A RESEARCH SYNTHESIS OF PRESERVICE TEACHERS' KNOWLEDGE OF MULTIPLYING AND DIVIDING FRACTIONS

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This paper describes a synthesis conducted to determine what research says regarding preservice teachers' understanding of fractions and identify the gaps in their existing knowledge basis. Specifically, this paper will address a smaller portion of the synthesis and report the findings from fraction multiplication and division topics. Results indicated that preservice teachers' understanding of fraction multiplication and division is limited and largely based on rote procedures. Implications for teacher education programs and future research studies are provided.

Keywords: Teacher Education–Preservice; Rational Numbers

Objectives

Elementary teachers need a "solid understanding of mathematics so that they can teach it as a coherent, reasoned activity and communicate its elegance and power" (Conference Board of the Mathematical Sciences (CBMS), 2001, p. xi). However, research studies on preservice teachers' mathematics knowledge have shown that many possess a limited knowledge of mathematics in key content areas such as number. For example, Thanheiser (2009) found that only 3 of 15 preservice teachers held a conception of place value that allowed them to explain how and why the subtraction algorithms with three-digit numbers work. The National Mathematics Panel affirmed the "proficiency with fractions" as a major goal for K–8 mathematics education because "such proficiency is foundational for algebra and, at the present time, seems to be severely underdeveloped" (p. xvii). Therefore, developing such proficiency in preservice elementary teachers is a critical task for mathematics educators. As the authors of *The Mathematical Education of Teachers* suggest, "The key to turning even poorly prepared prospective elementary teachers into mathematical thinkers is to work from what they *do* know" (CBMS, 2001, p. 17). There is still much to be learned about preservice teachers' (PST) conceptions in a wider array of topics and how we might use such knowledge in designing mathematics courses for PSTs. In this paper, we discuss the main findings from a research synthesis of existing studies on preservice elementary teachers' fraction knowledge to identify critical directions for future research specifically in the area of fraction multiplication and division.

Theoretical Framework

Shulman (1986) proposed three categories of content knowledge for teachers: (a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge. For Shulman, subject matter content knowledge includes knowing a variety of ways in which "the basic concepts and principles of the discipline are organized to incorporate its facts" and "truth or falsehood, validity or invalidity, are established" (p. 9). Pedagogical content knowledge refers to the knowledge of useful forms of representations (e.g., analogies, illustrations, explanations) of subject-matter ideas that make it understandable to others, and an understanding of the conceptions and pre-conceptions students bring to the learning processes. The third type of knowledge, curricular knowledge, includes knowledge of a "full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances" (p. 10).

Shulman's ideas on pedagogical content knowledge sparked a huge interest in knowledge for teaching, eliciting over a thousand studies throughout a number of content areas with a large number of these studies

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focusing on teachers' knowledge of mathematics (e.g., Davis & Simmt, 2006; Ball, Thames, & Phelps, 2008; Hiebert, 1986; Ma, 1999). Deborah Ball and her colleagues introduced the term "mathematical knowledge for teaching" (e.g., Hill, Ball, & Schilling, 2008), which focused on the work that teachers do when teaching mathematics.

Building on Shulman's (1986) categories of knowledge, Ball, Thames, and Phelps (2008) introduced a framework for mathematical knowledge for teaching. This framework broke subject matter knowledge into three categories: common content knowledge (CCK), the mathematical knowledge that should be known by everyone; specialized content knowledge (SCK), the knowledge of mathematics content that is specific to the work of teachers; and horizon content knowledge, which involves understanding how different mathematical topics are related. Pedagogical content knowledge was similarly broken into: knowledge of content and students (KCS), which dealt with understanding how students relate to different topics; knowledge of content and teaching (KCT), which involves the sequencing of topics and the use of representations; and knowledge of the curriculum as a whole. While a number of different frameworks look at mathematical knowledge for teaching, this framework proposed by Ball and her colleagues is typically looked at by groups focusing on what teachers know about mathematics and served as a framework for our study as well.

Background and Research Questions

This work was initiated at a PME-NA working group in 2009 and 2010. The members of the working group all taught specially designed mathematics courses for elementary school teachers in the United States and sought to improve their practice by building on PSTs' current knowledge. The working group was formed with a goal of summarizing the prior research addressing PSTs' content knowledge and its development with the idea that we could both improve our teaching and design further research to extend what we know about PSTs' mathematical knowledge. We broke into smaller groups by content area, (whole-numbers, fractions, decimals, geometry, and algebra), and attempted to synthesize the current research in each of these fields.

This paper reports a synthesis of the research that has been done to this point on fraction multiplication and division. These are the areas that came up most frequently in the literature. Our goals for the research synthesis were to: (1) to identify what we already know about preservice elementary teachers' knowledge of fractions in both the domains of subject-matter and pedagogical content knowledge, and (2) to identify the knowledge gap in the existing knowledge basis to help guide future research endeavours.

Methods

The first step of conducting this research synthesis was to identify the existing literature. Initially, we decided to restrict our search to only published research journal articles to maintain the quality of the findings. However, recognizing the time lag required for publication, we decided to expand the search to also include proceeding papers published in 2007, 2008 and 2009, and dissertations published since 2005. With key words such as "preservice teachers," "preservice elementary teachers," "fraction," "fraction concepts," "fraction operations," "fraction multiplication," "fraction division," and "rational numbers," we searched the ERIC, Google Scholar, Dissertation Abstract and Rational Number Reasoning databases (*gismo.fi.ncsu.edu/database)*. We also manually searched through the recent PMENA and PME proceedings because we were not sure about the time lag for a proceeding paper to be included in the above databases. This search yielded 42 journal and proceeding articles and 3 dissertations between 1988 and 2011.

The second step required the research team to locate these papers and skim through them to determine if they had a research question focusing on preservice elementary teachers' fraction knowledge. Fifteen papers were rejected because they did not meet this criterion. For example, some papers were about curriculum sequence or instructional activities that would facilitate preservice elementary teachers' learning of fractions, while others were about preservice elementary teachers' teaching of fractions. So the synthesis we report on in this paper is based on 30 papers and dissertations.

Careful readings of these documents were carried out during the third step. To assist the comparison across these documents, an entry in a synthesis table was filled with information such as "research questions," "research design," "descriptions of participants," "content foci," "data collection," "data analysis," "findings," and "implications" for each one.

We soon noticed the following main trends in our synthesis table. First, all but two papers were published in two distinct time periods: 1988 to 1994, and 2005 to 2011, with 21 articles falling in the latter time period. This shows a growing interest in preservice elementary teachers' fraction knowledge. Second, the majority of the papers focused on preservice teachers' subject matter content knowledge, with a handful on pedagogical content knowledge such as their ability to construct valid fraction story problems and representations, and their ability to predict students' errors, and none focused on preservice elementary teachers' fraction curricular knowledge. Third, the majority of the studies (17 of them) focused on preservice elementary teachers' fraction knowledge in a single sub-concept, for example, fraction multiplication or fraction division, with six studies focused on two or more fraction sub-concepts. Only four focused on the development of fraction knowledge. In the following section, we will report the main findings about preservice elementary teachers' fraction knowledge related to multiplication and division.

Results

We organized our findings around the major themes that we found in the research. These dealt with PSTs' common content knowledge of fraction procedures, their specialized content knowledge of being able to write story problems modeling situations with fractions, their knowledge of content and students, in relation to common student errors in regards to fractions, and of different instructional interventions designed to help improve this knowledge.

Preservice Teachers' Understanding

Research illustrates that preservice elementary teachers are most uncertain about dividing fractions, followed by subtracting, multiplying, and then adding fractions (Newton, 2008). This becomes problematic especially when the ability to represent an operation with diagrams and story contexts has been identified as an important type of specialized mathematics knowledge for teaching (Ball, Thames, & Phelps, 2008). Such ability even becomes more important in the context of the Core State Standards of Mathematics implementation. Grade 5 and 6 students are expected to solve real word problems involving fraction multiplication and division through visual fraction models (e.g., a tape diagram, number line diagram, or area model) and equations to represent them.

However, studies have shown that the majority of preservice elementary teachers do not have a strong ability to represent fraction multiplication and division with story problems (Ball, 1990; Luo, 2009; Simon, 1993). While preservice teachers have an easier time on writing story problems when one of the two numbers involved in multiplication and division computation is a whole number, many of them were not able to do the same when mixed fractions are involved or when both given numbers are fractions. For example, Luo (2009) found that preservice teachers struggled to provide an appropriate context and representation given a symbolic expression of fraction multiplication. In agreement with Goodson-Epsy (2009), Luo concluded that whole number by fraction multiplication is easier than problems with two fractions. Of the 127 preservice early childhood and elementary education students in Luo's study, only 27% could construct a valid word problem to represent $1/2 \times 1/3$, while 58% could construct a valid word problem to represent $1\ 2/3 \times 4$. In addition, Luo found that the majority of the preservice teachers used a "multiplication as repeated addition" construct which can be problematic when working with non-whole numbers.

Findings from Ball (1990) and Simon (1993) indicated that many preservice teachers were unable to generate a valid story problem for fraction division problems such as $1\ 3/4 \div 1/2$ or $3/4 \div 1/4$. Many of them wrote story problems that were actually for multiplication of fraction by either the given fractions or by the reciprocal of the divisor. It also appeared that preservice teachers who attempted to use measurement division contexts were more successful than those that used partitive division. The field has just began to examine preservice teachers' proficiency of using diagrams to represent fraction

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multiplication and division. One study that compares Taiwanese and U.S. preservice elementary teachers' fraction knowledge contains a multiple-choice item asking them to choose the diagram that can not be used to model $3/4 \times 4/5$ or $4/5 \times 3/4$ (Luo, Lo, & Leu, 2011). The finding suggested that the majority of preservice elementary teachers from both populations were unable to identify the correct answer which simply showed a diagram of 3/4 and a diagram of 4/5 with a multiplication sign listed in between.

Preservice Teachers' Knowledge of Student Errors

When teachers enter the classroom, they need to have an understanding of mathematics content as well as student thinking (Ball, Thames, & Phelps, 2008). By understanding how students think, teachers can establish classrooms where discussions focus on the validity of students' responses. Knowing how preservice teachers interpret student responses before they enter a classroom can provide a foundation for the types of activities needed in teacher education programs.

In the context of fraction division, research has shown that preservice teachers' analyze student responses at a surface level (Son & Crespo, 2009; Tirosh, 2000). For example, when shown a correct student's method that included dividing the numerators and denominators, preservice teachers argued that the method works but only because the answer matched to what they got by inverting the second fraction and multiplying (Son & Crespo, 2009). The participants in this study did not delve deeper into the concepts underlying the methods and their beliefs about teaching and learning strongly correlated with their responses toward the non-traditional algorithm.

Other research has shown that similar conceptions hold true when preservice teachers analyze students' incorrect methods for dividing fractions (Tirosh, 2000). Tirosh found that participants were able to identify common problems that students would have, but they generally attributed these errors to students not understanding the algorithm for dividing fractions. Thus, preservice teachers had some understanding of the types of difficulties students may have, but were not able to justify why those methods are incorrect (Tirosh, 2000).

Improving Preservice Teachers' Understanding

Several recent studies have examined the effects of special instructional strategies on preservice teachers' procedural and conceptual knowledge for fraction: for example, use of manipulatives (Green, Piel, & Flowers, 2008); web-based instruction (Lin, 2010) and problem posing (Toluk-Ucar, 2009). All three studies used the experimental design with control and experimental groups. All showed significant better improvement by the preservice teachers in the experimental groups.

For example, in a study conducted by Green, Piel, and Flowers (2008) only 15% of the preservice teachers in the experimental group were able to illustrate the fraction division 1 1/2 ÷ 3/4 during the pre-test. After working with manipulatives for four weeks, 66% of them were able to do so. These studies also pointed out that it was more difficult for preservice teachers to illustrate fraction multiplication and division situations than it was for them to write story problems. This pattern remained true after the treatment. For example, while the percent of preservice teachers in the experimental group who were able to write story problems for fraction division increased from 2% to 88% between the pre- and post-test in the study by Toluk-Ucar (2009); the corresponding result for drawing diagram for fraction division were 2% and 80%.

Discussion

Research regarding preservice teachers' mathematical content knowledge illustrates that they have a rule-based conception of fraction multiplication and division. Misconceptions result from overgeneralized rules from other number systems, such as multiplication always makes bigger, or result from not understanding algorithms for multiplying and dividing fractions. Other difficulties preservice teachers have with fractions stem from not having a conceptual understanding of the mathematics. Thus, when asked to provide contextualized situations, they tend to create situations not related to the original problem or are unable to generate a situation at all.

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Preservice teachers' conception of fraction multiplication is based off of the part-whole meaning of fractions. Studies suggest that in the context of multiplication, preservice teachers are more successful when the problem contains fractions less than one and whole numbers (Goodson-Epsy, 2009; Luo, 2009). Thus, more experiences with fractions greater than one as well as more problems not incorporating whole numbers are needed in teacher preparation programs.

Preservice teachers' conception of fraction division is largely focused on the sharing meaning of division. In addition, fraction division understandings are procedurally or algorithmically based. As a result, preservice teachers have difficulty with interpreting fraction division situations and struggle with representing the situation with an appropriate context.

Recent reports have begun to document the ways in which preservice teachers' develop an understanding of fractions (Tobias, 2012). Tobias (2012) found that their fraction understanding does not develop linearly in that knowledge of one topic may not be fully developed before they start to learn another. Thus, classroom instruction may need to focus on multiple fraction concepts before preservice teachers can develop an understanding of one idea.

Conclusion

By understanding preservice teachers' knowledge of fraction multiplication and division, future studies and improvements in teacher education programs can start to investigate the ways in which preservice teachers overcome their misconceptions to develop the mathematical understandings needed to be an effective teacher. Virtually every study suggests that strong teacher education programs and improvements to teacher education courses are needed, however little has been done to document the types of experiences preservice teachers need. With current recommendations suggesting that mathematics teacher education programs design instruction around what preservice teachers do know, and with the majority of studies focusing on fraction division, research is needed regarding preservice teachers' understanding of other fraction topics. Though there has been a recent increase in the number of publications pertaining to preservice teachers' knowledge of fractions, this is still not enough for teacher educators to have an adequate understanding of how preservice teachers think.

Acknowledgments

The authors would like to thank Cindy Edgington for her work on previous versions of this manuscript.

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