PROSPECTIVE TEACHERS' TRANSITION FROM THINKING ARITHMETICALLY TO THINKING ALGEBRAICALLY ABOUT EVEN AND ODD NUMBERS

Kadian M. Callahan
Kennesaw State University
kmcallahan@kennesaw.edu

Amy F. Hillen
Kennesaw State University
ahillen@kennesaw.edu

We present findings from a study of prospective middle school teachers' reasoning as they transitioned from thinking arithmetically to thinking algebraically about even and odd numbers. Teachers were asked to make sense of and use two representations of even and odd numbers to model them and to make connections between the representations. Analysis of a whole-class discussion indicates that although teachers easily represented even and odd numbers using an algebraic generalization, they grappled to make sense of a given geometric model. As teachers worked to make sense of the geometric model, they transitioned back and forth among three ways of interpreting the model (two of which were incorrect; one of which was correct).

Keywords: Algebra and Algebraic Thinking; Classroom Discourse; Number Concepts and Operations; Teacher Education–Preservice

Purpose of the Study

Standards for K–12 mathematics emphasize the use of multiple representations (e.g., written words, diagrams, symbolic expressions, physical models, graphs) for making sense of and communicating mathematical ideas (Common Core State Standards Initiative [CCSSI], 2010; National Council of Teachers of Mathematics [NCTM], 2000). Scholars argue that examining different representations can make mathematics more meaningful by illuminating different aspects of a mathematical idea or relationship (Cuoco, 2001; NCTM, 2000). Representations may be particularly important in the middle grades as students make the transition from thinking *arithmetically*—for example, working with specific even and odd numbers—to thinking *algebraically*—for example, making sense of even and odd numbers as sets of numbers that can be generalized (CCSSI, 2010; NCTM, 2000). The difficulties that many students have in making this transition are well documented (e.g., Chazan, & Yerushalmy, 2003; Smith, 2003), and scholars suggest that considering both visual and numerical modes of generalizing may facilitate this transition by helping students understand the nature of variable and familiarizing them with the structure of algebraic expressions (Lannin, 2003; Rivera & Becker, 2009; Thornton, 2001).

Considering different mathematical representations has also proven to be beneficial for teachers. Research shows that making sense of and using different representations can strengthen teachers' content knowledge by requiring them to make connections among representations and can strengthen teachers' pedagogical content knowledge by providing them greater access to student thinking as students interpret different representations (Herbel-Eisenmann & Phillips, 2005; Izsák & Sherin, 2003). Thus, one way to strengthen teachers' ability to support middle school students' transition from arithmetic thinking to algebraic thinking is to provide them with opportunities to reason about and use different representations to make sense of mathematical ideas. In particular, the mathematics content courses taken during their teacher preparation programs might be a promising context for such learning experiences.

This research report presents findings from a study that investigated the ways in which a class of prospective middle school teachers (PMSTs) reasoned about different representations of even and odd numbers during their work on a number theory unit in a mathematics content course for PMSTs. The context of even and odd numbers was chosen for two reasons: (a) it is central to the number theory ideas that are studied in the middle grades (CCSSI, 2010; Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006); and (b) it is accessible, yet still presents some challenge, for teachers (Smith, 2011).

Theoretical Perspective

Much of the mathematics education literature in recent years conceptualizes the learning of mathematics as a collective act. The view that learning is a social endeavor is rooted in theories of cognitive development based on foundational ideas by Jean Piaget and Lev Vygotsky. Piaget argued that social interactions are central to knowledge development because they influence individuals' attempts to resolve conflicts between their perspective and the perspectives of others (Brown & Palincsar, 1989; Rogoff, 1998). Vygotsky also emphasized social interaction as an essential element of cognitive development. He asserted that there is fluidity between self and others, and cognitive exchanges at this boundary mitigate the process through which knowledge development occurs. Sociocultural theories of mathematics learning integrate the perspectives of both Piaget and Vygotsky and maintain that one cannot examine individual students' reasoning and cognitive development without considering how it is influenced by the social context or examine the social context without considering how individual students' reasoning influences that context (Cobb, 2001; Cobb & Yackel, 1996).

Particularly relevant to the present study is the notion that representation can be conceptualized as a collective act occurring within a specific social and mathematical context. Collective representation involves "negotiating individually constructed representations in the shared space of a group or classroom as well as the teacher's role in facilitating these interactions" (Stylianou, 2010, p. 327). This study investigated the extent to which considering different representations of even and odd numbers was helpful in PMSTs' collective transition from thinking arithmetically to thinking algebraically.

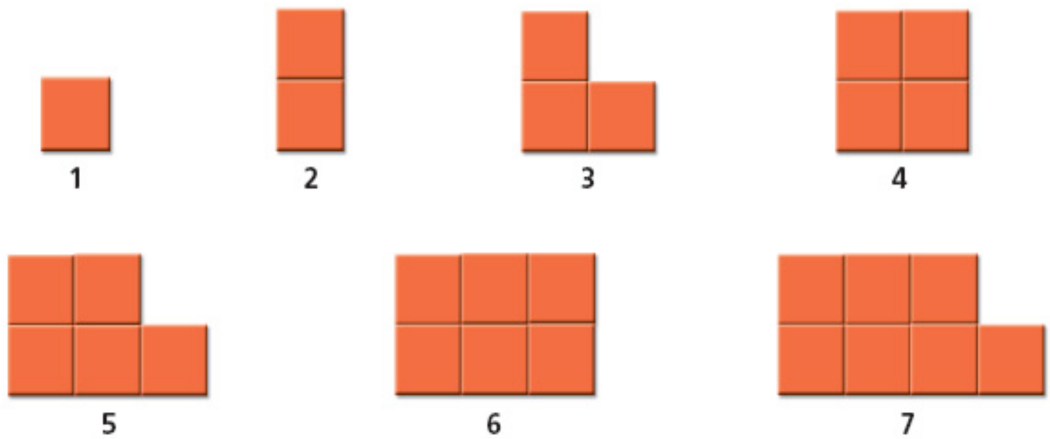
Methods

The study was conducted during Spring 2011 at a four-year, public university in the southeastern United States. The 22 participants (14 female; 8 male) were PMSTs enrolled in a required 16-week mathematics content course for prospective middle and secondary school teachers. (Since secondary school teachers at this university are certified to teach both middle and secondary school, all participants were considered prospective middle school teachers.)

The course met once a week for two hours and thirty minutes and focused on three content strands central to the middle grades: (a) number and operations; (b) algebra and functions; and (c) geometry and measurement (CCSSI, 2010; NCTM, 2000). Throughout the course PMSTs were asked to reason about and make connections between and among representations of mathematical ideas across these three content strands. The instructor (and first author) also immersed PMSTs in the processes of mathematical inquiry (CCSSI, 2010; NCTM, 2000) and modeled a type of teaching whose goal was learning with understanding (Carpenter & Lehrer, 1999). PMSTs were encouraged and expected to regularly engage in discussions with their peers as they shared their thinking about problems and solution strategies, reasoned about and made connections between mathematical ideas, made and evaluated conjectures, and developed and revised mathematical arguments.

Data collection occurred during a five-week unit on number theory that included considering numeric, algebraic, and geometric representations of even and odd numbers and exploring how to add and multiply even and odd numbers using these different representations. The primary data source was transcripts of video of the whole-class discussions that the second author filmed. Secondary data sources included: (a) the second author's field notes; (b) written work produced by PMSTs during each class; and (c) audio recordings of weekly meetings between the first and second authors in which they reflected on each class meeting, identified and discussed any ideas that the PMSTs seemed to be struggling with, and discussed the instructor's plans for the next class meeting.

The study reported herein focuses on the first idea that PMSTs grappled with during the data collection, which occurred during the second class meeting of the unit (the fifth class meeting of the semester): explaining how the different components ('2', 'n', and '+1') of the algebraic generalizations of even and odd numbers ('2n' and '2n+1', respectively) were represented geometrically in "Tilo's model" (shown in Figure 1). In particular, PMSTs had difficulty making connections between: (a) the '2' in the algebraic generalizations of both even numbers and odd numbers and the maximum number of tiles in each column of Tilo's model; (b) the 'n' in the algebraic generalizations of both even and odd numbers and the total number of 'complete' columns (i.e., columns that contain two tiles); and (c) the '+1' in the algebraic generalization of odd numbers and the one extra tile that makes an 'incomplete' column. (Although 2n-1 is also a valid generalization of the set of odd numbers, the instructor chose to focus on the 2n+1 generalization in order to be consistent with the source of Tilo's model, the *Connected Mathematics Project* curriculum.)



(Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006, p. 25)

Figure 1: Tilo's model

The transcript of the class meeting that is the focus of this study was analyzed to identify the ways in which PMSTs used the different representations of even and odd numbers to reason about the meaning of the components of the algebraic generalizations of even and odd numbers. Three milestones in PMSTs' thinking about the variable 'n' (shown in Figure 2) emerged from the data: (a) claiming that 'n' represented *one tile*; (b) claiming that 'n' represented *one complete column*; and (c) claiming that 'n' represented the *total number of complete columns*.

Thinking Arithmetically		***Thinking Algebraically***
Milestone a 'n' represents one tile (incorrect)	Milestone b 'n' represents one complete column (incorrect)	Milestone c 'n' represents the total number of complete columns (correct)

Figure 2: Milestones in PMSTs' transition from arithmetic to algebraic thinking

Conceptualizing 'n' as representing *one tile* (Milestone a) would result in a different algebraic expression for each natural number (e.g., 2=(2n), 3=(3n), 4=(4n), 5=(5n), 6=(6n), where n=1 tile) and reflects arithmetic thinking since it treats each number *individually* rather than as belonging to a *set* of numbers with shared characteristics. Conceptualizing 'n' as representing *one complete column* (Milestone b) also results in a different algebraic expression for each natural number (e.g., 2=2(n), 3=2(n)+1, 4=2(2n), 5=2(2n)+1, 6=2(3n) where n=1 complete column); however, it recognizes that even numbers have the

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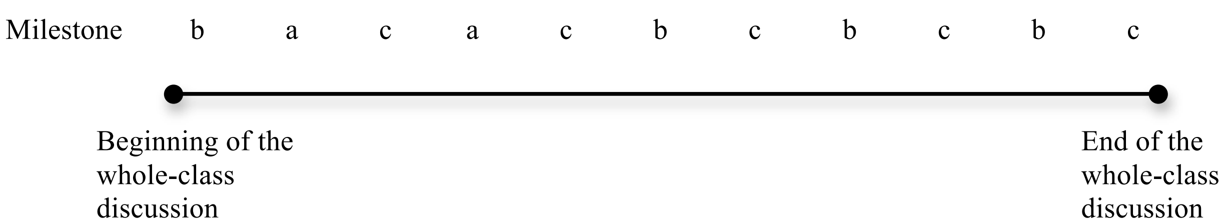
characteristic of being able to put the tiles into groups of two with no tiles left over, and that odd numbers have the characteristic of being able to be put the tiles into groups of two with one tile left over. Thus, Milestone b is considered to be a more algebraic way of thinking because it suggests that there are some shared characteristics among numbers within the same *set*. However, this way of thinking is not completely algebraic because it does not allow for a single algebraic generalization to represent even numbers and another algebraic expression to represent odd numbers. Conceptualizing ‘n’ as representing *the total number of complete columns* (Milestone c) reflects algebraic thinking by recognizing the shared characteristic among even numbers as being able to put the tiles into groups of two with no tiles left over, the shared characteristic among odd numbers as being able to put the tiles into groups of two with one tile left over, and results in one algebraic expression for the *set* of even numbers (2n) and another algebraic expression for the *set* of odd numbers (2n+1).

After identifying and coming to agreement on the extent to which the three milestones reflected arithmetic and algebraic thinking, the authors reanalyzed the transcripts to trace PMSTs’ ideas as they moved back and forth between the three milestones in their transition from thinking *arithmetically* about *individual* even and odd numbers to thinking *algebraically* about *generalizations* of the *set* of even numbers and the *set* of odd numbers.

Results

The transition from thinking arithmetically about *individual* even and odd numbers to thinking algebraically as *sets* of numbers that can be generalized was challenging for this group of PMSTs and did not occur in one direction (from thinking arithmetically to thinking algebraically). Instead, in their efforts to make connections between the geometric and algebraic representations, PMSTs’ understanding of the meaning of the variable ‘n’ in the algebraic generalizations of even and odd numbers (‘2n’ and ‘2n+1’, respectively) shifted back and forth between the three milestones: (a) ‘n’ representing *one tile*; (b) ‘n’ representing *one complete column*; and (c) ‘n’ representing the *total number of complete columns*. As shown in Figure 3, all three milestones were considered at least twice during the whole-class discussion. That Milestone a was abandoned fairly early in the discussion suggests that PMSTs realized that putting tiles into groups of two to make complete columns helped to make some distinction between the *set* of even and the *set* of odd numbers. However, the movement back and forth between Milestones b and c indicates that PMSTs were not convinced that grouping the complete columns together would allow them to clearly distinguish the *set* of even numbers from the *set* of odd numbers and connect to the algebraic generalizations they had identified. Thus, although the discussion ended on the (correct) Milestone c, PMSTs’ understanding of ‘n’ as representing the *total number of complete columns* may have remained fragile. Field notes from the subsequent class, in which PMSTs began operating on even and odd numbers, also reflected this instability.



Milestone a: ‘n’ representing *one tile*
Milestone b: ‘n’ representing *one complete column*
Milestone c: ‘n’ representing the *total number of complete columns*

Figure 3: PMSTs’ transition among the three milestones

The excerpts that follow are illustrative of the struggle that PMSTs had as they collectively made sense of the different representations and transitioned from thinking arithmetically to thinking algebraically about even and odd numbers. In the first section, PMSTs work to make sense of the variable 'n' in the algebraic generalizations and to find commonality within the *set* of even numbers and within the *set* of odd numbers – thus making the transition toward thinking more algebraically, and moving away from Milestone a. In the second section, PMSTs focus on making explicit connections between the features of Tilo's model and the specific components of the algebraic generalizations and move back and forth between Milestones b and c. All names used in the excerpts that follow are pseudonyms.

Finding Commonality Within the Sets of Even and Odd Numbers

Prior to introducing Tilo's model, the instructor asked PMSTs to describe the set of even numbers. PMSTs easily identified commonalities across all numbers in this set. For example, they described even numbers as being *divisible by 2, having no remainder after dividing by 2, being a multiple of 2, being an integer, and including 0*. Despite these ways of describing the set of even numbers, PMSTs struggled to make sense of Tilo's model (shown in Figure 1) when the instructor introduced it. In particular, PMSTs struggled to make connections between individual representations of even and odd numbers in Tilo's model and the algebraic generalizations of even and odd numbers that they identified (Even=2n and Odd=2n+1, where n is a whole number). The source of difficulty was in making sense of the meaning of the variable 'n' in both the algebraic and geometric representations of even and odd numbers. This difficulty is illustrated in the following excerpt where the PMSTs grapple with whether 'n' represents *one tile*, *one complete column*, or *the total number of complete columns* in Tilo's model:

Uberto: Every, every n is a column.
Olive: Yeah, every n is a column.
Uberto: [Tilo's] saying there [are] no sets- there's no columns and then there's one extra.
Kaila: Every two n would have to be a column.
Uberto: Yeah. Every two n is a column.
Instructor: Every two n is a column, ok. Help me see where that is here. How do you see that?
Uberto: Actually isn't it just the n? It is just the n because, look at two. If you plug it [into] the two n equation, you have one [complete] column, and therefore two tiles.

In this excerpt, Uberto and Olive initially state that every 'n' is a column – suggesting that 'n' represents each column. This understanding of the variable is not consistent with the algebraic generalizations of the sets of even or odd numbers. Instead, even and odd numbers were being represented differently and treated *individually* rather than as belonging to a *set* of numbers with common attributes (e.g., 2=2(n), 3=2(n)+1, 4=2(2n), 5=2(2n)+1, 6=2(3n) where n=1 column). Kaila argues that every '2n' would be a column – suggesting that 'n' represents an individual tile. Again, this understanding of the variable would result in different expressions for even and odd numbers (e.g., 2=(2n), 3=(3n), 4=(4n), 5=(5n), 6=(6n), where n=1 tile). While both of these ways of making sense of even numbers suggest that the PMSTs were thinking more arithmetically than algebraically, there seems to be a qualitative difference in Uberto and Olive's thinking as compared to Kaila's thinking. Whereas the expressions aligned with Kaila's thinking showed no commonality within the sets of even or odd numbers, expressions for Uberto and Olive's thinking suggested that the components of the algebraic expressions for even and odd numbers (i.e., '2', 'n', '+1') needed to be considered. This became more evident after Uberto responded to Kaila's idea about the meaning of 'n' by drawing the PMSTs' attention to the number of tiles in a column compared to the total number of columns. As such, Uberto was extending the PMSTs' reasoning beyond thinking about individual even and odd numbers to beginning to find commonality within these two sets of numbers.

Explicitly Connecting the Geometric and Algebraic Models

As the PMSTs continued to reason collectively about the meaning of the variable 'n' in the algebraic generalizations of even and odd numbers, they frequently used the geometric representations to make further distinctions between the components of the algebraic expression. This movement back and forth between representations is reflected in Bobbie's comment:

> How about we just define n and say for two, you have two tiles on top of each other and that's a column, that's one [complete] column. So if you define n as the [total] number of [complete] columns, if you put [it] in the formula, two n, two times one, it'd give you two. So we have seven, right, to get eight, you would add another tile on top of the one that's out we would end up with one, two, three, four columns, n representing the number of [complete] columns, you do two times four, it would give you eight. So we should just have- n should just represent the [total] number of [complete] columns, and it'll give you the number of- um even number.

In this excerpt, Bobbie was thinking algebraically and making an argument for why 'n' represents the *total number of complete columns* rather than representing *one complete column*. By comparing specific even and odd numbers in the geometric representation to the algebraic generalization of even and odd numbers, she was able to determine that the '2' represents the number of tiles in a column, and that the 'n' represents the total number of complete columns. Furthermore, Bobbie identified a difference between columns that are complete (by explicitly stating, "you have two tiles on top of each other and that's a column, that's one [complete] column") and those that are not complete for an odd number (by stating, "you would have to add another tile on top of the one that's out.") As such, Bobbie's way of thinking was consistent with the algebraic generalizations of even numbers and odd numbers and allowed for even numbers to be expressed as '2n' and odd numbers to be expressed as '2n+1'. Isabel clarified the meaning of the components even further by noting:

> I think what [Bobbie and others] are trying to say is that two is a constant; there are two tiles in a column, so that doesn't change. So that the only number that changes is n, and it says how many [total complete columns]. So if you have like eleven [complete columns], it'd be two times eleven which would put you at twenty-two tiles.

Despite PMSTs' transition toward thinking about even and odd numbers algebraically (the discussion ended with Milestone c), as the class continued to make connections between the different representations, the meaning of the variable 'n' in the algebraic generalization was often revisited. The pattern of Milestones c, b, c, b, c, b, c (shown in Figure 3)—shifting back and forth between Milestones b and c—suggests that PMSTs' understanding of the meaning of 'n' in the algebraic expressions for even and odd numbers remained unstable.

Discussion

This study examined PMSTs' reasoning as they transitioned from thinking arithmetically to thinking algebraically about even and odd numbers while making connections between algebraic and geometric representations of even and odd numbers. Flexibility in connecting multiple representations is a critical aspect of mathematical understanding (e.g., Lesh, Post, & Behr, 1987; NCTM, 2000), and the results of this study suggest that this is challenging for teachers. The geometric model (Tilo's model) seemed to be a catalyst for helping the PMSTs in this study reason about the different components of the algebraic generalizations of even and odd numbers (i.e., the '2', the 'n', and the '+1') as they transitioned from thinking about *individual* even and odd numbers to thinking about the *set* of even and the *set* of odd numbers. Furthermore, although PMSTs easily generated algebraic representations of even and odd numbers—that included the variable 'n'—they had difficulty making sense of what 'n' represents when asked to *connect* the algebraic representations to the geometric one. Thus, the introduction of the geometric model revealed PMSTs' conceptions in ways that working exclusively with the algebraic model may not have.

The results of this study also suggest that the transition from thinking about *individual* numbers to thinking about *sets* of numbers is not uni-directional, but rather involves several conceptual milestones—some of which are incorrect—that teachers move back and forth between as they made sense of generalizations of even and odd numbers. It is also important to note that teachers reverted back to the incorrect conceptual milestones throughout the whole-class discussion – suggesting that their misconceptions were rather stable. Thus, the results of this study indicate that mathematics teacher educators face a serious challenge: how to support prospective or practicing middle school teachers in making sense of and connecting multiple representations of mathematical ideas. Future work will investigate how PMSTs used these different representations to add and multiply even and odd numbers and how the mathematics teacher educator who was the instructor of this course supported them in making sense of the representations.

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