THE ROLE OF PICTURES IN READING MATHEMATICAL PROOFS: AN EYE MOVEMENT STUDY

Jana T. Beitlich, Andreas Obersteiner, Gabriele Moll, Julio G. Mora Ruano, Jiafang Pan, Sarah Reinhold, Kristina Reiss

TUM School of Education, Technische Universität München, Germany

To support university students' understanding of mathematical proofs, pictures accompanying text are frequently used in textbooks as well as in lectures. However, it is unclear if such pictures influence the individual's reading behaviour. By recording the eye movements of eight mathematicians, we investigated whether and how adults with high expertise in mathematics pay attention to additional pictures when reading a written mathematical proof. We found that all participants paid attention to the pictures. As expected, in two out of three items, the text was fixated upon significantly longer than the picture. The data suggest that the participants tried to integrate information from text and picture by alternating between these representations.

THEORETICAL FRAMEWORK

The transition from secondary school to university mathematics and the first semesters of studies in mathematics are considered challenging for many students. Mathematics at the university level makes use of axioms, definitions and theorems that are not easy to understand for novices. In particular, mathematical proof is a central obstacle for students of mathematics. In order to improve students' understanding, pictorial information might be a useful supplement to the written text. Accordingly, the present study focuses on the role of pictures in reading mathematical proofs.

Mathematical proof

Mathematics is a science of proof (Hilbert, Renkl, Kessler, & Reiss, 2008). The deductive structure of (university) mathematics demands dealing with mathematical proofs. In a mathematics lecture, the lecturer typically writes a sequence of definitions, theorems and proofs on a blackboard. Similarly, textbooks typically provide such a definition-theorem-proof structure as well. Accordingly, reading and comprehending proofs is a central activity of studying mathematics (Mejia-Ramos & Inglis, 2009). Dealing with mathematics in this way at the university level differs greatly from the secondary school level, where mathematics is typically presented in a much more concrete and for the most part non-deductive way. For this reason, many students struggle especially with the working with proofs.

In spite of its high relevance for university mathematics, there is only little research on how individuals read proofs (Mejia-Ramos & Inglis, 2009). Inglis and Alcock (2012) asked first-year undergraduate students and academic mathematicians to evaluate mathematical proofs on a computer screen while their eye movements were recorded. The results revealed that the students spent proportionately more time on the formulas

2014. In Nicol, C., Liljedahl, P., Oesterle, S., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 2, pp. 121-128. Vancouver, Canada: PME.

(compared with the non-formula parts of the proof) than did the mathematicians. Furthermore, the mathematicians shifted their attention back and forth between the lines of the proof more often than the students, suggesting that the mathematicians spent more effort on searching for between-line warrants than the students.

While the study by Inglis and Alcock (2012) investigated how experts and learners read proofs to *evaluate* the proof, there is almost no existing research on how individuals read proofs to *comprehend* them (Mejia-Ramos & Inglis, 2009). Reading proofs for comprehension plays the dominant role in early university studies, and is thus the topic of our study. As a first step, we involved adults with high expertise in mathematics in the current study. In a next step we plan to examine the behaviour of novices (that is students at the beginning of their studies) and to compare the findings of these two groups. This way we want to determine ideal reading strategies to adapt our teaching to student needs.

The combination of text and pictures

In university lectures as well as in textbooks, written mathematical proofs are frequently accompanied by pictures to visualize the main information provided in the text with the intention of facilitating the learning process. Cognitive psychological theories on multimedia learning (e.g. Mayer, 2001; Schnotz, 2005) support this idea and there is empirical evidence that students generally learn better from a combination of text and pictures than from text alone (for a review see e.g. Levie & Lentz, 1982).

The combination of text and pictures seems to be particularly beneficial if the representations are semantically related to each other or presented closely together (Schnotz, 2005). However, there is also evidence that this effect occurs only under specific conditions. For instance, according to Schnotz (2005) pictures can be not beneficial when they visualize the text in a task-inappropriate way, as the form of visualization influences the structure of the mental model which is built from the picture.

Yet, in the field of mathematics, there is little empirically based research about the effect of the combination of text and pictures. In a recent study by Dewolf, Van Dooren, Hermens, and Verschaffel (2013), pictures had seemingly no effect *at all* on higher-education students' behaviour when solving mathematical word problems. The authors showed the students word problems on a computer screen and recorded their eye movements. In the experimental group, the text of every task was accompanied with a picture. The students' answers to the word problems did not differ between the groups with or without the pictures. One possible reason for that was that the students barely looked at the pictures. Only around 1% of all fixations were on the area where the illustrations were presented. In view of these results, it is no matter of course that individuals pay any attention to the pictures presented as part of a proof.

RESEARCH QUESTIONS AND HYPOTHESES

The aim of our study was to find out whether and how experts look at a picture given with a mathematical proof while reading the proof to comprehend it. Following the findings of Schnotz (2005) mentioned above, we used pictures which visualize part of the information given in the text and complete the text without providing any other information than given in the text, so that the text and the picture are semantically related to each other. Such pictures have been referred to as “representational pictures” (Elia & Philippou, 2004).

Measuring individuals’ reading behaviour is a methodological challenge. One way is to show the items to participants and to ask them afterwards if they looked at the picture (retrospective reporting). Another way is to ask the participants to think aloud while working on the items (concurrent reporting). A drawback of both methods is, however, that they are highly subjective and do not reliably assess the actual behaviour.

A more objective way to examine reading behaviour is eye tracking. Eye tracking is a technique with which the eye movements of a person, consisting of fixations and saccades, can be made visible. A fixation is the status when the eyes remain still (for example on a word during reading). A saccade is the very fast movement between two fixations where no information is perceived. The underlying idea of analysing eye movement data is that people are mainly processing the information they are looking at (Just & Carpenter, 1980). Although this assumption may not hold true in general, it is arguably reasonable to use eye fixations as a proxy for information processing during reading.

The specific research questions and hypotheses of this study were: 1) Do academic mathematicians look at the representational picture provided along with a written proof at all? 2) If so, how long do they fixate the picture compared to the text? We assumed that the participants would indeed look at the picture (hypothesis 1), because this would help them understand the information provided in the text (see Schnotz, 2005). We further expected that the participants would fixate the text longer than the picture (hypothesis 2), because although the picture would help them understand the text, it was not essential to understand it and it did not reflect the whole content of the text. 3) Furthermore, we were interested if the participants look at the text and the picture in a specific sequence. We assumed that the participants would alternate between text and picture (hypothesis 3), in order to integrate information from the text and the picture. This behaviour was shown by experts in a study by Inglis and Alcock (2012). Here, the experts tried to integrate the information given in consecutive lines.

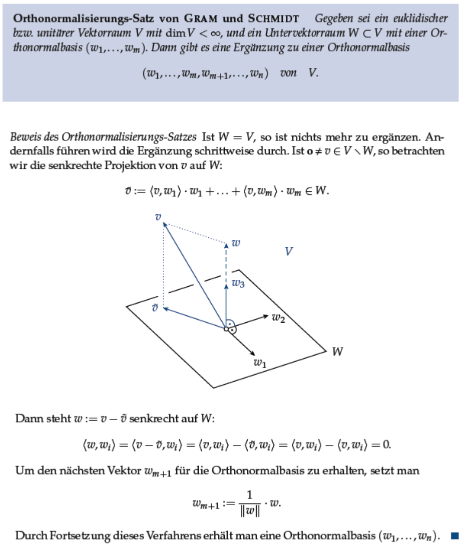
METHODOLOGY

The participants were six staff members of a German university who had an academic degree in mathematics and two university students majoring in mathematics. The mean age of these eight participants (five female) was 26 years (*SD* = 3.9).

The participants sat in front of a computer screen, which was connected to a binocular remote contact free eye tracking device (SensoMotoric Instruments) with a sampling rate of 500 Hz. The eye tracking device was placed underneath the screen. The participants were asked to avoid head and body movements as far as possible, so that their eye movements could be recorded reliably. First, calibration was performed through fixations of nine small dots on the screen. After that, the participants were instructed that they should try to comprehend the information provided by the items shown on the screen, so that they would be able to answer subsequent questions on the content. They were also informed that there was a time limit of five minutes per item, but that they could proceed earlier by pressing the space bar. In fact, on average the participants spent only 2.5 min on each item.

Then the experiment started and the participants saw the first out of three items. After reading the item, they had to answer two multiple-choice-questions related to the content of the given item by clicking on the correct answer on the screen. The same procedure occurred for the second and third item.

The three items were chosen from German mathematical textbooks that are commonly used for undergraduates. Thus, the items represented highly valid learning materials for students. Every item consisted of a theorem, its proof and a representational picture (see figure 1). The picture was arranged between the text and visualized the written proof without presenting any other information than the text. The contents were selected so that the participants were expected to understand them easily, but did at the same time not include mathematical knowledge that is typically learned by heart.



Orthonormalisierungs-Satz von GRAM und SCHMIDT *Gegeben sei ein euklidischer bzw. unitärer Vektorraum V mit $\dim V < \infty$, und ein Untervektorraum $W \subset V$ mit einer Orthonormalbasis $(w_1, \ldots, w_m)$. Dann gibt es eine Ergänzung zu einer Orthonormalbasis*

$$(w_1, \ldots, w_m, w_{m+1}, \ldots, w_n) \quad \textit{von} \quad V.$$

Beweis des Orthonormalisierungs-Satzes Ist $W = V$, so ist nichts mehr zu ergänzen. Andernfalls führen wird die Ergänzung schrittweise durch. Ist $\mathbf{o} \neq v \in V \setminus W$, so betrachten wir die senkrechte Projektion von v auf W:

$$\tilde{v} := \langle v, w_1 \rangle \cdot w_1 + \ldots + \langle v, w_m \rangle \cdot w_m \in W.$$

Dann steht $w := v - \tilde{v}$ senkrecht auf W:

$$\langle w, w_i \rangle = \langle v - \tilde{v}, w_i \rangle = \langle v, w_i \rangle - \langle \tilde{v}, w_i \rangle = \langle v, w_i \rangle - \langle v, w_i \rangle = 0.$$

Um den nächsten Vektor w_{m+1} für die Orthonormalbasis zu erhalten, setzt man

$$w_{m+1} := \frac{1}{\|w\|} \cdot w.$$

Durch Fortsetzung dieses Verfahrens erhält man eine Orthonormalbasis $(w_1, \ldots, w_n)$. ■

Figure 1: Sample item containing a theorem (shaded paragraph) and its proof, including a representational picture.

RESULTS

Data from one participant had to be excluded from the analysis due to low calibration quality. To analyse the eye movements, we defined three areas of interest (AOIs) for each item. These were fitted around the text above the picture, the picture itself and the text below the picture. As we were only interested in the proof part of the items, the gazes on the theorem itself were not considered. In the following, the values for the text above and below the picture are summarized as values for "text".

As can be seen from table 1, the fixation times for the pictures were always larger than zero, which is in line with hypothesis 1 and indicates that the participants paid attention to the picture.

To compare fixation times on text and picture, we decided to divide the fixation times (in ms) for the AOI "text" and the AOI "picture" by the size (in pixel; px) of the respective AOI to account for the different areas of the AOIs (AOI sizes: $text_{item\_1}$ = 190498 px, $picture_{item\_1}$ = 78001 px; $text_{item\_2}$ = 158096 px, $picture_{item\_2}$ = 66674 px; $text_{item\_3}$ = 204824 px, $picture_{item\_3}$ = 68214 px). Table 1 displays the fixation times per pixel for each participant and the two AOIs of each item.

	Item_1		Item_2		Item_3	
Participants	Text ms | ms/px	Picture ms | ms/px	Text ms | ms/px	Picture ms | ms/px	Text ms | ms/px	Picture ms | ms/px
P01	120133 | .63	86807 | 1.11	86144 | .54	30660 | .46	141536 | .69	15518 | .23
P02	83385 | .44	49659 | .64	42013 | .27	9786 | .15	55931 | .27	5363 | .08
P03	58428 | .31	48133 | .62	26885 | .17	11104 | .17	43197 | .21	15824 | .23
P04	71020 | .37	16888 | .22	62035 | .39	21261 | .32	67596 | .33	20402 | .30
P06	53572 | .28	28619 | .37	38102 | .24	16449 | .25	63172 | .31	9110 | .13
P07	64469 | .34	23324 | .30	46359 | .29	10862 | .16	71429 | .35	8502 | .12
P08	19869 | .10	60 | .00	14148 | .09	701 | .01	79554 | .39	7797 | .11
M (SD)	67268 (30520) | .35 (.16)	36213 (28246) | .46 (.36)	45098 (23528) | .29 (.15)	14403 (9557) | .22 (.14)	74631 (31700) | .36 (.15)	11788 (5471) | .17 (.08)

Table 1: Fixation times in ms and ms/px for each participant and the two AOIs of each item. Note: M = mean, SD = standard deviation.

We conducted a paired t-test for each item to compare the fixation times of text and picture. There was no significant difference between the fixation times of text and picture in item 1, $t(6)_{item\_1} = -1.26$, $p = .25$. For item 2 and item 3 there were significant differences, $t(6)_{item\_2} = 3.49$, $p = .013$, $t(6)_{item\_3} = 3.17$, $p = .019$. In both cases, the text was fixated upon longer than the picture. Even if there were comparatively large inter-individual differences in the fixation times, this tendency is found within almost all participants and items (see Table 1). This result supports hypothesis 2 for items 2 and 3, but not for item 1.

To illustrate the fixation times, Figure 2 shows the heat map for fixation times of item 2. From blue to green via yellow to red a heat map shows the least and most fixated areas. As can be seen from this figure, participants looked at the text as well as at the picture, but focused longer on the text than on the picture.

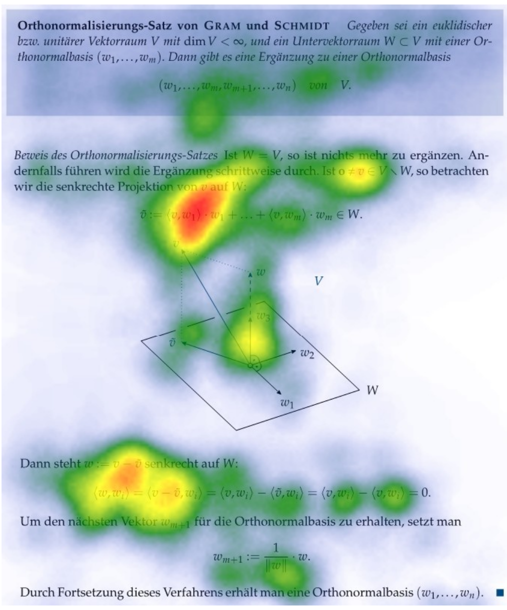


Figure 2: Heat map for fixation times of item 2.

To answer the question if participants looked at the text and picture in a specific order, we analysed the sequence charts which show the order and the duration of fixations of both AOIs for each participant. Figure 3 exemplarily shows the sequence chart for item 2. The gaps result for example from the gaze to the theorem or to regions of the page where no AOIs were defined or from the loss of the eye contact.

Figure 3: Sequence chart for item 2;
fixations of text are coloured green, fixations of the picture red.

The sequence chart for item 2 shows that every participant switched back and forth between the text and the picture frequently. The only exception was participant P08 who merely had short glances at the picture at the end. The sequence charts of item 1 and item 3 looked similar. All in all we can state that the participants alternated between the text and the picture in each item, which supports hypothesis 3.

DISCUSSION

The aim of our study was to find out whether and how academic mathematicians look at a representational picture given with a mathematical proof while reading the proof in order to understand it. We recorded eye movements of eight participants with high expertise in mathematics and analysed fixation times on text and picture, as well as the sequence charts.

We found that all participants paid attention at the pictorial information. This is not in line with the study by Dewolf et al. (2013), who found that students barely looked at pictures presented with word problems, no matter if these pictures were representational or only decorative. One reason for that might be that the picture in our study was positioned in the middle of the text so that it was unlikely to overlook it. However, as the participants switched back and forth between the text and the picture during reading the proof, it is not likely that they looked at the picture just because of its position.

As expected, in two out of three items the relative fixation times for the text were significantly longer than for the picture. As the picture visualized the written proof without presenting any other information than the text, looking at the picture was not essential for understanding the proof. Furthermore, most of the participants were academic mathematicians who were certainly familiar with proofs in general and with the presented topics (chosen from undergraduates' textbooks) in particular. This could explain why shorter fixations at the picture might have been enough to comprehend the proof. For the first item, there was no difference in the fixation times of text and picture. One reason for this unexpected result could be that only after working on the first item, the participants learned that the subsequent questions would not explicitly refer to the picture, so that they payed less attention to the picture in the following two items.

Furthermore we could show that the participants alternated between the text and the picture during reading the proof, which suggests that they tried to integrate the information given in the text and in the picture. This is plausible as the text and the picture were semantically related. It might also be an indicator of mathematical expertise (see Inglis & Alcock, 2012). An interesting question for a follow-up study will be to see if individuals who are not familiar with reading proofs, such as first-year students of mathematics, show the same fixation pattern.

We used eye movements as a relatively new method to assess mathematical tasks. We could show that this method is feasible to analyse whether and how participants look at a picture while reading a proof in order to understand it. Based on eye movement data, we could draw reliable conclusions, which would not have been possible through verbal reports (retrospective or concurrent).

A limitation of the study is the small number of participants, which restricts the generalizability of our findings. Certainly, further studies with a larger sample size are necessary to replicate the present results. Moreover, we aim to examine the reading behaviour of novices (that is students in their first year at university) and compare these data to our present findings, to trace students' problems. On the long run, these studies could help developing learning materials tailored to student needs when learning university mathematics.

References

Dewolf, T., Van Dooren, W., Hermens, F., & Verschaffel, L. (2013). Do students attend to and profit from representational illustrations of non-standard mathematical word problems? In A. M. Lindmeier & A. Heinze (Eds.), *Proc. 37th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 217-224). Kiel, Germany: PME.

Elia, I., & Philippou, G. (2004). The functions of pictures in problem solving. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 327-334). Bergen, Norway: PME.

Hilbert, T. S., Renkl, A., Kessler, S., & Reiss, K. (2008). Learning to prove in geometry: Learning from heuristic examples and how it can be supported. *Learning and Instruction*, *18*(1), 54-65.

Inglis, M., & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education*, *43*(4), 358-390.

Just, M. A., & Carpenter, P. A. (1980). A theory of reading: From eye fixations to comprehension. *Psychological Review*, *87*(4), 329-354.

Levie, W. H., & Lentz, R. (1982). Effects of text illustrations: A review of research. *Educational Communication and Technology Journal, 30*(4), 195-232.

Mayer, R. E. (2001). *Multimedia learning*. New York: Cambridge University Press.

Mejia-Ramos, J. P., & Inglis, M. (2009). Argumentative and proving activities in mathematics education research. In F.-L. Lin, F.-J. Hsieh, G. Hanna, & M. de Villiers (Eds.), *Proceedings of the ICMI Study 19 conference: Proof and proving in mathematics education* (Vol. 2, pp. 88-93). Taiwan: The Department of Mathematics, National Taiwan Normal University Taipei.

Schnotz, W. (2005). An integrated model of text and picture comprehension. In R. E. Mayer (Ed.), *The Cambridge handbook of multimedia learning* (pp. 49-69). Cambridge: Cambridge University Press.