PROSPECTIVE ELEMENTARY TEACHERS DEVELOP IMPROVED NUMBER SENSE IN REASONING ABOUT FRACTION MAGNITUDE

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We report on results of a classroom teaching experiment in a mathematics content course for prospective elementary teachers. A local instruction theory for the development of number sense, which was previously applied to whole-number mental computation, was extended to inform instruction concerning reasoning about fraction magnitude. We found that students' reasoning in fraction comparison tasks improved in several ways. Their performance improved, they became more flexible in their reasoning, and they came to use less conventional and more sophisticated strategies. These changes parallel those that we previously saw around mental computation.

We report on results of the implementation of a local instruction theory (LIT) for number sense development in a mathematics content course for prospective elementary teachers. Our previous research showed that students involved in an earlier teaching experiment developed improved number sense, particularly in terms of flexible mental computation (Whitacre, 2007). The previous research was informed by a conjectured local instruction theory and informed the refinement and elaboration of that local instruction theory (Nickerson & Whitacre, 2010). In a recent iteration of the classroom teaching experiment (CTE), the local instruction theory was extended from the whole-number portion of the course to the rational-number portion. We found that interview participants' reasoning about fraction magnitude improved over the course of the semester. Their performance improved, they became more flexible in their reasoning, and they came to use less conventional and more sophisticated strategies. These changes parallel those that we previously saw around mental computation.

Background

This study represents the latest phase in an ongoing design research effort (Gravemeijer, 1999) concerning the number sense of prospective elementary teachers. Our previous research involved the design and elaboration of a local instruction theory for students' development of number sense with a focus on whole-number mental computation (Nickerson & Whitacre, 2010). In the recent CTE, the local instruction theory was applied to reasoning about fraction magnitude. Our research involves both documenting collective classroom activity and analyzing student learning. Our focus in this paper is on the analysis of interview data.

Previous Research

One area of focus in the number sense literature has been the computational strategies that students use. Good number sense is associated with flexibility, which is exhibited in the use of a variety of computational strategies. In mental computation, inflexibility often manifests in the use of the mental analogues of the standard paper-and-pencil algorithms. While the standard algorithms can be useful, skilled mental calculators tend to select a strategy for an operation based on the particular numbers at hand. Furthermore, the strategies that these individuals use often stray far from standard, as in reformulating computations or rounding and compensating (Carraher, Carraher, & Schlieman, 1987; Greeno, 1991; Hierdsfield & Cooper, 2002; 2004;

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Hope & Sherrill, 1987; Markovits & Sowder, 1994; Reys, Rybolt, Bestgen, & Wyatt, 1982; Reys, Reys, Nohda, & Emori, 1995; Sowder, 1992; Yang, Reys, & Reys, 2009).

In order to teach mathematics effectively, elementary teachers need to understand elementary mathematics deeply (Ball, 1990). However, prospective and practicing elementary teachers often know the procedures of elementary mathematics, but do not understand the material conceptually (Ball, 1990; Ma, 1999). Studies of preservice elementary teachers have found that this population tends to exhibit poor number sense, even after having completed their required college mathematics courses (Tsao, 2005; Yang, 2007; Yang, Reys, & Reys, 2009). In light of these findings, we have focused on improving the number sense of prospective elementary teachers.

In 2005, we conducted a CTE in a mathematics course for prospective elementary teachers. In that study, we focused on mental computation as a microcosm of number sense. Thirteen students participated in pre/post interviews in which they were given story problems to be solved mentally. In addition to coding for the particular mental computation strategies that participants used, we coded these as belonging to more general categories of strategies. We used a scheme of Markovits and Sowder (1994) to categorize participants' strategies as *Standard*, *Transition*, *Nonstandard without Reformulation*, and *Nonstandard with Reformulation*. The essential criterion in this scheme is the extent to which the person's approach is tied to (or departs from) the standard algorithm for the operation. Nonstandard strategies are those that diverge substantially from the standard algorithms. The use of such strategies suggests an understanding of the operation that is not bound to any particular algorithm; these nonstandard strategies are associated with number sense (Markovits & Sowder, 1994; Yang, Reys, & Reys, 2009).

The Standard-to-Nonstandard framework revealed a rather dramatic shift in the strategies used by the 13 interview participants. Participants shifted from using the most standard to the least standard strategies, which suggests that their understanding of the operations moved from being bound to the standard algorithms to being unconstrained by these (Whitacre, 2007; Whitacre & Nickerson, 2006). These results were encouraging and led us to pursue further research concerning the number sense development of prospective elementary teachers.

Local Instruction Theory for Number Sense Development as Applied to Fraction Magnitude

The previous teaching experiment was reflexively related to the development of a local instruction theory. A *local instruction theory* (LIT) consists of "the description of, and rationale for, the envisioned learning route as it relates to a set of instructional activities for a specific topic" (Gravemeijer, 2004, p. 107). We have described elsewhere our local instruction theory for the development of number sense (Nickerson & Whitacre, 2010). Here, we briefly list the three major goals around which this LIT is organized: (1) Students capitalize on opportunities to use number-sensible strategies; (2) Students develop a repertoire of number-sensible strategies; (3) Students develop the ability to reason with models.

We sought to extend the LIT to the rational number domain, with a focus on reasoning about fraction magnitude. This area relates to mental computation in that a variety of strategies can be used, including traditional procedures, as well as nonstandard strategies. Behr, Wachsmuth, Post, and Lesh (1984) touted the importance of reasoning about fraction magnitude as a prerequisite to reasoning meaningfully about operations involving fractions. Fraction estimation and comparison tasks have been used in assessments of students' number sense (Hsu, Yang, & Li, 2001; Reys & Yang, 1998; Yang, 2007).

Our thinking concerning reasoning about fraction magnitude is informed by the framework of Smith (1995). Smith groups fraction comparison strategies into four categories, which he calls *perspectives*. These perspectives serve not only to categorize strategies but also to highlight commonalities in reasoning across groups of strategies. Strategies such as converting to a common denominator or converting to a decimal belong to the *Transform* perspective. They involve transforming one or both fractions in some way in order to facilitate the comparison. One can also compare fractions without performing any sort of transformation. One way to do this is to apply the *Parts* perspective, wherein the fractions are interpreted in terms of parts of a whole. This perspective alone is sufficient for relatively simple cases, such as comparing fractions that have the same numerator or same denominator.

The *Reference Point* perspective involves reasoning about fraction size on the basis of proximity to reference numbers, or *benchmarks* (Parker & Leinhardt, 1995). For example, using the *residual strategy* for comparing fractions, one compares the difference of each fraction from a common benchmark number, typically 1: To compare 7/8 and 6/7, we can notice that 7/8 is 1/8 away from 1, whereas 6/7 is 1/7 away from 1. Since 1/8 is less than 1/7, 7/8 is closer to 1, and therefore larger (Yang, 2007). (Smith refers to this as the *reference point* strategy, as do Behr, et al., 1984.) The *Components* perspective involves making comparisons within or between two fractions, as in coordinating multiplicative comparisons of numerators and denominators. For example, in order to compare 13/60 and 3/16, we can notice that $13 \times 5 = 65 > 60$, whereas $3 \times 5 = 15 < 16$. It follows that 13/60 is greater since its numerator-denominator ratio is less extreme.

In designing instruction, these perspectives informed our decisions relative to tasks, number choices, and anticipated student reasoning. We mapped out the envisioned learning routes described in our LIT in terms of the evolution of these perspectives and of particular strategies within each category.

Although Smith (1995) does not describe the perspectives or particular strategies belonging to his framework in a hierarchical way, we view the Reference Point and Components perspectives as generally more sophisticated categories of reasoning about fraction size. There is support for this in the literature. For example, Yang (2007) considers the residual strategy to be Number sense-based, as opposed to Rule-based. We posit that there is a general correspondence between Smith's perspectives and the Standard-to-Nonstandard framework, described earlier. In particular, the Transform and Parts perspectives correspond more or less to the Standard and Transition categories of strategies, while the Reference Point and Components perspectives correspond to Nonstandard strategies (with or without reformulation). We do not intend by this a one-to-one mapping of categories, but a more general grouping into Standard (including Transition) and Nonstandard.

The general number sense literature, as well specific studies of researchers such as Yang (2007) and Newton (2008), suggested that prospective elementary teachers would come to the first course with limited number sense and would tend to apply standard algorithms for comparing fractions. Pilot interviews that we conducted with preservice elementary teachers who had completed their mathematics content courses confirmed this expectation. In our instructional sequence, we aimed for the more sophisticated strategies to eventually be used by students and established for the class by mathematical argumentation. In particular, we sought to engage students in reasoning about fraction size from Smith's Reference Point and Components perspectives. Tasks were designed and sequenced so as to begin with students' current ways of reasoning and to provide opportunities for reasoning about fraction size in new ways.

Instruction

Topics in the curriculum include quantitative reasoning, place value, meanings for operations, children's thinking, standard and alternative algorithms, representations of rational

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numbers, and operations involving fractions. Our work in the CTE involved identifying in the curriculum particular opportunities to engage students in authentic mental computation and reasoning about fraction size, and to share and justify their strategies. Over time, a shared set of strategies was established via mathematical argumentation. These strategies were given agreed-upon names, and the class maintained a list of strategies with examples of each. Students came to the course with a Parts conception of fractions, and Parts reasoning served as a foundation upon which more sophisticated strategies came to be established. Students also engaged in activities involving placing fraction markers on a string representing a number line, and distance along the number line often featured prominently in students' arguments concerning fraction comparisons.

Methods

This study took place at a large, urban university in the Southwestern United States. The participants in the study were students enrolled in a first mathematics content course of a four-course sequence. There are multiple sections of the course, and a common final exam is used. The second author taught the section of the course in which data was collected.

Seven of the students participated in pre/post interviews concerning their rational number sense. The interview participants were female undergraduates. Participants were asked to evaluate the relative sizes of pairs of fractions, amongst other tasks. Nine pairs of fractions were presented, one at a time. The particular pairs of fractions that participants were asked to compare appear in Figure 1. The same pairs of fractions were used in both interviews. The fractions were presented visually, and participants were asked to read them aloud. Participants solved the tasks mentally and explained their reasoning verbally, as shown in Figure 2. Participants were not allowed to do any written work for this portion of the interviews.

2/8	3/8
3/4	3/5
6/7	7/8
14/13	13/12
8/24	13/39
13/60	3/16
7/28	13/50
2/7	12/43
35/832	37/834

Figure 1. Fraction comparison tasks

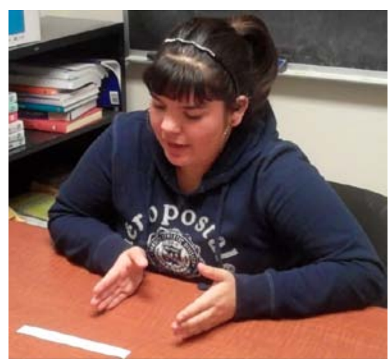

Figure 2. Maricela comparing 13/60 with 3/16 in her second interview

We modified the framework of Smith (1995) for use as a coding scheme in order to account for differences in student population, the particular set of tasks that we used, and the foci of our research. Certain strategies were removed from the scheme because they never occurred in our data set, and there were pairs of strategy codes that we collapsed into a single code. The refined scheme was used to code the pre/post data.

In addition to coding for strategies and perspectives, we coded responses as Valid or Invalid. This is a researcher's assessment of the mathematical validity of a participants' strategy. Fraction comparisons are multiple-choice tasks with only three options (one fraction is larger, the other fraction is larger, or the two are equal). As a result, it is not uncommon for students to give correct answers on the basis of invalid or unclear approaches. We were interested in identifying valid strategies that led to correct responses. Thus, we also coded responses for correctness. In analyzing the pre/post data, we focused on those responses that were Valid and Correct (VC). We also coded students' strategies as Standard or Nonstandard, as discussed earlier.

As an assessment of change in performance, we compared the number of VC responses pre and post for each student. To assess change in flexibility, we compared the number of distinct VC strategies used by each student pre and post. Finally, we compared the numbers of Standard and Nonstandard VC responses pre and post.

Results

A comparison of interview participants' pre/post responses reveals improved performance and increase flexibility. Furthermore, participants' reasoning shifted from predominantly employing Standard strategies to a more balanced range of Standard and Nonstandard strategies. We discuss the case of one particular participant in more detail.

For six of the seven interview participants, the numbers of VC responses increased from the first to the second interview. The mean VC score increased from 5.86 to 7.7 (of a total of 9 responses). The additional VC responses were often the result of new strategies used by the participants. The number of distinct VC strategies used also increased from the first to the second interview for six of the seven participants. The mean number of distinct strategies used increased from 4.86 to 7.57. Thus, the participants used VC strategies more often in the second interview, *and* they used a wider variety of VC strategies. These data appear in the tables below.

Student	VC Pre	VC Post
Angela	4	8
Brandy	4	7
Maricela	7	8
Nancy	8	9
Trina	7	8
Valerie	5	8
Zelda	6	6
Mean	*5.86*	*7.7*

Table 1. Counts of Valid-Correct Responses Pre and Post

Student	Distinct VC Pre	Distinct VC Post
Angela	4	7
Brandy	4	7
Maricela	3	8
Nancy	6	9
Trina	6	8
Valerie	5	8
Zelda	6	6
Mean	*4.86*	*7.57*

Table 2. Numbers of Distinct Valid-Correct Strategies Pre and Post

Maricela's Transformation from Inflexible to Flexible

Maricela was particularly inflexible in her reasoning about fraction magnitude in the first

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interview. For seven of the nine comparisons, she attempted to solve by finding a common denominator. In some cases, this approach was manageable and led her to the correct answer. However, she also attempted this approach for comparisons for which it was extremely unwieldy, including to compare 35/832 and 37/834. Such conspicuous inflexibility suggests that her reasoning about fraction magnitude was bound to the standard algorithms for comparing fractions. Even though pencil and paper were unavailable to her, she entertained no alternative other than attempting to mentally compute and retain multiple multi-digit products.

In her second interview, Maricela used a different strategy for each of the nine comparison tasks, and eight of these were VC responses. In her first interview, she had compared 13/60 with 3/16 by converting to a common denominator. In her second interview, she compared these by comparing their distance from 1/4. We coded her primary strategy as *Distance Below* under the Reference Point perspective. She identified a benchmark fraction that would be useful for comparison. This required an initial estimation of the size of the fractions as being roughly close to that benchmark of 1/4. She then identified fractions equivalent to 1/4 but with denominators of 60 and 16, which facilitated her distance comparison by making it easy to find differences. She identified the distances from 1/4 as 2/60 and 1/16. She simplified 2/60 to a unit fraction, which then enabled her to compare the "gaps" of 1/16 and 1/30. She stated that 1/16 was the larger of these. She then correctly concluded that 13/60 was larger than 3/16 because it was closer to 1/4. The reasoning that Maricela displayed here contrasts starkly with her very procedural approach to this and other fraction comparison tasks in her first interview.

Shift from Standard to Nonstandard

Maricela's contrasting pre/post responses suggest a shift in perspectives. Her responses to eight of the nine comparison tasks in the first interview reflected a Transform perspective: She approached the tasks by attempting to apply a procedure for converting one or both fractions to an equivalent form in order to make the comparison. In her second interview, Maricela's responses reflected a range of perspectives. Four of her nine responses were coded as Parts, two as Reference Point, two as Components, and only one as Transform.

This shift in perspectives was also a trend across the interview participants. In the first interview, 30 of the 41 VC responses involved a Standard strategy, such as converting to a common denominator. In the second interview, there were 54 VC responses, and 30 of these involved Nonstandard strategies. (See Figure 3.) We note that the responses in the first interview were weighted largely toward Standard, which reflects a lack of flexibility. In the second interview, Nonstandard responses outnumbered Standard ones, but by a relatively small margin. We view this more balanced set of strategies used as a desirable picture. There is nothing wrong with the Standard strategies in principle. In fact, four of the nine comparison tasks that we used lend themselves well to Standard approaches. Thus, we would expect flexible, skilled individuals to use such approaches approximately as frequently as the interview participants did for the given set of tasks. In fact, exactly 24 of 54, or 4/9, of the VC responses were Standard.

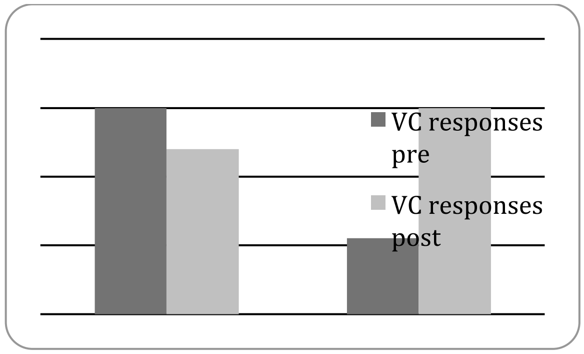

Figure 3. Shift from Standard to Nonstandard strategies

Figure 3 actually provides a concise summary of the interview results: We see the shift from Standard to Nonstandard, the increase in VC responses demonstrates improved performance, and the increase in Nonstandard VC responses coincides with improved flexibility.

Conclusion

We began this study with the idea of extending a local instruction theory for the development of number sense to the domain of rational numbers. Our focus shifted from whole-number mental computation to fraction comparison tasks. Smith's framework informed our thinking about strategies and perspectives involved in comparing fractions. The research literature, together with our pilot interviews, enabled us to anticipate prospective elementary teachers' initial reasoning about fraction magnitude. Our pre-instruction interviews enabled us to refine our expectations. We designed the instructional sequence in terms of students moving from the Transform and Parts perspectives toward the Reference Point and Components perspectives. Specific instructional activities were crafted with this envisioned learning route in mind.

Results from pre/post interviews with seven of the students show that the participants' performance and flexibility in comparing fractions improved, and that they shifted from using predominantly Standard strategies to a balance of both Standard and Nonstandard. These results provide evidence that the prospective elementary teachers developed improved rational number sense. The results also parallel those that we saw in a previous study in which the LIT was applied to mental computation. Taken together, these findings suggest to us that the LIT does serve its purpose in shaping instruction design in such a way that substantial number sense development occurs. The fact that these results have been achieved with prospective elementary teachers is especially significant since the literature tells us that this student population tends to have poor number sense, even after having completed their mathematics content courses.

Having found further evidence that the implementation of our local instruction theory supports number sense development, the current phase in our ongoing design research seeks to better understand *how* that development occurs by analyzing collective classroom activity.

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