. Thus, students may have a question; "For what do we generalize it?". It's a natural question from the *viewpoint of students*. In fact, students do not always make any endeavors to generalize in learning mathematics (cf. Tatsis & Tatsis, 2012), though our human mind has an ability of generalize something and a tendency to generalization since very young age (cf. Vinner, 2011).

However, in mathematics education, the authors of this paper believe in the value of that students find its meaning, purpose, and usefulness of generalization by themselves through mathematics activities. Therefore, we will investigate and clarify an epistemological motivation of generalization for students. In this paper we use the term "*function of generalization*" as the meaning, purpose, and usefulness of generalization for students in learning mathematics, and discuss the following two research questions:

RQ1: What are specific and characteristic functions of generalization for students in learning mathematics?

RQ2: How do the functions of generalization improve students' mathematics learning?

2014. In Oesterle, S., Liljedahl, P., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 3, pp. 257-264. Vancouver, Canada: PME.

For RQ1, previous studies pointed out mainly two suggestions. First, for example according to Davydov (2008), generalization means that one investigate invariant(s) and associate the invariants with a label. As a result, generalization yields useful structures or systematization (pp.74-75). It's no doubt a function of generalization, and the view is commonly shared among some researchers (cf. Radford, 1996). However, this function is not specific to mathematics but common in all scientific disciplines. Furthermore, as Davydov (2008) pointed out, this function is that generalization *functioned* as a result identified when one observer makes an analysis of a completed and static mathematical (and scientific) knowledge. Hence, a student as a learner may not think "I associate the invariants with the label for systematization!" The interest for us is the function of generalization in students' activities of learning mathematics. The function of generalization must be identified from the students' viewpoint, though it is not contradicted with the Davydov (2008). Second, previous studies on generalization in mathematics education pointed out the function of *variablization* that is to extending a range of reasoning and/or communication (Ursini, 1990; Dörfler, 1991; Iwasaki & Yamaguchi, 1997; Radford, 2001). This function is an important function of generalization. However, variablization is one of functions of generalization, because some researchers pointed out other functions of generalization.

DISTINCTION BETWEEN GENERALIZATION AND EXTENSION

In this study, we use the term of generalization as "recognition that has epistemological direction from the particular to the general". The necessity of this definition is derived from the fact that similar recognition called extension does not have this direction. The authors (Hayata & Koyama, 2012) make a distinction between generalization and extension, and formalize them as following in Figures 1 and 2 respectively:

D is a field. D' is a wider field than D. M is a meaning in the field D. M' is an established meaning in the field D'.

Generalization: Recognition establishing M in D, and extending D to D' without changing M

Extension: Recognition incorporating D into D' such that if D' is limited to D, M' is equivalent to established M

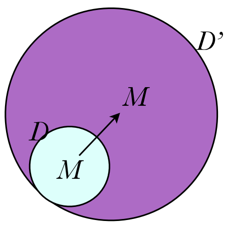

Figure 1: Model of generalization

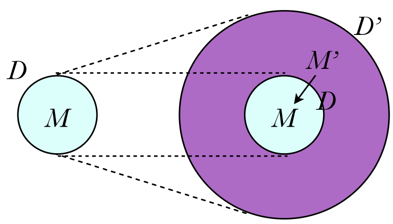

Figure 2: Model of extension (Tomosada, Himeda, & Mizoguchi, 2006, p. 9)

For example, when students noticed that the sum of interior angles is straight angle (M) in concrete triangles (D), thereby they suppose that it is case of all triangles (D'). This recognition is generalization because M is not changed. On the other hand, for example, when students work on multiplication of decimal numbers (D') for the first time, they cannot solve the multiplication by using the meaning of multiplication as repeated addition (M) in natural numbers (D). The decimal number multiplication can be solved with the meaning of proportion (M'), and this meaning is equivalent to repeated addition in natural numbers. So, by its definition, this recognition is extension.

In this study, we make the above distinction between generalization and extension, and focus on the function of generalization in terms of its meaning, purpose, and usefulness of generalization for students in learning mathematics. On the other hand, we do not distinguish between algebraic generalization and geometrical generalization for the purpose of this study in spite of that there are important cognitive differences between them (Iwasaki & Yamaguchi, 1997), because in both generalizations one must consciously see algebraic/geometrical symbols as general symbols (e.g. *n* is general natural number, and triangle ABC is general triangle).

METHODOLOGY

As mentioned above, previous studies mainly discussed the function of generalization identified in the static and completed mathematics knowledge. Thus, their method is, for example, to analyse the history of mathematics (cf. Radford, 1996). However, there is no whole picture/framework for the function of generalization in mathematics activities. Without a framework, we cannot see and analyse any students' actual learning activities of mathematics in school classroom practices. For this reason, in this study the authors adopt the methodology of analyzing previous studies on generalization in terms of its meaning, purpose, and usefulness in order to extract implicit functions of generalization from the studies, carefully consider them, and organize them in a framework. In this paper, we analyze Polya (1954), Dörfler (1991), Ito (1993), and Tatsis and Tatsis (2012), because all of them epistemologically consider generalization in mathematics from the learner's viewpoint, and reveal the nature of generalization without restricting generalization to any specific mathematical context. In the following, as a result of the analysis, six identified functions of generalization (*variablization, purification, unification, discovery, association*, and *socialization*) are presented with their examples, and a new theoretical framework consisted of the six functions and their structure is proposed.

SIX FUNCTIONS OF GENERALIZATION IN LEARNING MATHEMATICS

Variablization

In short, the widely accepted meaning of generalization is to extending the range of reasoning. When one intends to extend the range, some attributes of the particular at hand are ignored and abstracted to become variables. For example, when students find out that area of a concrete rhombus ABCD with diagonals of AC (9cm) and BD (6cm) can be calculated by $9 \times 6 \div 2$, and from it they infer that the area of all rhombuses can be calculated by "diagonal $\times$ another diagonal $\div 2$". In this case, the students see length of sides, inner angles, and so on as not essential attributes, thus these attributes will become variables, while the angle between two diagonals is not variable. The variablized attributes are dealt as algebraic variables, and can be substituted by any concrete values. As a result, students can know all objects in a set (ex. set of rhombus) nevertheless there is infinite number of objects in the set. We call this characteristic function *variablization*. This function leads to construct new class. The variablized objects are more or less isolated from physical objects. Thus, some symbols are needed to deal with the objects. For this reason, some researchers emphasized the importance of generalization in algebra. The variablization is important function of generalization, but it is not enough for learning algebra (cf. Dörfler, 2008).

Unification

There is another case of generalization as "extending the range of reasoning". One recognizes that known various particulars are integrated by single notion, and therefore elements in a set are increased. As a result, in such case, the range of reasoning also is extended. For example, let's consider the same example of area of rhombus used in explaining the variablization. The area formula "diagonal × another diagonal ÷ 2" for rhombus is also applied to kite, because the formula depends only on the condition that angle between two diagonals makes a right angle, and because that angle between two diagonals is also right angle. In this case, two particulars (area of rhombus and area of kite) are unified by single notion (area formula). Thereby the range of the formula that calculates area of rhombus is extended. We call this characteristic function of generalization *unification*. According to Polya (1954), sometimes we can surprisingly unify different objects by single notion through generalization. We need pay attention to this different function unification from the function *variablization*.

Purification

In actual problem solving, there are many situations where one does not always intend to work the function variablization of generalization. In such situation, for solving the problem easily one removes the attributes appeared unnecessary from the original problem. For example, let's think about the problem to find. $\sqrt{103\times102\times101\times100+1}$ .If students must solve this problem without using any devices, they have to work on a quixotic challenge to find $\sqrt{106110601}$. Thus, some students are motivated to conjecture that generalizing the problem may be useful for solving it. They express it generally, and try to factorize $\sqrt{(n+3)(n+2)(n+1)n+1}=\sqrt{n^4+6n^3+11n^2+6n+1}$. In this case, to factorize the generalized ($\sqrt{n^4+6n^3+11n^2+6n+1}=\sqrt{(n^2+3n+1)^2}=n^2+3n+1$) is easier than to find $\sqrt{106110601}$. Finally, they substitute n=100 for the equation, and get the answer to original problem is 10301. In this case, the generalization of "extending the range of reasoning" is not purpose but means. We call this characteristic function *purification*. Dirichlet and Dedekind (1999) and some researchers pointed out "As it often happens, the general problem turns out to be easier than the special problem would be if we had attacked it directly (p. 13; quoted in Polya (1954: 29))" .

Discovery

According to Giusti (1999), new mathematical knowledge is invented implicitly while solving a problem and subsequently discovered as valuable object. In deed, Giusti (1999) pointed out that method of solving problem of planetary orbit (i.e. differential) invented the notion of limit implicitly. In school mathematics, we can find similar examples of generalization leading to a "discovery". For instance, in the above example used for the *purification* (find $\sqrt{103 \times 102 \times 101 \times 100 + 1}$), one can discover new proposition; "the value of $\sqrt{(n+3)(n+2)(n+1)n+1}$ is always natural number $n^2 + 3n + 1$" by generalizing the original problem. This proposition was not expected when students tried to solve the problem. We call this characteristic function *discovery*. According to Tatsis and Tatsis (2012), the function *discovery* of generalization is for students to "grasp" the deeper underlying structure of mathematics.

This function is closely related to the Dörfler's notion of "symbols as objects". According to Dörfler (1991), at first the abstracted something is associated with cognized particular(s), then, they are separated in the process of generalization. As a result, the abstracted something with symbols become independent object. He called this process as "symbols as objects". As the above example indicates, the function *discovery* is interpreted as our conscious evaluation of the independent object.

Association

In learning mathematics, new mathematical objects (knowledge, concepts, and so on) are constructed in mathematical activities. The something new should be meaningful for students. According to Howson (2005), there are two methods to create meaning. The first is to construct geometrical (graphical) model such as Poincaré Disk Model in mathematics, and number line for arithmetic operations in school mathematics. The second is to associate known objects with new object. The second has two methods in detail; to investigate and organize the connection between known objects with new object, and to construct new object by using known objects and inference rules.

Here, if we interpret "known objects" in the latter method of the second as "the particular", we can say, new object that is constructed by using the particular and inference rules has meaning. We call this characteristic function *association*. For example, according to Howson (2005), one can meaningfully construct integer (the general notion) by using natural number (the particular) and inference rules. In school mathematics, for example, students have their meaning for general triangle that is invisible and inexistent, because they construct general triangle by being based on particular triangles. Ito (1993) focused on this function and developed his learning theory, and analyzed elementary school students. As a result, he pointed out that the students had spontaneous attitude to use this function in order to construct new objects. In mathematics classroom, usually the function *association* does not become obvious. Rather, the function seems work implicitly in students' mind in learning mathematics.

Socialization

For example, if one says that the next term in the number sequence of 1, 2, 3, 4, 5, 6,… is 727, most people may not agree to it, and say that answer is 7. If those who wants to convince others that the next term is 727, they must present that the sequence $\{a_n\}$ can be generalized such as $a_n = (a-1)(a-2)(a-3)(a-4)(a-5)(a-6) + n$. In this example, both 7 and 727 are correct. As this example shows, however, other people do not always accept an individual subjective cognition even if the individual cognition (e.g. 727) is reasonable for the person without making the reason public. Thus, if the person wants to make one's own cognition be socially acceptable knowledge among other people, the generalization is required. Typically, we can say that Euclid described *The Element* with the intention to generalize the known and accepted propositions for socialization. This social aspect of generalization is emphasized by Dörfler (1991). We call this characteristic function *socialization*. Because the socialization means to open own cognition to other people, the function plays a very important role in constructing sound mathematical knowledge.

The function *socialization* of generalization usually works implicitly, especially from students' viewpoint. When students' cognition meet the counterintuitive, the function may become obvious. Nevertheless, the function socialization always plays very fundamental roll in the activity of learning mathematics in school classroom.

A FRAMEWORK FOR THE FUNCTION OF GENERALIZATION

In this section we will organize the six functions for making a theoretical framework. The above examples and consideration suggest that there is epistemological order from *variablization* to *unification*, and from *purification* to discovery respectively. In fact, Polya (1954) argued that *unification* is more higher than *variablization*, and likened it to the proverb; "To dilute a little wine with a lot of water is cheap and easy. To prepare a refined and condensed extract from several good ingredients is much more difficult, but valuable (p.30)". On the other hand, their examples imply that *association* and *socialization* are usually functioning implicitly, but both play fundamental rolls in learning mathematics. In addition, *association* and *socialization* are in their nature different from other four functions. They are not exclusive, and play different roll in constructing meanings for oneself or other people. Therefore, we propose a hypothetical structure of the six functions of generalization in Figure 3.

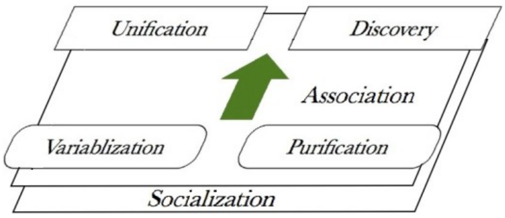

Figure 3: A hypothetical structure of the six functions of generalization

The framework consisted of six functions and their structure implies three didactical suggestions. First, teacher should design didactical situations where students can discern meaning of the six functions of generalization (ex. purification and/or unification). It is the answer to the students' question; "For what do we generalize it? ". Second, the structure shown in Figure 3 has possibility of a principle of didactics for designing mathematics classes. For example, if teacher intends to promote the unification in a mathematics class, the unification should be set up after the variablization or the purification. If teacher intends to promote students discern the socialization and/or the association, it is latent until after teacher expose students to other functions in a mathematics class. In a mathematics class, when one function of generalization is changed, teacher should give students the needed didactical support for making them be aware of the change "for what we do generalize it". Third, the most important suggestion is that the structure may become a guideline in forming mental habit for students through their experiencing the functions of generalization in mathematics classes. For example, Figure 3 shows that after activity of variablization, students do the activity of unification, and then reflecting on the association. However, it is difficult for students at the beginning do these activities without any didactical supports by teacher. Hence, if mathematics classes are usually planed based on the structure in Figure 3, it may become a guideline in forming mental habit for students, for example, "we have variablized this notion, so maybe we can unify other objects!" We expect that the formed mental habit could support students use the functions of generalization, leading to enjoy and endeavor their generalization as genuine mathematics activity.

CONCLUDING REMARKS

In this paper, as the answer to RQ1, we identified six functions with their examples of generalization; *variablization, purification, unification, discovery, association*, and *socialization*. We proposed the new theoretical framework consisted of the six functions and their structure for generalization in learning mathematics. Then, as the answer to RQ2, we implied three didactical suggestions for teaching and learning mathematics in classroom. First, teacher should design didactical situations where students can discern meaning of the six functions of generalization. Second, the structure has possibility of a principle of didactics for designing mathematics classes. Third, the structure may become a guideline in forming mental habit for students through their experiencing the functions of generalization in mathematics classes.

The following are main tasks to be tackled in the future research. First, we need to plan and practice mathematics classes based on the framework for the function of generalization in classrooms. Second, the functions and their structure need to be more refined with empirical data and philosophical consideration. Third, we need to investigate and sequence in detail the differences in the function of generalization for students in learning mathematics from elementary to secondary school mathematics.

References

Davydov, V. (2008). *Problems of developmental instruction: A theoretical and experimental psychological study*. NewYork: Nova Science Publishers.

Dirichlet, P. G. L., & Dedekind, M. R. (1999). *Lectures on number theory.* (J. Stillwell, Trans.). USA: American Mathematical Society.

Dörfler, W. (1991). Forms and means of generalization in Mathematics. In A. J. Bishop, S. Mellin-Olsen, & J. V. Dormolen (Eds.), *Mathematical knowledge: its growth through teaching* (pp. 63-85). Dordrecht, Netherlands: Kluwer.

Dörfler, W. (2008). En route from patterns to algebra: comments and reflections. *ZDM*, *40*, 143-160.

Giusti, E. (1999). *Where numbers come from – Hypothesis about the nature of mathematical objects* [in Japanese]. Tokyo, Japan: Kyoritsu.

Hayata, T., & Koyama, M. (2013). Generalization, mental object, and stability of import. In A. M. Lindmeier & A. Heinze (Eds.), *Proc. 37th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 17-24). Kiel, Germany: PME.

Howson, G. (2005). "Meaning" and school mathematics. In A. J. Bishop (Ed.), *Meaning in mathematics education* (Vol. 37, pp. 17-38). New York: Springer.

Ito, S. (1993). *A study on the constructive method in mathematics education part 1* [in Japanese]. Tokyo, Japan: Meijitosho.

Iwasaki, H., & Yamaguchi, T. (1997). The cognitive and symbolic analysis of the generalization process: the comparison of algebraic signs with geometric figures. In E. Lehkonen (Ed.), *Proc. 21st Conf. of the Int. Group for Psychology of Mathematics Education* (Vol. 3, pp. 105-112). Lathi, Finland: PME.

Polya, G. (1954). *Mathematics and plausible reasoning: Volume I, Induction and analogy in mathematics.* USA: Princeton University.

Radford, L. (1996). Some reflections on teaching algebra through generalization. In N. Bernarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra* (Vol. 18, pp. 107-111). Netherland: Springer.

Radford, L. (2001). Factual, contextual and symbolic generalizations in algebra. In M. V. D. Hauvel-Panhuizen (Ed.), *Proc. 25th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, pp. 81-88). Utrecht, Germany: PME.

Tatsis, B., M., & Tatsis, K. (2012). *Generalization in mathematics at all educational levels.* Rzeszów, Poland: Rzeszów University.

Tomosada, S., Himeda, K., & Mizoguchi, T. (2006). Aspects of mathematical activity in mathematics learning through generalization or extension and its design [in Japanese]. *Tottori Journal for Research in Mathematics Education*, *9*(1), 1-10.

Ursini, S. (1990). Generalization processes in elementary algebra: Interpretation and symbolization. In G. Booker, P. Cobb, & T. N. Mendicuti (Eds.), *Proc. 21st Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 149-156). Lahti, Finland: PME.

Vinner, S. (2011). The role of examples in the learning of mathematics and in everyday thought processes. *ZDM*, *43*, 247-256.