VARIABLE PARTS: A NEW PERSPECTIVE ON PROPORTIONAL RELATIONSHIPS AND LINEAR FUNCTIONS

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We present a mathematical analysis that distinguishes two quantitative perspectives on ratios and proportional relationships: Multiple Batches and Variable Parts. We argue that (a) existing research on proportional relationships has addressed Multiple Batches but has largely overlooked Variable Parts, (b) Multiple Batches makes the co-variation aspect of proportional relationships more explicit, while Variable Parts makes the fixed multiplicative relationship between two quantities more explicit, (c) the distinction between Multiple Batches and Variable Parts is orthogonal to the within-measure-space versus between-measure-space ratio distinction, and (d) Variable Parts affords promising new approaches for addressing linear relationships.

PAST RESEARCH ON PROPORTIONAL RELATIONSHIPS

Ratios and proportional relationships are critical mathematics in elementary and secondary grades (e.g., Kilpatrick, Swafford, & Findell, 2001; National Council of Teachers of Mathematics, 1989, 2000). Although traditional instruction has emphasized applying rote procedures like cross multiplication to solve missing-value and comparison problems, *a robust understanding of proportional relationships involves (a) attending to co-variation between two quantities and (b) forming multiplicative relationships between those quantities*. Despite a significant body of empirical and theoretical research on proportional relationships, understanding how to support students' and teachers' understandings of *both* aspects of proportional relationships remains a significant challenge for the field.

Empirical research has documented numerous difficulties that students, and sometimes teachers, experience with proportional relationships. One line of research has analyzed factors that influence the difficulty of proportion problems for students—including whether students are familiar with problem contexts (e.g., Tourniaire, 1986), whether quantities are discrete or continuous (e.g., Behr, Lesh, Post, & Silver, 1983), and whether ratios are integral, nonintegral, or unit ratios (e.g., Hart, 1981, 1988; Karplus, Pulos, & Stage, 1983; Noelting, 1980a, 1980b). A second line of research has examined students' and teachers' capacities to distinguish missing-value problems that describe proportional relationships from ones that do not (e.g., Cramer, Post, & Currier, 1993; Fisher, 1988; Freudenthal, 1983; Van Dooren, De Bock, Vleugels, & Verschaffel, 2010.) A third line of research has examined difficulties that students and teachers have conceiving of a ratio as a measure of a physical attribute, such as steepness or speed (Simon & Blume, 1994; Thompson & Thompson, 1994). A fourth line of research has examined strategies that students use to solve problems about

2014. In Nicol, C., Liljedahl, P., Oesterle, S., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 2, pp. 113-120. Vancouver, Canada: PME.

proportions successfully, often before any substantial instruction in these topics. These include forming progressively elaborate unit structures (e.g., Lamon 1993a, 1994; Lobato & Ellis, 2010) and double counting strategies (e.g., Hart 1981, 1988; Lamon, 1993b).

Theoretical research has identified various ways to think about multiplicative relationships in terms of quantities (see Greer, 1992, for a review). There is widespread agreement among mathematics education researchers that ratios and proportional relationships are part of the *multiplicative conceptual field*—a web of interrelated ideas that includes multiplication and division, fractions, linear functions, and more (Vergnaud, 1983, 1988). Furthermore, much of the theoretical work on proportional relationships has been informed by Vergnaud's (1983) identification of isomorphism of measures as one of three fundamental multiplicative structures. Isomorphism of measures covers direct proportions between two measure spaces, and Vergnaud distinguished forming multiplicative relationships within measure spaces from forming such relationships between measure spaces (e.g., Freudenthal, 1973; Lamon, 2007; Noelting, 1980b).

We present an analysis that contributes to the theory of proportional relationships, identifying an overlooked perspective that promises new avenues for reasoning about proportional relationships and foundations for understanding slope and rate of change, among other subsequent topics.

THE TWO PERSPECTIVES ON PROPORTIONAL RELATIONSHIPS

Beckmann and Izsák (2013) identified two distinct, complementary perspectives on how quantities vary together in a proportional relationship. The two perspectives follow from consistently distinguishing the multiplier, M, from the multiplicand, N, in the equation $M \bullet N = P$ (M denotes number of groups, N denotes the number of units in each/whole group, and P denotes the number of units in M groups).

Figure 1 uses ***Punch Problem 1*** to illustrate the two perspectives, which conceptualize and depict covariation and fixed multiplicative relationships in complementary ways. *Multiple Batches* has been widely studied among children—for instance, Lamon, (1993b) and Lobato and Ellis (2010) have referred to it as composed unit reasoning. In this perspective (Figure 1a), a mixture of 3 cups peach juice and 2 cups grape juice is fixed as 1 batch. One *varies the number of batches* to produce different amounts in the same ratio, which corresponds to operating on M. Vertical alignment on the double number line indicates amounts in the same 3-to-2 ratio. Covariation is made visually explicit as movement of that vertical alignment up and down the double number line, but the fixed multiplicative relationship between the quantities—the amount of peach juice is always 3/2 times the amount of grape juice—remains implicit. *Variable Parts* has been largely overlooked in past research and teaching on proportional relationships. In this perspective (Figure 1b), one fixes numbers of parts of peach juice (3) and grape juice (2), and all parts are the same size. One *varies the size of the parts* to produce different amounts in the same ratio (throughout, any one part remains equal to all the other parts), which corresponds to operating on *N*. The numbers of parts show explicitly that the amount of peach juice is always 3/2 times the amount of grape juice, but variation within parts remains implicit.

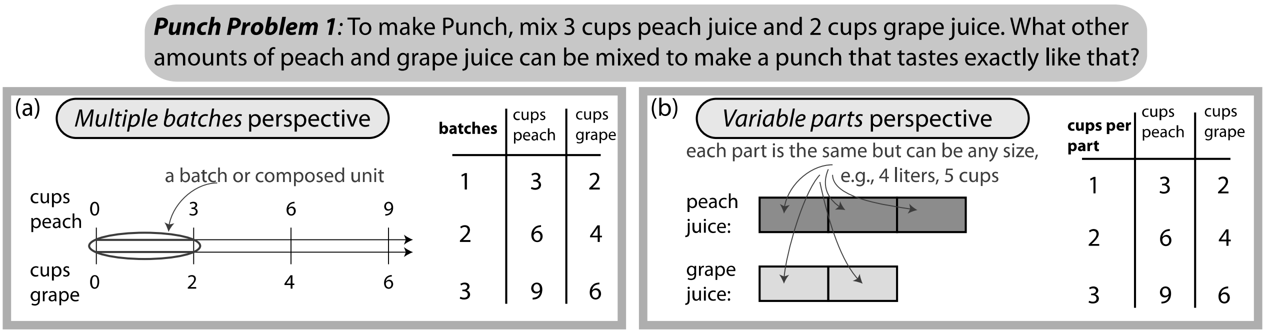

Figure 1: (a) The multiple batches perspective. (b) The variable parts perspective.

The two perspectives on proportional relationships are orthogonal to the within-measure-space versus between-measure-space ratio distinction mentioned previously (Vergnaud, 1983): One can use each perspective to relate quantities within measure spaces or between measure spaces. To illustrate within-measure-space reasoning from the two perspectives, consider the following problem that continues to use the punch scenario. *You made a mixture of 3 cups peach juice and 2 cups grape juice. Now you want to make a mixture in the same ratio using 1/4 as much peach juice. How much grape juice should you use?* Using Multiple Batches, one might view the 1/4 as operating on 1 batch and therefore reason that 1/4 should operate on the cups of peach and grape juice (1/4 batch•3 cups of peach juice in each batch; 1/4 batch•2 cups grape juice in each batch). In this case, multiplying by 1/4 *changes the number of batches* (*M*). Using Variable Parts, one might start with 1 cup of juice in each of the 5 parts and view 1/4 as operating on the size of each part. Here, one needs 3 parts peach juice•1/4 cup in each part and 2 parts grape juice•1/4 cup in each part. In this case, multiplying by 1/4 *changes the size of all 5 parts* (*N*). Beckmann and Izsák (2013) explain how both Multiple Batches and Variable Parts can support between-measure-space reasoning.

Research on proportional relationships has emphasized Multiple Batches, which facilitates within-measure-space reasoning using a variety of strategies, including a "building up" strategy and iterating and partitioning a composed unit (e.g., Kaput & West, 1994; Lamon, 1994, 2007; Lobato & Ellis, 2010; Vergnaud, 1988). Although Lobato and Ellis showed how iterating and partitioning a composed unit can be used to derive a fixed multiplicative relationship between measure spaces, Kaput and West noted that some multiplicative relationships are not well handled by iterating and partitioning within measure spaces:

> A major question not addressed in this chapter is how to deal with multiplicative change situations that are not well modeled [sic] by build-up patterns, change situations that are not inherently replicative. These include the geometric similarity problems handled poorly by our students. The larger, rescaled figure is not the join of several smaller ones. Rather,

> each of the infinitely subdivisible parts of the smaller figure "grows" by the same amount to produce the larger as discussed by Confrey (this volume). This form of multiplicative growth likely has different primitive conceptual roots and is likely to require a different curriculum strand and different types of concrete representations. (p. 284)

We hypothesize that Variable Parts and strip diagrams can provide the needed complementary perspective on multiplicative relationships. In particular, in the next section, we argue that adding Variable Parts to the study of proportional relationships may provide a more robust foundation for the study of linear functions than Multiple Batches alone. Thus, Variable Parts deserves consideration in research on cognition around proportional relationships.

TWO PERSPECTIVES AS A FOUNDATION FOR LINEAR FUNCTIONS

An important theme in the extensive literature on students' and, to a lesser extent, teachers' understandings of algebra is the role of prior experience with arithmetic, including with rational numbers, in supporting and constraining reasoning about linear relationships (e.g., Carraher & Schliemann, 2007; Hackenberg, 2010, 2013; Kieran, 1992). For instance, Kieran (p. 394) argued one source of difficulty is that using algebraic notation to model problem situations requires students to modify their interpretations of symbols like the equal sign and to use arithmetic operations that invert operations they have learned to use almost automatically, while Hackenberg has argued that experience reasoning with fractions in terms of quantities provides a critical foundation for interpreting equations that relate quantities. We focus on the persistent challenge of forming fixed multiplicative relationships between quantities, including slope.

Confusion about meanings of slope, rate of change, and steepness have been found among students using either reform or more traditional curricula (Lobato, Ellis, & Munoz, 2003; Teuscher, Reys, Evitts, & Heinz, 2010), as well as among future teachers (e.g., Simon & Blume, 1994). As one example, Lobato et al. (2003) reported on U.S. high school students' understandings of slope after instruction using a reform curriculum that emphasized slope as a rate of change between covarying quantities in multiple real-world settings and that used multiple representations. The researchers' reported examples of students' persistent difficulties understanding slope as a multiplicative relationship between changes in values of x and y, even when students reasoned about partitioning and iterating Multiple Batches. Such results raise as a question whether other perspectives on covarying quantities might better support appropriate multiplicative conceptualizations of slope (or constants of proportionality). Next, we return to the ***Punch Problem 1*** (Figure 1) and examine how Multiple Batches and Variable Parts can support such conceptualizations.

In a Multiple Batches approach to slope, one thinks of having 3 cups peach juice for every 2 cups grape juice. The value 3/2 specifies the number of cups of peach juice needed for every 1 cup of grape juice (a unit rate) (Figure 2a). This view foregrounds slope as the coordinated variation within the grape juice and peach juice measure spaces: For every new cup of grape juice, the amount of peach juice increases by 3/2 cups. This perspective evokes repeatedly moving to the right 1 unit and up 3/2 units, but the general multiplicative relationship, $y = (3/2)\,x$, is less evident. In a Variable Parts approach to slope, the value 3/2 is a direct multiplicative comparison between the numbers of parts of grape and peach juice: The number of parts peach juice is 3/2 the number of parts grape juice (Figure 2b). Put another way, the value 3/2 is the factor that multiplies the number of parts of grape juice to produce the number of parts of peach juice, regardless of amounts. Figure 2b shows how strip diagrams can be coordinated with Cartesian graphs to support such an interpretation of slope. This view foregrounds slope as a multiplicative relationship: The y-coordinate is 3/2 of the x-coordinate, so $y = (3/2)\,x$.

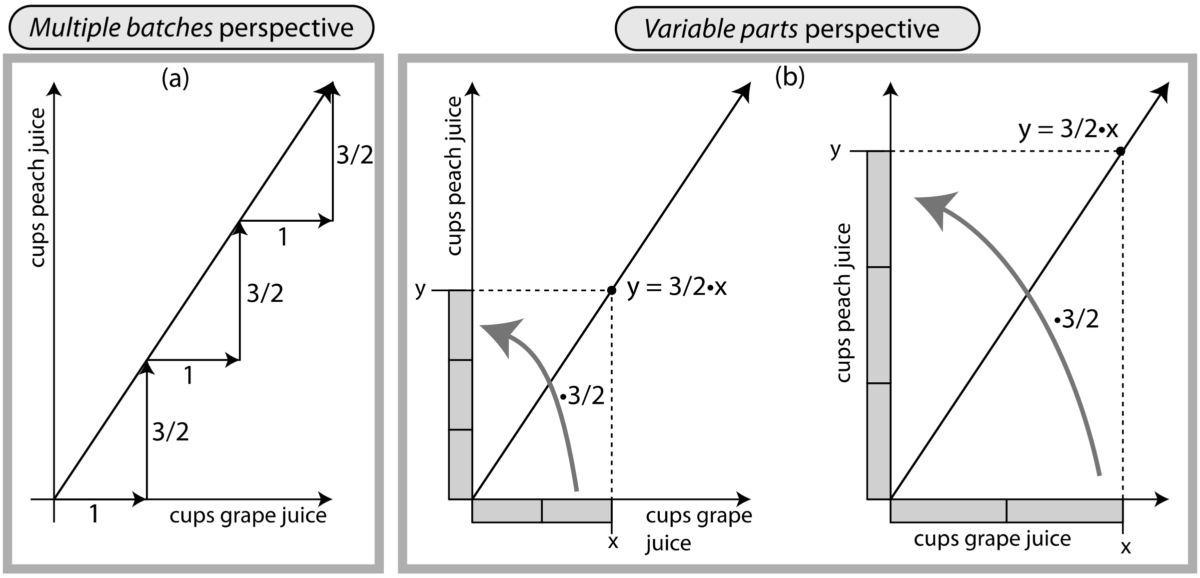

Figure 2: Two perspectives on slope. (a) Multiple batches. (b) Variable parts.

CONCLUSION AND DISCUSSION

An important question for future empirical research is whether introducing Variable Parts as a complementary perspective to Multiple Batches might help both students and teachers develop both key features of proportional relationships between two quantities and help them apply what they learn about proportional relationships to subsequent, central topics, such as slope. Our presentation of the two perspectives on ratios and proportional relationships suggests that adding a Variable Parts perspective may benefit students and teachers.

One benefit is that Variable Parts may support forming direct multiplicative comparisons of two quantities. Past research has shown that children and adults can have difficulty making such comparisons when using Multiple Batches (e.g., Vergnaud, 1980; Schliemann & Nunes, 1990).

A second benefit is that Variable Parts may support understanding not just slope and rate of change as multiplicative relationships but also equations that relate variables.

Numerous studies have demonstrated students' difficulties forming equations (e.g., Clement, 1982; Koedinger & Nathan, 2004). The process of deriving equations from strip diagrams by relating quantities may highlight a relational rather than computational interpretation of the equal sign (e.g., Kieran 1992) and support generating linear equations. Investigating this possibility would be consistent with Kieran's (2007) recommendation for additional research on how students could be assisted to make connections between verbal problem solving activity and generating equations (p. 729).

Finally, an important question for future research is how students and teachers might develop understandings of the two perspectives. It might be that Multiple Batches better supports initial coordination of two varying quantities but that Variable Parts better supports subsequent applications, such as applications to linear functions. Furthermore, it might that students and teachers understandings of the two perspectives could support one another: Seeing covariation explicitly in Multiple Batches might support seeing covariation in Variable Parts and seeing multiplicative comparisons explicitly in Variable Parts might support seeing such comparisons in Multiple Batches. Thus, in combination, the two perspectives on proportional relationships are promising for supporting students' understandings of a central mathematical domain. Therefore, the two perspectives deserve further investigation.

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