REFERENTIAL COMMUTATIVITY: PRESERVICE K-8 TEACHERS' VISUALIZATION OF FRACTION OPERATIONS USING PATTERN BLOCK

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This paper examines ten K-8 preservice teachers' visual representations of fraction operations using the four main pattern blocks. Data consist of figures made using the pattern blocks, drawn colored representations, and detailed written comments and algebraic formalism. The theoretical framework is drawn from representational theories and analyses of fraction operations, and work on coordination of different levels of units. The main result is that only those teachers meaningfully coordinating the different referent units in the fraction situations, were the ones consistent in their representations and reasoning, and in successfully establishing referential commutativity for multiplication of fractions.

Introduction

The usual way of representing a fraction numerically, is the expression $\frac{a}{b}$ where a is a whole number and b is a nonzero whole number. This representation has several interpretations such as part-whole, quotient, operator, and measurement (Kieren, 1980; Skemp, 1986; Olive, 1999; Olive & Steffe, 2002). In the *part-whole* meaning, the referent unit 1 is defined, the denominator indicates the number of congruent pieces into which the unit 1 is partitioned, and the numerator indicates how many of those congruent parts are selected. In the *sharing equally (partitive) division* model, the fraction is interpreted as the equi-partitioning (Olive & Steffe, 2002) of the quantity, a, into b congruent parts (shares), with the fraction, $\frac{a}{b}$ being the share of one person, relative to the referent unit for quantity, a. For example, if we share 3 chocolate bars among 5 friends, each friend gets 3/5 of ONE chocolate bar.

The repeated subtraction (*measurement* or *quotitive division*) interpretation attends to the instruction "How much (or how many) of quantity b is (or are) there in quantity a?" or "What is the measure of quantity a in units of size b?"

Children need to be aware of how the same quantity can be represented by many fractions (i.e. *fraction equivalence*) before the exploration of +, −, ×, and ÷ operations with fractions. Children should be able to recognize and create fractions equivalent to a given fraction, because they will frequently need to determine an equivalent fraction in order to add, subtract, multiply, or divide, in a way that makes sense to them (Sowder et al., 1998). For example, in adding or subtracting fractions, in the process of obtaining fractions of equal denominator, students must be able to refer to their knowledge about fraction equivalence.

In dealing with multiplication of two fractions, the understanding of what the multiplier, the multiplicand, and the product refer to is of paramount importance. The referent units for the multiplier, the multiplicand, and the product respectively are the multiplicand, the whole unit, and the whole unit. The algorithm $\frac{a}{b}\times\frac{c}{d}=\frac{a\times c}{b\times d}$ is effortless to memorize and to perform; however, to render fraction multiplication meaningful, children must be aware of the referent

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units for these fractions and what the product really indicates. Moreover, while both $\frac{a}{b} \times \frac{c}{d}$ and $\frac{c}{d} \times \frac{a}{b}$ yield the same numerical answer, the order matters if we want to conceptualize the referent units involved, namely the fact that the referent units are being swapped between the multiplier and the multiplicand. As will be explained in the results section of this study, although fraction multiplication is *algebraically* commutative, that commutativity is definitely not so obvious to construct referentially. Construction of referential commutativity requires proficiency in simultaneously coordinating various fraction relations and different levels of units meaningfully.

As for the division of fractions, the understanding that the referent unit for both the dividend and the divisor is the same unit whole, is necessary in order to make sense of the division operation. Moreover, it is equally important to realize that the quotient has no reference to the original unit whole; the quotient must be seen as a relation between the dividend and the divisor in order to develop an in-depth understanding of fraction division. For example, when dividing ¾ by ½ in order to find out how many ½ lb-bags of coffee we could make from ¾ lb of coffee, the answer is one and a half *bags* (not pounds of coffee). While fraction division has traditionally been related with the crude invert and multiply algorithm, most children and adults do not make sense of how this algorithm works (NCTM, 2000). Awareness of referent units for fraction division is crucial in order to develop any meaning for the algorithm.

This study investigates the above assertions (with special emphasis on referential commutativity of fraction multiplication) by analyzing the work of ten pre-service K-8 teachers.

Theoretical Framework

The theoretical framework of this study is drawn from the work by Steffe and Olive involving representations of operations with fractions and referent unit coordination (Olive & Steffe, 2002; Steffe & Olive, 2010). These researchers postulated a series of fractional schemes as a foundation for the construction of fraction operations (Olive & Steffe, 2002, p. 436). They also reported facility with whole-number sense as one of the main prerequisites of fractional thinking and reasoning (the *Reorganization Hypothesis*, Steffe, 2010). Their work with children in grades three through five involved children's representations and actions in fractional situations using electronic manipulatives called TIMA (Tools for Interactive Mathematical Activity)(Olive, 2000).

The idea of a representational system is also relevant to the research presented in this paper. This construct comprises written symbols, thinking aloud, physical manipulatives, and drawn representations (Behr et al., 1983). In what follows, we focus on K-8 pre-service teachers' drawn and physical representations of five main fraction operations modeled with pattern blocks, and on their abilities to connect their visual and written formalism.

Context and Methodology

This study investigates pre-service K-8 teachers' construction of fraction operation problems (equivalence, addition, subtraction, multiplication, division) using physical manipulatives (the four main pattern blocks). Ten pre-service K-8 teachers, whom the first author met weekly for two weeks in two-hour sessions, were selected from his "algebra for teachers" class to participate in this study. They demonstrated their solutions for each problem with both the actual pattern

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blocks and colorful drawings on a triangular grid (Figure 1). They also explained their reasoning in detail for each task with reference to their physical and drawn representations.

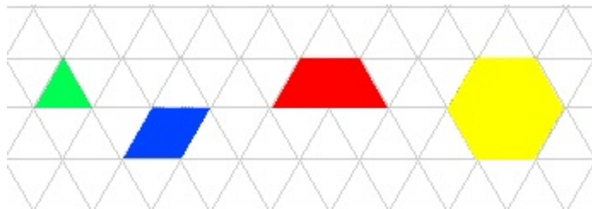
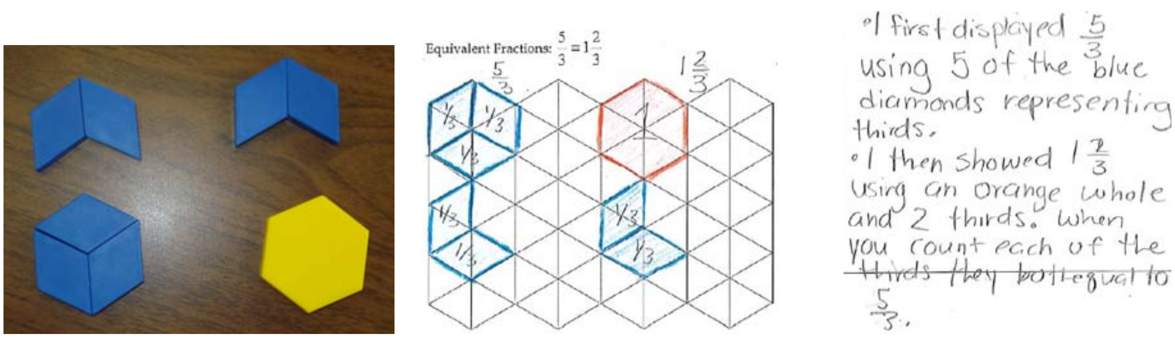


Figure 1 - The Four Main Pattern Blocks on Isometric Grid

Our data consists of photographed physical representations, and scanned drawn representations along with written arithmetic formalism and detailed comments. The purpose of creating these scanned versions was to conduct a retrospective, preliminary thematic analysis in order to find possible themes for a detailed analysis. The dataset was then revisited multiple times in order to generate a thematic analysis (using constant comparison methodology) from which the following results emerged.

Results & Analysis

For the equivalence problem $\frac{5}{3} \equiv 1\frac{2}{3}$, all ten teachers pretty much came up with the same physical and drawn representations. Several of them even clarified their construction using a *counting the thirds* strategy (see Figure 2). We can infer that Edie relied on this strategy by explicitly labeling the thirds that are being counted and also by referring to the whole unit, the yellow (or orange) hexagon, at each step of her counting. Lauren used a similar reasoning by writing *3 "one thirds" equal one, so 5 thirds will have 2 "one thirds" left over with one yellow whole*.



2a 2b 2c

Figure 2: Edie's Physical (2a) & Drawn (2b) Representations & Written Work (2c)

For the addition and subtraction tasks, not all teachers were as explicit with reference to the referent unit, the yellow hexagon, at each step of their construction. Moreover, although the instructions required explanation for each step, not all teachers, for instance, thought about using the idea of common denominator and involving that in their physical and drawn representations. As depicted in Figure 3, both Lauren and April arrive at the correct final answer; however, April's thinking seems to be more sophisticated than Lauren's in that she not only refers to the referent unit yellow hexagon (for the addends in the addition task, and for the minuend and

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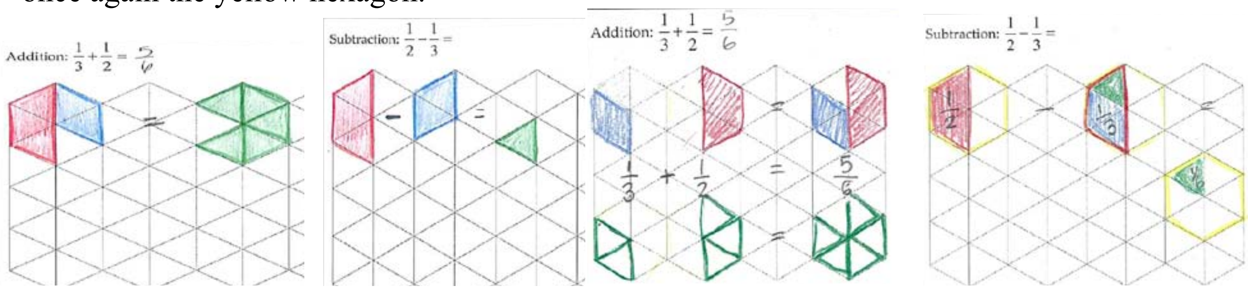
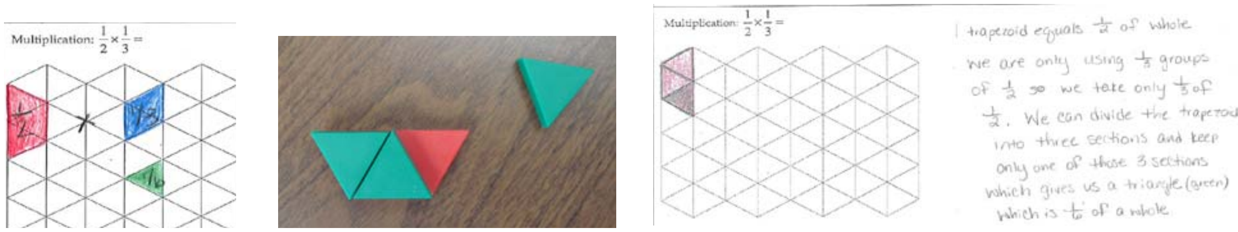
subtrahend in the subtraction task), but she also decomposes each fraction into smaller pieces (green triangles representing sixths), thus making sense of the situation. She also clarifies both in her writing and her physical and drawn representations that the referent unit for those sixths is once again the yellow hexagon.



3a 3b 3c 3d

Figure 3: Lauren's (a b) and April's (c d) Drawn Representations for the + and − Tasks

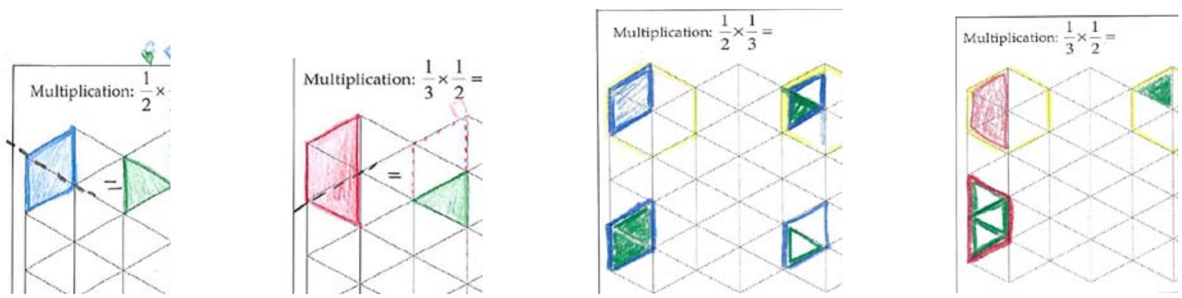
All teachers succeeded in establishing the referential commutativity of addition in dealing with the problems $\frac{1}{3}+\frac{1}{2}$ and $\frac{1}{2}+\frac{1}{3}$ in constant comparison of their physical and drawn representations. They also invalidated the commutativity of subtraction with very creative constructions. Multiplication and division tasks, on the other hand, were rather cumbersome for many of the participants. There were some who overcame this difficulty by appropriately relating the multiplier, the multiplicand, and the product (the dividend, the divisor, and the quotient in the division tasks) to their referent units. Only two participants were able to induce a referential commutativity for fraction multiplication, the most sophisticated behavior resulting from this research study. For the multiplication problems $\frac{1}{3}\times\frac{1}{2}$ and $\frac{1}{2}\times\frac{1}{3}$, many teachers constructed and drew both the multiplier and the multiplicand, the former being irrelevant, a non-operative interpretation of the problem situation. Several teachers swapped the role of multiplier and multiplicand. We begin the discussion on multiplication with these problematic representations. Emma modeled the first multiplication problem as the product of a red trapezoid and of a blue rhombus (Figure 4a). Kristie not only included both $\frac{1}{2}$ and $\frac{1}{3}$ in her constructions, but also swapped their roles as well (Figure 4b-c). Her referent units for $\frac{1}{2}$ and $\frac{1}{3}$ respectively were the whole unit (yellow hexagon) and $\frac{1}{2}$ (the red trapezoid). Her interpretation of $\frac{1}{2}\times\frac{1}{3}$ is actually $\frac{1}{3}\times\frac{1}{2}$, and vice versa. While she is very clear in establishing the referent unit of the product $\frac{1}{6}$ (green triangle) as the yellow hexagon, we postulate that Kristie failed to establish referential commutativity of multiplication due to the interchange of referent unit roles. Both Kristie and Emma used their same approaches for the other multiplication problem as well.



4a 4b 4c 4d

Figure 4: Emma's (4a) and Kristie's Constructions for $\frac{1}{2} \times \frac{1}{3}$

Lauren was one of the few who established the referential commutativity of multiplication in a meaningful and appropriate manner. For the half of a third problem, she started by constructing the third as the blue rhombus. She then bisected this third using a dashed line, as depicted in Figure 5a. She was aware of the fact that constructing a half (red trapezoid) was irrelevant in the problem situation. She also understood that the referent unit for the multiplier (the half) was the multiplicand (the third). It is also worth noting that she did not specify whether the product (the sixth) has the whole unit as the referent unit. She followed a similar approach for the third of a half problem, as depicted in Figure 5b.



5a ***5b*** ***5c*** ***5d***

Figure 5: Lauren's (a, b) and April's Constructions for Fraction Multiplication Tasks

We can say that April followed a reasoning pattern similar to Lauren's, in that she was aware of the fact that constructing a half (red trapezoid) was irrelevant in the problem situation (Figure 5c). She also successfully interpreted the referent units of the multiplier and the multiplicand. As can be detected in Figures 5c-d, April took the whole thing one step further by also specifying the referent unit of the product "the sixth" as the yellow hexagon unit. Both Lauren and April established referential commutativity, but April's representations can be considered to be more sophisticated than Lauren's.

Teachers overall seem to have successfully applied the "How many of … are there in …" (or "how much of … is there in …") view in their representations of fraction division problems. Some teachers were very explicit in their reference to the yellow hexagon as the referent unit for the dividend and the divisor. Some others perhaps over-generalized this reference for the quotient by attempting to construct the quotient using the physical or drawn representations. What was the quotient's referent unit then? Was it the yellow hexagon, the dividend, or the divisor? Or something else? We begin our discussion on fraction division with Emma's drawn representations for the problems $\frac{1}{2} \div \frac{1}{6}$ and $\frac{1}{6} \div \frac{1}{2}$. Emma not only constructed both the dividend

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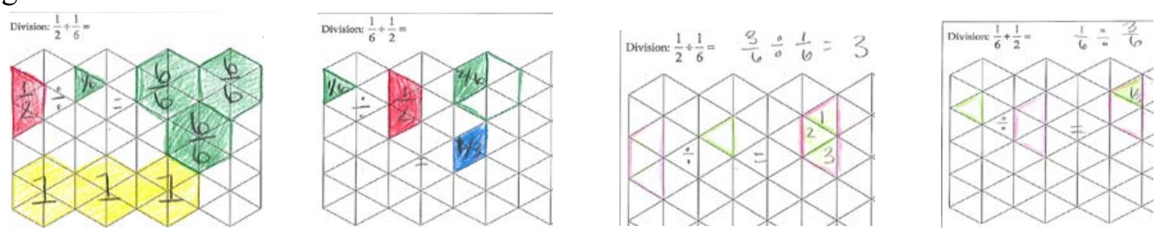
and the divisor, but she also drew the quotient as well (Figures 6a-b). She basically arrived at this construction with reference to her purely algebraic formalism using the invert-and-multiply method. In her interpretation, the quotient of the first problem, 3, is referred to the 3 yellow hexagon whole units. In fact, this corroborates our theory about Emma's view of fraction multiplication in which she constructed both the multiplier and the multiplicand (Figure 4a). For Emma, all these elements have to be represented using a pattern block.

Edie, on the other hand, seemed to have meaningfully constructed all the constituents in fraction division task. She explained "*How many sixths are there in a half? Equivalently, how many green triangles are in a red trapezoid? There are three sixths in a half.*" In her drawing, she also illustrated her way of counting those three sixths by labeling them as 1, 2, and 3, respectively (Figure 6c). Edie followed a similar reasoning for the other division problem by stating ""*How many halves are in a sixth? There is* $\frac{1}{3}$ *of a half in a sixth*." In her drawing, Edie used the idea of labeling, in an attempt to count, for which this time she used a fraction (Figure 6d). And that fraction, 1/3, the quotient, has nothing to do with the blue rhombus. It is true that the blue rhombus represents one third of the yellow hexagon whole unit, but in the context, as constructed by Edie, it refers to the quotient 1/3 with referent unit 1/2 (the red trapezoid), which also happens to be the divisor. We also observe that Edie meaningfully divides both the half by the sixth, and the sixth by the half algebraically, without reference to the invert-and-multiply algorithm.

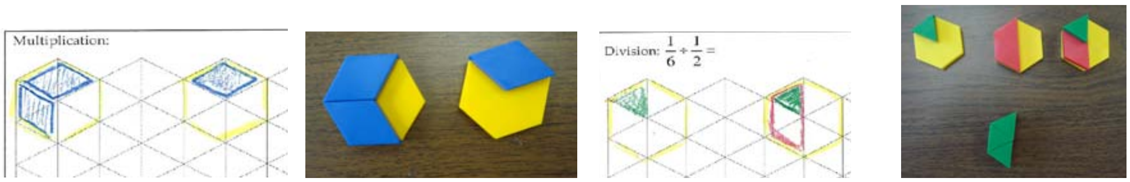


6a 6b 6c 6d

Figure 6: Emma's (a, b) and Edie's (c, d) Constructions

We conclude our findings with April's performance on fraction division tasks. We consider April's reasoning as the most sophisticated one in that she makes use of a variety of meaningfully connected fraction ideas. April was one of the few making sure to refer to the yellow hexagon whole unit and including it in her drawn and physical representations, whenever relevant (Figures 3d, 5c-d). For the division (of the half by the sixth, and the sixth by the half) problems, she proceeds in a manner very similar to Edie's (Figure 6c-d). The only difference is that April also includes the yellow hexagon in her drawing, which is an indication that she is aware that the dividend's referent unit is the yellow hexagon whole unit (She also uses the idea of labeling the sixths the same way Edie does). This stacking approach, which can be thought of coordination of referent units at different levels, is a powerful tool in making sense of fraction division. We look at April's coordination of referent units through her physical and drawn representations for the multiplication task $\frac{1}{2}\times\frac{2}{3}$ and the division task $\frac{1}{6}\div\frac{1}{2}$ simultaneously. For the multiplication task, she explains "*There are two 1/3 portions in 2/3; 1/2 of that is one 1/3 portion*." She also relates the multiplier, namely the 1/2, to its referent unit 2/3 (two blue rhombi); the multiplicand, namely the 2/3, to its referent unit yellow hexagon; and the product, namely the 1/3, to its referent unit yellow hexagon (Figures 7a-b). For the division task, she explains "*Division is how many or how much of a given will go into another given portion. 1/3 of*

1/2 will divide into 1/6 of a whole. Therefore 1/6 ÷ 1/2 = 1/3." She also relates the dividend, namely the 1/6, to its referent unit yellow hexagon; the divisor, namely the 1/2, to its referent unit yellow hexagon; and the quotient 1/3 to its referent unit red trapezoid (Figures 7c-d).



7a 7b 7c 7d

Figure 7: April's Constructions

April's referent unit coordination scheme for the fraction multiplication and fraction division tasks can be tabulated as follows:

Fraction Multiplication		**Fraction Division**	
Components	*Referent Units*	*Components*	*Referent Units*
Multiplier	Multiplicand	Dividend	Whole
Multiplicand	Whole	Divisor	Whole
Product	Whole	Quotient	Divisor

Table 1. April's Referent Unit Coordination Scheme

Conclusions and Discussion

This study aimed to investigate the visual representations of five main fraction operations (equivalence, addition, subtraction, multiplication, and division) created by K-8 pre-service teachers using pattern blocks. Analysis of these participants' physical and drawn representations, accompanied by their algebraic formalism and verbal reasoning, helps us to determine important insights into their sense-making of the mathematics they are exploring. These insights have direct implications for the teaching of fractions in a hands-on-activity based environment. Mathematics teachers should be more conscious and explicit in modeling problems because their models may lead to a misinterpretation of the problem situation, or even the solution to the problem, as depicted in this present study. For example, although some students were confident with their algebraic solutions for the multiplication tasks (using the multiply the numerators and denominators algorithm) and division tasks (using the invert-and-multiply algorithm), their interpretation of the processes differed considerably, when they were asked to represent these tasks using the drawn and physical representations. In particular, Emma's representations of fraction multiplication and fraction division indicate that she may not have a meaningful concept for these operations with fractions. She appears to lack the necessary three levels of units (Olive & Steffe, 2010) to mentally coordinate the relations among multiplier, multiplicand and product, with respect to their roles in the situation and their respective referent units. This lack of coordination is even more apparent in her representation of fraction division (see Figure 6a-b). These necessary relations are depicted in April's referent unit coordination (Table 1 above).

Although fraction multiplication is algebraically commutative, the representation of that commutativity requires sophisticated reasoning. Construction of referential commutativity requires proficiency in simultaneously coordinating various fraction relations meaningfully. Awareness of the referent units for each component (multiplier, multiplicand, multiplier), ability to recognize which fractions are operators and which are quantities, and ability to connect the representations of these to the written explanations and algebraic formalism are essential in

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establishing referential commutativity of multiplication within a representational system (Behr et al., 1983).

Research indicates that the multiplicative conceptual field is very complex and includes many concepts of mathematics, other than multiplication itself (Behr, Harel, Post, & Lesh, 1992; Harel & Behr, 1989; Harel, Behr, Post, & Lesh, 1992). "*Additive reasoning develops quite naturally and intuitively through encounters with many situations that are primarily additive in nature*" (Sowder, Armstrong, Lamon, Simon, Sowder, & Thompson, 1998, p. 128). Building up multiplicative reasoning skills, on the other hand, is not obvious; schooling and teacher guidance are essential to acquire a profound understanding and familiarization with multiplicative situations, especially with respect to fractions (Hiebert & Behr, 1988; Resnick & Singer, 1993). This present study indicates the importance for teachers (and students) to develop three levels of units structures and the skill to coordinate those units (Olive & Steffe, 2010).

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