TRUE OR FALSE? PRE-SERVICE SECONDARY MATHEMATICS TEACHERS' STRATEGIES FOR EVALUATING STATEMENTS

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To implement current reforms regarding proof and reasoning in secondary school mathematics successfully, pre-service secondary mathematics teachers must have adequate understandings of concepts across mathematical domains for their future teaching. This paper examined the strategies that pre-service secondary mathematics teachers had for evaluating statements in the domains of algebra, analysis, geometry, and number theory. The results suggest that most pre-service teachers rely on examples and their (partially) correct understandings of mathematical facts when determining the statement's veracity. The results also suggest that the majority of pre-service teachers accurately evaluate the validity of statements.

Introduction

Reasoning and proof have been receiving an increasing level of attention in mathematics because they are considered important components of understanding concepts mathematically. Recent reform documents from the National Council of Teachers of Mathematics (NCTM, 2000) and the Mathematical Association of America (MAA, 2004) have placed emphasis on teaching and learning about verifying statements in secondary school mathematics and undergraduate mathematics. The *Principles and Standards for School Mathematics* (NCTM, 2000) suggests that students should be able to evaluate conjectures by the end of secondary school. The *Undergraduate Programs and Courses in the Mathematical Sciences: CUPM Curriculum Guide* (MAA, 2004) recommends that undergraduate students need to "learn a variety of ways to determine the truth or falsity of conjectures" (p. 45). Research investigating secondary students' and pre-service secondary mathematics teachers' abilities to decide the truth and falsity of statements, however, suggests that many students and teachers have difficulty doing such tasks (Hoyles & Kuchemann, 2002; Ko & Knuth, 2009; Riley, 2003). Hoyles and Kuchemann (2002) found that 40% of 1,984 eight-grade students inaccurately determined the geometric conjecture's veracity. Riley's (2003) result indicated that roughly 57% of 23 prospective secondary mathematics teachers believed that a false statement in geometry was true. Ko and Knuth (2009) reached a similar finding, reporting that 20% of 35 pre-service secondary mathematics teachers who answered believed one false statement about differentiation to be true.

The aforementioned findings should come as no surprise because students are usually asked to prove a true statement rather than to disprove a false one in the mathematics classroom (Buchbinder & Zaslavsky, 2007). Under such a learning environment, it is likely that most students are easily convinced that mathematical propositions are true (Smith, 2006). Limited research has documented evidence that undergraduate students might rely on examples, deductive reasoning, both example-based reasoning and deductive reasoning, or their familiarity with content when asked to evaluate the statement to be true or false (Gibson, 1998; Goetting, 1995; Weber, 2009). No studies to date have investigated the mathematical thinking behind the strategies pre-service secondary mathematics teachers use to determine the validity of statements in different mathematical domains. Such practices are particularly important for secondary mathematics education majors, as they need to understand the conventions of proving in a variety of domains to implement current reform recommendations about reasoning and proving in secondary school mathematics. Thus, the main purpose of this study was to examine strategies

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pre-service secondary mathematics teachers had for evaluating various statements in the domains of algebra, analysis, geometry, and number theory. This study was guided by two research questions: (1) In what ways did pre-service secondary mathematics teachers verify each given conjecture to be true or false? and (2) Which strategy did lead pre-service secondary mathematics teachers to make a correct determination of the statement's veracity?

Theoretical Framework

Evaluating the truth and falsity of statements accurately is a complex problem-solving process, as individuals should have adequate understandings of mathematical concepts related to problems and be able to apply such knowledge flexibly. Although individuals need to be able to correctly decide the truth or falsity of a given proposition before constructing a proof or generating a counterexample, such a decision really depends on whether or not individuals believe the statement to be true or false. When evaluating purported statements, some individuals are inclined to base their determination on example-based reasoning strategies. Moreover, they tend to use random, general, or specific examples related to statements during the processes of verifying the statement's validity (Alcock & Inglis, 2008; Gibson, 1998; Goetting, 1995; Harel & Sowder, 1998).

When asked to evaluate the validity of statements, others use their understandings of true known definitions, theorems, or axioms involved in problems (Alcock & Inglis, 2008) or start constructing a proof and then find a counterexample if they get stuck in the proof (Weber, 2009). Still others make judgments based on their past memories of similar conjectures, so they begin producing a proof, looking for a counterexample, or testing a couple of numbers and then making an attempt at a proof (Goetting, 1995). Indeed, verifying statements across mathematical domains is a vital practice for pre-service teachers to foster their reasoning and comprehend the concepts. Particularly, undergraduate mathematics is an important period for pre-service teachers to build knowledge to advance mathematical reasoning for learning more advanced mathematics in secondary-school teaching. The goal of the paper reported here is to shed light upon the processes of evaluating practices associated with conjectures.

Methods

Eight secondary mathematics education majors from a large Midwestern university in the United States participated in this study. One was a third-year, five were fourth-year, and two were fifth-year undergraduate students. All of the participants had taken a number of upper-level, non-proof intensive mathematics courses that included the topics of analysis, combinatorics, differential equations, linear algebra, modern algebra, number theory, probability, or statistics. They had also taken at least two courses involving mathematical proofs about the topics of analysis, linear algebra, geometry, modern algebra, or transition to proofs. Since number theory, geometry, continuous functions, differentiation, and one-to-one functions were addressed in undergraduate mathematics courses as well as in pre-calculus and calculus courses in high school, all of the pre-service teachers participating in this study had some relevant domain knowledge.

The primary source of data was one-on-one semi-structured interviews. Interviews were audio-recorded, lasted approximately 90-120 minutes, and included a focus on the evaluation of the statement being true or false and the productions of proof and counterexample. Because the focus of this article is on pre-service teachers' strategies for verifying the validity of statements, the results presented and the subsequent discussion focus exclusively on data about their proof and counterexample productions. During the interview, the pre-service teachers were handed

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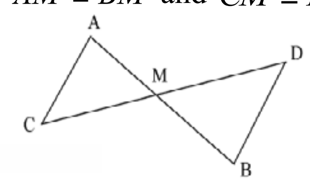
each statement (see Table 1), one at a time, and were asked to think aloud how to determine each statement to be true or false. The instrument, comprised of six mathematical statements that were adapted from existing literature and textbooks, was designed to assess pre-service teachers' abilities to evaluate the veracity of statements about algebra, analysis, geometry, and number theory. The instrument was finalized after receiving feedback from three mathematics professors as well as one mathematics education professor and pilot testing with engineering and mathematics graduate students.

The interview transcripts and participants' written responses were summarized initial impressions and highlighted interesting issues regarding the participants' statement verifications. Coding of the data began with a set of external codes that were derived from the theoretical framework. By examining the data and reviewing the transcripts, themes emerged in participants' statement verifications. After proposing these internal (data-grounded) codes, each transcribed interview was reexamined and recoded to incorporate these new codes.

Table 1. The instrument.

Question	Statement	Domain	True or False
1.	If $n \in N$, then GCD(n, $6n-5$) $= 1$.	Number Theory	False
2.	If $\overline{AB}$ and $\overline{CD}$ intersect at the point M, $\overline{AM} \cong \overline{BM}$ and $\overline{CM} \cong \overline{DM}$, then $\overline{AC} // \overline{DB}$.  (Adapted from Yang & Lin, 2008, p.65)	Geometry	True
3.	Let $f: D \to R$ be a function and $x_0 \in D$. If $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x)$, then f is continuous at x_0.	Analysis	False
4.	If $n \in N$, then $n^3 + 44n$ is divisible by 3. (Adapted from Smith, Eggen, & St. Andre, 2006, p. 109)	Number Theory	True
5.	If $h: A \to C$ and $g: B \to D$ are both 1-1 functions, $A \cap B = \phi$, and $C \cap D = \phi$, then $h \cup g: A \cup B \to C \cup D$ is a 1-1 function. (Adopted from Smith et al., 2006, p. 204)	Algebra	True
6.	Let $f: D \to R$ be a function and $a \in D$. If $f: D \to R$ is differentiable at a, then f is continuous at a.	Analysis	True

Results

The results reported in this section are organized by two research questions concerning pre-service teachers' strategies for evaluating various conjectures.

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Pre-Service Teachers' Strategies for Evaluating the Statement

Based on the theoretical framework regarding strategies for verifying conjectures, four specific categories—the example-based strategy, the mixed reasoning strategy, the naïve reasoning strategy, and the sophisticated reasoning strategy, as listed in Table 2, were proposed through the coding process to access the participants' responses to the ways they used for doing these tasks.

Table 2. Strategies for evaluating statements.

Strategies	Description
Example-Based Reasoning	● Individuals rely on numbers or diagrams to verify the statement.
Mixed Reasoning	● Individuals both use examples to identify relevant patterns and structures, and manipulate (partially) correct properties, definitions, and/or theorems to identify a reasonable example to attempt to prove or disprove the statement.
Naïve Reasoning	● Individuals manipulate partially correct properties, definitions, and/or theorems from their intuitive understanding or past experience to verify the statement.
Sophisticated Reasoning	● Individuals manipulate relevant true properties, definitions, and/or theorems to attempt to prove or disprove the statement.

Table 3 displays the distribution of strategies the pre-service teachers had for deciding each statement to be true or false. As shown in the table, the mixed reasoning strategies (33 cases out of 46) were used more often than the other three strategies (5 example-based reasoning strategies, 4 sophisticated reasoning strategies, and 4 naïve reasoning strategies) by prospective teachers when determining the validity of statements.

Table 3. Pre-service teachers' strategies for evaluating the statement.

Strategy	True								False				Frequency Count*
	Question 2 (Geom-etry)		Question 4 (Number Theory)		Question 5 (Algebra)		Question 6 (Analysis)		Question 1 (Number Theory)		Question 3 (Analysis)		
	n	%	n	%	n	%	n	%	n	%	n	%	n
Example-based	0	0	1	13	0	0	0	0	3	38	1	13	5 (4)
Mixed	8	100	7	88	6	86	5	71	5	63	2	25	33 (8)
Naïve	0	0	0	0	0	0	2	29	0	0	2	25	4 (3)
Sophisticated	0	0	0	0	1	14	0	0	0	0	3	38	4 (4)

Note. Pre-service secondary mathematics teacher 8 did not decide Questions 5 and 6 to be true or false.

*The frequency count is the number of each strategy that each participant used for evaluating the validity of six given statements.

Due to decimal rounding, the total percentage of Questions 1, 3, and 4 is greater than 100.

Data described in Table 3 also illustrates that each pre-service teacher used the mixed reasoning strategies at least once when evaluating the statement's veracity. Although the mixed reasoning strategies was the most commonly strategies used for verifying the validity of Questions 1, 2, 4, 5, and 6, substantially fewer pre-service teachers used the example-based reasoning, naïve reasoning, or sophisticated reasoning strategies for evaluating Questions 1, 3, 4, 5, and 6. Considering the statements across different domains, it is interesting to note that the false question in Number Theory was the most likely to be verified by examples-based reasoning strategies. Also, all pre-service teachers tended to employ the mixed reasoning strategies for determining the true geometric statement. The section that follows describes each strategy used by pre-service teachers when deciding the statement's veracity.

Example-based reasoning strategies. Four pre-service teachers based their determination of the statement's validity merely on numbers or diagrams related to the problem. One pre-service teacher stated what he did when determining a statement's (Question 1) veracity: "I went through a few possibilities, […], and I got up to five. […] Six times five is twenty five, so the greatest common divisor between twenty five and five is five" (PSMT1). Another participant used diagrams to evaluate a statement's (Question 3) veracity: "I was thinking like x, and it would be continuous. And then I was thinking about x^2, and it would be continuous as well" (PSMT5). Thus, these pre-service teachers tested a few numbers or diagrams to evaluate the statement's validity.

Mixed reasoning strategies. In general, every pre-service teacher used the mixed reasoning strategies at least once when determining the veracity of statements. Moreover, these teachers decided the statement's validity on the basis of examples and their (partially) correct understandings of definitions or theorems involved in the problem. One teacher labeled the figure first and said, "Angle *AMS* and angle *BMD* are equal because they are vertical angles, so [triangle *AMC* and triangle *BMD* are] congruent triangles, [so the] corresponding angles [angle *CAM* and angle *DBM*] are congruent, and they are alternative interior angles which make line *AC* and line *DB* be parallel." (PSMT3, Question 2). In explaining why he determined the statement to be true, one teacher said, "I plug in a couple of numbers just to see if the statement works. […] When I factor out n, I can see that any natural number that is a multiple of 3 means that is also divisible by 3" (PSMT2, Question 4). Still another teacher drew a picture and explained why she agreed with the statement to be true, "Continuous functions mean that you have a graph, it is a smooth line, and there is no hole on the graph. [T]he limit from the left and the limit from the right is the same that gonna be continuous" (PSMT8, Question 3). In summary, these pre-service teachers who applied the mixed reasoning strategies for verifying the validity of statements either relied on examples to identify the structure of the statement or based on their (partially) correct understandings of mathematical properties along with examples related to the problem.

Naïve reasoning strategies. With this strategy, three per-service teachers determined the statement's veracity based on their partially correct understandings of mathematical facts related to the problem. When evaluating a statement's (Question 3) veracity, one teacher stated, "[The continuous function] was defined at some point x on the function, or x_0 on the function. […] [T]he limit from the left is the same as the limit from the right, so since it's defined at that point and those limits are the same, then it has to be continuous" (PSMT1). Another teacher explained, "What I can remember, if a function is differentiable at a, then that means that the corresponding y value exists at a, so it doesn't necessarily mean the function to be continuous" (PSMT7, Question 6). Thus, these pre-service teachers seemed to possess fragile mathematical knowledge to verify the statement's validity.

Sophisticated reasoning strategies. Four pre-service teachers decided the validity of statements on the basis of true mathematical definitions, theorems, or properties. For example, one teacher explained his strategy for evaluating a statement's (Question 5) validity:

I know the definition of one-to-one function means that for exactly one, there is one x value for only one y value, so from the domain to the range. Umm, since both of those are one-to-one *functions and since, umm, the domains of each function h and g, respectively, are mutually exclusive as well as the range of mutually exclusive, the intersection is the no set, and the union between the two, umm, will still be one-to-one. (PSMT1).*

In responding to what he did when verifying the validity of a statement (Question 3), one teacher stated, "The statement is false because the definition of the continuity is the limit of the function at a point is also equal to the value of the function at that point" (PSMT2). In short, these pre-service teachers attempted to employ relevant mathematical facts logically when determining the statement's veracity.

Pre-Service Teachers' Strategies for Evaluating the Statement and Their Determination of its Validity

Table 4 presents the frequency count of strategies for verifying statements used by pre-service teachers along with their correct and incorrect decisions on the validity of statements.

Table 4. Frequency count of pre-service teachers' strategies for verifying the statement and their decisions on its validity.

Strategy	True								False				Frequency Count*	
	Question 2		Question 4		Question 5		Question 6		Question 1		Question 3			
	Co	In	Co	In	Co	In	Co	In	Co	In	Co	In	Co	In
Example-based	0	0	1	0	0	0	0	0	3	0	0	1	4(3)	1(1)
Mixed	8	0	7	0	5	1	4	1	4	1	1	1	29(8)	4(3)
Naïve	0	0	0	0	0	0	1	1	0	0	0	2	1(1)	3(3)
Sophisticated	0	0	0	0	1	0	0	0	0	0	3	0	4 (4)	0(0)

Note. Pre-service secondary mathematics teacher 8 did not decide Questions 5 and 6 to be true or false.
Co indicated that the correct decision regarding a statement's validity made by pre-service teachers.
In indicated that the incorrect decision regarding a statement's validity made by pre-service teachers.

[a]The frequency count is the number of occurrences that a particular strategy was used. Totals may include multiple counts for a single pre-service teacher (i.e., a teacher may have used the example-based reasoning strategy for verifying the validity of more than one statement). The number of different teachers citing a particular strategy is provided in the parenthesis.

As seen in Table 4, the participants who used the sophisticated reasoning strategies accurately made a decision on the statement's veracity. It is clear from the table that the pre-service teachers were more likely to make an incorrect determination of the statement's validity by employing the naïve reasoning strategies. If we consider the most effective strategy used in allowing a correct decision on a statement's validity, then the mixed reasoning strategies (29 cases out of 46) far out-numbered the other three strategies (4 sophisticated reasoning strategies, 4 example-based reasoning strategies, and 1 naïve reasoning strategy). These findings show that most pre-service teachers did not solely rely on examples when testing the proposition to be true or false. These findings also show that only a few pre-service teachers were able to employ true mathematical properties and theorems to decide the validity of statements correctly. Considering the statements across various domains, it is interesting to note that all participants accurately evaluated the geometric statement (Question 2) by using the mixed reasoning strategies. Perhaps relying on pictures serves as a helpful means for pre-service teachers to identify the structure of the given statement in Geometry.

Discussion

This study examined the processes through which pre-service secondary mathematics teachers evaluated the truth and falsity of given statements across domains. The overall findings indicate that all participants employed the mixed reasoning strategies—using both examples and their (partially) correct understandings of mathematical concepts—for determining the statement's veracity at least once. The fact that the pre-service teachers in this study overwhelmingly used examples when asked to verify the validity of statements is not surprising considering the literature on undergraduate students' tendencies to rely on example-based reasoning strategies (Gibson, 1998; Goetting, 1995; Harel & Sowder, 1998). This study, however, shows that the participants used examples to identify patterns and structures rather than conclusively determined the truth of statements. Considering the strategy used in allowing a correct decision on the statement's validity, the mixed reasoning strategies were more effective ways (29 cases out of 46) than the other three strategies (4 sophisticated reasoning strategies, 4 example-based reasoning strategies, and 1 naïve reasoning strategy). There were a few cases in which the pre-service teachers inaccurately made a decision on the validity of statements by employing the example-based reasoning strategy (1 cases out of 46), the mixed reasoning strategies (4 cases out of 46), or the naïve reasoning strategies (3 cases out of 46). These teachers seemed to use their partially correct understandings of limits, continuous functions, one-to-one functions, and the greatest common divisor to determine the statement's veracity. This result is reminiscent of Weber and Alcock's (2004) suggestion that "students' [concept] images of mathematical concepts are often inconsistent with the corresponding formal definitions" p. 232). In order to help individuals develop concept images—defined as mental pictures—to be consistent with mathematical definitions, mathematics instructors need to draw attention to pre-service teachers' misconceptions of concepts and seek ways to refine their content understanding.

Another feature of the results is that the majority of participants determined the validity of statements accurately (38 cases out of 46). Yet, the wealth of existing studies investigating pre-service secondary teachers' conceptions of proof show that many teachers have considerable difficulty understanding and producing proofs and counterexamples (e.g., Goetting, 1995; Harel & Sowder, 1998; Weber, 2001). The results of this study suggest that providing pre-service teachers with opportunities to experience determining the validity of statements across domains might be a way to enhance their development with proof and counterexample. Such practices can help pre-service teachers not only see the logic behind a statement (Tall, 1992) and explain why something is true or false, but also foster their mathematical reasoning and understanding (Buchbinder & Zaslavsky, 2007). If engaging learners in verifying the truth and falsity of statements can illuminate underlying concepts of propositions as well as promote learners' mathematical reasoning, more research is needed on what curricular tasks can better develop pre-service secondary school mathematics teachers' conceptions of proof and counterexample.

In summary, four strategies—example-based reasoning, mixed reasoning, naïve reasoning, and sophisticated reasoning—used by pre-service teachers to evaluate the validity of statements in different domains cannot make generalizations due to the small number of participants in this study. While most pre-service teachers relied on examples to test Question 1 (Algebra), the majority of participants were inclined to use numbers or diagrams along with their (partially) correct understandings of mathematical facts when verifying various statements across domains. These findings suggest a need to help pre-service teachers apply example-based and deductive reasoning strategies flexibly to evaluate the statement's veracity. These findings also suggest that designing instructional approaches by drawing from individuals' strategies for evaluating statements to help pre-service secondary mathematics teachers learn mathematics meaningfully and foster their mathematical reasoning is needed.

References

Alcock, L. & Inglis, M. (2008). Doctoral students' use of examples in evaluating and proving conjectures. *Educational Studies in Mathematics, 69*, 111-129.

Buchbinder, O. & Zaslavsky, O. (2007). How to decide? Students' ways of determining the validity of mathematical statements. In D. Pitta-Pantazi, & G. Philippou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education* (pp. 561-570). Cyprus, Larnaca.

Committee on the Undergraduate Program in Mathematics. (2004). *Undergraduate programs and courses in the mathematical sciences: CUPM curriculum guide*. Washington, DC: Mathematical Association of America.

Gibson, D. (1998). Students' use of diagrams to develop proofs in an introductory analysis course. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education. III* (pp. 284-307). Providence, RI: American Mathematical Society.

Goetting, M. (1995). *The college students' understanding of mathematical proof*. Unpublished doctoral dissertation, University of Maryland, Maryland.

Harel, G., & Sowder, L. (1998). Students' proof schemes. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), *Research on collegiate mathematics education* (Vol. III, pp. 234–283). Providence, RI: American Mathematical Society.

Hoyles, C., & Kuchemann, D. (2002). Students' explanations in geometry: Insights from a large-scale longitudinal survey. *Proceedings of the International Conference on Mathematics: Understanding Proving and Proving to Understand*, 9-22.

Ko, Y. Y., & Knuth, E. (2009). Problems manifested in prospective secondary mathematics teachers' proofs and counterexamples in differentiation. In F. L. Lin, F. J. Hsieh, G. Hanna, & M. de Villiers (Eds.), *Proceedings of the ICMI Study 19 conference: Proof and Proving in Mathematics Education* (Vol. 1, pp. 262-267). Taipei, Taiwan.

National Council of Teacher of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Riley, K. J. (2003). *An investigate of prospective secondary mathematics teachers' Conceptions of proof and refutations*. Unpublished doctoral dissertation, Montana State University, Bozeman.

Smith, D., Maurice, E., & St. A., R. (2006). *A transition to advanced mathematics* (6th ed.). Monterey, CA: Brooks Cole.

Smith, J. C. (2006). A sense-making approach to proof: Strategies of students in traditional and problem-based number theory courses. *Journal of Mathematical Behavior, 25*(1), 73-90.

Tall, D. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity, and proof. In Grouws D.A. (ed.), *Handbook of research on mathematics teaching and learning* (pp. 495-511). New York: Macmillan.

Weber, K. (2009). How syntactic reasoners can develop understanding, evaluate conjectures,

and generate counterexample in advanced mathematics. *Journal of Mathematical Behavior, 28*, 200-208.
Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, 48(1), 101-119.
Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics, 56*, 209-234.
Yang, K. L., & Lin, F. L. (2008). A model of reading comprehension of geometry proof. *Educational Studies in Mathematics, 67*, 59-76.