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Implementing a Framework for Early Algebra[1]

Blanton, M., Brizuela, B., Stephens, A., Knuth, E., Isler, I., Gardiner, A., Stroud, R., Fonger, N., & Stylianou, D.[2]

In this chapter, we discuss the algebra framework that guides our work and how this framework was enacted in the design of a curricular approach for systematically developing elementary-aged students' algebraic thinking. We provide evidence that, using this approach, students in elementary grades can engage in sophisticated practices of algebraic thinking based on generalizing, representing, justifying, and reasoning with mathematical structure and relationships. Moreover, they can engage in these practices across a broad set of content areas involving generalized arithmetic; concepts associated with equivalence, expressions, equations, and inequalities; and functional thinking.

Keywords: Algebraic thinking, randomized study, early algebra, learning progressions, qualitative methods, curriculum

1. Introduction

When tasked with the open question of measuring the impact of *early algebra*[3] on children's algebra-readiness for middle grades, our first challenge was to identify the "early algebra curriculum" from which impact could be measured. Essentially, such a curriculum as we envisioned it—that is, an instructional sequence that integrated core algebra concepts and practices across the elementary school mathematics curriculum through a research-based, multi-year approach—did not exist in curricular resources in the United States [US]. At best, we found that mainstream arithmetic curricula offered only a random treatment of "popular" algebraic concepts (e.g., a relational understanding of the equal sign, finding the value of a variable in a linear equation, finding a pattern in sequences of numbers), often buried in arithmetic content in ways that allowed one to potentially ignore or marginalize their treatment in instruction. This curricular challenge presented us with an obvious corollary: What is the *algebra* that we want

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[2] Author affiliations: Maria Blanton, Maria_Blanton@terc.edu, TERC, 2067 Massachusetts Ave., Cambridge, MA 02140; Bárbara M. Brizuela. Barbara.Brizuela@tufts.edu, Tufts University, 12 Upper Campus Rd., Medford, MA 02155; Ana Stephens, acstephens@wisc.edu, Wisconsin Center for Education Research, 683 Educational Sciences Building, 1025 W. Johnson Street, Madison, WI 53706; Eric Knuth, eric.knuth@austin.utexas.edu, University of Texas at Austin, 1912 Speedway STOP D5700, Austin, TX 78712; Isil Isler, iisler@metu.edu.tr, Middle East Technical University, Orta Dogu Teknik Universitesi, Egitim Fakultesi, Matematik ve Fen Bilimleri Egitimi Bolumu, 06800 Cankaya, Ankara, Turkey; Angela Murphy Gardiner, Angela_Gardiner@terc.edu, TERC, 2067 Massachusetts Ave., Cambridge, MA 02140; Nicole Fonger, nmlfonger@gmail.com; Syracuse University, 103 Carnegie Library, Syracuse, NY 13244-1170; Rena Stroud, Rena_Stroud@terc.edu; Despina Stylianou, dstylianou@ccny.cuny.edu, The City College of New York 160 Convent Ave, NAC 6/207 New York, NY 10031.

[3] By *early algebra* we mean algebraic thinking in the elementary grades (i.e., Grades Kindergarten–5).

young children to learn and that will suitably prepare them for a more formal study of algebra in the middle grades?

These challenges led us on a lengthy journey to apply a widely-acknowledged framework for algebra (Kaput 2008) as a conceptual basis for designing an early algebra curriculum for Grades 3–5. Such a curriculum would allow us to measure elementary grades students' potential for algebraic thinking as well as their readiness for algebra in the later grades. In a separate line of work, we also began exploratory research that would allow us to back this approach down into the lower elementary grades (i.e., Grades K–2). We share part of this journey here on three fronts: (1) we characterize the algebra framework that has informed our approach; (2) we describe the curricular approach and its components designed using this framework for Grades 3–5; and (3) we share evidence of the impact of this approach on children's algebraic thinking.

2. The Emergence of Early Algebra in the US

Research shows that, historically, algebra education in the US—an "arithmetic-then-algebra" approach in which an arithmetic curriculum in the elementary grades was followed by a formal treatment of algebra in secondary grades—was unsuccessful in terms of students' mathematical achievement (e.g., Stigler et al. 1999) and led to a widespread marginalization of students in school and society (Kaput 1999; Schoenfeld 1995). Algebra's resulting status as a gateway to academic and economic success (Moses and Cobb 2001) led to calls for identifying new approaches to algebra education. As part of this effort, scholars worked to develop new recommendations for school algebra instruction that would provide students with the kind of sustained experiences necessary for building informal notions about algebraic concepts and practices into more formal ways of mathematical thinking. Importantly, algebra education was re-framed as a longitudinal effort that would span Grades K–12 rather than one that began abruptly in high school (e.g., National Council of Teachers of Mathematics [NCTM] 2000; RAND Mathematics Study Panel 2003).

Recent US reform initiatives such as the *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices [NGA] and Council of Chief State School Officers [CCSSO] 2010) have reiterated the significant and increasing role algebra is now expected to play across school mathematics by outlining content standards and mathematical practices for algebraic thinking beginning at the start of formal schooling (i.e., kindergarten). While these efforts have strengthened the national discourse on the role of early algebra in school algebra reform, the development of a research-based approach to early algebra that would guide the systematic, long-term development and assessment of young children's algebraic thinking has been lacking. In this sense, we hope that the approach we share here might provide one route for clarifying and deepening the role of algebra in the elementary grades.

3. A Conceptual Framework for Early Algebra

The early algebra perspective that guides our work is based on Kaput's (2008) content analysis of algebra as a set of *core aspects* across several mathematical *content strands*. We discuss each of these here and how they are enacted in our work.

3.1 Core Aspects and the Algebraic Thinking Practices Derived from Them

Kaput (2008) proposes that algebraic thinking involves two core aspects: (a) making and expressing generalizations in increasingly formal and conventional symbol systems; and (b) acting on symbols within an organized symbolic system through an established syntax, where conventional symbol systems available for use in elementary grades are interpreted broadly to include "[variable] notation, graphs and number lines, tables, and natural language forms" (p. 12). While Kaput acknowledges differing views on whether and how acting on symbolizations (Core Aspect (b)) such as variable notation should occur in elementary grades, he and others (e.g., Blanton et al. 2017a; Brizuela and Earnest 2008; Carraher et al. 2008) maintain that interactions with all of these symbol systems early on can actually deepen students' algebraic thinking. In our work, we also adopted this broad interpretation of symbol systems, along with the view that incorporating such diverse systems throughout children's algebraic work would be a potentially productive route to developing their algebraic thinking.

We derive four essential practices from Kaput's (2008) core aspects that define our early algebra conceptual framework: *generalizing, representing, justifying,* and *reasoning with* mathematical structure and relationships (see also Blanton et al. 2011). We see the activities of *generalizing* and *representing* generalizations as the essence of Core Aspect (a). Furthermore, from Core Aspect (b), we take *justifying* generalizations and *reasoning with* established generalizations in novel situations as two principal ways of acting on conventional symbol systems, broadly interpreted. A critical component of these four practices is that they are centered around engagement with mathematical structure and relationships. For example, we take the view that the activity of justifying is not, in and of itself, algebraic, but it serves an algebraic purpose when the context is justifying generalized claims. In what follows, we elaborate on each of these four algebraic thinking practices as we interpret them in our work.

3.1.1 Generalizing. Generalizing is central to algebraic thinking (Cooper and Warren, 2011; Kaput 2008) and the very heart of mathematical activity (Mason 1996). It has been characterized as a mental process by which one compresses multiple instances into a single, unitary form (Kaput et al. 2008). For example, in simple computational work, a child might notice after several instances in which she adds an even number and an odd number that the result is an odd number. In this, the child is starting to "compress" all of the instances of adding a specific even number and a specific odd number and getting an odd number as a result into the generalization that the sum of *any* even number and *any* odd number is odd. Engaging elementary-aged children in the activity of generalizing is vital because it strengthens their ability to filter mathematical information from common characteristics and to draw conclusions in the form of generalized claims.

3.1.2 Representing generalizations. The activity of representing mathematical structure and relationships is as important as generalizing (Kaput et al. 2008). As a socially mediated process whereby one's thinking about symbol and referent is iteratively transformed (*ibid.*), the act of representing not only gives expression to the generalizations children notice in problem situations, but also shapes the very nature of their understanding of these concepts. As Morris (2009) notes, the practice of representing generalizations builds an understanding that an action applies to a broad class of objects, not just a particular instance, thereby reinforcing children's view of the generalized nature of a claim. In the example of evens and odds given earlier, a child might represent what they notice in their own words as "the sum of an even number and an odd number is odd." They might represent generalizations in other ways, such as with variable notation. For example, a child might represent the Commutative Property of Addition as $a + b =$

$b + a$, where, for the young child, a and b represent the counting numbers. Later, as students become more sophisticated, this number domain expands to include all real numbers.

3.1.3 Justifying generalizations. In justifying generalizations, students develop mathematical arguments to defend or refute the validity of a proposed generalization. In elementary grades, the forms of arguments students make are often naïve empirical justifications. Research shows, however, that they can develop more sophisticated, general forms that are not based on reasoning with particular cases (Carpenter et al. 2003; Schifter 2009). For example, students might build "representation-based arguments" (Schifter 2009) where they use drawings or manipulatives to justify the arithmetic relationships they notice. In building an argument as to why the Commutative Property of Addition is reasonable[4], a child might construct a snap-cube "train" of 3 red cubes followed by 4 blue cubes and visually demonstrate that the sum of the cubes (i.e., the length of the train) does not change when one flips the train around to become a 4-blue-cube, 3 red-cube "train." In a representation like this, the actual number of cubes is treated algebraically as a place-holder for any number of cubes. That is, the "3" and "4" become irrelevant in the more general justification the child is making.

There are long-term dividends for engaging children in the practice of justifying the mathematical generalizations they make. For instance, Morris (2009) notes that the development of children's capacity to justify relationships about generalized quantities can help prepare children for a more formal study of proof in later grades. As such, justifying generalizations is an important act of algebraic reasoning.

3.1.4 Reasoning with generalizations. Finally, algebraic thinking involves reasoning with generalizations as mathematical objects themselves. In this practice, children act on the generalizations they have noticed, represented, and justified to be true as objects of reasoning in new problem scenarios. For example, elsewhere we have observed young children building functional relationships that they represent with variable notation and with which they can reason as objects in solving new problem situations (Blanton et al. 2015a). Returning again to the example of evens and odds, a child might use previously noticed generalizations such as "the sum of an even number and an odd number is odd" to reason about the sum of three odd numbers. Cognitively, we see this type of reasoning as signifying an advanced point of concept formation in which the generalization has been reified in the child's thinking (Sfard 1991). Thus, cultivating this practice represents an important objective in learning to think algebraically.

3.2 Content Strands and Their Relation to our Framework

Kaput (2008) further argued that Core Aspects (a) and (b) occur across three content strands:

1. Algebra as the study of structures and systems abstracted from computations and relations, including those arising in arithmetic (algebra as generalized arithmetic) and quantitative reasoning.
2. Algebra as the study of functions, relations, and joint variation.
3. Algebra as the application of a cluster of modeling languages both inside and outside of mathematics. (p. 11)

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[4] Technically, such properties are axioms and assumed to be true without proof. However, it is productive for children to think about why such properties are reasonable.

Early algebra research has matured around several core areas relative to these content strands. Elsewhere (e.g., Blanton et al. 2015b), we have parsed these core areas, with three predominant areas being (1) generalized arithmetic; (2) equivalence, expressions, equations, and inequalities; and (3) functional thinking. We take generalized arithmetic to involve generalizing, representing, justifying, and reasoning with arithmetic relationships, including fundamental properties of operations (e.g., the Commutative Property of Multiplication) as well as other types of relationships on classes of numbers (e.g., relationships in operations on evens and odds). We take equivalence, expressions, equations, and inequalities to include developing a relational understanding of the equal sign and generalizing, representing, and reasoning with expressions, equations, and inequalities, including in their symbolic forms. Finally, we take functional thinking to include generalizing relationships between co-varying quantities and representing, justifying, and reasoning with these generalizations through natural language, variable notation, drawings, tables, and graphs.

Areas 1 and 2 align with Kaput's Strand 1, while Area 3 aligns with Strands 2 and 3. Although Kaput's content analysis—and our interpretation of it in our research—is not the only way to organize the content strands (or, our core areas) in which algebraic thinking practices occur, we do see this framework as reasonable and consistent with other perspectives (e.g., Carraher and Schliemann 2007; Cooper and Warren 2011).

4. Designing an Early Algebra Curricular Approach Using Our Algebra Framework

We expanded our algebra framework to establish an approach to teaching and learning early algebra that included an articulation of a curricular progression with associated learning goals, an instructional sequence to accomplish these goals, assessments to measure learning within the instructional sequence, and a characterization of students' ways of thinking as a result of their learning within the instructional sequence. This initial work, characterized in this chapter as Project LEAP[5], focused on Grades 3–5. In particular, our approach built on the body of work concerning learning progressions and learning trajectories in educational research (Barrett and Battista 2014; Clements and Sarama 2004; Daro et al. 2011; Duncan and Hmelo-Silver 2009; Shin et al. 2009; Simon 1995) which uses an integrated approach to both supporting students' learning and characterizing learning in the context in which it is supported. In what follows, we briefly elaborate the theoretical foundations, methods, and design principles that guided our curricular approach to early algebra across Grades 3–5. Broadly, we followed a learning progressions approach to developing coherent curricular products (Battista 2011; Shin et al. 2009), instruction that targets students' development of understandings over a large span of time (Schwartz et al. 2008), and assessments to measure sophistication in student thinking over time (Battista 2011). In what follows, we describe the components of our learning progression.

4.1 Curricular Progression

Our curricular progression elaborates finer grain sizes of the algebraic concepts and practices to be learned within each core area and at each grade level.[6] We conducted a research

[5] We use the term "LEAP" (Learning through an Early Algebra Progression) here in reference to our Grades 3–5 suite of projects that focused on understanding the impact of a systematic, multi-year approach to teaching and learning algebra in the elementary grades.

[6] We elaborate on this curricular approach in Fonger et al. (2017).

synthesis and textbook analysis to specify (1) appropriate algebraic concepts and practices (e.g., a relational understanding of the equal sign; generalizing a functional relationship between two quantities) within our core areas, and (2) learning goals that characterized the depth of understanding that might reasonably be expected at each grade level and which could guide the design of learning activities for our instructional sequence. Finally, we sought external review of our proposed curricular progression to validate its consistency with empirical research and teaching and learning standards.

A guiding design principle for our curricular progression is to build sophistication in learning goals over time, starting from students' experiences and prior knowledge. Following Battista (2004) and as elaborated in Fonger et al. (2017), we balanced a dual lens on empirical research on students' understandings with an eye toward the canonical development of algebra over time in accordance with mathematical sophistication. This lens supported our specification of how we sequenced and ordered content across the grades. Our curricular progression served as a blueprint for designing an instructional sequence.

4.2 Instructional Sequence

Our instructional sequence is an ordered set of lessons across Grades 3–5[7] designed to build in complexity over time and to weave together the core areas (e.g., generalized arithmetic, functional thinking) and algebraic thinking practices (e.g., generalizing and representing generalizations) to support teaching and learning early algebra in an integrated way. Each grade level sequence consists of approximately 18 one-hour lessons that are intended to be taught along with the regular mathematics curriculum. While we follow a proposed sequence during implementation for our research purposes, there is flexibility with how teachers might incorporate lessons into their existing curriculum to accommodate their needs.

Using the curricular progression as a framework, we designed tasks or modified existing tasks from research that showed potential to facilitate students' construction of algebraic ideas (Clements and Sarama 2014), then built a core sequence of lessons using these tasks. We refined our instructional sequence through cycles of testing and revision. Moreover, we sequenced the introduction of core areas to generally start from equivalence and a relational understanding of the equal sign, transition to generalized arithmetic and a study of fundamental properties of number and operation as well as other arithmetic generalizations, then progress to a study of generalized (indeterminate) quantities as a gateway for representing and reasoning with relationships between quantities through equations, inequalities, and functional relationships. Table 1 illustrates the lesson sequence and learning goals for Grade 3. Instructional sequences for Grades 4 and 5 were similar.

Lesson Sequence and Focus	**Learning Goals**
Relational understanding of the equal sign *(Lessons 1 – 2)*	• Identify meaning of '=' as expressing a relationship between quantities • Interpret equations written in various formats (e.g., other than $a + b = c$) to correctly assess an equivalence relationship (true/false number sentences)

[7] Ultimately, our aim is to develop a Grades K–5 sequence. Our decision to focus initially on Grades 3–5 was guided largely by the more extensive early algebra research base available in upper elementary grades.

	• Solve missing value problems by reasoning from the structural relationship in the equation (open number sentences)
Fundamental Properties: Additive Identity, Additive Inverse, Commutative Property of Addition, and Multiplicative Identity Arithmetic relationships involving classes of numbers (e.g., evens and odds) *(Lessons 3 – 6, 11)*	• Analyze information to develop a generalization about the arithmetic relationship • Represent the generalization in words • Develop a justification to support the generalization's truth; examine *representation-based arguments* (Schifter 2009) vis-à-vis empirical arguments • Identify values for which the generalization is true • Represent the generalization using variables • Examine the meaning of repeated variable or different variables in an equation representing a generalization • Examine values for which the generalization is true • Identify a generalization in use (e.g., in computational work)
Modeling problem situations with (linear) algebraic expressions *(Lesson 7)*	• Identify a variable to represent an unknown quantity • Informally examine the role of variable as a varying quantity • Represent a quantity as an algebraic expression using variables • Interpret an algebraic expression in context • Identify different ways to write an expression
Modeling and solving a problem situation involving one-step, single variable linear equation (additive and multiplicative) *(Lessons 8 – 10)*	• Model a problem situation to produce a linear equation ($x + a = b$ or $ax = b$) • Identify different ways to write the representative equation • Analyze the structure of the equation to determine the value of the variable • Check the solution to an equation or determine if the solution is reasonable given the context of the problem • Informally examine the role of variable as an unknown, fixed quantity
Modeling a problem situation involving linear functions of the form $y = x + b$, $y = mx$, or $y = mx + b$ with diverse representations (e.g., variables, words, graphs) and exploring function behavior *(Lessons 12 – 18)*	• Generate data and organize in a function table • Identify variables and their roles as varying quantities • Identify a recursive pattern, describe in words, and use to predict near data • Identify a covariational relationship and describe in words • Identify a function rule and describe in words and variables • Use a function rule to predict far function values • Examine the meaning of different variables in a function rule • Justify why a function rule accurately represents the problem data • Recognize that corresponding values in a function table must satisfy the function rule • Construct a coordinate graph to represent problem data • Given a value of the dependent variable and the function rule for a one-operation function, determine the value of the independent variable

Table 1. Overview of the instructional sequence for Grade 3.

4.3 Assessments

We developed grade level assessments across Grades 3–7 to measure progress in the development of students' algebraic thinking in response to their participation in the Grades 3–5 instructional sequence and to monitor retention of that knowledge after the intervention (i.e., in Grades 6–7). Key algebraic concepts and practices identified in our curricular progression were used to design tasks that formed the basis for these grade-level, one-hour assessments. We designed assessment items to have multiple points of entry (e.g., students might use different strategies to solve a particular problem) and to include common items across several grades as a means to track growth over time. To strengthen the validity of our assessments, experts on teaching and learning algebra evaluated the extent to which the proposed assessment items aligned with algebraic concepts and practices in each of the core areas, and assessments were administered to elementary grades students and tested for psychometric soundness. The assessments have provided a critical means to measure effectiveness of our instructional sequence (see Sections 5.1–5.3).

4.4 Student Thinking

We characterize students' thinking according to levels of sophistication, or qualitatively distinct ways of thinking, as evidenced in the strategies students use in written assessments and individual interviews. To strengthen the validity of our classification of student thinking, we accrued evidence of and distilled patterns in students' thinking over the span of several years (Stephens et al. in press). In our approach, the levels of sophistication observed in students' thinking is inseparable from the curricular and instructional context in which the learning was supported (see also Clements and Sarama 2004, 2014). In other words, the learning goals established in our curricular progression (and, subsequently, our instructional sequence) guide and support learning, while assessments measure that learning and levels of sophistication are the means by which we qualitatively characterize the nature of learning in that context over time.

5. Evidence of Growth in Students' Algebraic Thinking

It is reasonable to ask whether young children can successfully engage with a curricular approach such as that described here, that is, one that captures such a broad expanse of algebraic concepts and thinking practices across the elementary mathematics curriculum. This seems to be a tall order in an already crowded general mathematics curriculum, at least in the US. Our perspective, however, is that early algebra is not an "add-on" to existing school mathematics, but a means to help children think more deeply about that very content (Kaput and Blanton 2005). Early algebra has the potential to embed arithmetic concepts in rich algebraic tasks in ways that can deepen children's understanding of arithmetic concepts. In this sense, early algebra does not introduce a dichotomy in school mathematics (i.e., arithmetic *or* algebra), but is a means by which children—some of whom may already be struggling with arithmetic—can build deep mathematical knowledge with understanding. Our tasks are often designed to highlight this nexus between algebraic and arithmetic thinking by using arithmetic work as a springboard for noticing, representing, and reasoning with structure and relationships in number and operations. Moreover, we aim to facilitate the development of algebraic thinking—and mathematical understanding more broadly— through learning environments that rely on both small-group investigations of open-ended tasks where students represent their ideas in different ways (for example, through drawings, written language, variable notation, and graphs) and rich classroom discourse that supports developing fluency with algebraic concepts and practices.

In this context, we examine next some of the evidence from several studies conducted by our project team that supports the viability of our approach. We look at evidence from two lines of research: quantitative studies conducted in Grades 3–5 (Project LEAP) as well as exploratory studies in Grades K–2 aimed at characterizing the cognitive foundations of children's algebraic thinking at the start of formal schooling. As described earlier, Project LEAP goals included the design of an instructional sequence for Grades 3–5, and we report here on studies addressing its effectiveness. We view the exploratory Grades K–2 work as prerequisite to the kind of systematic design and development that occurred in Project LEAP. Both serve our broader goal of developing a Grades K–5 instructional approach to early algebra education that has been rigorously tested for its ability to develop children's algebraic thinking and their readiness for a formal study of algebra in middle grades.

5.1 Project LEAP: Grade 3 Intervention

In Blanton et al. (2015b) we reported on our first quasi-experimental study designed to measure the effectiveness of the Grades 3–5 instructional sequence developed as part of Project LEAP[8]. We compared the algebra learning of third-grade students who were taught the Grade 3 sequence to students in a demographically and academically comparable control group. Approximately 100 students participated in the study. The Grade 3 sequence used in the intervention consisted of 19 one-hour lessons taught over the course of the school year by a member of our research team. Each lesson involved a preliminary small group activity that either reviewed previously taught concepts or previewed concepts addressed in the upcoming lesson. The remainder of the lesson focused on small group explorations in which students discussed a problem activity, collected and organized their data, looked for relationships, and represented the relationships through words, drawings, or variable notation. This was followed by whole-class discussions that revolved around teacher questioning designed to engage students in discussing their thinking about the generalizations they noticed, the nature of their representations, and why they viewed their observations as valid. Lessons focused on eliciting students' higher order thinking through both written and oral communication.

Control students were taught only their regular mathematics curriculum. All students were given our written, one-hour LEAP algebra assessment as a pre/post measure of shifts in their understanding of core algebraic concepts and practices.

From our analysis of student responses to the pre/post-assessment reported in Blanton et al. (2015b), we found that there were no significant differences between the two groups in terms of overall performance (percent correct) at pretest ($M = 18.22$, $SD = 12.36$ for the experimental group; $M = 14.99$, $SD = 10.58$ for the control group; $F = 2.01$, $p = 0.16$, $d = 0.28$). However, the experimental group showed significantly greater pre-to-post gains than the control group ($M = 65.51$, $SD = 21.01$ for the experimental group; $M = 21.97$, $SD = 15.37$ for the control group at post-assessment; $F = 143.6$, $p < 0.001$, $d = 2.37$). At the item level, the experimental group showed statistically significant pre-to-post gains for all but two of the pre/post-assessment's 19 questions. The control group did not show statistically significant pre-to-post gains on any of the

[8] The LEAP Grades 3–5 instructional sequence and associated assessments are available upon request to Maria_Blanton@terc.edu.

assessment items. These results suggest that, overall, students as early as Grade 3 (approximately 9 years old) can successfully engage with core algebraic thinking concepts and practices over a broad expanse of algebraic ideas—as reflected in the algebra framework used in our approach—far beyond the occasional algebraic concept that they might otherwise see in their regular curriculum. At the same time, the business-as-usual curriculum control students received seemed to do little by way of developing students' algebraic understanding.

Moreover, as we reported in more detail in Blanton et al. (2015b), we also coded students' strategy use in their assessment responses so that we could more closely detail shifts in students' thinking. We found that experimental students exhibited more algebraic approaches to problem solving than did their control peers. This included that experimental students were more likely to interpret the equal sign relationally rather than operationally (Carpenter et al. 2003), correctly solve linear equations using strategies that invoked inverse operations, recognize varying quantities and represent operations on such quantities as algebraic expressions, recognize structural characteristics of equations (e.g., the Commutative Property of Addition) and develop arguments that invoked this structure, and recognize and represent with both words and variable notation relationships between two co-varying quantities.

5.2 Project LEAP: Grades 3–5 Intervention

Given the results of our Grade 3 study, we conducted a second quasi-experimental study with the goal to more extensively test the effectiveness of our Grades 3–5 instructional sequence.[9] In this sequel study, we compared the algebraic thinking of students who participated in a 3-year, longitudinal implementation of our Grades 3–5 instructional sequence to students in more traditional (arithmetic-focused) classrooms. Additionally, we followed these students into Grade 6 in a follow-up study to assess retention of or shifts in their algebra knowledge (no intervention was provided in Grade 6).

Participants (n = 165) in the study were from two schools, one designated control and one designated experimental. One member of our project team taught the 3-year intervention in the designated experimental school, beginning with a Grade 3 cohort and continuing with this cohort for 3 years. Approximately 18 lessons were taught at each of Grades 3–5 as part of students' regular mathematics instruction. Students in both experimental and control schools were assessed at the beginning of Grade 3 (baseline data) and at the end of Grades 3, 4, and 5 using the one-hour, grade-level written algebra assessments developed in our curricular progression (see Section 4.3).

Students' performance (correctness) on common assessment items[10] was compared over time (Grades 3–5) and by group (experimental and control). Results of a two-factor, mixed-design ANOVA showed significant main effects for both experimental condition, $F(1,144)=137.03$, $p<.01$, $h^2=.49$, and grade level, $F(3,432)=736.66$, $p<.01$, $h^2=.78$, as well as a significant interaction between the two, $F(3,432)=70.29$, $p<.01$, $h^2=.15$. Simple main effects tests revealed that there were no significant differences between experimental and control students at baseline (beginning of Grade 3), $F(1,144)=1.46$, $p=.23$. However, experimental students significantly outperformed control students at each subsequent time point: Grade 3 post-test,

[9] See Blanton et al. (2017b) for a more detailed account of this study.

[10] Grade-level assessments contained between 12 and 14 items. Nine of these were common across all assessments.

$F(1,144)=205.88$, $p<.01$; Grade 4, $F(1,144)=99.74$, $p<.01$; and Grade 5, $F(1,144)=103.28$, $p<.01$ (see Figure 1).

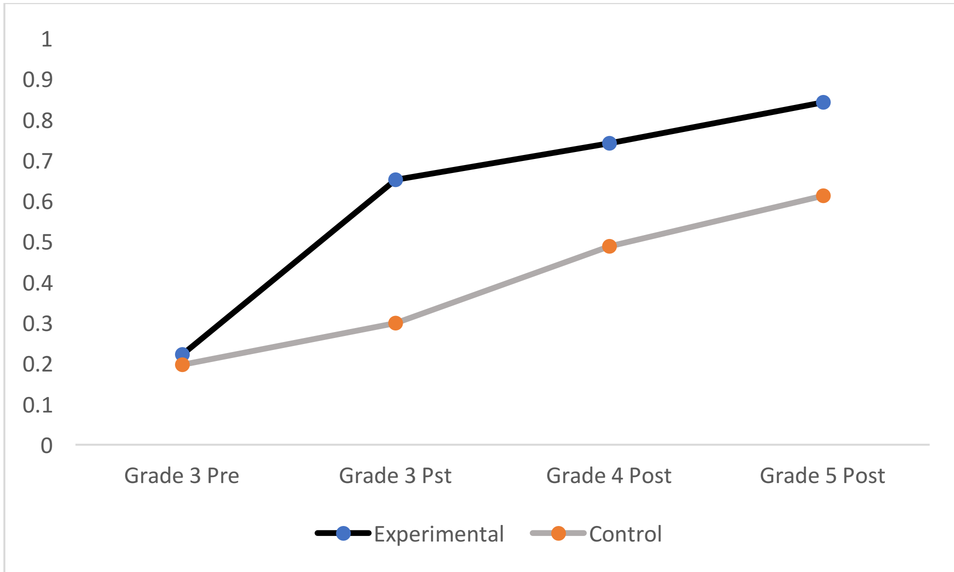

Figure 1. Comparison of overall percent correct on Grades 3–5 common assessment items.

We note that the intervention had the most impact at Grade 3, as indicated by the decreasing rate of performance of experimental students after Grade 3 (although experimental students' performance still improved year to year). We also note that by Grade 4, control students were being introduced to some of the algebraic concepts that were addressed in the intervention as part of their regular classroom instruction. As such, we think it is reasonable that there is a jump in their performance beginning in Grade 4. However, shifts in experimental students' overall performance (correctness) on the Grade 3 pre-assessment to the Grade 5 post-assessment from 22% to 84% offers perhaps even stronger evidence that elementary-aged students' can successfully engage in a broad expanse of algebraic practices and concepts, as reflected in our algebra framework. Moreover, we suggest that the absence of a sustained, multi-year approach to fostering algebraic thinking leaves students significantly less prepared for algebra in middle grades, as indicated by control students' shifts on overall correctness from 20% (Grade 3 pre-assessment) to 61% (Grade 5 post-assessment). It is a positive result that there were shifts in control students' algebraic thinking by Grade 5 and, in our view, this reflects long-term efforts to integrate algebraic thinking into elementary grades. However, the difference in gains for the two groups shows that significant opportunities stand to be missed in current educational practice.

To unpack these results further, we look here at students' performance (correctness) across 4 timepoints—Grade 3 pre/post and Grades 4–5 post—on an item that captures how students were able to represent generalized quantities, an important transition point in the development of algebraic thinking. Although generalizing has rightfully received much attention as the heart of algebraic thinking (Cooper and Warren 2011; Mason 1996), Kaput (2008) argues for the equal importance of representing, or *symbolizing*, a generalization. On this item, students were asked to represent and reason with generalized (varying) quantities in the following item[11]:

> ***Piggy Bank Problem.*** *Tim and Angela each have a piggy bank. They know that their piggy banks each contain the same number of pennies, but they don't know how many. Angela also has 8 pennies in her hand.*
>
> *a.* *How would you represent the number of pennies Tim has?*
>
> *b.* *How would you represent the total number of pennies Angela has?*
>
> *c.* *Angela and Tim combine all of their pennies. How would you represent the number of pennies they have all together?*

Results (see Figure 2) show that experimental students made greater gains in representing Tim's and Angela's numbers of pennies (parts a and b), as well as their combined number of pennies (part c), than did their control peers. We considered a correct response[12] to these items to be a letter to represent Tim's number of pennies (e.g., n), a related algebraic expression for Angela's number of pennies (e.g., $n + 8$), and a related expression such as $n + n + 8$ for the combined number of pennies. Experimental students correctly represented Tim's number of pennies with variable notation (part a) at a rate of 1%, 87%, 83%, and 92% across the 4

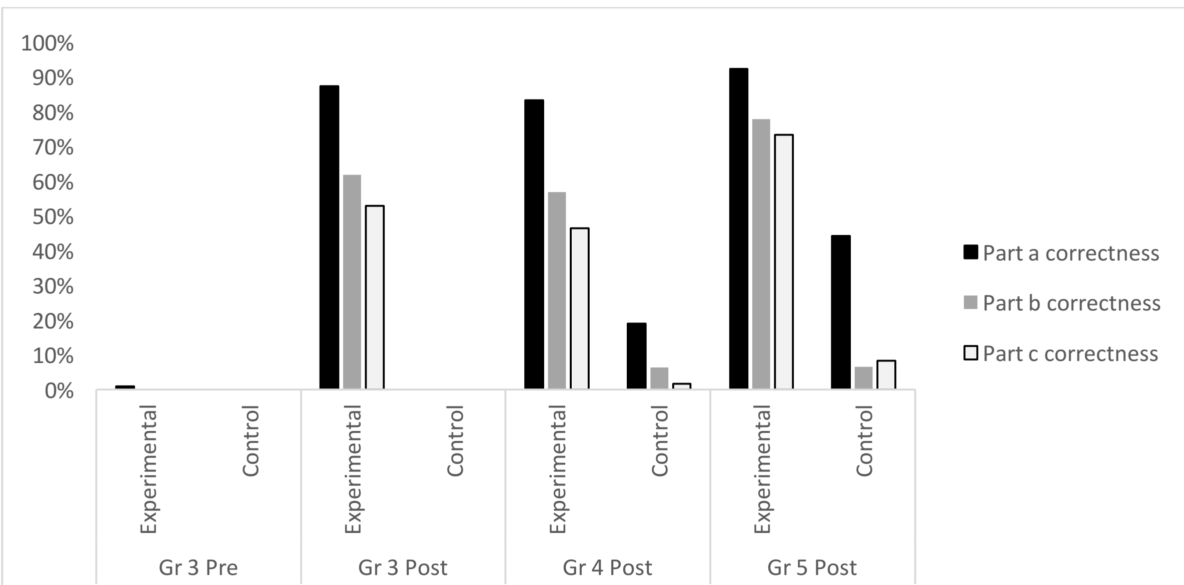

Figure 2. Comparison of experimental and control performance (correctness) on the Piggy Bank Problem, parts a–c.

[11] Adapted from Carraher et al. (2008).

[12] We recognize that a child might give a response such as n, m, and $n + m$, for parts a, b, and c, respectively. In a further analysis of strategy, we considered such responses. However, for overall correctness, we considered only the most stringent case in which students accounted for the fact that Angela and Tim had the same number of pennies in their banks in their representations.

assessments, respectively. By contrast, only 0%, 0%, 19%, and 44% of control students could do so. Students who could not correctly represent Tim's number of pennies with variable notation typically assigned this quantity a numerical value.

Similarly, experimental students made greater gains than control students in representing Angela's number of pennies as an algebraic expression (part b) across the 4 assessments (0%, 62%, 57%, and 78% respectively). Meanwhile, only 0%, 0%, 6%, and 7% of control students could correctly represent Angela's number of pennies across the 4 assessments. Finally, experimental students made greater gains in representing the combined number of pennies with an algebraic expression (part c) across the 4 assessments (0%, 53%, 46%, and 73%, respectively) than did control students, whose overall percent correct was 0%, 0%, 2% and 8% across the 4 assessments, respectively.

We find these results to be compelling for various reasons. First, this is a particularly complex problem for young children because in an arithmetic-saturated experience, they have not learned to "see" and mathematize variable quantities in problem situations (see e.g., Blanton et al. [2015a] for a treatment of progressions in young children's understanding of variable and variable notation.) As such, even a simple task such as representing Tim's number of pennies is often beyond their perceptual field, as indicated by their action of assigning a numerical value to a varying quantity. In our view, a first step in understanding algebraic concepts such as those addressed in the Piggy Bank Problem is learning to perceive and represent a variable quantity (i.e., part a), after which students might notice and represent relationships between quantities (as in parts b and c).

Secondly, these results show that, unlike control students, experimental students were very successful at representing generalized quantities with variable notation. Moreover, experimental students were able to use variable notation in meaningful ways (e.g., they understood that the same letter was to be used to represent the number of pennies in each of Tim's and Angela's bank, since the number of coins was the same but unspecified). This calls into question the conventional wisdom that younger students are not "ready" for variable notation and should use those representational systems that are already available to them—particularly, natural language and drawings—to represent variable quantities, rather than variable notation (e.g., Nathan et al. 2002; Resnick 1982).

Finally, to test the claim that we set at the beginning of the study regarding whether participating in our instructional sequence would impact students' algebraic thinking in middle grades, we followed Grades 3–5 students into middle school and administered our Grade 6 algebra assessment (see Section 4.3) at the end of Grade 6. No intervention was given. We found that the experimental students ($n = 46$) outperformed the control students ($n = 34$), with an overall correctness of 52% (experimental) vs. 44% (control) on this assessment[13] one year after the early algebra intervention ended. In Isler et al. (2017), for example, we found that experimental students remained more successful in generalizing functional relationships and representing them in words and variables than did control students. Experimental students were able to correctly generalize and represent a functional relationship in words and variable notation at a rate of 48% and 65%, respectively, while control students were able to do so only at a rate of 26% and 41%, respectively. Such results suggest that when students experience a broad, sustained approach to early algebra instruction, they are better positioned for success in algebra in middle grades.

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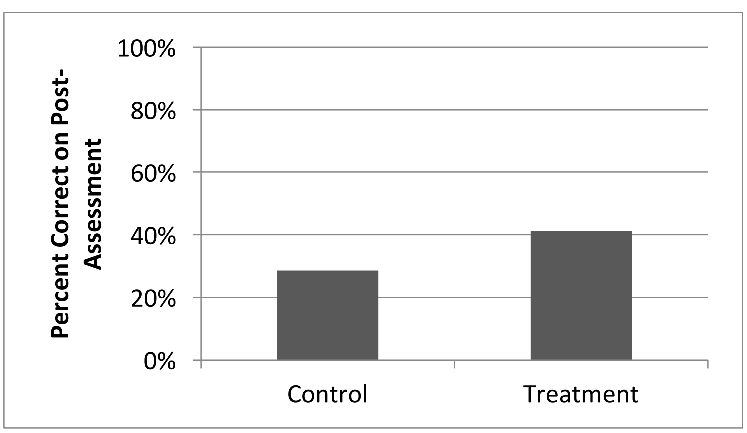
[13] It should be noted that the analysis for Grade 6 data was for all items on the assessment (not just items common with the Grades 3–5 assessments) and that it included new, more difficult items.

5.3 Project LEAP: Examining a Teacher-Led Grades 3–5 Intervention

Findings from our previous Project LEAP studies, summarized above, have led to a longitudinal, randomized study in 46 participating schools where we are currently following a Grade 3 cohort across Grades 3–5 as experimental students receive the intervention and control students receive their regular instruction. A key difference in this study was its experimental design (randomized) and the fact that teachers led the intervention as part of their regular classroom instruction. The utilization of classroom teachers to lead instruction is a core component of testing the efficacy of our intervention. It holds unique challenges that lie in the fidelity with which teachers might implement the sequence across different instructional settings, given their own varied professional experiences. To increase their fidelity of implementation, we provided all participating teachers with long-term professional development to strengthen their knowledge of algebraic concepts and practices, as well as their understanding of students' thinking about these concepts and practices and how to craft classroom discourse that engaged students in dialogue around them.[14]

Results thus far show that, although there was no significant difference between experimental and control groups on the Grade 3 algebra assessment given at pre-test, experimental students significantly outperformed control students ($p < .001$) in overall performance on this assessment administered at post-test (see Figure 3). In particular, participation in LEAP 3 was associated with a 13% increase in post-test score compared to the control group, suggesting that the Grades 3–5 instructional sequence we designed using Kaput's (2008) conceptual framework shows potential to positively change the way children think algebraically in elementary grades and their potential for success in middle grades. We note, however, that improvements in overall performance for Grade 3 experimental students in this teacher-led study are not as robust as those for our previous interventions led by our research team (e.g., see Figure 1). One possible explanation for this difference could be the diverse fidelity with which teachers implemented the intervention as opposed to the fidelity of implementation for a researcher with extensive knowledge of and instructional experience with the intervention.



[14] For of our analysis of teachers' fidelity of implementation, see Cassidy et al. (to appear).

Figure 3. Comparison of Grade 3 students' overall performance (correctness) on the Grade 3 algebra post-test.

5.4 Extending our Work into Earlier Grades

Ultimately, early algebra is intended to be a focus of mathematics curriculum and instruction in the US across all of elementary grades, beginning in kindergarten (NGA and CCSSO 2010). A natural progression of research for us, then, is to consider how the conceptual framework we applied in our Grades 3–5 work might translate into earlier grades. We have initiated exploratory, qualitative studies on this, with the goal of understanding the genesis of algebraic thinking practices in children's thinking in Grades K–2. We provide a brief overview of some of our findings here.

In Blanton et al. (2015a), we provided evidence that Grade 1 (age 6) students participating in an 8-week classroom teaching experiment that focused on functional thinking could generalize, represent, and reason with (linear) functional relationships. As reported in this study, we developed a learning trajectory to describe first-grade participants' thinking about generalizing functional relationships, analyzing data from children's pre-, mid-, and post-instruction interviews. In particular, we identified eight different levels of thinking, ranging from *pre-structural* to *function-as-object*, exhibited by participants as they advanced through the teaching experiment.

At the pre-structural level, children could not describe or even implicitly use any kind of mathematical relationship in talking about function data. At the function-as-object level, some children had progressed to an ability to generalize and represent a functional relationship with words and variable notation and reason with their symbolic rule as an object for exploring novel scenarios. For example, by the end of the teaching experiment some children were able to generalize a relationship between the number of cars in a train and the number of stops the train made, where it was assumed the train picked up two cars at each stop and the engine (the only "car" on the train before the first stop) was not counted.

One child represented this relationship as $R + R = V$ and described R as representing the number of stops and V as representing the number of cars. When asked how the relationship would change if the engine was counted, she noted that she would "just add 1" and represented this as $+1+ R + R = V$. In other words, she was able to reason with her first function rule as an object in order to solve the new problem and did not have to reconstruct a new function table and find a new relationship independently of her original one. In essence, at this advanced level students who were able to reason in this way no longer viewed the original rule as a *process* (Sfard and Linchevski 1994) of operating on numbers but instead were able to transform a function rule as an *object* (Cottrill et al. 1996; Gilmore and Inglis 2008). Moreover, children who exhibited thinking at this level understood boundaries concerning the generality of the relationship and conditions under which the generalization would not hold.

Elsewhere (Blanton et al. 2017a), we reported on how Grade 1 students participating in this study understood variable quantities and variable notation in the context of functional relationships. Again, as students progressed through the teaching experiment we found that their thinking advanced from what we characterized as a *pre-variable/pre-symbolic* level to a *letter as representing variable as mathematical object* level. We argue that at the most primitive level, students did not recognize a variable quantity in a mathematical situation and could not use or did not accept the use of any symbolic notation to represent such a quantity. As students progressed through the sequence, some were ultimately able to use variable notation to represent functional relationships and to reason with these symbolic rules.

In a related study, Brizuela et al. (2015) illustrate the variety of understandings about variable and variable notation held by Grade 1 children, including that (1) variable notation can signify a label or object; (2) variable notation can represent an indeterminate quantity; (3) quantitative relationships can be expressed through the ordinal relationships between letters in the alphabet; and (4) the inclusion of both letters and numbers in a single equation should be avoided. We also observed that these children were able to act on a mathematical expression that includes variable notation as a mathematical object. Our findings illustrate that given the opportunity, even very young children can use variable notation with understanding to express relationships between varying quantities. We argue that the early introduction of variable notation in children's mathematical experiences can offer them opportunities to develop familiarity and fluency with this convention. This raises an interesting question relative to prior research that has documented secondary school students' difficulties with variables and variable notation (e.g., Knuth et al. 2011; Küchemann, 1981) and whether such difficulties might be ameliorated by a sustained introduction to variable and variable notation from the start of formal schooling.

6. Conclusion

Our goal here has been to describe the conceptual framework of our approach to early algebra, how we are enacting that framework through the design of a curricular approach to algebra instruction in the elementary grades, and a brief overview of some of our findings, reported in detail elsewhere, regarding the impact of this approach on children's algebraic thinking. Ultimately, our program of research aims to outline a curricular approach to teaching and learning algebra across Grades K–5 that can positively impact students' readiness for and success in algebra in middle and high school grades. Collectively, our studies contribute evidence to the perspective that elementary-aged children can engage in sophisticated practices of algebraic thinking—generalizing, representing, justifying, and reasoning with mathematical structure and relationships—across a broad set of core content areas involving generalized arithmetic; concepts associated with equivalence, expressions, equations, and inequalities; and functional thinking.

We have found that a research-based, comprehensive early algebra intervention across upper elementary grades (i.e., Grades 3–5) can statistically improve children's algebraic understanding and potentially improve their algebra-readiness for middle grades. Further, we have found that in lower elementary grades students exhibit a capacity for algebraic thinking beyond what we had originally hypothesized as possible. Our observations of children's algebraic thinking have been perhaps most striking in the early elementary grades (particularly, Grades K–1). Indeed, prior to our studies, we assumed that children in these early grades might have even more difficulty with the algebraic concepts with which adolescents so often struggle—for example, the object–quantity confusion associated with variables (McNeil et al. 2010) or the difficulty in shifting students' perspectives away from recursive thinking towards functional thinking (Cooper and Warren 2011). We have found instead that, in these early grades, children are far more able to think algebraically than we anticipated. In our view, providing sustained experiences, *from the start of formal schooling*, with the conceptual approach to early algebra described here holds promise for ameliorating the deeply held difficulties and lack of success that students have historically had with high school algebra in the US.

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