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Individual Differences in Fraction Arithmetic Learning

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Abstract

Understanding fractions is critical to mathematical development, yet many children struggle with fractions even after years of instruction. Fraction arithmetic is particularly challenging. The present study employed a computational model of fraction arithmetic learning, FARRA (Fraction Arithmetic Reflects Rules and Associations; Braithwaite, Pyke, and Siegler, 2017), to investigate individual differences in children's fraction arithmetic. FARRA predicted four qualitatively distinct patterns of performance, as well as differences in math achievement among the four patterns. These predictions were confirmed in analyses of two datasets using two methods to classify children's performance—a theory-based method and a data-driven method, Latent Profile Analysis. The findings highlight three dimensions of individual differences that may affect learning in fraction arithmetic, and perhaps other domains as well: effective learning after committing errors, behavioral consistency versus variability, and presence or absence of initial bias. Methodological and educational implications of the findings are discussed.

Keywords: fractions; arithmetic; individual differences; computational model; Latent Profile Analysis

1. Individual Differences in Fraction Arithmetic Learning

Two central goals for the science of learning are to explain the processes that lead to learning and to explain why different learners display different learning outcomes. These two goals have often been pursued separately, by studying learning processes without considering individual differences or by studying individual differences without an accompanying theory of learning. In the present study, we addressed the two goals simultaneously, by using a theory of learning to generate predictions about individual differences in learning outcomes and then testing the predictions.

We focused on the domain of fractions because of their importance in children's mathematical development. Knowledge of fractions in middle school predicts knowledge of algebra and general mathematics achievement in high school, even after statistically controlling for whole number arithmetic skill, IQ, working memory, reading ability, and socio-economic status (Siegler et al., 2012). Fractions are also important in the workplace: Over two-thirds of a large, nationally-representative sample of working adults in the U.S. reported using fractions in their jobs (Handel, 2016).

Despite the importance of fractions, many children struggle with them (Byrnes & Wasik, 1991; Fuchs et al., 2014; Jordan et al., 2013; Newton, Willard, & Teufel, 2014; Siegler & Lortie-Forgues, 2015; Siegler & Pyke, 2013; Siegler, Thompson, & Schneider, 2011). Fraction arithmetic is particularly challenging. For example, U.S. sixth and eighth graders in Siegler and Pyke (2013) scored 41% and 57% correct, respectively, on 16 fraction arithmetic problems including all four arithmetic operations and operands with equal and unequal denominators. However, large individual differences in fraction arithmetic proficiency exist. Despite children's poor average performance, some children reach high levels of proficiency. For example, 14% of children in Siegler and Pyke (2013) answered at least 14 out of 16 problems (88%) correctly.

These findings suggest several questions. First, in addition to the differences between children who are more and less successful in learning fraction arithmetic, are there also substantial individual differences within the less successful group? Specifically, do children who perform poorly overall display different patterns of errors? Second, what factors could plausibly explain individual differences in fraction arithmetic learning? In particular, what factors could enable some learners to reach high levels of proficiency, and if individual differences among less successful learners exist, what factors might cause those differences?

We addressed these questions in the context of a theory of fraction arithmetic learning that was implemented as a computational model, FARRA (Fraction Arithmetic Reflects Rules and Associations; Braithwaite, Pyke, & Siegler, 2017). FARRA was previously used to simulate children's typical—that is, poor—performance. In the present study, we used the model to generate predictions about individual differences; then we empirically tested the predictions in two behavioral datasets. This model-driven approach allowed us to test the theoretical assumptions underlying FARRA. If the model generated substantive predictions regarding individual differences that were consistent with the data, the results would provide further support for the underlying theory; if the model failed to generate substantive predictions, or generated predictions that were disconfirmed by the data, the results would indicate a need to modify the theory. This approach extends previous research applying cognitive process models to the study of individual differences in knowledge of whole number magnitudes (Prather, 2014), counting (Lee & Sarnecka, 2010), and whole number arithmetic (Siegler, 1988) to the more advanced topic of fraction arithmetic.

Below, we briefly review previous research on children's fraction arithmetic, focusing on findings regarding strategy use, which is central to FARRA. We then describe FARRA and discuss the model's implications for individual differences among children.

1.1. Children's Strategy Use in Fraction Arithmetic

Children's errors in fraction arithmetic involve using incorrect strategies more often than they involve incorrect execution of correct strategies (Siegler & Pyke, 2013; Siegler et al., 2011). One common strategy error involves performing the arithmetic operation (e.g., addition) separately on the numerators and denominators of the operands (e.g., 3/5+1/4 = 4/9). We refer to this strategy as the "whole number/multiplication strategy," because it often reflects thinking of a fraction as two separate whole numbers, but also is a correct strategy for multiplying fractions (e.g., 3/5×1/4 = 3/20). Children commit errors involving incorrect use of the whole number/multiplication strategy as early as fourth grade (Gabriel et al., 2013), and these errors continue to appear years later, for example on 14% of trials among sixth and eighth graders in Siegler and Pyke (2013).

Another common type of mistake, termed "wrong fraction operation" errors (Siegler & Pyke, 2013; Siegler et al., 2011), involves using a strategy that would be correct for one type of problem to solve a different type of problem for which the strategy is incorrect. For example, the correct strategy for adding and subtracting fractions is to perform the given operation on the operands' numerators while passing their common denominator into the answer, after first converting the operands to a common denominator if necessary (e.g., 3/5+1/5 = 4/5). However, using this strategy on multiplication and division problems yields errors (e.g., 3/5×1/5 = 3/5). Children commit errors involving wrong fraction operation strategies, excluding errors involving incorrect use of the whole number/multiplication strategy, as early as fifth grade (Byrnes & Wasik, 1991); such errors occurred on 26% of trials among children in Siegler and Pyke (2013).

The majority of middle school children's fraction arithmetic errors involve incorrect use of the whole number/multiplication strategy or other wrong fraction operation strategies (Hecht, 1998; Newton et al., 2014). For example, the percentage of sixth and eighth graders' errors that belonged to one of these two types was 87% in Siegler and Pyke (2013) and 82% in Siegler et al. (2011). For a more comprehensive analysis of fraction arithmetic errors including other, less common errors, see Braithwaite et al. (2017).

The frequencies of different types of strategy errors vary depending on the types of numbers involved (Siegler & Pyke, 2013). Errors involving incorrect use of the whole number/multiplication strategy are more common on problems involving unequal denominator fractions than on equal denominator problems (e.g., 3/5+1/4 = 4/9 is more common than 3/5+1/5 = 4/10). In contrast, errors involving incorrect use of the addition/subtraction strategy are more common on equal than on unequal denominator problems (e.g., 3/5×1/5 = 3/5 is more common than 3/5×1/4 = 12/20×5/20 = 60/20), although both types of error occur.

Many children display variable strategy use even on highly similar problems. In Siegler and Pyke (2013), 61% of children used different strategies for at least one pair of problems involving the same arithmetic operation and either both equal denominators (e.g., 3/5×1/5 and 4/5×3/5) or both unequal denominators (e.g., 3/5×1/4 and 2/3×3/5).

1.2. A Model of Fraction Arithmetic Learning

To explain these findings and other aspects of children's fraction arithmetic, Braithwaite, Pyke, and Siegler (2017) developed FARRA, a computational model of fraction arithmetic learning. The model is summarized below; technical details are provided in the Appendix.

FARRA simulates the process of solving fraction arithmetic problems by selecting and executing problem-solving strategies. The model simulates both correct solutions and errors. Errors result primarily from strategy over-generalization—that is, selecting a strategy that would be correct for one arithmetic operation to solve a problem involving a different operation. For example, FARRA can use the addition/subtraction strategy to solve multiplication problems and can use the whole number/multiplication strategy to solve addition problems.

FARRA selects which strategy to use stochastically based on the reinforcement each strategy has received during previous learning. The model learns by solving problems and receiving feedback indicating whether each answer is correct or incorrect. When an answer is correct, the model positively reinforces the strategy used to generate the answer, making it more likely to use the strategy in the future. Incorrect answers can result in either positive reinforcement, albeit less than the reinforcement that occurs after correct answers, or negative reinforcement; Braithwaite et al. (2017) reported simulations involving both types of reinforcement. Because most students display weak conceptual understanding of fraction arithmetic (Braithwaite, Tian, & Siegler, 2018; Siegler & Lortie-Forgues, 2015), conceptual understanding plays no role in the model's learning.

Simulations conducted with FARRA displayed all eight phenomena identified in Braithwaite et al.'s (2017) review of the literature, including those described in Section 1.1: low overall accuracy; most errors involving incorrect use of the whole number/multiplication strategy or wrong fraction operation strategies; incorrect use of the addition/subtraction strategy being more common with equal denominators, and incorrect use of the whole number/multiplication strategy being more common with unequal denominators; and variable strategy use by individual children.

Although Braithwaite et al. (2017) focused on using FARRA to explain children's typical performance in fraction arithmetic, the model also has implications regarding individual differences. FARRA's learning and strategy selection mechanisms are fixed, reflecting an assumption that these underlying mechanisms do not vary among children. However, these mechanisms are governed by four free parameters, which are assumed to reflect attributes that do vary among children.

First, the *learning rate* parameter determines the amount of positive reinforcement given to a strategy each time that it yields a correct answer. This parameter is intended to capture variation in how much children learn from a given amount of practice (see, e.g., Garlick, 2002).

Second, the *error discount* parameter determines the reduction in reinforcement that is given to a strategy when using the strategy results in an incorrect answer. If the error discount is greater than 1, strategies receive negative reinforcement when they generate errors; otherwise, strategies still receive positive reinforcement even when they generate errors. Thus, larger error discounts imply more effective learning from errors. Children differ in how effectively they learn from incorrect examples that are labeled as incorrect (Große & Renkl, 2007; Heemsoth & Heinze, 2014), suggesting that they may also differ in how effectively they learn from errors.

Third, the *decision determinism* parameter governs how much previous reinforcement of strategies influences strategy selections. High decision determinism makes the model more consistently select whichever strategy previously received the most reinforcement, whereas low decision determinism leads to more varied strategy selection. Some children choose problem-solving strategies more carefully than others (Siegler, 1988), which could lead to more consistent strategy choices.

Finally, the *whole number bias* parameter determines the initial (positive) reinforcement given to the whole number/multiplication strategy before the model has solved any fraction arithmetic problems. This parameter is intended to reflect the fact that many children employ this strategy before receiving instruction in fraction arithmetic (Byrnes & Wasik, 1991), perhaps because they think of a fraction as two separate whole numbers (Ni & Zhou, 2005). Between fourth and eighth grade, whole number bias varies substantially among children within each grade (Braithwaite & Siegler, 2018b).

1.3. Predictions About Individual Differences Based on the Theory

We predicted that variation in the values of FARRA's parameters would cause the model to display the four distinct behavior patterns summarized in Table 1. The rationale for predicting each pattern is explained below. Study 1 tested whether the model does in fact generate the predicted patterns and whether these patterns account for most of the model's behavior. To preview the results, it does. Therefore, we predicted that children would display the same four patterns. This prediction was tested in Studies 2 and 3 using both theory-driven and data-driven approaches.

Table 1. Fraction Arithmetic Behavior Patterns Hypothesized Based on Analysis of FARRA.

Behavior Pattern	Description
Correct Strategies	Correct strategies used on most or all problems
Whole Number Perseveration	Whole number strategy used on most or all problems
Addition/Subtraction Perseveration	Addition/subtraction strategy used on most or all problems
Variable Strategies	Multiple strategies used for most or all arithmetic operations

"Correct Strategies" refers to the pattern of using correct strategies on most or all problems. We predicted that large values of the error discount parameter would enable FARRA to learn to select the correct strategy for each type of problem consistently. Our rationale was that error discounts greater than 1 lead to negative reinforcement of strategies that generate errors, and therefore should enable the model to learn not to use these incorrect strategies. Although all of FARRA's parameters might affect whether the model learns to use correct strategies, we expected the error discount parameter to be particularly important.

"Perseveration" refers to using a single strategy for most or all problems, including problems for which the strategy is incorrect. Error discounts less than 1 should often lead to perseveration, because for these parameter values, any use of a strategy is self-reinforcing regardless of whether it yields a correct answer or an incorrect answer. Over time, this self-reinforcement could amplify a small initial preference for a strategy into a much stronger preference for that strategy above all others.

If Perseveration results from self-reinforcement of initial preferences, then initial conditions should influence the strategy on which children perseverate. Strong whole number bias creates an initial preference for the whole number/multiplication strategy that, when combined with a small error discount, should lead to perseverative use of that strategy. We refer to this behavior pattern as "Whole Number Perseveration."

On the other hand, weak or zero whole number bias should lead to an initial preference for the addition/subtraction strategy. The reason is that addition and subtraction are typically introduced before multiplication and division (CCSSI, 2010), so the addition/subtraction strategy should initially have the greatest opportunity to receive the large positive reinforcement that results from generating correct answers. Thus, if Perseveration emerges under the condition of small or zero whole number bias, it would most likely involve perseverative use of the addition/subtraction strategy, a pattern we refer to as "Addition/Subtraction Perseveration."

Finally, "Variable Strategies" refers to using multiple strategies for most or all arithmetic operations. We hypothesized that this behavior pattern would emerge when the learning rate and/or decision determinism parameters have low values. Low decision determinism makes FARRA's strategy choices more random, and therefore should preclude any consistent pattern of strategy use, including the Correct Strategies pattern and both Perseveration patterns. Low learning rates should have a similar effect, because decreasing the reinforcement gained from practice should reduce the degree to which FARRA learns to prefer one strategy over another.

To summarize, we predicted low learning rate or decision determinism parameters would lead to the Variable Strategies pattern, regardless of the values of the other parameters. If learning rate and decision determinism were both at least moderately high, we predicted that high error discounts would lead to Correct Strategies, whereas low error discounts would lead to Perseveration. Finally, we predicted that if Perseveration emerged, high whole number bias would lead to Whole Number Perseveration and low whole number bias would lead to Addition/Subtraction Perseveration.

2. Study 1

An advantage of implementing a theory of learning as a computational model is that the predictions of the theory can be determined objectively by running the model, rather than subjectively by interpreting the theory. In Study 1, we attempted to determine the predictions of Braithwaite et al.'s (2017) theory regarding individual differences in fraction arithmetic by running simulations with FARRA in which the model's parameters were systematically varied over a wide range of values. The predictions that resulted from these simulations were then tested empirically across two behavioral data sets in Studies 2 and 3.

We hypothesized that the model would generate the four behavior patterns in Table 1, and that these four patterns would jointly account for most or all of the model's behavior. However, this was not a foregone conclusion. For example, FARRA might not learn to use correct strategies consistently regardless of the values of its free parameters. FARRA might also display behavior patterns other than those in Table 1, such as perseverative use of the "invert-and-multiply" procedure for fraction division. The simulations were essential to confirm whether FARRA's output matched what we hypothesized based on our analysis of the model.

Another goal of the simulations was to gain insight into possible causes of differences in learning outcomes. To achieve this goal, we analyzed how variation in each of FARRA's parameters affected the probability that each of the four hypothesized behavior patterns would emerge. This analysis generated predictions regarding how individual differences in fraction arithmetic performance should relate to other aspects of individual differences—particularly, differences in general mathematics achievement. These predictions were also tested in Studies 2 and 3.

2.1. Method

2.1.1. Learning and Test Sets

In each simulation, FARRA was trained on a *learning set* of fraction arithmetic problems and subsequently tested on a *test set*. The learning set was the same one used in Studies 1, 3, 4, and 5 of Braithwaite, Pyke, and Siegler (2017). It consisted of 659 fraction arithmetic problems extracted from the fourth, fifth, and sixth grade textbooks of a commercial math textbook series, enVisionMath (Charles et al., 2012). Problems in the learning set were presented in the same order as they appeared in the textbooks: addition and subtraction with equal denominators, then addition and subtraction with unequal denominators, then multiplication, and finally division.

The test set consisted of the 16 fraction arithmetic problems presented to children in Siegler and Pyke (2013; these problems also served as the test set in Braithwaite, Pyke, & Siegler, 2017). The set involved four pairs of operands, two with equal denominators and two with unequal denominators, each presented once with each of the four arithmetic operations, resulting in four problems for each operation. For example, the four addition problems were 3/5+1/5, 4/5+3/5, 3/5+1/4, and 2/3+3/5; the four multiplication problems were 3/5×1/5, 4/5×3/5, 3/5×1/4, and 2/3×3/5; and so on.

2.1.2. Simulation Procedure

Each of FARRA's parameters were independently varied over a wide range of values[1], resulting in 6,600 combinations of parameter values. For each combination, 60 simulations were conducted, for a total of 396,000 simulations. In each simulation, the model solved the problems in the learning set and received feedback after each problem. Then, the model solved each problem in the test set without feedback. The simulation procedures were identical to those employed in Study 1 of Braithwaite et al. (2017), except that in the current study, the values of FARRA's parameters were systematically varied rather than selected randomly.

2.1.3. Classification of Simulations

Simulations were classified based on the model's strategy use on the test set. (1) A simulation was classified as Correct Strategies if the standard correct strategy was used on at least 12 of the 16 test trials. (2) A simulation was classified as Whole Number Perseveration if

[1] Learning rate took the values [0.005, 0.01, 0.015, 0.02, 0.025, 0.03, 0.035, 0.04], error discount took the values [0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0], decision determinism took the values [0.50, 0.75, 1.00, 1.25, 1.50, 1.75, 2.00, 2.25, 2.50, 2.75, 3.00], and whole number bias took the values [0.0, 0.1, 0.2, 0.3, 0.4]. 60 simulations were run for each of the 6,600 possible combinations of these values.

the whole number/multiplication strategy was used on at least 10 test trials, and as Addition/Subtraction Perseveration if the addition/subtraction strategy was used on at least 12 test trials. These criteria ensured that a simulation would be classified as displaying Perseveration if a strategy was used on all problems for which the strategy was appropriate and half of the other problems—for example, if the whole number/multiplication strategy was used on the four multiplication problems and six of the other 12 problems, or if the addition/subtraction strategy was used on the eight addition and subtraction problems and four of the other eight problems. (3) A simulation was classified as Variable Strategies if, for at least three of the four arithmetic operations, the model used more than one strategy across the four problems involving that operation (e.g., the addition/subtraction strategy on two problems and the whole number/multiplication strategy on the other two). (4) Simulations that did not meet any of the above criteria were classified as None.

The categories were applied preferentially in the above order. For example, simulations meeting the criteria for both Correct Strategies and Addition/Subtraction Perseveration were classified as Correct Strategies.

2.2. Results

2.2.1. Percentage of Simulations Matching Each Behavior Pattern

Substantial numbers of simulations matched the criteria for each of the four hypothesized behavior patterns. Specifically, 24% of simulations were classified as Correct Strategies, 26% as Whole Number Perseveration, 15% as Addition/Subtraction Perseveration, and 34% as Variable Strategies. Only 2% of simulations were classified as None. (Details regarding the 17% of simulations that met the criteria for multiple patterns are provided in Supplement S1.)

2.2.2. Effects of Parameter Values on Percent of Simulations Matching Each Behavior Pattern

For each of the model's parameters, the percent of simulations classified as matching each pattern was calculated for each value of the parameter while collapsing over values of the other three parameters. The results are shown in Figure 1.

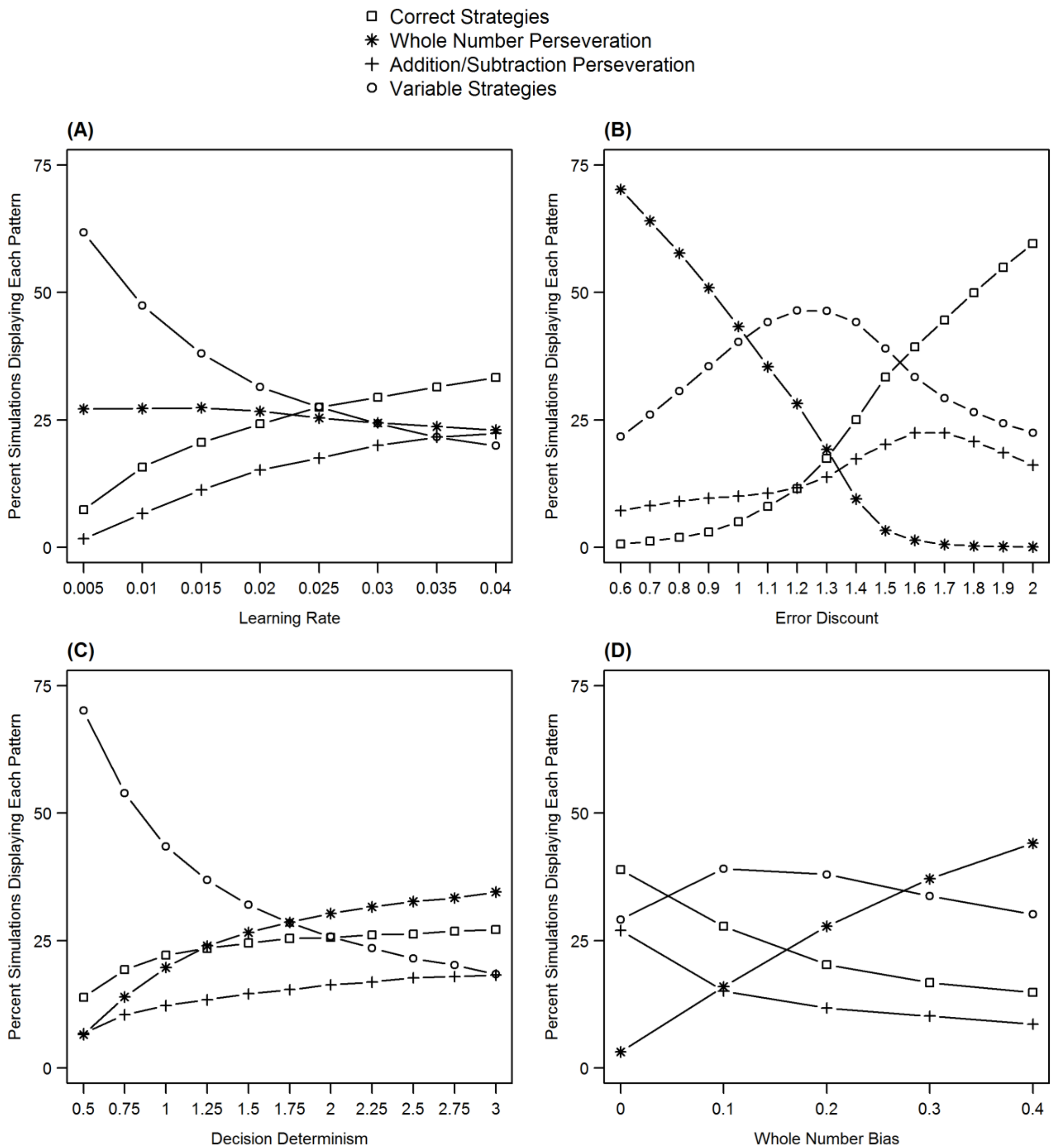

Figure 1. Percent of simulations that yielded each of the four behavior patterns for different values of (A) learning rate, (B) error discount, (C) decision determinism, and (D) whole number bias (Study 1).

The Correct Strategies pattern appeared with appreciable frequency only when the error discount was above 1; high error discounts yielded this pattern over half the time (Figure 1B).

Thus, as predicted, effective learning from errors was essential for the model to achieve high accuracy. High learning rate, high decision determinism, and low whole number bias also increased the proportion of simulations displaying Correct Strategies (Figures 1A, 1C, and 1D).

Whole Number Perseveration was common predominantly when the error discount was low and whole number bias was high (Figures 1B and 1D). Thus, FARRA predicts that initial bias to use the whole number/multiplication strategy, combined with positive reinforcement (or only weak negative reinforcement) in response to negative feedback, results in perseverative use of that strategy. High decision determinism also increased the proportion of simulations displaying Whole Number Perseveration (Figure 1C). The learning rate had little effect (Figure 1A), possibly because Whole Number Perseveration results primarily from persistence of initial bias rather than effects of learning from practice.

Addition/Subtraction Perseveration was common primarily when whole number bias was zero or near zero (Figure 1D), as predicted. However, in contrast to our prediction that this pattern would appear mainly when the error discount was less than 1, it was most common for relatively high error discounts (between 1.5 and 1.8, Figure 1B); the pattern also appeared more often when the learning rate and decision determinism were high (Figures 1A and 1C). Thus, the parameter values that yielded Addition/Subtraction Perseveration most often were quite similar to those that yielded Correct Strategies, the only difference being that Addition/Subtraction Perseveration appeared most often with slightly lower—though still high—error discounts than did Correct Strategies.

Finally, the Variable Strategies profile was dominant for low values of the learning rate and decision determinism (Figures 1A and 1C), as predicted. A slow rate of learning from practice or a highly “noisy” decision-making process apparently prevent FARRA from displaying consistent strategy choices and thereby work against the previous three behavior patterns. Intermediate values of the error discount and whole number bias also increased the proportion of simulations displaying Variables Strategies (Figures 1B and 1D); these parameter values may have increased variability in strategy choices because they fell in between the parameter values that tended to generate Whole Number Perseveration and Addition/Subtraction Perseveration.

2.3. Discussion

The simulation results confirmed that FARRA generates the four hypothesized behavior patterns and essentially no others. Thus, FARRA predicts that children should display some or all of these behavior patterns and that the behavior of the great majority of children should fit one of them. Testing these predictions with empirical data from children would therefore constitute a strong test of the model and the theory on which it was based (see Studies 2 and 3).

Unexpectedly, some parameter values that tended to increase Addition/Subtraction Perseveration—specifically, high learning rate and error discount—resembled the parameter values that generated the Correct Strategies pattern more than those that generated Whole Number Perseveration. High learning rate and error discount may have generated Addition/Subtraction Perseveration because they enabled FARRA quickly to unlearn initial whole number bias (if present) and reinforce the addition/subtraction strategy through initial practice with addition and subtraction problems. Possibly due to the sequence of problems in the training set, this initial reinforcement of the addition/subtraction strategy may have sometimes caused that strategy to become so strongly entrenched that subsequent practice with fraction multiplication and division was insufficient to dislodge it.

Both Correct Strategies and Addition/Subtraction Perseveration appeared most often when the learning rate and error discount were high, whereas Whole Number Perseveration and Variable Strategies appeared most often when these parameters were medium or low (Figure 1). High learning rate and error discount represent fast learning of correct procedures and unlearning of incorrect procedures that generate errors, attributes typical of strong students. Therefore, the results suggested that children who display Correct Strategies or Addition/Subtraction Perseveration should have higher math achievement than children who display Whole Number Perseveration or Variable Strategies. This prediction was also tested in Studies 2 and 3.

3. Study 2A

The main goal of Study 2A was to test whether children display the patterns of behavior predicted by FARRA. We re-analyzed data from a previous study of children's fraction arithmetic, Siegler & Pyke (2013). The data from this study included coded trial-by-trial strategy self-reports, which allowed us to classify participating children according to the criteria used to classify simulations in Study 1. We predicted that substantial numbers of children would display each of the four behavior patterns and that few children would display none of them.

Another goal of Study 2A was to test the prediction that children displaying Correct Strategies or Addition/Subtraction Perseveration would have higher math achievement than children displaying Whole Number Perseveration or Variable Strategies. To do so, we compared math achievement test scores of children matching the four behavior patterns.

A third goal of Study 2A was to describe children matching the four behavior patterns with respect to aspects of fraction arithmetic performance other than strategy use. Separately for each behavior pattern, we analyzed percent correct on eight types of fraction arithmetic problems representing the factorial combinations of the four arithmetic operations and operand pairs with equal or unequal denominators. We also compared the behavior patterns with respect to within-subject strategy variability.

3.1. Method

3.1.1. Participants

The 120 children who participated in Siegler and Pyke's (2013) study included 60 sixth graders and 60 eighth graders recruited from a middle school in Pittsburgh, PA.

3.1.2. Materials and Procedure

Children were presented the 16 fraction arithmetic problems that served as the test set in Study 1. Cronbach's alpha for the problem set within this sample was 0.85. Children solved the problems on paper, entered their answers into a computer, and immediately after explained how they solved that problem. The explanations were audio recorded.

With parental consent, children's teachers provided their scores on the math section of the Pennsylvania System of School Assessments (PSSA). Cronbach's alpha for this test is 0.94 in sixth grade and 0.93 in eighth grade (Pennsylvania Department of Education, 2013). Math PSSA percentile ranks relative to state norms for each grade served as our measure of children's math achievement.

3.1.3. Classification of Children

Children's descriptions of how they solved each problem were previously coded according to whether they used the addition/subtraction strategy, the whole number/multiplication strategy, the division strategy, or any other strategy (Braithwaite et al., 2017; Siegler & Pyke, 2013). Based on the results of this coding, children were classified into the four behavior patterns, or none of them, using the same criteria as in Study 1. Patterns were assigned preferentially in the same order as in Study 1; details regarding the 5% of children that met the criteria for two patterns are provided in Supplement S2.

3.2. Results

3.2.1. Percentage of Children Matching Each Behavior Pattern

Of children in this sample, 90% met the criteria for (at least) one of the four hypothesized patterns. The percentage of children meeting the criteria for the patterns were: Correct Strategies 31%, Whole Number Perseveration 12%, Addition/Subtraction Perseveration 25%, and Variable Strategies 22%. Ten percent of children did not match the criteria for any of the patterns.

We analyzed differences between grade levels in the distributions of different strategy use patterns, as well as differences between grade levels in the distributions of different latent profiles in the following studies. Because the results of these analyses were not consistent across experiments, and because they neither supported nor disconfirmed our predictions, these results are reported in Supplement S3 for this and the following experiments.

3.2.2. Differences in Math Achievement Among Behavior Patterns

ANOVA revealed that math PSSA percentile ranks differed among the four behavior patterns, $F(3, 98) = 14.9$, $p < .001$. Pairwise comparisons with a Holm correction found that math achievement percentile was higher among children fitting the Correct Strategies pattern (mean = 65.3, *SD* = 20.2) and the Addition/Subtraction Perseveration pattern (mean = 59.5, *SD* = 27.1) than among children matching the Whole Number Perseveration pattern (mean = 28.7, *SD* = 24.9) or the Variable Strategies pattern (mean = 30.0, *SD* = 25.5), *p*s < .001. Math achievement did not differ between children meeting the Correct Strategies and Addition/Subtraction Perseveration patterns or between those meeting the Whole Number Perseveration and Variable Strategies patterns, *p*s > .7. The 8 children for whom PSSA scores were not available were excluded from these analyses, as were the 12 children who did not match any of the four behavior patterns.

We also compared the groups with respect to other domain-general and number-specific measures collected by Siegler and Pyke (2013), including response inhibition, working memory, whole number division, fraction magnitude comparison, and fraction number line estimation. ANOVA revealed effects of group on all of these variables except inhibition. Pairwise comparisons with a Holm correction revealed that whole number division accuracy was higher in the Correct Strategies group (93%) than in all other groups, *p*s < .05, and was higher in the Addition/Subtraction Perseveration group (77%) than in the Whole Number Perseveration (52%) or Variable Strategies (52%) groups, *p*s < .05. Group differences in the other variables were directionally consistent with the differences in math achievement and whole number division, but pairwise comparisons were not significant, excepting some comparisons involving the Correct Strategies group. Therefore, these analyses are reported in Supplement S3.

3.2.3. Fraction Arithmetic Performance Among Children Matching Each Behavior Pattern

The percentage of correct answers among children matching each behavior pattern on problems involving each combination of arithmetic operation and denominator equality/inequality is shown in Figure 2A. Accuracy was above average on all problem types within the Correct Strategies pattern, higher on multiplication problems than on all other problems within the Whole Number Perseveration pattern, and higher on addition and subtraction than on other problems within the Addition/Subtraction Perseveration pattern. Within the Variable Strategies pattern, accuracy was higher on equal denominator addition and subtraction and unequal denominator multiplication than on all other problem types. This pattern was also true for the sample as a whole (equal denominator addition: 66%, equal denominator subtraction: 73%, unequal denominator multiplication: 57%, all other problem types: 50% or lower), but it was not evident within any other behavior pattern.

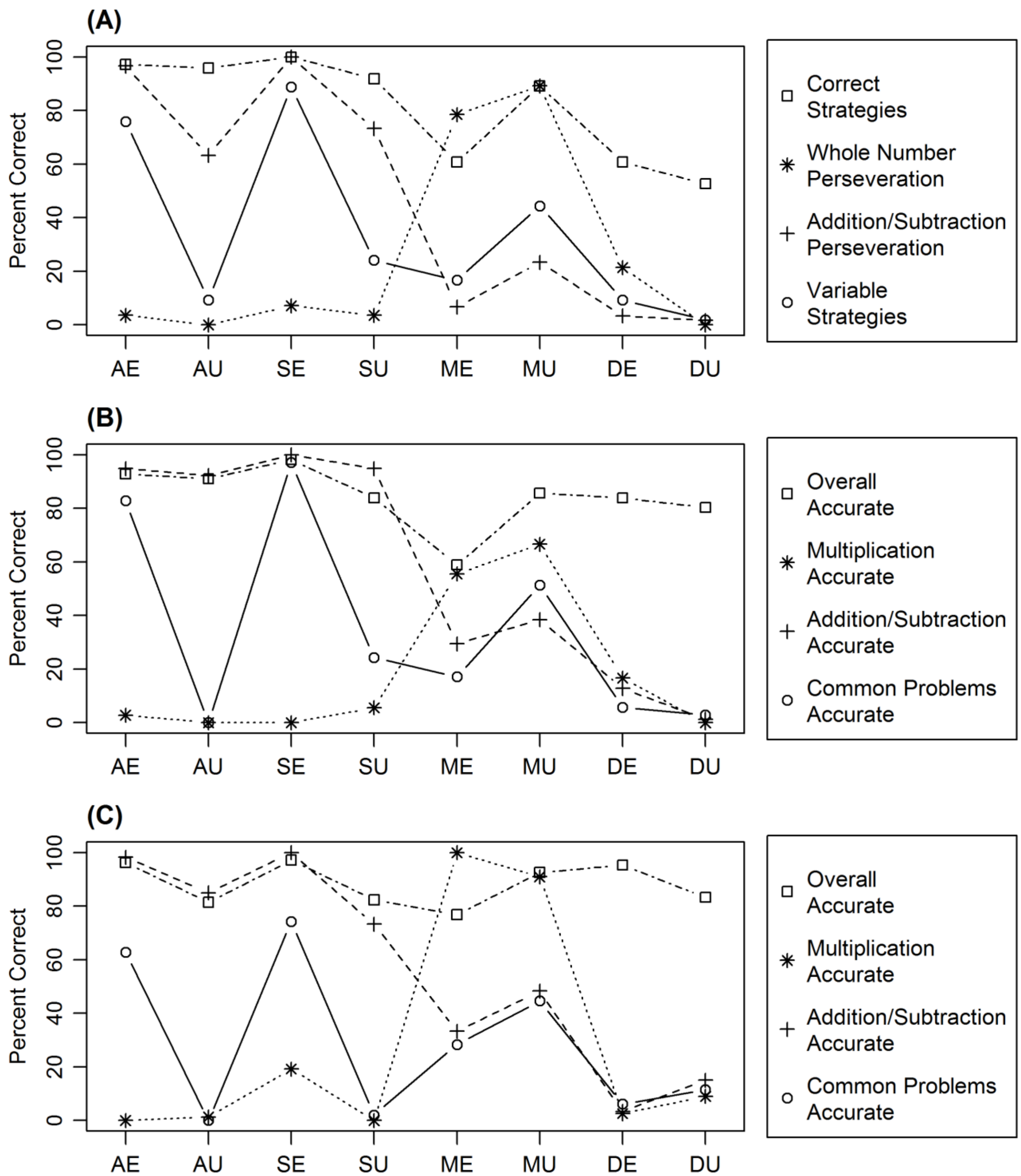

Figure 2. Percent correct on each combination of arithmetic operation and denominator equality/inequality within (A) each behavior pattern in Study 2A, (B) each latent profile in Study 2B, and (C) each latent profile in Study 3. “A,” “S,” “M,” and “D” represent addition, subtraction, multiplication, and division; “E” and “U” represent equal and unequal denominators.

Siegler and Pyke (2013) observed that 61% of children used different strategies for at least one pair of near-identical problems, such as 3/5×1/5 and 4/5×3/5. Doing so was neither logically required nor logically prohibited by the criteria for any of the four behavior patterns. Nevertheless, the proportion of children who used different strategies for at least one pair of near-identical problems was much higher within the Variable Strategies pattern (78%) than the Correct Strategies (16%), Whole Number Perseveration (36%), or Addition/Subtraction Perseveration (20%) patterns, $\chi^2(3) = 30.4$, $p < .001$.

3.3. Discussion

All four behavior patterns predicted by FARRA in Study 1 appeared among children in Study 2A, and 90% of children fit one of the four patterns. Further, the results confirmed FARRA’s predictions regarding the relation between math achievement and the four behavior patterns. These results lend support to FARRA as a model of children’s fraction arithmetic learning.

The findings highlight the importance of examining individual differences in addition to analyzing aggregated data. In the aggregate, children displayed higher accuracy on equal denominator addition and subtraction and unequal denominator multiplication than on all other types of problem (Siegler & Pyke, 2013; the same result was obtained by Siegler et al., 2011). In the present analysis, however, accuracy was highest on these three problem types only among children who fit the Variable Strategies pattern, 22% of the sample. Further, in the aggregate, children displayed variable strategy use, but children who fit the Variable Strategies pattern displayed much higher variability than those who fit the other three patterns. Thus, fraction arithmetic appears to be a domain in which aggregating across children can conceal substantial between-child variability. This conclusion dovetails with previous findings revealing between-child variability in whole number arithmetic strategies (Siegler, 1987, 1989) and fraction magnitude representations (Braithwaite & Siegler, 2018b).

The findings also suggest that superficially similar patterns of strategy use may have different origins. Addition/Subtraction Perseveration and Whole Number Perseveration both involve perseverative strategy use but were associated with very different levels of math achievement. Addition/Subtraction Perseveration may be associated with higher math achievement because this pattern reflects successful learning—though also over-generalization—of the strategy for adding and subtracting fractions. In contrast, Whole Number Perseveration may be associated with lower math achievement because it results from persistence of an initial bias derived from whole number arithmetic, rather than from any learning of fraction arithmetic. These considerations highlight the importance of distinguishing between these two patterns, rather than regarding them merely as alternative forms of Perseveration.

4. Study 2B

Classifying children according to strategy use patterns that were defined *a priori* based on theoretical considerations, as was done in Study 2A, invites the question of how much the results depend on the theoretical assumptions underlying the analysis and the classification criteria employed in the analysis. In other words, would an analysis without the same assumptions and classification criteria yield similar conclusions? To address this question, in Study 2B, we re-analyzed results of Siegler and Pyke (2013) using Latent Profile Analysis (LPA;

Hickendorff, Edelsbrunner, McMullen, Schneider, & Trezise, 2018). LPA is a data-driven technique for separating a group of individuals into subgroups, or profiles, according to their patterns of performance to maximize similarity between individuals within a profile while minimizing similarity between profiles.

An additional consideration was our reliance, in Study 2A, on strategy use data as the basis for classifying children's behavior patterns. To further test the robustness of our findings, in Study 2B, children's accuracies on different types of problems—rather than strategy use—served as input to the LPA. Thus, children within each latent profile generated by the analysis displayed a distinctive pattern of accuracies on different problem types. We compared these patterns of accuracies to those of the four strategy use patterns (Figure 2A) in order to identify which profile, if any, resembled each strategy use pattern.

We tested three predictions. (1) LPA using children's accuracies would generate profiles with characteristics similar to those of the strategy use patterns in Study 2A. (2) Most children who matched a given strategy use pattern in Study 2A would be classified into the analogous profile in Study 2B. (3) Children belonging to the profiles analogous to Correct Strategies and Addition/Subtraction Perseveration would display higher math achievement than children belonging to the profiles analogous to Whole Number Perseveration and Variable Strategies.

4.1. Analysis

LPA was used to model children's accuracies across the eight problem types: addition, subtraction, multiplication, and division with equal and unequal denominators. Percentages of correct answers for each problem type were used as the observed variables.

The LPA was modeled in Mplus version 8.0 (Muthén & Muthén, 1998-2017). Models were estimated using maximum likelihood estimation with robust standard errors (MLR), a full information approach that can handle missing-at-random data. To ensure the validity of the solution, the LPA was carried out with at least 200 and 20 random start values in the first and second steps of model estimation, respectively (Geiser, 2013). To determine the suitable number of latent profiles and best fitting models, model fit was evaluated with several statistical indicators (Nylund, Asparouhov, & Muthén, 2007): entropy, Bayesian Information Criterion (BIC), Lo-Mendell-Rubin (LMR) test, and Parametric Bootstrapped Likelihood Ratio Test (BLRT). Entropy values that approach 1 indicate more certainty in the classifications; following Tein, Coxe, and Cham (2013), entropy greater than .80 was considered acceptable. Lower values for BIC indicate a better fit. Significant results of the LMR test and BLRT suggest the *k*-profile solution is more appropriate that the (*k*-1)-profile solution.

4.2. Results

4.2.1. Model Selection

Table 2 details the statistical indicators for the LPA models with two to four profiles[2]. BIC was lowest for the four-profile model. Entropy was acceptably high for all three models. LMR indicated a significant advantage of the three-profile model over the two-profile model, but not the four-profile model over the three-profile. BLRT indicated advantages of the four-profile model over the three-profile one and of the three-profile model over the two-profile one. Although LMR did not provide evidence favoring the four-profile model, the BIC and BLRT results both favored this model. Simulation studies conducted by Nylund et al. (2007) suggested

[2] The log-likelihood for the five-profile model could not be replicated after increasing the number of random starts to 1000 (first step) and 100 (second step), suggesting critical issues with this model, so results are not presented for models with five or more profiles.

that BLRT is a more accurate indicator than LMR for these analyses. Therefore, the four-profile model was selected.

Table 2. Statistical Indicators for LPA Model (Study 2B). BIC indicates Bayesian Information Criterion, LMR indicates Lo-Mendell-Rubin test, and BLRT indicates Bootstrapped Likelihood Ratio Test. The LMR and BLRT columns display *p* values for the comparisons between each model and the model with one fewer profiles.

Number of Profiles	Entropy	BIC	LMR (*p*)	BLRT (*p*)
2	.99	552	.37	.001
3	.99	347	.006	.001
4	.99	250	.38	.001

4.2.2. Fraction Arithmetic Performance Within Each Latent Profile

The mean accuracy of children within each latent profile for each of the eight problem types is shown in Figure 2B. Each profile was assigned a name reflecting the types of problems on which it displayed relatively high accuracy. Each profile's accuracies on different problem types clearly resembled those of one of the four strategy use patterns in Study 2A (Figure 2A). (1) The Overall Accurate profile (23% of children), like the Correct Strategies pattern, was characterized by above-average accuracy on all problem types. (2) The Multiplication Accurate profile (15% of children), like the Whole Number Perseveration pattern, displayed higher accuracy on multiplication problems than on all other problems. (3) The Addition/Subtraction Accurate profile (32% of children), like the Addition/Subtraction Perseveration pattern, was characterized by higher accuracy on addition and subtraction problems than on all other problems. (4) The Common Problems Accurate profile (29% of children), like the Variable

Strategies pattern, displayed higher accuracy on equal denominator addition and subtraction and unequal denominator multiplication problems than on all other problems. This profile was named “Common Problems Accurate” because the types of problems just mentioned are relatively common in math textbooks (Braithwaite et al., 2017; Braithwaite & Siegler, 2018a).

4.2.3. Relations Between the Four Behavior Patterns and the Four Latent Profiles

The proportion of children who were classified into the latent profile that was analogous to their strategy use pattern was 71%, excluding children who were not classified into any of the strategy use patterns. This proportion ranged from 62% for children in the Correct Strategies pattern to 93% for children in the Whole Number Perseveration pattern (Table 3). Chi-square tests of independence revealed dependencies between membership in a strategy use pattern and membership in the analogous latent profile. The dependencies were present for the Correct Strategies pattern and Overall Accurate profiles, $\chi^2(1) = 42.0$, $p < .001$; for the Whole Number Perseveration pattern and Multiplication Accurate profiles, $\chi^2(1) = 68.6$, $p < .001$; for the Addition/Subtraction Perseveration pattern and Addition/Subtraction accurate profiles, $\chi^2(1) = 19.3$, $p < .001$; and for the Variable Strategies pattern and Common Problems Accurate profile, $\chi^2(1) = 36.9$, $p < .001$. No children in the Whole Number Perseveration pattern were assigned to the Addition/Subtraction Accurate profile, and no children in the Addition/Subtraction Perseveration pattern were assigned to the Multiplication Accurate profile.

Table 3. Proportion of children within each strategy use pattern in Study 2A that were classified into each latent profile in Study 2B. **Boldface** indicates the proportion of children within a strategy use pattern that were classified into the analogous latent profile.

	Behavior Pattern (Study 2A)			
Latent Profile (Study 2B)	Correct Strategies	Whole Number Perseveration	Addition/ Subtraction Perseveration	Variable Strategies
Overall Accurate	**62**	0	0	0
Multiplication Accurate	0	**93**	0	7
Addition/ Subtraction Accurate	38	0	**67**	15
Common Problems Accurate	0	7	33	**78**

4.2.4. Differences in Math Achievement Among Latent Profiles

Analogous to the results of Study 2A, math PSSA percentile ranks differed between the four latent profiles, $F(3, 108) = 12.0$, $p < .001$. Likewise, pairwise comparisons with a Holm correction found that math achievement percentile was higher among children in the Overall Accurate profile (mean = 61.5, $SD = 20.8$) and Addition/Subtraction Accurate profile (mean = 62.9, $SD = 20.4$) than among children in the Multiplication Accurate profile (mean = 33.6, $SD = 30.2$) or Common Problems Accurate profile (mean = 34.2, $SD = 29.3$), *p*s < .01. Also analogous to the results of Study 2A, math achievement did not differ between Overall Accurate and

Addition/Subtraction Accurate or between Multiplication Accurate and Common Problems Accurate, ps > .9.

4.3. Discussion

The similarity between the results of Studies 2A and 2B (Figures 2A and 2B) was striking, given that these results were generated using different analytical methods (*a priori* classification criteria or LPA) and applied to different types of data (strategy use or accuracy data). Over 70% of children whose behavior fit one of the *a priori* behavior patterns in Study 2A belonged to the analogous latent profile in Study 2B. Moreover, math achievement varied in the same way among the strategy use patterns (Study 2A) and the latent profiles (Study 2B). These findings lend support to the conclusions of Study 2A and show that they were not dependent on particular theoretical assumptions, a particular analytical method, or a particular dependent measure. However, it is not yet clear if the results were dependent on the particular group of children who participated in Study 2.

An important difference between accuracy and strategy use data is that accuracy can be coded automatically, whereas strategies usually require hand-coding. The fact that our classification can be obtained from accuracy data alone increases its potential utility as a diagnostic tool in educational settings, a subject to which we return in the General Discussion.

5. Study 3

The goals of Study 3 were to replicate the findings of Study 2B using different arithmetic problems with a different, larger sample of children from a different part of the U.S.; and to provide an additional test of whether the four-profile model provided a superior fit to the three-profile model. We predicted that despite the differences in samples and problems, the analysis would yield latent profiles similar to those in Study 2B.

Besides solving fraction arithmetic problems, a subset of the children completed a standardized measure of math achievement. We predicted that, if the expected profiles were found, children belonging to the Overall Accurate and Addition/Subtraction Accurate profiles would again display higher math achievement than those belonging to the Multiplication Accurate and Common Problems Accurate profiles.

5.1. Method

5.1.1. Participants

Participants were 394 seventh (*n* = 232) and eighth (*n* = 162) grade students (53% female) from Gainesville, FL. The population of the school was 51% white, 28% African American, 11% Hispanic, and 5% Asian; 43% of students received free or reduced lunch in the year the study was conducted. All participants had parental consent to participate in the study and gave their own assent; the ethics board of the last authors' institution approved the study, as did the district and school administrations.

5.1.2. Materials

The fraction arithmetic problem set consisted of twelve fraction arithmetic problems. Eight of the problems had only proper fractions as operands and included one problem for each combination of arithmetic operation with denominator equality or inequality; these problems were 7/8+2/8, 1/5+2/3, 2/3−1/3, 5/7−1/2, 3/5×1/5, 3/4×1/5, 5/8÷3/8, and 4/7÷1/2. The other four problems had two mixed numbers or one mixed number and one proper fraction as operands; these problems were 2 3/4 + 4 1/8, 2 6/7 + 5 1/2, 3 2/3 – 3/4, and 8 1/2 ÷ 4 1/8. Our analysis was limited to the eight problems that had only proper fractions as operands, because these were the only types of problems included in Studies 1 and 2. Cronbach's alpha for these problems within the sample was 0.77.

We assessed math achievement using the Woodcock-Johnson III Calculation sub-test (WJC; Woodcock, McGrew, Mather, & Schrank, 2011). Only problems that did not involve fractions were used to calculate children's math achievement. Cronbach's alpha for these problems was 0.73 within the sample of children who completed the WJC (see below).

5.1.3. Procedure

Both the WJC and the fraction arithmetic test were administered in whole-class, paper-and-pencil format. The WJC was completed in the previous year by 215 of the children when they were in sixth grade (N = 119) or seventh grade (N = 96). The test items were presented on two sequential paper sheets, which the students worked through independently. Students had 15 minutes to complete as many items as they could. Because we did not identify basal and ceiling levels for each student, we could not evaluate performance relative to published norms. Instead, we used percentile rank within the sample of number of items answered correctly as our measure of math achievement.

The fraction arithmetic test was administered about one year later, when the children were in seventh or eighth grade. At that time, the sample was expanded to include all willing participants in the 215 children's classrooms, resulting in the final sample of 394. Petrill et al. (2012) found that scores on the WJC were moderately stable over a one-year period (r = .70).

5.1.4. Analysis

Students' responses for both tasks were scored as correct (1) if they were numerically equivalent to the correct answer and as incorrect (0) otherwise. Their scores for each of the eight fraction arithmetic problem types were used as the observed variables for the model. The LPA[3] and accompanying model selection used the same procedures as in Study 2B.

5.2. Results

5.2.1. Model Selection

Statistical indicators for the LPA models with two to five profiles are shown in Table 4. Entropy was sufficiently high in all models. BIC, LMR, and BLRT all indicated that the four-profile model was superior to the two- and three-profile models. However, the two indicators that favored the chosen model in Study 2B (BIC and BLRT) offered partially discrepant results, with BIC favoring four over five profiles and BLRT favoring five over four profiles. We resolved this discrepancy by following the results of the LMR, which favored the four-profile model. Doing so facilitated comparison of the results to those of Study 2B, but by no means guaranteed that the profiles would have analogous characteristics to those found in Study 2B. Comparison of the four- and five-profile models within Study 3 revealed that three profiles were virtually identical in the models, and the remaining two profiles in the five-profile model could be viewed as sub-groups within the remaining profile in the four-profile model; details are provided in Supplement S4.

[3] Because there was only one problem per type, children's scores (0 or 1) on the eight problem types were treated as categorical variables in the analysis. Thus, our analysis would typically be described as Latent Class Analysis (LCA). However, for conceptual consistency, we refer to the analysis as LPA.

Table 4. Statistical Indicators for LPA Model (Study 3). BIC indicates Bayesian Information Criterion, LMR indicates Lo-Mendell-Rubin test, and BLRT indicates Bootstrapped Likelihood Ratio Test. The LMR and BLRT columns display p values for the comparisons between each model and the model with one fewer profiles.

Number of Profiles	Entropy	BIC	LMR (p)	BLRT (p)
2	.90	3336	.001	.001
3	.83	3237	.001	.001
4	.84	3230	.006	.001
5	.81	3244	.22	.001

5.2.2. Fraction Arithmetic Performance Within Each Latent Profile

The four latent profiles' patterns of accuracy on the different problem types in Study 3 are shown in Figure 2C. These patterns were clearly analogous to those of the latent profiles in Study 2B (Figure 2B), so the profiles were given the same names. (1) The Overall Accurate profile (27% of children) was characterized by above-average accuracy on all eight problem types. (2) The Multiplication Accurate profile (20% of children) was characterized by higher accuracy on multiplication problems than on all other problems. (3) The Addition/Subtraction Accurate profile (18% of children) was characterized by higher accuracy on addition and subtraction problems than on all other problems. (4) The Common Problems Accurate profile (38% of children) was characterized by higher accuracy on equal denominator addition and subtraction and unequal denominator multiplication problems than on all other problems.

5.2.3. Differences in Math Achievement Among Latent Profiles

The 179 children who did not complete the WJC were excluded from analyses involving math achievement. The distribution of latent profiles differed between children who did or did not complete the WJC; details are provided in Supplement S5. However, we had no reason to expect this fact to bias the relation between math achievement and the distribution of latent profiles among the children who did complete the WJC.

Percentile rank on the WJC differed between the four latent profiles, $F(3, 211) = 39.9$, $p < .001$. Pairwise comparisons with a Holm correction found that math achievement percentile was higher among children in the Overall Accurate profile (mean = 68.8, $SD = 23.4$) and Addition/Subtraction Accurate profile (mean = 60.5, $SD = 23.3$) than among children in the Multiplication Accurate profile (mean = 26.1, $SD = 22.5$) or Common Problems Accurate profile (mean = 36.7, $SD = 22.9$), ps < .001. Math achievement did not differ between the Overall Accurate and Addition/Subtraction Accurate profiles or between the Multiplication Accurate and Common Problems Accurate profiles, ps > .05. Thus, the differences in achievement test performance among children in different profiles were analogous to those in Studies 2A and 2B.

5.3. Discussion

Despite using different fractions problems and different children from a different part of the U.S., the LPA in Study 3 data yielded strikingly similar results to those of the LPA in Study 2B. Four very similar latent profiles appeared, and—despite use of a different measure of math achievement—math achievement varied among the four profiles in a similar manner. Thus, the findings of Study 2B were not dependent on the particular children, problems, or measure of math achievement employed in that study.

6. General Discussion

FARRA, a model previously employed to explain children's aggregate performance in fraction arithmetic, correctly predicted individual difference patterns as well. The findings support a major theoretical assumption of the model—that children employ similar cognitive processes to learn and perform fraction arithmetic but differ with respect to the parameters governing those cognitive processes. Further, the findings demonstrate that distinct patterns of strategy use exist in the domain of fraction arithmetic, and suggest that these qualitatively different patterns may result from continuous parametric variation among individuals (Schunn & Reder, 2001; Siegler, 1988).

The present study highlights three dimensions of individual differences that appear likely to affect learning outcomes in fraction arithmetic and quite likely in other areas of mathematics as well: effective learning from errors, consistency versus variability of strategy choices, and presence or absence of initial bias. We discuss each of these dimensions in turn; then we discuss methodological and educational implications of the findings.

6.1. Effective Learning from Errors

A major theoretical assumption underlying FARRA is that children vary in how effectively they learn from their own errors. The theory posits that when children use a strategy to solve a problem that generates an incorrect answer and elicits negative feedback, some individuals avoid using the same strategy for similar problems in the future, whereas others continue using it or even increase their use of it despite the negative feedback. This assumption contrasts with models of learning in which errors always lead to negative reinforcement (e.g., Thomas & McClelland, 2008) as well as with models in which using a strategy always leads to positive reinforcement, even when the strategy generates an error (e.g., Siegler & Shipley, 1995).

Variation in how effectively children learn from errors is captured in FARRA by the error discount parameter. In the simulations in Study 1, Whole Number Perseveration appeared primarily when the error discount was small, whereas the Correct Strategies pattern appeared almost exclusively when the error discount was larger than one. The fact that substantial numbers of children displayed each of these patterns supports the assumption that children vary in how effectively they learn from their own errors. Further, the simulation results suggest that failure to learn effectively from errors may be a cause of Whole Number Perseveration in children, whereas effective learning from errors may be an important condition for children to achieve proficiency in fraction arithmetic.

These considerations invite the question of what factors cause such individual differences. One possible cause is differences in domain-general cognitive abilities. Consistent with this possibility, individual differences in working memory capacity (Fyfe, DeCaro, & Rittle-Johnson, 2015) and inductive reasoning ability (Schunn & Reder, 2001) predict differences in learning from feedback. Alternatively, or additionally, conceptual knowledge of the domain (such as fraction magnitude knowledge, discussed in Section 6.5) could affect learners' ability to reflect on and make sense of errors. Some studies have found more effective learning from incorrect worked examples among individuals with greater domain knowledge (Große & Renkl, 2007; Heemsoth & Heinze, 2014), though others have found the opposite (Barbieri & Booth, 2016). Finally, teachers differ in their approaches to discussing student errors in the classroom (Schleppenbach, Flevares, Sims, & Perry, 2007), suggesting that instructional factors could also play a role. However, little research has directly assessed children's ability to learn from their own errors. The present findings suggest that this topic deserves greater study.

6.2. Consistency versus Variability

Variable strategy use is a hallmark of children's fraction arithmetic performance when viewed in the aggregate (Siegler & Pyke, 2013; Siegler et al., 2011). However, the present study revealed that variability itself varies among children. Children in the Variable Strategies pattern in Study 2A, and the Common Problems Accurate profiles in Studies 2B and 3, displayed high variability; they were not consistently correct or consistently incorrect for any one arithmetic operation. Other children displayed greater consistency in the form of consistently high accuracy for one or more arithmetic operations. Although high accuracy required consistency, consistency did not guarantee high accuracy. The findings identified children's perseverative use of a single strategy, resulting in high accuracy on some arithmetic operations and low accuracy on others, as a distinct, maladaptive performance pattern.

The simulation results reported in Study 1 suggest that whether a given child displays variable or consistent strategy use may depend on the child's learning rate and decision determinism, captured in FARRA by the learning rate and decision determinism parameters. Many formal cognitive models include similar parameters that are assumed to vary among individuals (e.g. learning rate parameters in connectionist models, Thomas & McClelland, 2008; precision parameters in mathematical models of decision-making, Friedman & Massaro, 1998). The present study demonstrates that variation in these parameters can explain qualitatively different patterns of strategy use among children in an educationally-relevant domain.

As in the case of the error discount parameter, variation in learning rate and decision determinism could reflect individual differences in domain-general cognitive attributes, domain-specific factors such as conceptual knowledge of fraction arithmetic, or instructional factors. A question deserving of further investigation is whether differences in learning rate and decision determinism are stable within an individual across domains. If they are, then an individual who exhibits either variable or perseverative behavior in one domain would exhibit analogous behavior in other domains. Stability of strategy use patterns across domains would greatly increase the diagnostic utility of the patterns, in that a diagnosis of a child's pattern of strategy use in one domain could predict difficulties in other domains.

6.3. Presence or Absence of Initial Bias

The results suggest that initial bias can have lasting effects on fraction arithmetic learning. In the Study 1 simulations, high values of the whole number bias parameter, reflecting an initial tendency to use the whole number strategy, made Whole Number Perseveration the most likely pattern. Even when Whole Number Perseveration was not the outcome, intermediate levels of whole number bias increased the proportion of simulations that yielded the Variable Strategies pattern and decreased the proportion of simulations that yielded the Correct Strategies or Addition/Subtraction Perseveration patterns. Apparently, the need to un-learn an initial tendency to use the whole number strategy for adding and subtracting fractions made it more difficult for FARRA to learn correct procedures for fraction addition and subtraction. We hypothesize that the same is true for children.

The present analysis also suggested that in the absence of whole number bias, Perseveration—when it emerged—would likely involve the addition/subtraction strategy. Given that fraction addition and subtraction are presented prior to multiplication and division in mathematics curricula, the addition/subtraction strategy initially has the greatest opportunity to receive positive reinforcement, which would give it a first-mover advantage not unlike the primacy effects observed in memory research (Digirolamo & Hintzman, 1997). Indeed, when whole number bias was set to zero in Study 1, Whole Number Perseveration almost never appeared, whereas Addition/Subtraction Perseveration appeared in about one-fourth of simulations. The Addition/Subtraction Perseveration pattern appeared among a substantial proportion of children in Study 2A, as did the analogous latent profile—Addition/Subtraction Accurate—in Studies 2B and 3. These findings deserve emphasis because previous literature on children's fraction arithmetic learning has devoted much less attention to incorrect use of the addition/subtraction strategy to multiply and divide fractions (though, see Siegler & Pyke, 2013; Siegler et al., 2011) than to children's incorrect use of the whole number strategy to add and subtract fractions (Byrnes & Wasik, 1991; Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980; Mack, 1995; Ni & Zhou, 2005). The fact that some children not only commit the former type of strategy error but do so consistently years after studying the correct procedures for fraction multiplication and division, suggests that this type of error merits more attention in both research and pedagogy.

Our distinction between two types of Perseveration in fraction arithmetic dovetails with recent findings from a Latent Class Analysis of fourth- to sixth-graders' responses on a fraction magnitude comparison task (Rinne, Ye, & Jordan, 2017; see also Resnick, Rinne, Barbieri, & Jordan, 2018). The analysis revealed that some children judged fractions with larger whole number components to be larger than those with smaller whole number components, a pattern called "Large Number Bias"; other children judged fractions with smaller whole numbers—either numerators or denominators—to be larger, a pattern called "Small Number Bias." Like Whole Number Perseveration, Large Number Bias involves persistence of a bias that appears before fractions instruction; like Addition/Subtraction Perseveration, Small Number Bias involves overcoming that initial bias and learning, though also over-generalizing, a rule—the rule that smaller denominators make fractions bigger. Large Number Bias was associated with lower math achievement than Small Number Bias, paralleling differences in math achievement between the two types of Perseveration in the present study. Together, these findings highlight the possibility that different persistent errors reflect individual differences in learning parameters and therefore call for different styles of intervention, an issue we address below.

6.4. Methodological Implications

The present study illustrates the advantages of using multiple converging approaches to study cognition and learning. Two methods of characterizing individual differences—one based on a cognitive process model and the other employing a data-driven technique, LPA—yielded remarkably convergent results. This convergence lent credence to the model-based results and suggested a theoretically-meaningful interpretation of the LPA results. Tenison, Fincham, and Anderson (2016) recently offered an excellent example of this converging methods approach: a data-driven analysis of brain imaging data suggested the existence of three phases of cognitive skill acquisition, for which an ACT-R model provided a principled theoretical explanation. We believe that combining theory-driven and data-driven analyses in a mutually supportive manner represents a promising direction for future research.

6.5. Educational Implications

The types of learner profiles identified in the present study could be a useful diagnostic tool in educational settings for tailoring instruction to the needs of individual students. Children's performance on a small set of fraction arithmetic problems provided sufficient information to classify the children into meaningfully different groups. Moreover, classifications using different types of data—either strategy use or accuracy—yielded consistent results for individual children. These analyses suggest that children with different patterns of strategy use or accuracy would likely benefit from different instructional interventions.

First, children who display Addition/Subtraction Perseveration are probably least in need of special intervention and most likely to improve from additional practice with feedback. These children have already partially succeeded in learning fraction arithmetic, in that they have learned the strategies for adding and subtracting fractions with both equal and unequal denominators. Our simulations suggest that this partial success reflects their learning effectively from errors, an interpretation that is consistent with the children's relatively high math achievement. Additional practice with feedback should allow these children to self-correct, leading to improved performance.

Children who display Whole Number Perseveration are less likely to benefit merely from additional practice. Our simulations suggest that these children learn relatively little from their own errors, an interpretation that is consistent with their relatively low math achievement. Failure to learn from errors can reflect learners not understanding why correct solutions are correct (Metcalfe, 2017). Ensuring that children reflect on their errors, attend to correct solutions, and hear explanations of why the correct solutions are correct, could be particularly beneficial for children who display Whole Number Perseveration.

Children who display Variable Strategies are also less likely to benefit merely from additional practice if, as our simulations suggest, this pattern results from slow learning and relatively random strategy selection. These children could benefit from interventions that compensate for these characteristics. For example, such students could be encouraged to attend to relevant features of each problem, and explicitly state before they start to solve the problem which strategy they plan to use and why. This approach could encourage deliberative rather than impulsive strategy selection, and thereby help children to select strategies more systematically. Practice in this more deliberate mode, paired with appropriate feedback, could result in more efficient learning (Ericsson, Krampe, & Tesch-Römer, 1993; Lehtinen, Hannula-Sormunen, McMullen, & Gruber, 2018).

We have previously recommended that students receive interleaved practice with different fraction arithmetic operations, on the grounds that interleaved practice could help children identify the conditions under which each strategy should be used (Braithwaite et al., 2017). If, as we have argued, children who display Addition/Subtraction Perseveration would benefit most from practice in general, they would seem particularly likely to benefit from practice that interleaves different types of problems in particular. In contrast, children who display Whole Number Perseveration or Variable Strategies might require interventions intended to encourage deliberate strategy selection and reflection after errors, as described above, in order to benefit from interleaved practice.

Finally, another approach to improving children's fraction arithmetic is to improve their understanding of fraction magnitudes and the effects of arithmetic operations on magnitudes (Dyson, Jordan, Rodrigues, Barbieri, & Rinne, 2018; Fuchs et al., 2013). Interventions that improve understanding of fraction magnitudes could enable children to notice that inappropriate use of the whole number strategy results in answers that are too small or too large and thereby reject these errors, such as rejecting $3/5+1/4 = 4/9$ because $4/9 < 3/5$ or rejecting $4/5-1/3 = 3/2$ because $3/2 > 4/5$. These interventions could be especially effective for children displaying Whole Number Perseveration. Interventions that focus on the effects of arithmetic operations, such as the principle that multiplying by a fraction smaller than one "makes smaller," could enable children to reject errors resulting from inappropriate use of the addition/subtraction strategy, such as rejecting $4/5 \times 3/5 = 12/5$ because the answer should be $<4/5$. These interventions could be especially effective for children displaying Addition/Subtraction

Perseveration. Children displaying Variable Strategies commit both types of error and therefore might benefit from both types of intervention. Of course, this proposal, and the others above, await empirical verification.

6.6. Conclusions

Qualitatively distinct patterns of strategy use and accuracy are evident in individual children's fraction arithmetic. Combining a computational model of learning processes with the assumption of continuous parametric variation across children is an effective approach to understanding the potential causes of individual differences in this, and probably other, domains. Future research should explore sources of variation in children's learning parameters, the degree to which individuals' tendencies towards consistent or variable strategy use are stable across domains, and how to apply findings about individual differences to help children learn.

Appendix: Technical Description of FARRA

FARRA consists of a short-term memory; a set of production rules; associative weights; a decision-making rule; and a learning rule.

Short-Term Memory. FARRA's short term memory contains a representation of the problem it is trying to solve, the features of the problem (i.e., whether the operands are fractions, mixed numbers or whole numbers; whether the operands have equal or unequal denominators; whether the arithmetic operation is addition, subtraction, multiplication, or division), the results of any intermediate calculations so far performed (e.g., calculation of a common denominator or the numerator of the answer), and the model's current goals.

Production Rules. A production rule is a condition-action pair. FARRA simulates the process of solving problems by selecting and firing production rules. At each step of this process, FARRA can select any of the production rules whose condition parts are met, and then take the action specified in the selected rule. Doing so modifies short-term memory by changing the problem representation, adding results of intermediate calculations, or creating/removing goals. These changes may satisfy the condition parts of other rules. Another rule is then selected and fired, and the process proceeds iteratively until an answer is obtained.

A complete list of FARRA's production rules is given in Braithwaite et al. (2017). These production rules can be classified as either *strategy rules*, whose condition parts permit them to be selected at the very beginning of solving a problem, or *execution rules*, whose condition parts cannot be satisfied until a strategy rule has been fired. Both strategy and execution rules include *correct rules*, which describe steps of correct procedures, and *mal-rules*, which represent deviations from correct procedures. The most important mal-rules are strategy mal-rules, which allow FARRA to use strategies that would be appropriate for one arithmetic operation to solve problems involving other operations. For example, one strategy mal-rule allows FARRA to apply the strategy for adding and subtracting equal-denominator fractions to any problem, regardless of the arithmetic operation.

Associative Weights. For each problem feature i and each production rule j, FARRA retains an associative weight w_{ij} connecting the feature to the rule. These associative weights are initially set to 0 and subsequently modified by learning. An exception is the weights for the whole number/multiplication strategy, whose initial values are determined by the whole number bias parameter, as described in the main text. The initial weights for the whole number/multiplication strategy were set to 0 in all simulations reported by Braithwaite et al. (2017).

Decision-Making Rule. When multiple rules' conditions are met—for example, if the model has a choice between a correct rule and a mal-rule—rule selection depends on the rules' activations. The activation η_j of rule *j* is the sum of the associative weights w_{ij} connecting each of the current problem's features to rule *j* (Equation A1; x_i is equal to 1 for each feature *i* that is present in the problem and is equal to 0 for all other features).

$$\eta_j = \sum_i w_{ij} x_i \quad \text{(A1)}$$

$$p_j = e^{\gamma \eta_j} \Big/ \sum_k e^{\gamma \eta_k} \quad \text{(A2)}$$

The probability of selecting each rule whose conditions are met is determined by a softmax decision rule (Equation A2). This decision rule is governed by the decision determinism parameter γ. When γ is large, the most highly-activated of the rules whose conditions are met is almost certain to be selected; when γ is near zero, all candidate rules are nearly equally likely to be selected regardless of their activations.

Learning Rule. FARRA learns by solving problems, receiving feedback, and adjusting its associative weights based on the feedback. After a problem-solving episode, for each feature *i* of the problem and each rule *j* that was used while solving the problem, the weight *wij* connecting feature *i* to rule *j* is adjusted according to Equation A3.

$$\Delta w_{ij} = \begin{cases} e & \text{if the answer was correct} \\ (1-d)\cdot e & \text{if the answer was incorrect} \end{cases} \quad \text{(A3)}$$

Correct answers always lead to increases in rule weights, making the model more likely to use the same rules to solve similar problems in the future; the amount of the increase in rule weights is equal to the learning rate parameter *e*. Learning after incorrect answers depends on the value of the error discount parameter *d*. Incorrect answers lead to increases in rule weights if *d* is less than 1—albeit by a smaller amount than after correct answers—but lead to decreases in rule weights if *d* is greater than 1.

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