USING A KNOWLEDGE-IN-PIECES APPROACH TO EXPLORE THE ILLUSION OF PROPORTIONALITY IN COVARIANCE SITUATIONS

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The non-normative application of proportional strategies in non-proportional covariance situations is widespread and documented in studies conducted in many countries and with participants across a wide range of ages. In the present study, we found that preservice middle-grades teachers have many of the same problems with proportional reasoning as those reported with other populations. We employed diSessa's (1993) knowledge-in-pieces perspective to track how pre-service teachers used knowledge resources before and after a unit on proportional reasoning in their methods course. Past research has often characterized this phenomenon as the result of intuitive or impulsive responses to familiar missing-value problem presentations. Our data show that even a detailed understanding of the relationship between covarying quantities by no means guarantees the normative use of the proportion equation.

Keywords: Teacher Knowledge; Teacher Education–Preservice; Rational Number

Purpose

Meno's slave famously told Socrates that a square of double area is obtained by doubling the side length of the given square. The inappropriate application of proportional reasoning strategies in non-proportional covariance situations (termed here *non-normative*) is widespread and documented in studies from many countries and among participants with a wide range of ages (for a review, see Van Dooren, De Bock, Janssens, & Verschaffel, 2008). However, knowledge of the root psychological sources of this tendency remain tentative and pedagogical remedies unsatisfactory (see the first issue of *Mathematical Teaching and Learning 12*).

In the present study, we found that preservice middle-grades teachers have many of the same problems with proportional reasoning as those reported with other populations. To gain further insight into these difficulties, we employed diSessa's (1993) knowledge-in-pieces perspective to track how pre-service teachers used knowledge resources before and after a unit on proportional and non-proportional reasoning in a methods course. We report two cases in which participants judged non-proportional relationships to be proportional even though they demonstrated clear understanding of the non-proportional covariance relationship.

Perspectives

Teachers and the Illusion of Proportionality

Fruedenthal (1983, p. 267) wrote, "Linearity is such a suggestive property of relations that one readily yields to the seduction to deal with each numerical relation as though it were linear." Teachers, even those certified for mathematics, are not exempt from its lure. Riley (2010) found that in a sample of 80 preservice elementary teachers, less than 50% solved *constant difference* and *inverse proportion* problems correctly. We illustrate these terms with example tasks from other studies with teachers. We used similar tasks in the present study.

Constant difference. Cramer, Post, and Currier (1993) report that 32 out of 33 preservice elementary teachers solved the following task using the proportion $9/3 = x/15$. "Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?" The relationship between these runners' laps is not proportional: They remain an equal distance apart: $9 – 3 = x – 15$.

Inverse proportion. Fisher (1988) reported that 12 out of 20 inservice secondary mathematics teachers did not solve the following problem correctly; "If it takes 9 workers 5 hours to mow a certain lawn, how long would it take 6 workers to mow the same lawn?" The common error was to assume a direct proportion such as $9/5 = x/6$. Instead, the relationship between men and minutes is inversely proportional: The amount of work is constant: $9(5) = 6x$.

Knowledge-in-Pieces and Conceptual Change

Prior work on this problem has been conducted within a traditional knowledge-as-theory perspective (Özdemir & Clark, 2007). Knowledge is understood as a unified and coherent structure and conceptual change is characterized as a paradigm shift: old structures are replaced with new ones. Instead, we use diSessa's (1993) knowledge-in-pieces epistemology that models knowledge as networks of loosely connected knowledge resources that are each highly sensitive to context. Conceptual change from this perspective is characterized as the piecemeal construction and reorganization of knowledge resources (e.g., diSessa & Sherrin, 1998) as learners gradually navigate the continuum from novice to expert.

Within the larger knowledge-in-pieces perspective, diSessa and Sherin (1998) proposed the *coordination class* as an empirically verifiable alternative to the black-box notion of concept in traditional work on conceptual change. A coordination class is made up of *readout strategies* by which one acquires information ("sees") in knowledge-use situations. The *causal net* of a coordination class is made up of the syllogism-like ways of inferring new information not directly available from readout. For example, someone might use the equation $F=ma$ to obtain information about acceleration from a situation that specified only force and mass. Different contexts require different readout strategies and result in different kinds of inferences in the causal net. *Coordination* in an expert's coordination class means (a) that one *integrates* all of the relevant information in a particular context, and (b) that inferences are *aligned* or consistent across the range of applicable contexts.

Under this analytic approach, knowledge and its use are not carefully distinguished, and knowledge per se is empirically linked at a fine grain size to contextual differences across situations of knowledge use. Wagner (2006) applied diSessa's framework and found that conceptual change happened as the participant's knowledge, "came to account for (rather than overlook) contextual differences" (p. 6). We follow Wagner in specifying terms for more precisely discussing the contextual differences that problem solvers encounter in activity. The *type* of a problem is defined by appealing to normative or expert judgment. The *aspects* of a problem are defined separately for each problem solver by obtaining empirical evidence for what features or details are perceived as relevant. The term *context* is broader, including type, aspect, and the cover story for the problem. For example, although the problems reported by Crammer et al. (1993) and Fisher (1988) are of a different type (one describes a constant difference and the other describes an inverse proportion), a problem solver might likely read out a similar aspect in both situations: the women run and the workers mow at equal rates.

Methods

Participants and Context

This study took place in the context of an 18-week content/methods course on number and operations for preservice middle-grades mathematics teachers that met for two 75-minute sessions each week. Students in the course ($N = 28$) were in their third year of college and had taken (at least) a course in introductory calculus prior to the study. Two weeks of the course were devoted to a unit on proportional reasoning. Students worked on tasks in small groups and participated in whole-class discussions. The instructors selected tasks and orchestrated discussion to focus on the big ideas for the unit such as: proportional situations are those in which covarying quantities maintain a constant multiplicative relationship (ratio).

Problem Situations

We gave students problem situations describing both proportional and non-proportional covariation during the unit and on assessments. The present study focuses on the following four problems that are variants of problems used in previous studies (e.g., Cramer et al., 1993; Fisher, 1988). The first two problems describe situations of constant difference. The last two are non-linear as well as being non-proportional. The Work Problem describes an inverse proportion situation, and the Interest Problem describes (an approximation to) an exponential situation.

The Running Problem (used in the pretest, posttest, and interviews). Determine whether the following problem is a *mathematically valid* illustration of the proportion A/B = C/D: Bob and Marty run laps together because they run at the same pace. Today, Marty started running before Bob came out of the locker room. Marty had run A laps by the time Bob had run B laps. How many laps C had Marty run by the time that Bob had run D laps?

The Combine Problem (used in the methods course unit on proportion). Two combines harvest grain at the same rate. The first combine starts harvesting 10 minutes before the second combine. After 20 minutes of operation, the second combine harvests 400 lbs of grain and the first harvests 600 lbs of grain. How many pounds will the second combine harvest by the time the first has harvested 1000 pounds of grain?

The Work Problem (used in the pretest, posttest, and interviews). Determine whether the following problem is a *mathematically valid* illustration of the proportion A/B = C/D: If A men paint the outside of a house in B minutes, then how many minutes D would it take C men to paint the same house, if all the men work at the same rate?

The Interest Problem (used in the methods course unit on proportion). Karl has a savings account that pays interest monthly at a rate of 5%. Three months ago, there was $300 in his account. If he did not withdraw any money from the account, how much is there now?

Data Collection

Each class session was video recorded using two cameras. One camera was stationary and positioned to capture the activity in the whole classroom. The second camera was hand-held and tracked the primary instructor (second author) and the written work of the students' with whom he was interacting. Students also took a pre-test and post-test. The primary data for this study were from video taped interviews (running 60 to 90 minutes) conducted with four pairs of students from the class. During the interviews, each pair was presented with a sequence of tasks and asked to solve the task together while verbally explaining their reasoning. The interviewer encouraged the students to talk freely and occasionally asked clarifying questions.

Analysis

Video data from each class was summarized, and student and instructor comments were time-stamped and supplemented with screen-shots to facilitate review. The interview data were fully transcribed. We watched all of the interview data and wrote summaries comparing each of the students' pre- and post-unit interviews. Then we reviewed the classroom video data for each student and looked for interactions or written work that might inform the observed changes. Finally, we reviewed students' written work on the pre-test and post-test, on the course midterm and final exam, and on the proportional reasoning homework assignments.

Results

We found that students who correctly explained relationships between quantities that were not proportional in the Running and Work problems (given above) still tried to set up and use proportion equations. To understand how this could take place, we examined the knowledge resources that students used to determine whether or not proportions could be set up to solve these problems. Our analysis led us to focus on (a) the consistent use of one knowledge resource (the necessary correspondence between quantities and their position in a proportion equation) by all interviewed students across all tasks, (b) the

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readout and use of the *same rate/pace* aspect of the tasks which varied among students, and (c) the various aspects of the non-proportional Interest and Combine problems (given above) that students found relevant for inferring proportionality.

First Interview, Work Problem

In this section, we use data from the Work Problem to describe a common aspect of the problems perceived as proportional—correspondence between quantities. *(Note on transcription:* Pauses are indicated by ellipses and interruptions by the em dash; overtalk is within double-slashes and action within square brackets; there are no deletions in the transcript provided.)

Lisa and Tess. These students immediately agreed that the Work Problem was proportional, and Lisa notably recognized the aspect of *same rate*.

Tess: I think [the proportion is] accurate because you have the number of men on top over //*Lisa*: and they're going at the same rate// the number of minutes and the same rate so you have your second number of men over your second number of minutes.

Alice and Clara. After reading the problem, Alice said, "So I guess like you would start off by setting it up like this," and wrote the proportion A/B = C/D. Then Clara used her pencil to point to A and then B while saying, "This amount of men paint a house in this many minutes." Alice touched her pencil to B at the same time as Clara, and then Alice said, "This many men [pointing to C] and minutes [pointing to D]."

Discussion. Across all pairs, students began almost all interview tasks by carefully reading out the correspondence between the quantities represented by variables and the position of these variables in the presumptive proportion equation. This is evident in Tess's initial comments ("... so you have your second number of men over your second number of minutes"). When asked about similar behavior during the second interview, Lisa said, "Proportion has to correspond." Alice and Clara also read out the correspondence between initial and final, men and minutes in the Work Problem, and this shared activity and its result had a shared interpretation: the proportion equation was applicable. Several other students we interviewed noted that a failure of correspondence would rule out the proportion equation in their view.

First Interview, Running Problem

In this section, we report how different interview pairs read out the aspect of *same pace* in the Running Problem, and how they made very different use of it.

Lisa and Tess. Tess established the correspondence between the quantities as she read the problem out loud by pointing to each variable in the proportion as she read its description. After a 30-second pause, Lisa began.

Lisa: Hmm. I mean if they keep on at same pace, //*Tess*: Right// isn't that going to be the same difference between the two //*Tess*: Right// so it would be an equal proportionality I would think ... kind of like equal fractions //*Tess*: Yeah// equivalent fractions?

They tried a numerical example, then both agreed the proportion was valid.

Interviewer: So run me through your reasoning one more time.

Tess: Okay. We just plugged in numbers to make sure that it was accurate and valid. So we said that if Marty had run 4 laps by the time that Bob had run 2 laps, we're looking for how many laps C, Marty ran by the time that Bob had run 8 laps. So we put 4 over 2 equals X over 8. And so our proportion, we're going to do 4 times 8, which is 32 equals 2 times X, just 2X, and then we're going to divide by 2. So we get 16 equals X and [rewriting the proportion] it's going to be 4 over 2 equals 16 over 8 and that is correct because you can simplify ... that's 2, 2 [writes 2=2]. So yes it is valid.

Alice and Clara. After working with a numerical example, the students made a discovery.

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Alice: So we're saying by the time Marty had ran 4 laps Bob would have ran 2 //*Clara*: Yes// because I feel like if they run it in the same pace—

Clara: —then it should have been only 1 more lap.

Within a few minutes, they agreed that the proportion equation was not valid.

Interviewer: Why are you questioning its validity?

Clara: Because if they're running the same pace ... if Bob had run 1 more lap than Marty should have run 1 more lap. He just started earlier but they're running at the same pace, so the same speed of 1 lap should just be in Marty's 1 more lap. If Marty went and started and he ran 1 lap and then Bob came and started and ran another lap, Marty is still running so Marty would have run 2 laps by the time Bob ran 1. And then when they run another lap, Marty would have run 3 laps by the time Bob ran 2, but with this proportion it's saying that Marty would have run 4 laps by the time Bob ran 2 because it's doubling.

Discussion. It is clear that the *same pace* aspect of the problem was quite salient for Lisa. She interrupted Tess on the Work Problem to point out a similar aspect of that problem, *same rate*. In the Running Problem, she used the *same pace* information to conclude (normatively) "the same difference." However, the information about *same difference* served as a warrant for Lisa's use of the proportion equation: "so it would be an equal proportionality, I would think." Clara made a different inference, "they're running at the same pace, so the same speed of 1 lap should just be in Marty's 1 more lap." These data are evidence of differences in these students' causal nets. That is, although both pairs had access to the same information about the problem, awareness of a constant difference led them to different conclusions about the applicability of the proportion equation.

The data presented thus far admit the alternative hypothesis that Lisa and Tess did not fully understand the problem situation or were simply failing to be attentive and careful. Certainly, it makes intuitive sense that students who possess or develop the kind of quantitative understanding that Alice and Clara displayed of the Running Problem would not apply the proportion equation in non-proportional situations. The unit on proportional reasoning in the methods course was designed to help students develop just this kind of quantitative understanding for several types of non-proportional covariance that are frequently viewed as proportional.

The Proportion Unit

The first and second interviews bracketed the two-week unit on proportional reasoning that included work distinguishing proportional and non-proportional relationships. The Interest Problem and the Combine Problem received prominent attention as explicit examples of non-proportional situations throughout the unit. Students in the methods course witnessed clear, normative explanations from their peers and had opportunities to work on these and similar tasks in class and on homework (including inverse proportion tasks). They also received feedback on their work from the instructors in class and on homework.

The unit appeared to have little effect on students' tendency to use the proportion equation on the non-proportional Work and Running problems. (These specific problems were not discussed during the unit.) Table 1 shows how many students maintained or changed their responses for the Work and Running problems on the posttest ($N = 27$, one student was absent for the posttest) and serves to contextualize the data from the second interview described in the next section. We used McNemar's (1947) test for matched data and found that there was no statistically significant change in students' responses on these items ($p_{\text{Work}} = 1.00$, $p_{\text{Running}} > .450$). Students were almost entirely agreed on the Work Problem, but the consensus response was non-normative. By contrast, the proportion of normative responses on the Running Problem was not much different than that expected by chance.

Table 1: Student Pretest and Posttest Responses on Two Non-Proportional Problems

	Posttest Response			
	Work Problem		*Running Problem*	
Pretest Response	Proportional	Non-prop.	Proportional	Non-prop.
Proportional	25	1	9 2	
Non-prop.	1	0	5	11

The Second Interview

In this section, we report two cases where students clearly understood quantitative relationships that were not proportional in the problem but still endorsed the proportion equation.

Clara. (Alice did not participate in this interview because of a family obligation.) Clara began the Work Problem with confidence and set out to justify her use of the proportion equation by using a numerical example: 2/15 = 4/? . Then she paused for 35 seconds.

Clara: It is proportional, but it's going to be a proportion going down this way if the men were increasing [draws a line with negative slope] because if more men are working on the house then it's going to take fewer minutes. But if it takes 2 men 15 minutes then it's going to take one man 7.5 minutes so it is going to be proportional. I mean ... not 7.5, that doesn't make sense ... takes one man 30 minutes. It's proportional, it's just a decreasing proportion.

After working for 3 more minutes, Clara rejected her graph but not the proportion equation. A changed tone of voice suggested decreased confidence, but she remained adamant.

Clara: What I'm thinking is that as the amount of people painting the house increases the amount of time it takes to paint the house would decrease. I'm just not sure how to illustrate that. It would be a proportion, I feel it's proportional ... I just don't know how to represent it proportionally [in a graph].

Lisa and Tess. Both Lisa and Tess immediately decided that the Running Problem was proportional.

Interviewer: How do you know it's proportional?
Tess: Because if they run at the same pace this says they run laps together because they run at the same pace. Even if Marty starts before Bob, however many ... they're going to run at the same pace. So it's going to in ... like the amount difference is going to stay the same the whole time because they're running at the same pace.
Lisa: Mm-hmm [yes]. It's a constant increase …
Tess: Right, if Marty starts he runs two laps, and Bob starts. So by the time that he runs 4, by the time Marty runs 4 laps, Bob will have run 2 laps ... then 6, 4 ... 8, 6 ... so the same amount of increase every time.
Lisa: Yeah, and if you know the lap difference between the two then you can give me any value of laps and I can figure out where they are //*Tess*: Right// without having to go step by step like you do with interest problems.
Tess: Right.
Interviewer: So that's what tells you it's proportional?
Lisa: Yeah, because you're not factoring in time or anything. You're like ... I mean if you were to ask at twenty minutes then you'd have to factor in the time difference.

A little later in the interview, during the Work Problem, Tess continued in the same vein.

Tess: Right, or like on interest problems like every month there's five percent interest. She starts with this amount; then she puts this amount in each month, and then you have to also think about your

> interest and how that's affecting your amount. But this is just a same rate you know, like there's nothing adding in.

Discussion. In both cases, the students provide normative reasoning about how the quantities in the situation covary using specific numbers. Clara said, "If it takes 2 men 15 minutes ... [it] takes one man 30 minutes." Tess gave a sequence of example values for the consecutive pairs of laps run; "Marty runs 4 laps, Bob will have run 2 laps ... then 6, 4 ... 8, 6" The students clearly demonstrated accurate knowledge of the relationships among the quantities in the problem situations, yet in both cases the students endorsed the proportion equation.

The data from Clara contrast with her work during the first interview on the Running Problem, where she used similar quantitative understanding to normatively reject the proportion equation. In this case, contextual differences led to non-normative knowledge use. Clara read out the aspect of *more men, fewer minutes* and interpreted the Work Problem as a negative proportion problem like those discussed in class and on the homework.

In the case of Tess and Lisa, some of the data are consistent with the first interview. For example, Tess read out the aspect of *same pace*, and used this to judge that "the amount difference is going to stay the same the whole time." Lisa apparently agreed, saying, "It's a constant increase." But rather than moving directly from the aspect of *constant increase* to the proportion equation (that is, in parallel to how Lisa moved from the aspect of *same difference* to the proportion equation during the first interview), this pair referenced their course experiences with the Interest and Combine problems.

Lisa and Tess read out and used aspects of the Combine and Interest problems that would likely go unnoticed by experts. The Running Problem had a functional aspect for Lisa, "You can give me any value of laps and I can figure out where they are." By contrast, Tess and Lisa's work on the Interest Problem was iterative; they used each monthly total to compute the next. In the class discussion of the Interest Problem, Tess said, "You need to know all of the previous months to find the next month." Several other aspects of the non-proportional problems discussed in class distinguished them from the interview problems. For example, Lisa read out the time interval of 20 minutes specified in the Combine problem and used this as a warrant for non-proportionality. She said that the Running Problem would not be proportional "if you were to ask at twenty minutes." Moreover, both students agreed that to solve the Interest and Combine problems, one had to "factoring in" an extra quantity like time or interest. In the Running and Work problems, Tess observed, "there's nothing adding in." Tess's and Lisa's knowledge resources were evidently refined and reorganized to incorporate their experiences in the course but their judgments remained non-normative.

Implications and Conclusion

The quantitative results of this study provide a partial replication with preservice middle-grades teachers of prior results with elementary and secondary teachers (e.g., Cramer et al., 1993; Fisher, 1988; Riley, 2010) showing that teachers face some of the same challenges as children when reasoning about situations involving non-proportional covariance. Recent research (e.g., Van Dooren, De Bock, Vleugels & Verschaffel, 2010) suggests that thinking about problems rather than answering reflexively is necessary if students are to avoid applying the proportion equation in non-proportional situations. The qualitative data reported here warrant a much stronger claim: Even a detailed understanding of the relationship between covarying quantities may not be sufficient for the normative use of proportionality. These results suggest a sobering assessment of the pedagogical challenge faced by teacher educators. Preservice teachers may need a broad and coordinated collection of fine-grained knowledge resources developed in a wide variety of contexts in order to distinguish between covariance relationships and to apply the proportion equation appropriately across contexts and problem situations.

References

Cramer, K., Post, T., & Currier, S. (1993). Learning and teaching ratio and proportion: Research implications. In D. T. Owens (Ed.), *Research ideas for the classroom: Middle grades mathematics* (pp. 159–178). New York: Macmillan.

diSessa, A. A. (1993). Toward an epistemology of physics. *Cognition and Instruction, 10*(2/3), 105–225.

diSessa, A. A., & Sherin, B. L. (1998). What changes in conceptual change? *International Journal of Science Education*, *20*(10), 1155–1191.

Fisher, L. C. (1988). Strategies used by secondary mathematics teachers to solve proportion problems. *Journal for Research in Mathematics Education, 19*(2), 157–168.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht, The Netherlands: Reidel.

McNemar, Q. (1947). Note on the sampling error of the difference between correlated proportions or percentages. *Psychometrika*, *12*(2), 153–157.

Özdemir, G., & Clark, D. B. (2007). An overview of conceptual change theories. *Eurasia Journal of Mathematics, Science & Technology Education*, *3*(4), 351–361.

Riley, K. R. (2010). Teachers' understanding of proportional reasoning. In P. Brosnan, D. B. Erchick, & L. Flevares (Eds.), *Proceedings of the 32nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1055–1061). Columbus, OH: The Ohio State University.

Van Dooren, W., De Bock, D., Janssens, D., & Verschaffel, L. (2008). The linear imperative: An inventory and conceptual analysis of students' over-use of linearity. *Journal for Research in Mathematics Education*, *39*(3), 311–342.

Van Dooren, W., De Bock, D., Vleugels, K., & Verschaffel, L. (2010). Just answering… or thinking? Contrasting pupils' solutions and classifications of missing-value word problems. *Mathematical Thinking and Learning*, *12*(1), 20–35.

Wagner, J. F. (2006). Transfer in pieces. *Cognition and Instruction*, *24*(1), 1–71.