AN ANALYTIC FRAMEWORK FOR REPRESENTATIONAL FLUENCY: ALGEBRA STUDENTS' CONNECTIONS BETWEEN MULTIPLE REPRESENTATIONS USING CAS

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To better support students' conceptual understanding of algebra in the information age, we need an improved understanding of how students interpret connections between multiple representations using CAS as a representational toolkit. This research centers on the development of an analytic framework for categorizing students' connections between multiple representations, an indicator of students' representational fluency. Videotaped task-based interviews with high school algebra students solving equations using CAS were analyzed to generate the proposed analytic framework, which has a hierarchical structure based on the direction and purpose of students' translations between representations.

Purpose of the Study

Each mathematical representation is a glimpse into a version or phase of a particular mathematical object, and when taken together, multiple representations offer complementary perspectives of a mathematical object, which can help to reveal its structure. Flexibility in multiple representational approaches is an indicator of more sophisticated mathematical competencies (Brenner et al., 1999). Despite an emphasis on multiple representations in standards and curricula, school algebra students' difficulties with translating between multiple representations have been well documented (e.g., Dreyfus & Eisenburg, 1996). In technology-intensive approaches, some researchers have found that school algebra students can use multiple representations in solving tasks and are successful in translating between multiple representations (Ruthven, 1990) while others attest to the persistent difficulties students face in using multi-representational approaches with flexibility (Huntley & Davis, 2008).

CAS-intensive trends in school mathematics, and algebra in particular, that were pioneered in the late 1980s and 1990s have finally infiltrated contemporary curricula and classrooms (e.g., Davis & Fonger, 2010). Congruous with the perspective that the coordination of multiple representations is an indicator of conceptual understanding, issues of linking or connecting mathematical representations are significant concerns. Indeed, although CAS environments can act as representational toolkits, Heid and Blume (2008) report that, "students do not necessarily connect representations when operating in a multiple representation environment" (p. 98). Indeed, Heid and Blume (2008) articulate a "need to better understand how students move between, connect, and reason from multiple representations" (p. 98).

With the integration of CAS in school algebra, researchers and practitioners alike need to know how high school students link multiple representations while using CAS technology (Arbaugh, et al., 2010). In addressing this gap, the purpose of this study is to gain insight into the ways in which students connect multiple representations during task-based partner interviews using TI-Nspire CAS. Specifically, this study seeks to answer the following research question: When solving equations using CAS as a representational toolkit, how can students' connections between multiple representations be characterized?

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

Theoretical Framework

Janvier (1987) defines the psychological conversion made from one type of representation to another as a translation process. In other words, the meaning of a source representation is interpreted in reference to a target representation perspective. Adapted from Huntley, Marcus, Kahan, and Miller (2007), a useful framework for investigating multiple representations is the "Rule of Four" model, which involves symbolic, verbal, graphic, and numeric representation systems, with arrows indicating translations between them. Morgan, Mariotti, and Maffei (2009) posit, "converting between different systems of representation is a critical cognitive activity for developing understanding of a mathematical object" (p. 247).

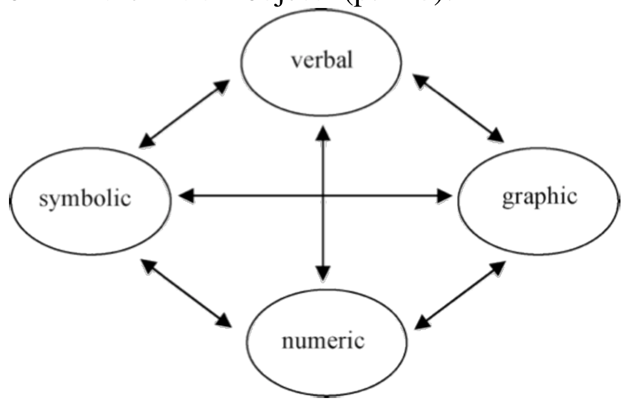


Figure 1: Rule of Four illustrating translations between representations.

The design of the TI-Nspire CAS Touchpad (OS v2.0) houses representations on separate types of "Pages" with a pre-determined structure and main representation for each (including Calculator, Notes, Graphs, Table). For clarity in determining students' use of representations, the Page type and corresponding prominent representation are integrated: symbolic-calculator (S), verbal-notes (V), graphical-graphs (G), and numeric-table (N).

In this context, the construct of representational fluency serves as a tool to characterize students' multi-representational activity. Inspired by Sandoval and colleagues (2000), representational fluency (RF) is the ability to construct, interpret, translate between, and link multiple representations. It is implicit in this definition that both the construction of representations on CAS (inscriptions) and discourse about these representations (e.g., interpretations) are of importance for accessing students' RF. Specifically, I focus on the connections (links) students make between representations to be an indicator of students' RF.

For purposes of this study, a student(s) is said to *make a connection between multiple representations using CAS as a representational toolkit* if they give a correct interpretation of multiple, mathematically equivalent, representations that are evident in their CAS activity or reflection on CAS activity or inscription(s). In other words, for a student to make a connection they must verbalize that they are coordinating information (i.e., invariant features of the object in question as evident in mathematically equivalent representations) in their interpretation of one representation in terms of another, or a pair of representations.

Methodology

Participants and Context

The study was conducted at a Midwest high school that drew accelerated students from several area high schools for mathematics and science coursework only. The school and classroom context was chosen based on the requirement that students have access to TI-Nspire CAS technology, and use this technology regularly in their second year algebra classroom. Of the 25 students in the targeted class, four volunteered to participate in the study and were interviewed. These ninth grade students had taken algebra in middle school and would take geometry after passing their current algebra class. By the beginning of the data collection for this study, students had been using CAS as a regular part of their mathematics instruction for seven weeks.

On three separate occasions prior to interviews with students, the researcher observed the enacted curriculum of students' classroom and took field notes on students' and teachers' use of CAS with regards to multiple representations and use of the adopted textbook, the third edition of the University of Chicago School Mathematics Project *Advanced Algebra* (Flanders et al., 2010). The main purpose of these site visits was to inform the design of instrumentation for the CAS-based task structured interviews; it was necessary to understand the enacted and written curriculum in this classroom to design tasks that involved accessible yet non-routine mathematics and familiar CAS functionality. The classroom observations, reviews of the written curriculum, and conversations with the teacher verified that all student participants had opportunities to learn mathematics through a variety of representations. Moreover, through the use of hypothetical or imperative language, the teacher and textbook often recommended or expected that students use CAS to create representations.

Data Sources and Instrumentation

The main data sources were digital video recordings of task-based interviews conducted with two pairs of students. The teacher determined pairs of students based on students' ability to work well together while engaged in mathematics tasks. Coincidentally, same sex pairs were formed (one male, one female), and the teacher reported that despite the high-ability of all students at this school, there was some variability in these four students' mathematical abilities. On two separate occasions during Fall 2010, each pair was interviewed for 50-minutes during class time. Partner interviews were conducted based on the rationale that richer data would be generated than what might transpire in interviews with individuals. In partner situations, students' interactions with each other and their CAS were perceived to be more authentic to their classroom experiences in which tablemates were observed to regularly communicate about mathematics and CAS technology.

During the interviews, students were prompted to solve two equations, Task R and L (see Figure 2). Both tasks were presented in an initial verbal representation, while Task L also included a graphic representation. These tasks were designed so that various constraints and affordances of the initial representation(s), the context, the mathematics, and/or the technology, might prompt students to construct and/or translate between representations.

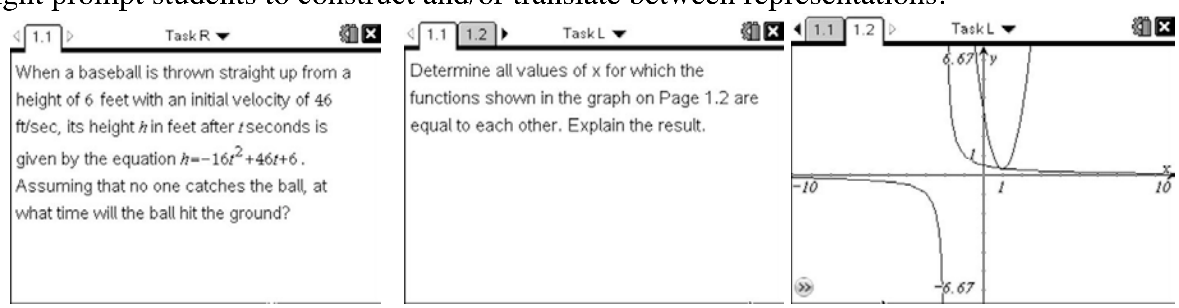


Figure 2. Tasks presented to pairs of students on CAS and in paper form.

Each task was stored and saved as a TI-Nspire document that each student accessed electronically on a handheld CAS. Participants were encouraged to use CAS for the entire interview, but they were also provided pencils and paper copies of screen shots of the tasks (as shown in Figure 2). Data from saved electronic documents of completed tasks and paper and pencil work (if any) were collected at the close of each interview.

During the task-based interviews, participants were encouraged to follow a "think aloud" protocol, openly conversing with one another throughout the interview, providing verbal explanations to accompany their CAS activity as they completed each task. To guard against the researcher changing the cognitive demand of the tasks, a collection of interview prompts was prepared to provide parameters for the interactions between the researcher and participants. The overall intent of these prompts was to elicit further explanations from the students regarding the meaning of their solution approaches with respect to multiple representations.

For instance, "Linking Probes" were given after students had considered the solution and/or solution process from multiple representations (e.g., "How is what you see/did here the same or different from what you see/did here?"). In some cases, students considered multiple representations on their own; in other cases, the researcher encouraged the use of multiple representations through probes (e.g., "Could you solve this in another way?"). In other words, if the students did not self-prompt the use of multiple representations or got stuck in an approach for several minutes, the researcher suggested students consider an alternative approach. Although it is possible that these prompts encouraged students to elicit connections between multiple representations that would not have otherwise been verbalized, this was deemed appropriate because the focus of the study was on the connections students were able to make, rather than on those they chose to verbalize.

Data Analysis

Two video files per interview—one per student in a pair that captured each student's CAS screen—were synchronized into one timeline for video analysis using Studiocode (SportsTec, 1997-2010). The merging of video files allowed for a data analysis method that accounted for both individual and taken-as-shared understandings (cf. Cobb & Yackel, 1996) of the pair of students while they solved equations using CAS. Seed ideas for an analytic framework for linking multiple representations were developed a priori to data analysis, yet the cyclical process of coding, developing, and refining an analytic framework occurred in several stages.

Initial rounds of analysis involved coding and memoing using a grounded-theory inspired approach in which the code categories and descriptions were developed in response to student data. Instead of inventing new terminology in all cases, a conflation of existing terminology was determined to be more beneficial. Specifically, the analytic framework was shaped by the analysis of data and was also purposefully crafted as an amalgamation of existing categorizations and descriptions of students' connections from the literature. A mathematics educator who was familiar with the study and related literature critiqued a refined version of the analytic framework. The entire data set was then reexamined and all instances of connections were coded, memoed, and transcribed, allowing segments of data to be revisited as the analytic framework was being developed. With all connections instances coded, emerging categories, sub-categories, and definitions for the framework were further refined into a hierarchical structure.

Results

The proposed framework for representational fluency is hierarchical in terms of the direction and number of students' translations between representations and the nature of their connections with respect to the perceived goals of students' problem solving activity. Specifically, connections are categorized to be *uni-directional*, *bi-directional*, *multi-directional*, or *abstract*. Both bi-directional and multi-directional connections necessarily involve uni-directional connections. The difference between bi-directional and multi-representational connections is that the former involve a pair of representations (to and from two distinct representations), and the latter involve more than two different representations (and may include a bi-directional pair). Abstract connections go beyond specific reference to types of representation to generalizing the underlying mathematical objects/principles (e.g., equations/equality). Table 1 outlines the framework levels with brief descriptions; connections can be categorized as *one* of: I, IA, IB, II, IIA, IIB, IIC, III, or IV. Transcript excerpts are discussed next to exemplify select levels and sub-levels including: *uni-directional justification*, *bi-directional reconciling*, *multi-directional connection*, and *abstract connection*. The examples were chosen to distinguish between justification and reconciling codes, and also to give more detail on each of the four directional categories, highlighting the hierarchical nature of the framework.

Level of Connection	Brief Description
I. Uni-directional Connection	Translation; interprets meaning of a given source representation in reference to a target representation (Janvier, 1987).
IA. Representational Resourcefulness	Uses a representation to overcome a barrier (Jon Davis, personal communication, 11/18/2010).
IB. Uni-directional Justification	"Use representations as justifications for other claims" (Sandoval et al., 2000, p. 6).
II. Bi-directional Connection	Translation and complementary translation (Janvier, 1987).
IIA. Bi-directional Justification	Pair of representations are used to (dis)confirm an approach (Sandoval et al., 2000).
IIB. Bi-directional Reconciling	Coordinated activity; checking the solution between two representations (Kieran & Saldanha, 2008).
IIC. Reflection on Reconciled Objects	Reflection on the compatibility of a result between a pair of representations (Kieran & Saldanha, 2008).
III. Multi-directional Connection	More than two representations are related by translation processes.
IV. Abstract Connection	A generalization is made across different representations.

Table 1. An analytic framework for representational fluency based on sophistication of connections between multiple representations.

In a *uni-directional justification* connection type, students "use representations as justifications for other claims" (Sandoval et al., 2000, p. 6). In other words, a representation is used to confirm or disconfirm a conjecture or solution approach in another representation. This code is not the same as checking the end result or product of a solution approach against another (i.e., reconciling), instead, the emphasis is that some information from a source representation is used to inform the solution approach in a target representation before the solution is obtained. For example, students from the first interview, attempted to solve Task R using the graph and relate the points where the graph crosses the x-axis to the verbal problem situation in which it only makes sense to have a positive value for time.

Researcher: So how can you use this graph to solve?

Student A: …There are there's two points where it crosses the x-axis [taps finger on desk

twice].
Student B: Right.
Student A: So it'd have to be the positive number and not the negative number because you can't have a negative time.
Student B: True.

In the exchange above, a contextually-based verbal representation is used to justify what is reasonable for a solution on the graph. This is not considered representational resourcefulness because the students were not stuck in using the graph, thus the reference to the verbal representation was to inform their graphic solution.

At the second level of the framework, instances of *bi-directional reconciling* are specific events of coordinated activity in which the student(s) is moving back and forth between two representations. For example, the results of equation solving processes are reconciled between a pair of representations in the second interview while students completed Task L.

(1) Student C: Oh, wow, there's more to the equation. [re-executes solve command on calculator page: solve(1/(x+2)=3*(x-1)^2+0.3,x), ENTER, yields x=-1.96245 or x=0.873499 or x=1.08895]
(2) Student C: … [traces x=1.09 on graph] Yeah look at that, it works now. I got, I got them to equal at 1.09 just like it does in the equation [looks at calculator page, mistakenly points to x=-1.96245].
(3) Student D: At x=1.09? [traces on graph]
(4) Student C: Oh wait. It's very close to 1.09, it just rounds up [comparing approximated values on calculator screen, x=1.08895 and rounded values on graph; continues to compare other values]
(5) Student D: [traces near x=1.11…x=1.08 on graph] So they're basically the same.

In the above excerpt, both students reconciled the results between the symbolic and graphic representations. Student C reconciled the solution from symbolic to graphic (lines 1-2), back to symbolic (line 4), a complementary translation. The utterance from Student D in line 5 is interpreted to mean that the solution of 1.09 was reconciled to be the same in both the graphic and the symbolic representations, taken as evidence of a bi-directional connection in which the solution is checked between two representations.

A student(s) is said to make a *multi-directional connection* when more than two different representations are related by translation processes. The example below from the first interview is a taken-as-shared multi-directional connection between graphic, verbal, and symbolic representations.

(1) Student B: It's [the graph] actually telling you it would hit the ground at 3 seconds, where as with the calculator you can make so many mistakes when figuring out the problem …
(2) Student A: I'm going back and I'm putting three in for the answer and I'm seeing if it comes out zero [typing 0=-16(3)^2+46(3)+6 then ENTER in the calculator page yields "true"]. Which it says it's true, so—
(3) Student B: [types h(3), ENTER, yields 0] Yep.
(4) Researcher: So what does that help you to understand about the problem?
(5) Student A: That the point that we got is the time it took for it to hit the ground.
(6) Student A: And then when we plugged it back into the calculator it told us that the equation is equal to zero and that's what we were looking for.
(7) Student B: Yup, and it basically reassured the fact that the graph, that that point was right.

Consistent with the definition, lower levels in the framework are evident within the dialogue of the above example. Student A interprets the graphical solution in terms of the verbal representation, a translation (line 1). Both students reconcile the solution obtained from the graphic representation with a symbolic representation (lines 2-3, 7). Third, the students reflect on the results that had been reconciled between the graphs and calculator pages and relate this to the verbal representation (lines 4-6). In sum, the students have related more than one pair of representations by a translation process. The fact that uni-directional and bi-directional connections are identified within the multi-directional connection is also evidence of the hierarchical nature of the framework, yet instances were not double-coded.

A student(s) is said to make an *abstract connection* when they make a generalization across different representations. In particular, a student demonstrates flexibility in solving equations from multiple representations and is able to generalize the process of solving equations from a functions-based perspective in which an equation is viewed as two expressions, interpreted as functions, which can be viewed from symbolic, graphical, and/or numeric representations. At this level of a connection, the notion of equality is understood from multiple representations. For example, the students in the first interview were asked to reflect on the fact that they had obtained a solution using the graphs page, but hadn't obtained a solution using the calculator page for Task R. This led Student B to generalize that "Because the graph is just a symbol of the equation [switches from Graphs page to Calculator page] or like the diagram of the equation so you should have been able to get the same thing." Moreover, when solving a given equation, Student B articulated that, "If you did it correctly you should have gotten the same answer." So at this point in the interview, even though the students had only successfully solved the task using the graphic representation, Student B was able to articulate that the same solution should be obtainable using a symbolic representation. This is evidence of a generalization across different representations.

Discussion

Students' difficulties in connecting representations of algebraic objects have been well documented. The definition of a connection and the analytic framework for representational fluency proposed here are the building blocks for future research aimed at understanding students' strengths in connecting representations and for instruction designed to foster richer connections among multiple representations using CAS. Using connections between multiple representations as an indicator, students' representational fluency can be categorized in a hierarchical manner per the direction and purpose of students' translations between representations. The results and examples discussed above can be illustrated using the Rule of Four framework, showcasing the four distinct levels in the proposed analytic framework (see Table 2). By teaching topics in algebra using pairs and sets of representations, and emphasizing bi-directional and multi-directional connections through the use and reflection on CAS inscriptions, students may develop representational fluency and in turn come to a more robust conceptual understanding of the mathematical object(s) in question.

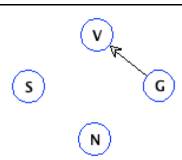
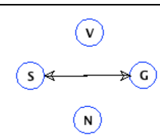
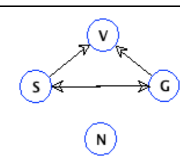
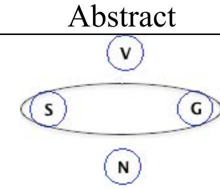
Uni-directional	Bi-directional	Multi-directional	Abstract
			

Table 2. Illustrations depicting levels of connections using Rule of Four framework.

The proposed framework and schematic representations mask some of the intricacies of the nature of students' connections as others have studied them. It would serve well to use the framework for analyzing data from a larger sample size. Specifically, the multi-directional connection category might be expanded to account for students' problem solving goals, analogous to the subcategories for the bi-directional level. Additionally, future research might employ the use of the analytic framework to elucidate types of opportunities afforded by written curricula to make connections between multiple representations with CAS. This framework would also be useful in the design of instructional intervention or tasks aimed to foster sophisticated connections between representations through the use of directionally linked dynamic technology environments.

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