“THAT SOUNDS GREEK TO ME!” PRIMARY CHILDREN’S ADDITIVE AND PROPORTIONAL RESPONSES TO UNREADABLE WORD PROBLEMS

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Both additive and proportional reasoning are types of quantitative analogical (QA) reasoning. We investigated the development and nature of primary school children’s QA reasoning by offering two missing-value word problems to 3rd to 6th graders. In one problem, ratios between given numbers were integer, in the other ratios were non-integer. These word problems were written in the Greek alphabet, and thus totally incomprehensible to the children. QA answers considerably increased with age. Younger children more frequently chose additive relations, whereas older children chose more proportional relations. The nature of the ratios between the given numbers also affected the answers, particularly in 5th grade.

THEORETICAL AND EMPIRICAL BACKGROUND

Solving proportional missing-value problems

In primary school, children frequently encounter proportion problems, mainly with a missing-value structure (Cramer & Post, 1993), in which three magnitudes are given and a fourth one has to be found by identifying the multiplicative relation between two given magnitudes and applying this relation to the third given magnitude (Kaput & West, 1994; Vergnaud, 1997). To illustrate the structure of missing-value word problems, and the two main approaches to solve them, we use the ‘placemat problem’ of Kaput and West (1994): “A restaurant sets tables by putting seven pieces of silverware and four pieces of china on each placemat. If it used thirty-five pieces of silverware in its table settings last night, how many pieces of china did it use?” (p. 254). Proportional reasoners using the external ratio assume a proportional relationship between silverware and china pieces (i.e. $7 \cdot 4/7 = 4$), and apply this relationship to the third magnitude (i.e. $35 \cdot 4/7 = 20$). Proportional reasoners using the internal ratio assume a proportional relationship between the first and second number of silverware pieces (i.e. $7 \cdot 5 = 35$), and apply this relationship to the third magnitude (i.e. $4 \cdot 5 = 20$).

From 4th grade on, children get ample instruction in, and practice with, the solution of proportional missing-value problems in a diversity of contexts (such as equal sharing, constant price, or uniform speed) (Vergnaud, 1983, 1988). However, previous research (e.g., Hart, 1988, Kaput & West, 1994; Karplus, Pulos & Stage, 1983) has shown that in the beginning children frequently give additive solutions instead of proportional ones. In the aforementioned ‘placemat problem’, those children would assume an additive relationship between pieces of silverware and pieces of china (i.e. ‘the external difference’, $7 – 3 = 4$), and apply it to the third known magnitude (i.e. $35 – 3 =$

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32). Additive reasoners could also assume an additive relationship between the two numbers of silverware pieces (i.e. 'the internal difference', 7 + 28 = 35), and apply this to the third magnitude (i.e. 4 + 28 = 32). Studies have pointed out that children use additive solution methods in proportional problems more frequently when the numbers in the problem form non-integer ratios (Kaput & West, 1994; Karplus et al., 1983; Vergnaud, 1983, 1988).

Solving additive missing-value problems

Of course, not every missing-value problem should be solved by means of proportional reasoning. In some missing-value word problems, another type of reasoning (e.g. quadratic, or exponential) is required. In this paper, missing-value problems where additive reasoning is required are of specific interest. An example is the one that Cramer, Post and Currier (1993) gave to pre-service elementary education teachers: "Sue and Julie were running equally fast around a track. Sue started first. When Julie had run 3 laps, Sue had run 9 laps. When Julie completed 15 laps, how many laps had Sue run?" (p. 159). Here, the relation is an additive one (i.e. a relation of difference). Sue is 6 laps ahead of Julie, so when Julie ran 15 laps, Sue ran 15 + 6 = 21 laps.

We are not aware of any mathematics curriculum where attention is spent to solving additive missing-value problems. Still, this could be valuable, given that (analogously to our overview of the incorrect use of additive reasoning to proportional missing-value problems as given above) many children erroneously use proportional solution methods to additive missing-value word problems. For instance, the most frequent erroneous answer to the aforementioned runner problem of Cramer et al. (1993) is "15 · 3 = 45". Previous research pointed out that the improper use of proportional reasoning is also strongly determined by task and subject characteristics, similar to those for the improper use of additive strategies (Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005; Van Dooren, De Bock & Verschaffel, 2010): First, the application of proportional methods occurs more frequently when the numbers in the word problem form integer ratios, and, second, the overuse of proportional methods to additive problems tends to increase with age during elementary school and the first years of secondary school. Moreover, between the stage where children overuse additive methods on proportional problems (as described in the previous paragraph) and the stage where they overuse proportional methods on additive problems, there is a stage of simultaneous overuse of additive and proportional methods. Children in this intermediate stage give additive answers to word problems with non-integer ratios and proportional answers to problems with integer ratios, independent of their actual mathematical structure. In Flanders (Belgium), this intermediate stage typically occurs in 5th grade of primary school.

Similar despite differences: quantitative analogical reasoning

Most research on the development of proportional reasoning considered children's additive reasoning as an indicator of not having reached the stage of proportional reasoning yet (or at least not yet completely). While we agree with this conclusion, a basic tenet of the present paper is that children who reason additively in those proportional word problems have already taken a valuable step in their development towards proportional reasoning, as compared, for instance, to children who just add all the given numbers. Kaput and West (1994) already emphasized that children who improperly use the additive approach for proportional reasoning problems of the missing-value type, still "distinguish the quantities, construct units, and correctly identify the unknown quantity" (p. 251). In other words, improper additive reasoners demonstrate insight into the different known and unknown magnitudes and the fact that these are analogously related. They focus on the quantitative relation between two magnitudes that are given in the word problem, and apply this relation to a third given magnitude in order to calculate the missing one. So, regardless of the correctness for a given problem, additive and proportional missing-value reasoning have in common that children focus on the analogical relations between the four magnitudes in the word problem. Thus, both additive and proportional missing-value reasoning are types of quantitative analogical reasoning (hereafter abbreviated as QA reasoning).

RATIONALE

In this study, we applied a novel approach to investigate the development of QA reasoning, namely by giving children word problems that were unreadable to them. We will explain the rationale for this – at first sight indeed strange – methodological choice. In all aforementioned previous studies into children's choice for an additive or proportional solution method, word problems with an underlying mathematical model that could be determined clearly and unquestionably by carefully reading and processing the word problem, were used. In the current study, besides the development of children's quantitative analogical reasoning per se, we also wanted to investigate children's choice for an additive or proportional approach in situations where they were not directed whatsoever by the mathematical structure of the word problem. This allowed us to get a view on children's general and spontaneous inclination towards QA reasoning, and, in case such reasoning occurred, which type of QA reasoning then would be used (additive or proportional). For this reason, we used an atypical kind of items, namely mathematically neutral word problems. We designed such neutral problems by posing them in Greek literal symbols which were completely inaccessible to the (Flemish) children involved in our study. The numbers were of course accessible as they were presented in their usual Arabic form. Still, children were asked to try to solve these 'incomprehensible' word problems. Our intention was thus to find out to what extent they would look for a quantitative analogical relation between the given numbers, and if so, if they would opt for an additive or a proportional one.

RESEARCH QUESTIONS AND HYPOTHESES

Our first research question was: To what extent do children apply quantitative analogical reasoning in neutral word problems, and how is this affected by age? Because of elementary school children's increasing classroom experiences with solving missing-value word problems, we expected that even those neutral word problems would elicit a substantial amount of QA reasoning (*hypothesis 1*), and that this amount would increase with age (*hypothesis 2*).

Our second research question was: What is the nature of children's QA reasoning, and how is it affected by age and by number characteristics of the neutral word problem? Given that both additive and proportional types of answers to missing-value problems were observed in previous research, we hypothesized that we would observe both types of QA answers to our neutral word problems (*hypothesis 3*). Furthermore, based on the aforementioned previous research results about clearly additive and proportional word problems, we anticipated that among the QA answers, there would be a development with age, from a dominance of additive answers towards a dominance of proportional answers for neutral word problems too (*hypothesis 4*). We also expected a reliance on the characteristics of the numbers in the word problem. More specifically, we predicted that problems containing non-integer ratios would lead to a higher number of additive answers than problems with integer ratios, and that the latter problems would lead to a higher number of proportional answers than problems with non-integer ratios (*hypothesis 5*). Finally, we anticipated that the sensitivity to the numbers in the problem would be the strongest in the intermediate stage of children's development, between the initial stage, with mainly additive answers, and the final stage, wherein mainly proportional answers were expected (*hypothesis 6*).

METHOD

Participants were 325 children from 3rd to 6th grade from two primary schools in Flanders (88 3rd graders, 78 4th graders, 81 5th graders and 78 6th graders). The number of boys and girls was approximately equal in the sample. The children solved two neutral word problems, that will be the focus of the current paper. These neutral word problems were part of two larger paper-and-pencil tests. Each of these tests contained one neutral word problem, along with some buffer items (related to various parts of the children's curriculum). Both neutral word problems were stated in Greek literal symbols, but the numbers were given in the usual Arabic form as shown in Figure 1. Flemish children could absolutely not read nor understand the text of these problems, so neither the proportional nor the additive solution method − nor any other solution method − could be considered as correct or incorrect. The two word problems only differed with respect to the numbers used in the problem: the given numbers formed integer (internal and external) ratios (e.g., 4, 16 and 8 as given magnitudes) for one problem, and non-integer (e.g., 4, 14 and 6 as given magnitudes) for the other one. To minimize the influence of the specific numbers in both problems, several sets of numbers forming integer and non-integer ratios were used.

The two tests were administered on two separate moments, with one week in between. The researcher told children that the test was aimed at assessing general mathematics achievement. For the neutral problems the test merely mentioned that the problems were in Greek but that children were nevertheless invited to try to fill them in.

This word problem is a Greek one. Try to fill in a number on the dotted line.

Αδα καλκα πορελαντορα λικτουν κοττορ.
Νοϖεργανιχα τινεσταρι 4 ποσστορ ιο χηιον ανπερα τον ϖορχον 16 στατον εστανο τυϖ μαγχανετο.
Προβαλεντι μογρονατεσ 8 ογροντ ο γνοστον καλκονο τοτ λινδεναν, ναγ κιφ νισϖορ κ σχκρινον λοπεναδο μαορν εωεινστ?

Answer:

Γελομαλ λοπανδορα ριτ .... νιφφ τοτο.

Figure 1: ‘Greek’ word problem.

RESULTS

Quantitative analogical reasoning

In a first step of the analysis, the responses to the two neutral word problems were classified as ‘QA answers’ when either proportional or additive operations were executed on given numbers (i.e. calculating x in $b / a = x / c$ or in $b - a = x - c$), or as ‘other answers’ when the given numbers were combined in another way than specified above, or when the problem was left unanswered.

While coding the responses, a third category, namely ‘sum of three’ answers was added for coding cases wherein the three given numbers were added (i.e. calculating x as $x = a + b + c$). This solution method is not of specific interest for the present study (as it is not a QA answer in the sense explained above), but was still included because a large number of children had used it.

Table 1 gives an overview of the percentage of all QA, other and sum-of-three answers in different grades. This table reveals that 20.5% of all answers were QA answers. Another 42.6% was of the sum-of-three type, and the remaining 36.9% were other answers. So, in line with hypothesis 1, we found a substantial number of QA answers, especially given that the two neutral word problems were completely incomprehensible to these children. However, even more interesting is the effect of age on the percentage of QA answers.

Grade	QA	Other	Sum-of-three
3	9.1	42.0	48.9
4	7.7	35.9	56.4
5	25.3	33.4	41.4
6	41.1	35.9	23.1
Total	20.5	36.9	42.6

Table 1: Percentages of quantitative analogical (QA), other and sum-of-three answers in different grades.

A generalized estimating equations analysis revealed that children’s age affected their answers. The percentage of QA answers significantly increased from 9.1% in 3rd grade to 41.1% in 6th grade ($\chi^2(3)$= 43.858, p < .001), which was in line with our second hypothesis. As shown in Table 1, the initially low percentage of QA answers was due to the remarkably large percentage of sum-of-three answers. Almost half of the answers (48.9%) was characterized as such in 3rd grade, and still almost a quarter in 6th grade ($\chi^2(3)$= 24.579, p < .001). The percentage of other answers also decreased with age, from 42.0% in 3rd grade to 35.9% 6th grade, but this decrease was much smaller and non-significant.

Proportional or additive quantitative analogical reasoning

In a second step, we focused on the subset of answers being coded as QA answers (20.5% of all answers, i.e. 133 out of 650), to answer our second research question about the precise nature of QA reasoning. All QA answers were further categorized as 'proportional answers' (when multiplicative operations were executed on given numbers, i.e. calculating x in the expression $b / a = x / c$) or 'additive answers' (when additive operations were executed on given numbers, i.e. finding x in $b - a = x - c$).

Table 2 gives an overview of the percentage of additive and proportional answers. As expected (hypothesis 3), the neutral word problems elicited both proportional and additive answers. Of all QA answers, half were additive (49.6%), whereas the other half were proportional (50.4%). Moreover, the percentage of additive and proportional answers differed depending on children's grade and on the nature of the numbers. The results of a generalized estimating equations analysis indicated, first, that the percentage of proportional answers significantly increased with age, from 25.0% in 3rd grade, to 64.1% in 6th grade ($\chi^2(3)$= 884.927, p < .001, see Table 2). Accordingly, the percentage of additive answers significantly decreased from 75.0% in 3rd grade to 35.9% in 6th grade. These findings were consistent with hypothesis 4. Second, the nature of the numbers affected the kind of QA answers, as expected in hypothesis 5. The integer problem evoked significantly more proportional answers than the non-integer problem (69.4% vs. 27.9%, $\chi^2(1)$= 1349.979, p < .001). Third, the number effect interacted significantly with the effect of grade ($\chi^2(2)$= 452.825, p < .001), which was in

Type of numbers	Grade	n	Additive	Proportional
Integer	3	12	60.0	40.0
	4	11	80.0	20.0
	5	20	26.1	73.9
	6	23	17.6	82.4
	Total	66	30.6	69.4
Non-integer	3	4	100.0	0.0
	4	1	100.0	0.0
	5	21	77.8	22.2
	6	41	56.7	43.3
	Total	67	72.1	27.9
All problems	3	16	75.0	25.0
	4	12	91.7	8.3
	5	41	48.8	51.2
	6	64	35.9	64.1
	Total	133	49.6	50.4

Table 2: Percentages of additive and proportional answers by grade and type of numbers.

line with hypothesis 6. The number effect was the largest in 5^{th} grade (leading to a difference of 51.7% between the percentage of proportional answers to the integer and non-integer variant), and decreased towards 6^{th} grade (39.1%). However, the difference in 3^{rd} grade (40.0%) and 4^{th} grade (20.0%) was not reliable, due to the very low absolute number of QA answers.

CONCLUSION AND DISCUSSION

This study focused on children's quantitative analogical (QA) reasoning in word problems that could be considered neutral in terms of their underlying mathematical model, given the completely unknown alphabet and language in which they were posed. In a first step, we analyzed children's tendency to give answers based on QA reasoning. This kind of analysis is rather unique, because previous research into this topic has mainly focused on either additive reasoning or proportional reasoning, without explicitly recognizing the common nature of these two types of reasoning. Our study indicated that the neutral word problems did elicit answers based on QA reasoning, in approximately one out of five cases. This percentage considerably increased with age. Consciously or not, older children more frequently looked for a relation between two given numbers in the word problem and applied this to the third number, in order to calculate a fourth one.

The finding that children became more focused on quantitative relations relates to the notion of 'spontaneous focus on relations' (SFOR) introduced by McMullen, Hannula-Sormunen and Lehtinen (2013). However, they studied this SFOR tendency by means of non-explicitly mathematical tasks, whereas we conceptualized QA reasoning in the context of missing-value word problems which are clearly mathematical. Future research should study the relation between these two notions.

In a second step, we investigated on which kind of quantitative relation the quantitative analogical reasoners relied. The same overall percentage of answers was additive or proportional, but the percentage of additive answers decreased with age, while that of proportional answers increased. Furthermore, problems with integer ratios evoked more proportional than additive answers, whereas there reverse was true for problems with non-integer ratios. This number effect was most prominent in 5^{th} grade.

The explanation for our findings is still open for discussion, but it may at least partly be found in the current elementary mathematics curriculum. Children encounter in their elementary mathematics lessons a restricted and stereotyped diet of word problems, and are taught to solve them by recognizing the problem type and activating the arithmetic solution method that is associated with it (e.g., Verschaffel, Greer, & De Corte, 2000). The majority of word problems with a missing-value structure with which children are confronted must be solved by focusing on the proportional relations. Moreover, when proportional reasoning is introduced, problems typically first involve numbers forming integer ratios (Van Dooren et al., 2010). This way, it is not surprising that older children increasingly reason proportionally, and that children connect superficial cues in the word problem (i.e. number characteristics) with concrete solution methods.

Regardless of the fact that additive analogical reasoning often inappropriately occurs in proportional missing-value problems, it is still an important and valuable step in children's development towards proportional reasoning. Additive reasoning is after all already a way of QA reasoning. Therefore, we suggest that both additive and proportional missing-value problems should be included in the elementary school curriculum, and that children repeatedly should be stimulated and helped to distinguish between them.

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