ANALOGIZING DURING MATHEMATICAL PROBLEM SOLVING – THEORETICAL AND EMPIRICAL CONSIDERATIONS

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The aim of the paper is to provide a process model to evaluate mathematical problem solving by analogy, in order to better determine at which point and under what conditions a learner is prompted to use analogies. The model is a theoretical construct. Qualitative results of an empirical study are used to underline and illustrate core aspects of the model.

INTRODUCTION

The ability to recognize and use analogies (Gick & Holyoak, 1983) is a key aspect of human cognition. If two situations are analogous, which means that there are the same relations between corresponding elements, knowledge transfer from the known situation (source) can help tackling the new situation (target). Thereby, analogical thinking can be used to assist us in understanding certain characteristics, relationships and mechanisms of unknown situations, or to construct plausible hypotheses. It can also play an important role in problem solving "when the solution to one problem suggests a solution to a similar one" (Holyoak & Thagard, 1989, p. 318).

Analogical reasoning is of particular importance in mathematics as the science of patterns, structures, and structure types: "Noticing higher order similarity relationships between such instances of structural similarity is at the core of complex mathematical thinking" (Richland et al., 2004, p. 38.). A closer look at the history of mathematics confirms that analogical reasoning has long played an important heuristic role in this field (Reed, 1985; Zimmermann, 2003).

Solving mathematical problems by analogy is a multi-step process involving higher-order cognitive skills, the first of which requires the identification of a source problem that can be retrieved from memory. A learner's ability to reason analogically is therefore very much dependent on their existing knowledge base (English, 2004). It also involves the mapping between the elements and relational structure of the source problem or known situation (source) and the new one (target). This requires the ability to change representations and therefore to abstract from concrete surface characteristics of the situations worked on (Novick, 1988). Finally, the modus operandi which is considered as appropriate has to transfer onto the new situation and therefore often adjust to its concrete requirements (Novick & Holyoak, 1991).

Although a number of studies have been conducted on using analogies during problem solving (e. g. on analogical transfer), not a great deal of research has been conducted in

2014. In Nicol, C., Liljedahl, P., Oesterle, S., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 2, pp. 73-80. Vancouver, Canada: PME.

mathematics education examining analogizing from a *qualitative* perspective. Rarely have mathematical problem-solving processes been examined from the point at which analogical reasoning occurs in learners and the conditions that can either facilitate, hinder or prevent this process. This paper has been written with the intent to broaden discussion in this area.

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PROBLEM SOLVING AS COGNITIVE MODELING

There is an extensive amount of research literature on mathematical problem solving and modeling, and more recently, on approaches that attempt to combine both in order to describe complex mathematical activities (e. g. Förster, 2000). This approach, derived and justified from the perspective of cognitive psychology and mathematics education, has been most recently taken also by Zawojeski and Lesh (2003). They argue that when students struggle with mathematical problem solving, this cannot always be attributed to a lack of heuristic tools and strategies in Pólya's sense alone. Rather, it is also due to the currently insufficient interpretation or modelling of the given situation. In 1998 Lesh accordingly defined the rather ambiguous term "problem" as follows: "the most important criteria that distinguishes 'non- routine problems' from 'exercises' is that the students must refine / transform / extend initially inadequate (but dynamically evolving) conceptual models in order to create 'successful' problem interpretations" (Zawojewski & Lesh, 2003, p. 318). The issue of "understanding the situation" is, of course, also addressed in classic problem solving models and heurisms a la Pólya can also be used to achieve an appropriate situation model. Moreover, the combination of elements of problem solving and modeling cycles to describe mathematical activity appear to be particularly fruitful when the construction and use of analogies in problem solving is to be analyzed, because analogizing bases upon mental models of mathematical situations.

EMPIRICAL BASIS

In order to examine the construction and use of analogies in mathematical problem solving, we conducted semi-structured clinical interviews (e.g. Beck & Maier, 1993) with 86 primary school pupils. 39 pupils came from regular primary classes of grade 3 to 6, the other 47 pupils participated in fostering projects for mathematically gifted students at university. Every pupil consecutively worked on two problems which were analogous to each other (party intermitted by a disturbing non-analogous exercise), and he was asked to describe what he is doing as far as possible (thinking aloud).

Also if the students failed to show appropriate signs of analogical thinking during the problem solving process, they were afterwards prompted by the interviewer to compare the problems and asked to describe any similarities they found (initiated review).

By now, we used altogether 12 different problems in our study. The most problem pairs tackled by the pupils were not only structurally analogous but also include the same numbers (see also P1 and P2 in Figure 1). This should make it possible to use the analogy for transferring also the results from one problem to the other. For more details of the whole study see Aßmus and Förster (2013a).

For explaining and empirically underpinning the process model presented in Figure 2 we only use two of these problems (cf. Figure 1). The problem combination P1+P2 was tackled by 12 fourth-graders. To ensure that the sample group was as heterogenic as possible, we had asked teachers from different schools to select an above average, average, below average, and if possible a mathematically gifted pupil from each of their classes to take part in the interviews.

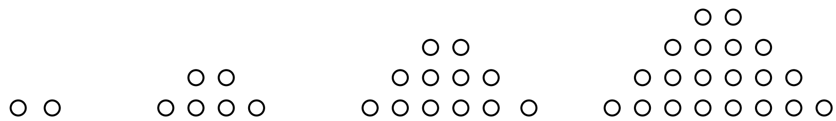
P1	 Paul makes groups of counters on the table. Each new group contains more counters than the last group in a certain way. How many counters do you think he will put in the 20th group?
P2	Anna starts to read a book. She reads two pages on the first day. She continues to read the book, reading 2 pages more than the day before each day. How many pages will she have read after 20 days in total?

Figure 1: Analogical problems (from an expert's view) used in the study.

Both problems were used in varying sequences to ensure that both the source and target points of the analogies to the real problems were empirically accessible.

MATHEMATICAL PROBLEM SOLVING AND CONSTRUCTION OF ANALOGIES

A Process Model

The process model in Figure 2 demonstrates possibilities for analogizing during mathematical problem solving. It attempts to combine the classical models of Pólya (1945) or Mason, Burton and Stacey (1982) to cognitive-modeling approaches while focusing simultaneously on the use of analogies in dealing with challenging mathematical situations.

The model demonstrates possible points in which analogies can be used. A distinction is made between how the situation is dealt with by the learners; whether or not the situation can be defined as the *source problem*, meaning that the ideas, approach or results of which can be transferred to another situation (*outgoing analogy*), or whether or not the situation can be defined as a *target problem*, meaning that a given learner's prior experience with a similar situation can be used (*targeting analogy*). The "reviewing" phase is marked by a dashed box, because in our setting it was, if necessary, initiated by the interviewer to stimulate analogizing processes.

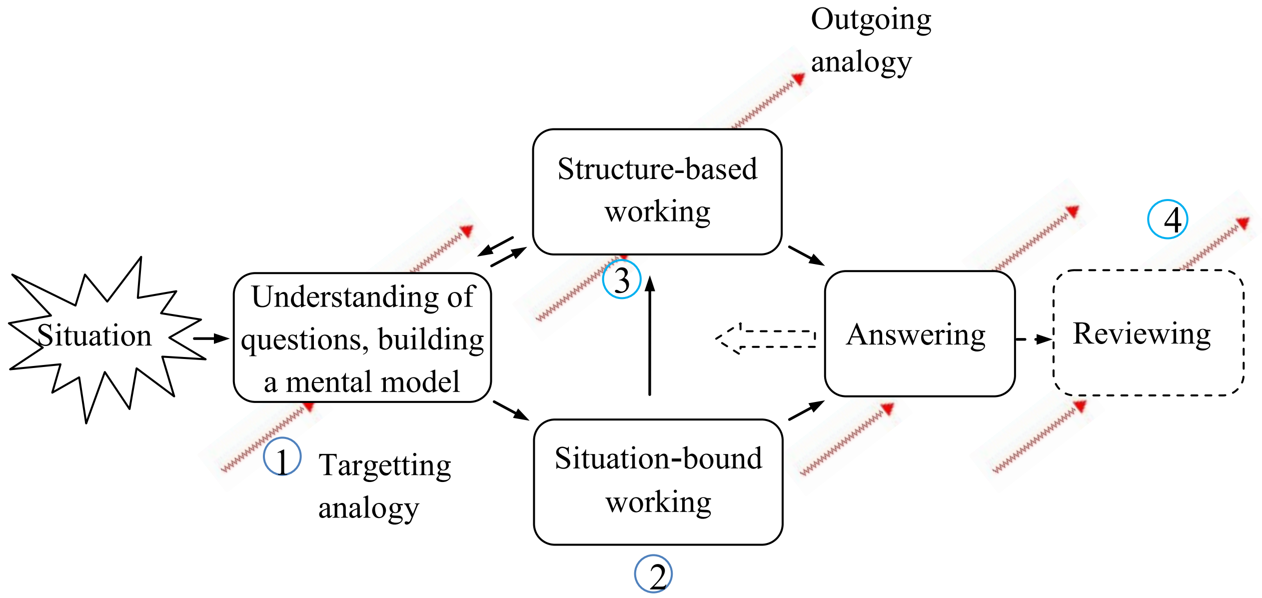


Figure 2: Process model of problem solving focusing possible uses of analogies.

The model also attempts to demonstrate the conditions necessary to facilitate analogizing. They to a large part justify the high cognitive demands of analogizing as a heuristic strategy: The pupil is thus required to understand the problem first and develop an appropriate cognitive model of the situation at hand. As long as the following steps only involve situative characteristics and elements, at most *pseudo-analogies* can be constructed, the level of which is constrained to the situation's surface – such as the same numbers. If the learner is however able to construct mathematical structures that fit the situation, i.e. a mathematical model, it also becomes possible to construct and use *structural analogies*. However, similar surface characteristics, approaches or (partial) results can also present triggers of structural analogies. This is the case particularly in the "Answering" and (initiated) "Reviewing" phases.

Where a learner indeed uses a (targeting) analogy, he may not pass though all phases shown in Figure 2, but other loops and setbacks are possible. In this sense they can be understood as descriptive modules which specifically take shape according to the individual learner's case not only in terms of their type but also in terms of their sequence.

Supportive and exemplifying examples

One the one hand, we are able to show empirical evidence for all possibilities of analogizing in the model depicted in Figure 2. On the other hand, it seems possible to classify all empirically found instances of constructing and using analogies according to this model. For the sake of brevity, we can only present few cases whose classification is also visualized by the numbers in Figure 2.

This first example shows, as referred to in the introduction, the "normal case" of analogizing during problem solving. While working on a problem (P2) analogies to a known problem (source problem, P1) are constructed and then used for solving the target problem P2 (targeting analogy).

<table>
<tr><td>1</td><td colspan="2">Jenny (10y5m) completes P1 successfully. Upon completion, she reads the instructions for P2 and writes down her answer (420) within 20 seconds.</td></tr>
<tr><td colspan="2">J: "The same as that one." [Jenny points to the answer sheet for P1.]
I: "Why?"
J: "20 rows there, it's 20 days here, and she always reads 2 pages more, which is why the answer is 420 pages."
I: … "How did you work that out?"
J: "…because she reads two pages more than the day before and the 20."</td><td>After Jenny had read the instructions for P2, she connect the corresponding elements of the two situations ("mapping": 20 rows --> 20 days; always two counters more --> always two pages more), thereby transferring the results from P1.</td></tr>
</table>

Figure 3: Targeting analogy during building a mental model.

The following example shows that a purely situation-bound working may lead to a false answer. Without using any mathematical structures there is no basis for analogizing.

<table>
<tr><td>2</td><td colspan="2">Ian (10y5m) makes a sketch for P1.</td></tr>
<tr><td colspan="2">
</td><td>Ian manages to transfer the predetermined fourth figure, and succeeds to enlarge it correctly across two lines. He then lets the figure become wider and wider somehow, by increasing size, numbers of or distances between counters. Geometric or arithmetic patterns play no role.</td></tr>
</table>

Figure 4: Situation-bound working.

Even a wrong approach can lead to the construction and use of analogies if it is based on a revised mental model and corresponding structural-based working.

<table>
<tr><td>3</td><td>Marc (10y2m) works on P1 successfully. To achieve his result, he adds up in parts the even numbers from 2 to 40. While working on P2 he expresses, that 40 pages would be read on the 20th day.
Thereafter he guesses a total number of 80 pages. He hesitates while explaining his answer. So the interviewer explains the problem.</td></tr>
</table>

<table>
<tr><td>I: “The question is how much she reads on all these days, taken together.”
M. writes down in lines 20 + 18 + 16. He leans back and mumbles “count down from 40”.
M: “It’s kind of just the same like this task [points at P1].”
I: “What is the same?”
M: “Well, I have to calculate, down from 40.”
I: “What do you have to calculate down?”
M: “I calculate 40 (…) plus 38 plus 36 plus 34”
I: “Okay, und what is the result?”
M: “420.”</td><td>Marc at first doesn’t understand the question completely. After clarifying the problem he begins to work and writes down some of the even numbers. During structural working he recognizes the one-to-one-correspondence between the summands of P1 and P2 and transfers the result of P1 onto P2.</td></tr>
</table>

Figure 5: Targeting analogy during structural-based working.

The final example shows that an analogy can be achieved in the phase of reviewing, even if the source and the target problem were tackled in different ways and with wrong results.

<table>
<tr><td>4</td><td>Michael works on P2 first. He tries to simplify his summation and checks his results by building patterns and subtotals, but doesn’t succeed. Due to some minor calculation errors he finally achieves 400 as a result. Subsequently he deals with P1 and describes several explicit and recursive connections. In order to determine the total number of counters of the 20th group efficiently, he has the following idea: “Within the 20th group one line always adds up to 40 with another. That is to say that all lines except of the 20th can be completed to 40 by another, 2 to 38, 4 to 36 and so on.” Michael supposes to get 19 pairs in this way. Finally he gets his result by calculating: $19 \cdot 40 + 20$.</td></tr>
</table>

<table>
<tr><td>After the computation the interviewer places the worksheets P1 and P2 in front of Michael and initiates the reviewing process:
I: “Do the two problems have something in common?”
Michael thinks for a few seconds.
M: “Somehow it’s nearly the same problem. Because, here she reads a book and … , and here it’s, each group increases by two counters. So only one of the results can be correct. Because both is up to 20. Mmmh, it’s difficult. I think I better check this result [points at P2] … there [points at P1] I may be wrong, too,”</td><td>Recognizing the analogy, Michael controls his results in the source problem (here P2). After achieving the third (now correct) result, he transfers this result to his current target problem (P1). Of course, the question, if and how far he would review his problem solving process self-contained or independently, remains unanswered.</td></tr>
</table>

While recalculating, Michael discovers his computation errors and finally gets the correct result. M: "I think, this should be the result of both [*points at 420*] … Because, I think, it's the same problem, just expressed differently. Because here it's also [*points at P1*], each time, it increases by two, just the same like here [*points at P2*]."	

Figure 6: Outgoing analogy in the phase of reviewing.

WHAT'S ALL ABOUT THIS MODEL?

The process model specified in this paper seems to be suitable for analyzing the construction and use of analogies during problem solving in many respects. On the one hand it can be used as an analysis tool for all problems investigated in this study. Differences regarding points of analogizing in problem-solving processes become comprehensible and describable. Moreover, based on the model phases, conditions that can facilitate or hinder the process of analogizing during mathematical problem solving can be carved out. Such conditions are in some extent already published (Aßmus & Förster, 2013b), but further research is required. This knowledge about constructing and using analogies would constitute an important basis for evaluating pupils abilities in problem solving and analogizing.

On the other hand the model enables us to classify and compare other studies on using analogies more precisely. This includes the arrangement of the studies as well as an assessment of their results. Studies can be systematically compared by determining the specific model phase investigated and by the perspective from which the construction and use of analogies is viewed.

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