COGNITIVE OBSTACLES AND MATHEMATICAL IDEAS RELATED TO MAKING CONNECTIONS AMONG REPRESENTATIONS

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This study investigates cognitive difficulties and mathematical ideas that are related to making connections among representations. A three-week intervention was designed and implemented to help prospective secondary mathematics teachers develop understanding of big ideas that are critical to connection of representations in algebra. This study finds that most participants had difficulties with the concept of variable, the Cartesian connection, and its related idea, graph as a locus of points, and held incomplete concept definitions and concept images of conic curves. During the intervention, however, many showed signs of progress. Many who were initially dependent on memorized forms of algebraic formulas made efforts to consider aspects of the Cartesian connection to make sense of their work.

Keywords: Teacher Education–Preservice; Teacher Knowledge; Algebra and Algebraic Thinking; Advanced Mathematical Thinking

Objectives

The purpose of this study was to investigate cognitive obstacles and mathematical ideas related to making connections among representations and how prospective teachers' thinking progressed while they worked on tasks that were designed to bring out these cognitive issues. The importance of representations in mathematical understanding is well documented (Brenner, Mayer, Moseley, et al., 1997; Knuth, 2000; Moschkovich, Schoenfeld, & Acavi, 1993), and students' learning of mathematics through representations is recommended in the Principles and Standards for School Mathematics (NCTM, 2000) and the Common Core State Standards (CCSI, 2010). However, not much is known about cognitive obstacles or ideas that are involved in making connections among representations or how to help students develop mathematical understanding through connection of representations (Bosse, Adu-Gyamfi, & Chetham, 2012; Even, 1998) except that even mathematically capable individuals have compartmentalized understanding of mathematical representations (Gagatsis & Shiakalli, 2004; Hitt, 1998; Vinner, 1989). Making connections among representations is one of the big ideas in mathematics education (Lacampagne, Blair, & Kaput, 1995; Knuth, 2000) and more research is needed to understand this complex, yet critical issue.

This study is a component of a larger project intended to develop and study a mathematics curriculum for prospective secondary mathematics teachers. This report specifically deals with a three-week unit designed to promote understanding of algebra through the connection of representations. Twenty prospective secondary mathematics teachers participated in this study. Qualitative research methods were employed in order to delve into cognitive difficulties and ideas that are related to the connection of representations and to document how student understanding evolved during the unit.

Theoretical Framework

In the mathematics education community, representations are regarded as critical tools for mathematical communications and problem solving (Brenner et al., 1997; Goldin & Shteingold, 2001; Hiebert & Carpenter, 1992; Hollar & Norwood, 1999; Mousoulides & Gagatsis, 2004; Thompson, 1994). However, it has known that mere representation of a mathematical concept in a certain mode of representation is not enough for mathematical understanding. In order to understand mathematical concepts or to be successful in problem solving, learners need to be able to connect representations. They need to be able to not only recognize ideas embedded in various representations but also convert a representation in one form to another and translate ideas from one representation to another within and across various representations (Borko & Eisenhart, 1992; Dufour-Janvier, Bednatz, & Belanger, 1987; Even, 1998; Hitt, 1998; Gagatsis & Shiakalli, 2004; Knuth, 2000; Lesh, Behr, & Post, 1987).

Many researchers have investigated how learners attempt to connect various representations of functions, a concept critical to much of algebra and higher mathematics. For example, Knuth (2000)) showed that the Cartesian connection—"a point is on the graph of the line *L* if and only if its coordinates satisfy the equation of *L*" (Moschkovich et al., 1993, p. 73)—is related to students' problem solving ability where they must connect symbolic and graphical representations of linear functions. Hitt (1998), Williams (1998), and Hansson (2005) have shown that teachers' inability to use subconcepts, such as variables and domain/range, affected their problem solving. Following up on the ideas posed by Knuth (2000) and other researchers cited above, a primary focus of this research was to examine how learners' understanding of critical mathematical ideas (that are related to connections among representations), such as the Cartesian connection, interacts with their abilities to connect representations and/or their problem solving ability.

In order to address this issue, we designed a unit focusing on representations in algebra. The unit, indeed the entire course, was developed and implemented along with the educational and philosophical principles of *realistic mathematics education* (Freudenthal, 1991; Gravemeijer, 1999) and *constructivist learning theory* (Cobb, Yackel, & Wood, 1992). As such, mathematical understanding in the course and hence in this study was regarded as social construction (or reconstruction) of ideas through mathematical communications and collaborations while learners are actively engaged with "realistic" (Gravemeijer, 1999) tasks.

For this study, we restrictively use the term, representations, in regard to only four types of representations in algebra—algebraic, spatial, numeric, and verbal representations. We also extended the notion of the Cartesian connection to "a point is on the graph of the mathematical relation, $R(x, y) = 0$ if and only if its coordinates satisfy $R(x, y) = 0$" in order to accommodate this idea to other mathematical relations with two variables, conic curves in this case.

The constructs of *concept definition* and *concept image* (Tall & Vinner, 1981; Vinner, 1991; Vinner & Dreyfus, 1989) are utilized as a tool to understand participants' cognitive structures relating to conic curves. According to Vinner and others, an individual's understanding of a concept is related to her verbal definition of the concept (concept definition) and her non-verbal image of the concept (concept image). When her concept definition and her concept images are aligned with the *formal concept definition* (the one accepted by the mathematics community), she can solve non-routine problems or prove theorems by successfully consulting both the definition and the image (Vinner, 1991). As with other studies concerning learners' understanding of mathematical concepts (Bangi, 2006; Bingolbali & Monaghan, 2008), this construct aided us to understand the mathematical thinking and understanding of the participants.

Methods

Research Setting

This research was conducted in an inquiry-based classroom during the winter quarter, 2009, when a three-week teaching unit focusing on Algebra was implemented. 20 prospective secondary mathematics teachers (PSMTs), mostly juniors and seniors majoring in mathematics, participated in this study. The Algebra unit started with the opening activity, a discussion of symbolic and spatial meanings of the solution of the algebraic equation $x^2 = 2$, followed by the major task, an interpretation of the historical work by Omar Khayyam (1048–1131). PSMTs were asked to figure out how Omar Khayyam's geometric approach to the solution of a cubic equation made sense, i.e., how an intersection point of the parabola $py = x^2$ and the circle $x^2 + y^2 = qx$, with p and q positive integers, determines the solution of the algebraic equation $x^3 + p^2x = p^2q$. This task involved three subtasks: (a) graphing the circle represented by algebraic representation $x^2 + y^2 = qx$ and proving why the formal concept definition of circle defines the equation of circle, (b) deriving the equation of a parabola based on the concept definition of parabola, and (c) explaining how the solution of $x^3 + p^2x = p^2q$ is represented spatially. During the Omar Khayyam task, two quick writes were administered to examine PSMTs' understanding of representations, the one examining PSMTs' abilities to connect the concept definition and the algebraic representation of circle, and the other examining their abilities to connect algebraic and geometric representations.

All sessions were videotaped using two video cameras. PSMTs' responses to quick-writes and group posters that they prepared for the presentations were also collected.

Methods of Analysis

For the analysis, we adopted and modified an analytic method by Powell et al. (2003), designed for research using video data. At first, we viewed the video recordings and prepared a brief, written record of the video content. The written record at this stage included rough transcriptions of some episodes and focused on mathematical activities, situations, and meanings (Powell et al., p. 416). We also roughly viewed PSMTs' quickwrites to get a sense of their responses. We then developed a priori codes, based on the research framework, research questions, and the problematic areas of the mathematical investigations that were found from the viewing of video recordings, quickwrites, and posters (Miles & Huberman, 1994). With these observational codes, we rewatched the video recordings and reexamined the written data. At this stage, we revised the codes by identifying more codes and constantly comparing with the existing codes (Glaser & Strauss, 1967), and identified *critical events*—significant moments showing learners' cognitive difficulties, conceptual leaps from previous understanding, or intuitive mistakes (Powell et al., 2003). At the next stage, we prepared word-to-word transcripts of the portion of the video data, including critical events and other episodes that "provided evidence for important theoretic or analytic matters to our guiding research questions" (Powell et al., 2003, p. 423). The new transcription data, combined with the written data, then, put through another phase of analysis for the accuracy and the consistency of the results.

Results

The Case of Circle

Most PSMTs knew the definition of circle correctly, as "the collection of points equidistant from a point". However, their algebraic concept image of circle, $(x - a)^2 + (y - b)^2 = r^2$, and the spatial concept image of circle were compartmentalized or existed without proper understanding of the roles of variables x and y or constant r.

Only one out of five groups (each group had 4 PSMTs) successfully transferred the algebraic representation, $x^2 + y^2 = qx$, to its graphical representation, the circle with the center $(q/2, 0)$ and the radius $q/2$. Three groups transferred the equation, $x^2 + y^2 = qx$, to a circle centered at the origin, with radius labeled $\sqrt{qx}$, by taking the left part of the equation, $x^2 + y^2 =$, as a process of drawing a circle with center (0, 0) and qx as the square of the radius, without paying attention to the variable x. In subsequent class discussions, they also showed lack of understanding of the Cartesian connection. Although these groups had an understanding that $x = 0$ and $y = 0$ satisfy the equation, $x^2 + y^2 = qx$, they were unable to translate this idea to the graphical representation in that they did not recognize that the graph of $x^2 + y^2 = qx$ had to pass through the origin. One group transferred $x^2 + y^2 = qx$ to a bow-tie figure passing through the origin (see Figure 1). Although this group's graph passes through the origin, they did so because they believed that their "radius" $\sqrt{qx}$ approached 0 as x approached 0. Understanding of the concept of variables in graphical representations or the Cartesian connection was absent in most of these students.

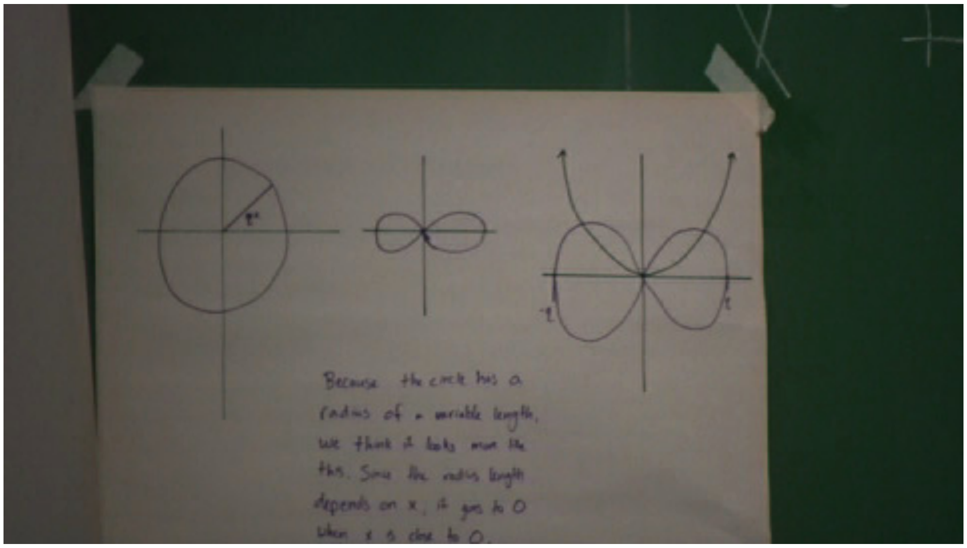

Figure 1: Representation of circle and parabola

The Case of Parabola

Only one out of 20 PSMTs knew the concept definition of parabola—the locus of points equidistant from a point, called focus, and a line, called directrix. When they were asked to find the equation of parabola with the focus $(0, f)$ and the directrix $y = -f$ given, the vast majority had no clue how to start. For them, a parabola was given by the equation $y = ax^2$ or $y = a(x - h)^2 + k$ where the only meaningful information they could draw from these expressions were the vertex (h, k) and the coefficient a, and their discussion was mainly about the concavity of the parabola, how wide the parabola was depending on the value of a, or how to shift $y = ax^2$ to $y = a(x - h)^2 + k$ using vertical and horizontal translations. Only after being reminded by the instructors that their job was to derive the equation of parabola using the definition of parabola and that they could name a random point that is equidistant from the focus and the directrix as (x, y) in the Cartesian plane, did they attempt to interpret the definition of parabola to come up with some kind of algebraic equation. Even then, a lengthy discussion within their group and with the instructors was required to reach to the algebraic representation, $x^2 = 4fy$. Although they seemed to understand that x, y represent variables in an algebraic relationship, $r(x, y) = 0$, they did not understand that (x, y) could represent varying coordinates in the geometric context. Most of them also did not understand or use the idea that a graph of a mathematical relation is a locus of points whose coordinates satisfy the relation even if they had heard the definition multiple times from their classmate and instructors. Many of them had difficulties in translating mathematical concepts, such as distance and equivalence, from verbal/geometric representations to algebraic representations.

Connecting the Solution of the Cubic to the Intersection Point of the Circle and Parabola

For PSMTs, finding the relationship between the real solution of the cubic $x^3 + p^2x = p^2q$ and the point of intersection of the parabola and the circle was difficult as well. Two groups falsely claimed that the solution of the cubic equation was the distance between the point of intersection and the origin or the distance between the point of intersection and the focus of the parabola (The actual solution of the cubic equation was the x-coordinate of the non-origin intersection point of two graphs). The examination of PSMTs' quickwrites also suggested that they fell short in understanding the relationship between algebraic and graphical representations. In one of the two quickwrites, only 5 out of 18 PSMTs specified that the solution of a system of linear equations, $2x + y = 10$, $x + 2y = 8$, was the x, y coordinates of the point of intersection (4, 2), discriminating the point (4, 2) on the plane from its coordinates $x = 4$ and $y = 2$, another evidence for their lack of understanding of the Cartesian connection. For them, splitting a single object, a point (x, y) in the Cartesian plane, into two objects, its coordinates x and y, or using x as a distance between the point (x, y) and the y-axis in their descriptions was very challenging.

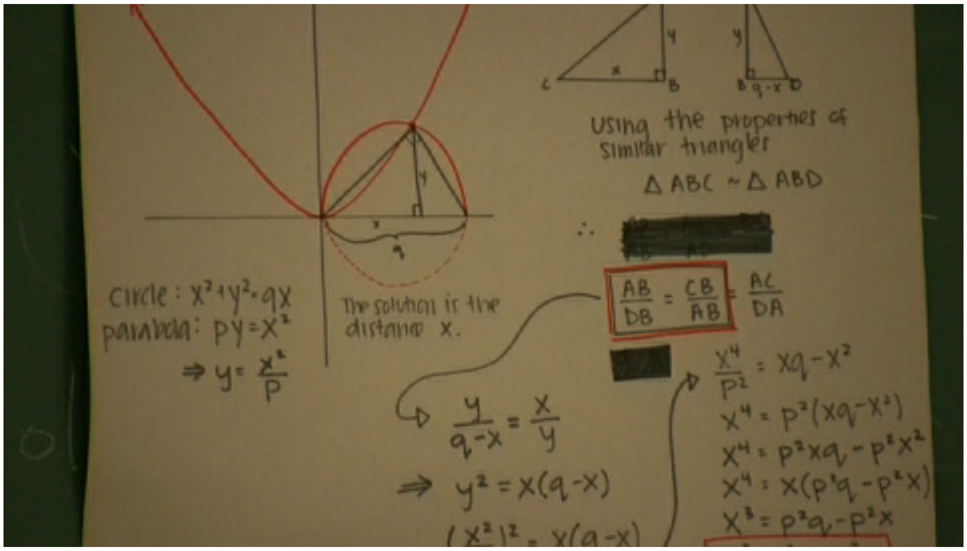

Figure 2: Graphical to algebraic transfer

Toward the end of the Omar Khayyam task, however, there were many signs that PSMTs were making progress in connecting representations. For example, one group showed the relationship between the algebraic representation of a circle $x^2 + y^2 = qx$ and the graph of circle with center ($q/2$, 0), by incorporating a concept in geometry—the proportionality of similar triangles—into the Cartesian coordinate system (Figure 2). This group first showed that $y/(q - x) = x/y$ and then derived the equation $y^2 = x(q - x)$ by cross-multiplying, which then can be transformed into $x^2 + y^2 = qx$. Their work, finding an analytic expression of circle using geometric properties on the Cartesian plane, as Descartes did, was remarkable progress, compared to their initial work on the Omar Khayyam task. In the beginning, they mostly focused on formulas and algorithms of which they did not make much sense. Two other groups also came up with explanations how Omar Khayyam could have represented his solution and idea in his period. The importance of verbal, spatial, and algebraic representations in understanding algebra was embedded in the groups' work.

Conclusion

This study found that without interventions prospective secondary mathematics teachers were largely dependant on memorized formulas or algorithms rather than focusing on meanings and ideas in connecting algebraic equations and their Cartesian graphs. Their problem solving was handicapped by incomplete concept images and definitions of circles and parabolas, similar to learners in the other studies on the concept of function (Even, 1993, 1998; Vinner, 1991; Vinner & Dreyfus, 1989; Williams, 1998). In the case of circle, although most of them acknowledged the formal definition of a circle, they were unable to prove why the definition gives the equation of circle. In the case of parabola, most of them had no knowledge of the definition of parabola or the subconcepts of focus and directrix. Even after they were reminded of the definition repeatedly by their classmate and by the instructors, they had difficulties in translating the ideas in the definition to an algebraic representation.

This study also affirms that the Cartesian connection, the idea that connects algebraic and graphical representations of a line (Moschkovich et al., 1993; Knuth, 2000), is a critical idea to connect representations in the concept of conic curves. The concept of variable and a graph as a locus of points—an extended idea of the Cartesian connection with the concept of variable intertwined—were also identified as critical ideas in making connections of representations.

Through the intervention, prospective teachers made some progress. During the three-weeks of instruction, we saw some positive changes in prospective teachers' mathematical thinking and behaviors and progress in understanding Omar Khayyam's solution to cubic equations. Many prospective teachers who initially were dependent on memorized forms of algebraic formulas made efforts to consider aspects of the Cartesian connection to make sense of their own work. Their exposure to these critical ideas and concepts also helped them analyze students' thinking. In the subsequent activities that dealt with students' understanding of algebra (which will be documented in a later article) they tried to relate many of these same issues that they experienced to students' cognitive difficulties and understandings in algebra.

An implication of this study is that teacher education programs might have to pay special attention to these critical ideas so that their graduates can better help their future students understand connections among algebraic equations and their Cartesian representations. This study shows one of those examples. Using a historical task of Omar Khayyam in accordance with the principles of realistic mathematics education (Freudenthal, 1991; Gravemeijer, 1999), we provided prospective teachers opportunities to reconstruct these critical ideas that are essential for their own understanding of algebra and for their future instruction. Further, by providing subsequent tasks with which prospective teachers could discuss student mathematical thinking around the same issues that they had experienced, we tried to provide them opportunities to develop pedagogical content knowledge along with subject matter knowledge.

Connection among representations is identified as one of the "big ideas" (Lacampagne, Blair, & Kaput, 1995; Schifter & Fosnot, 1993) in algebra by many researchers (Knuth, 2000). Understanding the role of the Cartesian connection and the idea that a graph is a locus of points that satisfy a relation in sense making about the connections between symbolic and graphical forms is a crucial piece of pedagogical content knowledge that secondary teachers need to be familiar with. Only when a teacher is aware of the importance of this big idea and hold the understanding of the big idea, can she teach for the big idea (Schifter & Fosnot, 1993; Schifter, Russel, & Bastable, 1999).

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