MATHEMATICAL KNOWLEDGE FOR TEACHING OF STUDENT TEACHERS AND ITS ENHANCEMENT THROUGH A SPECIAL FINAL COURSE

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In this article we describe the results of a special final course, at the main teachers' college in Mexico, which had two related main objectives: one was to find out the Mathematical Knowledge for Teaching (MKT) held by student teachers (ST) at the end of their instructional preparation. The other was to discern ways to improve this knowledge and to document the changes observed. In teachers' colleges in Mexico, math contents and pedagogical ideas are taught separately, so we aimed to help student teachers to integrate these. The analysis showed that their knowledge is mainly instrumental but that through discussions and reflection about the main issues, they were able fairly quickly to attain a significant improvement in all the contents included. Moreover, they also showed changes on some of their views about math and its teaching.

Introduction and Theoretical Framework

For two decades in Mexico, great effort had been placed on improving education at the basic levels. Study programs have been changed, text books have been replaced and computers have been brought to help out, but internal and external evaluations have shown at best very small improvements in students' achievements. The most likely explanation, supported by research in math education (Adler et al., 2005) is that a very important element has been overlooked: teachers' professional development. With new principles, standards and approaches brought everyday into education, the preparation of teachers had become even more crucial.

A very important sector of teachers, which also has to be taken into account, are the future teachers being prepared in the different pedagogical schools and colleges. Ponte and Chapman (2008) give an overview of the studies carried out with student teachers (ST) about their math knowledge, their teaching knowledge and their development.

A common practice in teachers' instruction in Mexico is to separate content and pedagogy in different courses, with the assumption that the future teachers will be able to integrate them in their practice. However, this and the formal procedural orientation of their math courses, leave the ST with limited skills and inadequate conceptual understanding. Thus, in the main teachers' college in Mexico for secondary education (Escuela Normal Superior de Mexico), it was felt necessary to introduce a special course at the end of their training that would give the ST an opportunity to build on their previous knowledge and to restructure their conceptions through group discussions and reflection. Within this context we initiated a research study to find out the general knowledge held by ST at the end of their instructional preparation and to assess to what extent their pedagogical content knowledge in several topics was modified by this didactical intervention. We also recognized the importance and close relationship between conceptions, beliefs and practice (Thompson, 1984; Leder, et al. 2002) and although it won't be described in

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here, another portion of this project looked into these issues.

Pedagogical Content Knowledge (Shulman, 1987) refers to a complex mixture with many components like content, pedagogy, organization of topics and problems, student conceptions, models, representations, activities, curriculum, etc. Some facets of this teachers' knowledge are more closely related to the mathematical content, like knowing the structure and connections of mathematical concepts and procedures, deconstructing one's own knowledge or understanding students' methods of solution. Ball and Bass (2000) associated this special knowledge with the term: Mathematical Knowledge for Teaching (MKT). Hill and Ball (2004) continue developing the concept of MKT and describe another of its components: profound math knowledge, which is a specialized knowledge that helps teachers understand and plan their classroom activities.

Professional development programs seek, in various ways, to enhance teachers' practices and therefore students' competence. Some are centered on pedagogical ideas, others on mathematical content and some others on the mix, Pedagogical Content Knowledge (PCK). Our orientation is along this last line, with an emphasis on MKT and following the view of many researchers (Ponte and Chapman, 2006) who stress that teachers' training should be connected to their practices, using the same tasks, materials and techniques that could be used in their classrooms.

Based on different frameworks and methods of inquiry, there have been a number of research studies connected to teachers' professional development projects in different countries. Amato (2006), within a mathematics teaching course for student teachers, conducted a study to improve their relational understanding (Skemp, 1976) of fractions, by playing games. In a study investigating the Pedagogical Content Knowledge (PCK) of elementary school teachers in the topic of decimals, Chick, et al. (2006) proposed a framework with three categories for their analysis: 1. *Clearly PCK*; 2. *Content Knowledge in a Pedagogical Context* and 3. *Pedagogical Knowledge in a Content Context*. Like several other authors, Seago and Goldsmith (2006) studied the possibility of using classroom artifacts like students' work and classroom videos to assess and promote MKT. Also, in a collaborative action research, Cooper, et al. (2006) uncovered some characteristics of instructional interactions that lead to positive results in students' learning.

In an article about cognitively guided instruction, Carpenter, et al. (2000) stressed the importance of teachers' knowledge about the mathematical thinking of children. The authors identified four levels of teachers' beliefs that correlate with their mode of instruction: I. They believe that math has to be taught explicitly and therefore they show procedures and ask the students to practice them. II. They start to question this explicit mode and therefore they give to the students some opportunity to solve problems by themselves. III. They believe that students can have their own strategies so they provide problems and the students report their solutions. IV. Teaching becomes more flexible, with the teacher learning from his students' productions and adapting his instruction to this knowledge.

Since we were interested in assessing the MKT, we based our analysis on the framework given by Ball, et al. (2008), who divides this knowledge into three main domains:

A. *Specialized content knowledge* is the math knowledge and skill needed almost exclusively for teaching. The teacher requires supplementary math knowledge in the multiple activities of his practice. Among other things, he must have a profound understanding of the fundamental concepts of each of the topics, knowing not only 'how' but also 'why'. In addition, he should be able to unpack math ideas to make them more visible to others. Moreover, he should know the connections of different concepts and topics and relations with other subjects.

B. *Knowledge of content and teaching* consists of knowing about teaching math. This knowledge is required in the planning, design and instruction in the classroom work to answer questions like: What would be an appropriate sequence of teaching? What example would illustrate this or that? What classroom methods should be followed? What representations are adequate? Each of these tasks requires a combination of specific math understanding and pedagogical ideas.

C. *Knowledge of content and students* combines knowing about students and math. This knowledge is related to students' thinking, strategies, difficulties and misconceptions. It is needed to infer and evaluate what the students say or do, their methods, their solutions, etc. These tasks require a blend of a specific math understanding and a familiarity with students' thinking.

Methodology

The special course was given during the last semester of formal courses to a group of 21 students in a teacher's college in Mexico City (ENSM – Escuela Normal Superior de Mexico), preparing them for teaching at the secondary school level. Each six hour session of the 16 given was divided into two thirds of selected contents and one third of additional pedagogical elements. In this article we will describe only the selected contents section of the course.

Before the course, we asked the ST to mention math topics from their previous courses which they found difficult or they think to be a source of conflict to students. Among the most cited answers were: fractions, decimals, mental calculation, algebra, variables, functions, probability… Thus, we decided to form three sequential blocks for discussion in the course: 1) Fractions and decimals, 2) Mental calculation and estimation and 3) Variables and functions.

Professional development programs use a variety of resources to make teachers reflect on the ideas involved. For example, Borko and collaborators (2008) suggested video analysis to motivate discussions between teachers and Chamberlin (2003) employed written students' productions. In our study however, our instruments for analysis and means of motivating reflection were a series of questionnaires and further inquiries during the sessions.

All the 16 sessions were audio taped for their analysis. At the beginning of each of the three blocks and also in most of the sessions, a questionnaire or a worksheet was handed out (or previously given to answer at home) with the objective of making the ST reflect about some important ideas and concepts of that particular content. This also had the purpose of steering their thinking about the three main domains of MKT described above. From the research point of view, their written answers gave us a small window of their knowledge. Then a full class discussion took place to argue their own ideas and to hear and evaluate the others'. Because of the dual character of the course, for the most part, we let them express their ideas and encourage interaction between them. In the latter sessions of each block, we gave them notes (taken home to read) synthesizing some important ideas of the topic being covered and then the ST discussed those ideas further. This second phase gave us another window of their knowledge and of the possible changes brought about. The main sources for these notes were research papers on mathematics education in each subject. The pedagogical support for the course was taken also from the research literature, for example Schwartz, et al. (2006), Chick and Baker (2005) and McDonough and Clarke (2003).

Results

To illustrate the two ways data was collected, here we will describe 1) the results of the questionnaire applied before and after the six sessions of the block of fractions and decimals and

2) the results of the observations of the four sessions of the block of mental calculation and estimation (the other block not discussed shared similar conclusions).

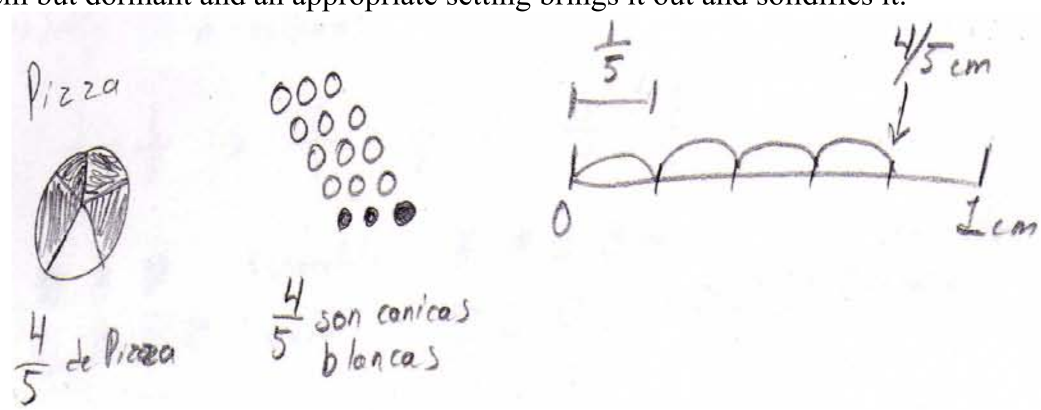
Results of the Questionnaire Applied Before and After the Block of Fractions and Decimals

The questionnaire consisted of 6 items (some with two parts). Here we show three of the six questions which seem representative (from the 21 ST, 18 took the initial questionnaire and 19 took the final).

Item 2. Consider the fraction 4/5. a) Describe some of its different meanings. b) Represent it in different ways.

Part a): *In the initial questionnaire*, 16 of the ST wrote only one or two meanings, without explanations: For example, "Four parts of five.", "Four over five.", "4/5 = .8", "Four of five.", "Four is to five. Four fifths." This exhibited a poorly established knowledge, extracted from what they vaguely remembered. *In the final questionnaire,* 15 of the ST gave much more complete answers, adding also the associated meaning. One short answer was: "As quotient → four over five; as ratio → four to five; as part-whole → four fifths…" Of course, this means that they learnt something, but the value we see in it is that, by briefly exposing them and letting them reflect on these ideas, the ST were able to demonstrate a more sound knowledge.

Part b): This is an important question that shows the amplitude and flexibility with different representations. *In the initial questionnaire*, 10 of the ST represented the fraction only as part-whole of continuous sets like rectangles and circles; 7 others used also part-whole in both types of sets (three of them include a third representation: an equivalent fraction, the numerical line or as a ratio). *In the final questionnaire,* we observed a multiplicity of answers. All of the ST employed at least three different representations like the one shown below in the figure. Five of them supported their answers with well elaborated explanations. Here we observe a richer *knowledge of content and teaching.* Again, this shows that much of this knowledge is held by them but dormant and an appropriate setting brings it out and solidifies it.



Item 4. a) Give a problem illustrating the operation: 1 ½ divided by 3. Explain its solution. b) Give a problem illustrating the operation: 1 ½ divided by ½. Explain its solution.

Part a): *In the initial questionnaire*, most of the ST gave a partitive context when posing the problem: "Juan wants to divide 1 ½ chocolate bars in equal parts for his 3 children. ¿How much would each one gets?" *In the final questionnaire,* we observed basically the same, but their symbolic solutions are more frequently supported with graphical representations and with more congruent explanations.

Part b): This obviously was more revealing than the first part. *In the initial questionnaire*, only 5 of the ST posed a correct problem, based on the interpretation "how many times it fits." For example: "I have a soda with 1 ½ liters and a glass of a ½ a liter. How many times will I be able to fill the glass without any leftover?" Another 6 ST posed an incorrect problem, either forcing the result of 3 of the original operation, not even using the numbers contained in it: "A car need gas every 50 km. How many times we have to stop for a trip of 150 km?" or posing instead the operation 1 ½ divided by 2: "If you have a chocolate bar and half of another, and we want to divide it between 2 people, how much does each one get?" The other 7 ST didn't answer. We can characterize this as a deficient *specialized content knowledge. In the final questionnaire,* more than half of the ST gave a correct problem, although we still observe the same two incorrect thinking mentioned. This seems to be a hard notion for them.

Item 6. Consider the numbers, 0.245, 0.2 and 0.1089. a) If a student says that 0.1089 is the biggest, what do you believe he is thinking? b) How would you show him which is bigger?

Part a): *In the initial questionnaire*, there were two types of similar but not equivalent answers; 11 of the ST mentioned that the decimal point was ignored and 7 others write that the student looks at the number of digits. This shows some *knowledge of content and students. The final questionnaire* showed similar results.

Part b): *In the initial questionnaire*, 7 of the ST drew a diagram showing the position value of each digit, 5 only proposed the use of the numeric line, another showed sequences of numbers without explanation and the rest didn't answer. This demonstrates a reduced *knowledge of content and teaching.* The main difference between the two questionnaires is that in the initial one, we observe mainly technical explanations based on the separation of the numbers into its digits, but in the *final questionnaire* there was an attempt to give semantic descriptions and more extended explanations. For example, one of them wrote: "In a table of positional values indicating integers, tenths, hundredths…" Shows the table and continues "We observe in the tenths, which is bigger, and see that in two numbers they are the same, so we compare the hundredths…" In general, they revealed afterwards a greater capacity for instruction.

Observations of the Sessions of the Block of Mental Calculation and Estimation

In the four sessions of mental calculation and estimation, the three issues addressed through diverse activities were: i) characteristics; ii) strategies; iii) comparison with the traditional algorithms. In the next subsections we will described the knowledge observed from the ST and their changes, according to the three main domains of the MKT proposed in the theoretical framework.

A. Specialized content knowledge. At the beginning of this block the ST were given three questions about mental calculation and then they argued their answers with the whole group. To the question "*Give two examples that show how people use mental calculation in real life.*" the great majority had difficulties identifying situations. There was a clear confusion between mental calculation and estimation since most of the examples were calling for estimations like: "In the kitchen mental calculation is used to estimate portions…" "Time is a factor that we estimate constantly…" "When we go shopping we perform estimations…" When they were told that in mental calculation we expect an exact answer, they gave very vague examples related to some operation and frequently mistaken like "When you multiply by 99 you add 1 to be 100 and then you subtract the result by 1." or "Rounding quantities, for example $25.65 × 4, can be converted to 25.50." Thus we observed a very small *specialized content knowledge*. The ST claimed that "In our classes they don't develop abilities for these subjects." and pointed out that the only strategies they know, they learnt as rules of memorization.

To the next question formulated to the ST "*Which strategies do you know of mental calculation?*" their responses were only a list of names like: "counting", "rounding", "basic operations", decomposition", "chopping" and "multiples of 10". In the follow up discussion, we

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requested an example illustrating the strategies, but the ST claimed they didn't know how to apply them. Only two ST gave an example: One was: "Decomposition: For 35 + 48 =, first add 30 and 40 to get 70 and the 5 and 8 to get 13. To finish we make the operation 70 + 13 = 83." and the other was, according to him about "counting", but was in effect another example of decomposition. We can appreciate again a very limited knowledge of this subject, with one single strategy known (decomposition) which is the closest in form to the conventional algorithms.

Later on, we applied a worksheet with 10 simple operations where the ST were asked to describe the procedures that they could follow to solve them. More than half of the responses were mental applications of the conventional algorithms. Another 35% were answered by applying a decomposition strategy (the way to differentiate between this one and the algorithms is that in the former the value of the quantities are preserved but in the later, only the digits are handled). Another strategy appearing in approximately 10% of the cases, it is called "by steps". To give an example, for the division 400÷25, one of the ST wrote "4×25=100; 100×4=400; 4×4=16". Rounding appeared only in 4% of the responses although a few operations were designed explicitly to apply this strategy. For example, for 37+46, one ST wrote "40+46=86, 86-3=83"; for 87÷3, another ST wrote "90÷3=30, 3÷3=1, 30-1=29" and for 693÷7, yet another ST wrote "700÷7=100, 7÷7=1, 100-1=99". So not only the ST didn't have a clear set of concepts but their familiarity with strategies was very poor.

Before the second session on mental calculation, the ST were given reading materials on the subject. With this new information, they were able to bring out many strategies during the oral exercises proposed. For example, for the subtraction 56–18, one proposed: "To 18, I take away 2 and to 56 I take away 16 to get me 40. Then I take away the other 2 to give 38" (although hard to follow, this is a correct strategy, using compatible numbers); another followed rounding of the 56 with compensation: "I add 56+4 to get 60 and add 4 to 18 to get to 22. Subtracting these we get 38" and a third one also used rounding but of the 18 with compensation: "I would add 2 to both to get 58–20 = 38". In this same exercise, we observed also decomposition and "by steps" strategies. We see once again here that the ST were capable of applying very rapidly the different strategies shown in the readings and developing their knowledge with the interactions with others. So what was hindering them was a lack of knowledge. As one of them expressed, "Now we have the strategies and know how to use them. We only need to direct them to each exercise."

On to the sessions on estimation, we observed too that the ST' initial conceptions were quite vague. They describe estimation as "It is something approximated." "It is what we use to approximate large quantities." Afterwards, the ST were given a worksheet consisting of 5 exercises related to this topic (which we will refer below as I, II, III, IV and V.)

In items I and II an operation (374 + 421 + 339 + 472 and 1797.50 – 635.10) was given to the ST to estimate the result and give an appropriate context of estimation. In items III and IV, the ST were given a problem (one of addition and one of multiplication) to estimate the result and to explain their solution. In the discussion that followed, they heard others' responses and learned from them. We observed, in general, a significant growth in the variety of their strategies, applying rounding of various types (with one or two significant figures or to units, tenths…) and translations like 400×4 for the first sum.

B. Knowledge of content and teaching. Another question posed in the initial session was: "*How would you teach mental calculation?*" The ST' answers were all along these lines "With a lot of exercises so the students practice their calculations." "Through fast operations without using paper." We observed several misconceptions and inappropriate approaches to the teaching of mental calculation like repetition and memory as opposed to reasoning and the construction of strategies. Furthermore, two of the ST indicated that mental calculation is something that cannot be taught: "It is something used outside the classroom and there is no way to instruct it." In general this shows a poorly founded knowledge for teaching.

In the next session, after reading the materials given, they were requested to think of advantages and disadvantages of mental calculation and the conventional algorithms. Here we observed that their initial views changed considerably. For the algorithms they mentioned: "They are not used in higher courses because the calculator is used, not to lose time." "The numbers lose their meaning." "They are general, but we don't have a feeling if it is right or not." "About mental calculation they stated that: "An advantage is that each one looks for his strategy and applies it." "It develops a numerical sense, for their studies and their life." The progress of the ST illustrated here is in their motivation to teach mental calculation and in knowing a reasonable approach for teaching it. As one of them stated "The development of mental calculations is not only to give the steps to follow but we should base our instruction on strategies so they can decide which to follow."

After this, we questioned the ST about the differences of mental calculation and estimation. Two of their answers were: "there is only one exact answer but many reasonable estimations." "In estimation we should use much bigger numbers." We can appreciate that they showed an increased understanding of these subjects for teaching.

C. Knowledge of content and students. In exercises III and IV mentioned before, we added in each, a possible student's solution. We requested the ST that they would explain the solution and also evaluate it. For example, question III "*For the construction of a special classroom, the school has to collect \$6,274 pesos. a) About how much does each of the 28 groups have to contribute? b) If a student writes: 6000÷20=300, explain and evaluate his answer."* Some of their explanations were "The student chopped. He didn't care if the last digits were 6, 7, 8 or 9." "He must take into account a smaller quantity to drop it." We appreciate here some dislike for the 'radical' way to carry out the estimation given. They showed in general very little tolerance for approximations. This is also shown in their evaluations of the estimation given: "this is correct although the error is very big." "It is right although his approximations were very far off."

Conclusions

The knowledge that a teacher should hold is very extensive and it has been characterized by what it is call MKT, which is the mathematical knowledge useful in the teacher practice, strongly conceptual in nature.

However, in many teachers' schools, math and pedagogy are separated in different compartments. ST in their last year of formal learning have a knowledge of mathematics and mathematics teaching, but our observations showed that the former is somewhat limited and forgotten, based mostly on procedural understanding. The latter is composed of conceptions formed through their own experiences.

In general, we observe important deficiencies in the three domains of the MKT. In the *specialized knowledge*, there was a lack of a variety of strategies of solution and the ability to unfold procedures. With respect to *knowledge for teaching*, they had a very small array of representations and illustrations. Their *knowledge of students* was based on their intuitions.

In this paper we described two rather different situations. In the case of fractions and decimals, the ST already received in previous courses, the same topics we covered in our sessions. In the case of mental calculation and estimation, although it is included in their programs of study, many times it was overlooked or the approach given is very mechanical without looking at strategies. In both cases, we observed a weak knowledge of the subjects. However, in the case of fractions and decimals, the six sessions given were able to bring back and solidify the main concepts and in the case of mental calculation and estimation the four sessions were able to provide a good background for further development. Furthermore, the ST learnt to ask themselves the proper questions about their practices within the three domains of MKT.

These results strongly suggest that it would be a very favorable strategy in teachers' colleges to implement a short special course at the end of the formal learning like the one described here, with the objective of ST discussing and reflecting about their strengths and weaknesses in some core subject matter. Currently in Mexico, training of ST is done in classrooms with in-service teachers. However this does not recuperate and refresh their knowledge and practices are often perpetuated by the same traditional ideas.

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