ARTICULATING A LEARNING SCIENCES FOUNDATION FOR LEARNING TRAJECTORIES IN THE CCSS-M

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The paper describes the history of how learning trajectories (LTs) were associated with the Common Core State Standards for Mathematics (CCSS-M) and discusses the degree to which the two correspond faithfully. It reports on a website, www.turnonccmath.com, which organizes the K–8 standards into 18 LTs describing the development of big ideas over time, informed by empirical studies of learners. The paper illustrates how descriptors for each LT identify: (1) conceptual principles, (2) strategies representations, and misconceptions, (3) meaningful distinctions and multiple models, (4) coherent structure, and (5) bridging standards. The design principles for the website are illustrated describing how the CCSS-M are related to a learning trajectory on division and multiplication.

Keywords: Standards, Cognition, Teacher Knowledge, Learning Trajectories

The Common Core State Standards for Mathematics (CCSSO, 2010) have been represented as "fewer, clearer, and higher," reflecting the view that revised standards should be: (1) focused, (2) rigorous and applicable, and (3) coherent. They offer "a more coherent progression of learning" described as "… clearly articulat[ing] how knowledge builds from year to year. Each standard extends previous learning while avoiding repetition and large leaps in instruction" (Hunt Institute, 2012, p. 8). Despite this intent, the progressions themselves are not immediately accessible to readers, so other documents are needed to articulate and display these relationships in different formats. Our research group has done this as a set of posters (www.wirelessgeneration.com/posters) and as a website (www.turnonccmath.com). After reviewing the history of how learning trajectories became foundational in the writing of the CCSS-M, I describe the elements of a learning trajectory analysis of the CCSS-M as a means to support implementation of standards and conduct of related professional development. The advantages of researchers working together, to create resources on learning trajectories built on empirical study are discussed, along with a warning of the likely costs of failing to do so.

History of Learning Progressions in the CCSS-M

In the summer of 2009, a meeting was held at the Friday Institute for Educational Innovation in North Carolina where researchers on learning trajectories hosted the writers of the Common Core State Standards (CCSS) and other leaders from the Council of Chief State School Officers (CCSSO).[2] The proposed standards were to be based on scientific evidence. While the College- and Career-Ready Standards (U.S. Department of Education, 2010) could be sufficiently justified with evidence of international benchmarking and studies of the needs and expectations of colleges and entry-level careers, the grade-level standards required a basis in the research on student learning. A number of learning sciences and mathematics education researchers gave presentations (including M. Battista, D. Clements, J. Confrey, G. Kader, and R. Lehrer) on learning trajectories (also called "learning progressions"). After the conference, many of the attendees were invited to participate on the CCSS-M writing teams. The use of these teams during the Standards development was perceived by many as more sporadic than systematic—and the teams were only one voice among many (including state departments, mathematics faculty, and teachers) in influencing the development of the Standards. However, their ideas contributed significantly to the final document. In sum, the CCSS-M incorporated a foundation in learning trajectories that can propel the country forward now, and be strengthened over time. In the period since the publication of the CCSS-M, at least three groups have engaged in efforts to delineate the trajectories in more detail (Confrey et al., 2011; Hess & Kearns, 2010; McCallum, 2011).

www.turnonccmath.com

Once the CCSS-M was validated and widely adopted, and in response to the need expressed in the field for urgent assistance, the DELTA research group at North Carolina State University (NCSU) decided to connect the Standards more directly with associated research on learning trajectories. Many state leaders had reported that teachers perceived little change from their old or current state standards to the new CCSS-M, and expected that "crosswalks" would provide a sufficient basis to support the transition to the CCSS-M and the related curriculum and assessment. In this scenario, teachers would only change the way they teach new topics at the grain size of the individual grade levels and otherwise continue teaching by making small adjustments to their lesson plans. A close reading of the CCSS-M document, my understanding of the CCSS-M from experience on the National Validation Committee, and our group's close comparison of the CCSS-M to previous state standards, however, told a different story. There are major changes in when and where mathematical topics are emphasized, namely the intensity of content treatment at earlier grades and major shifts in several topics that will radically change teacher preparation and professional development. The "higher" and "fewer" aspects of the CCSS-M mean, also, that there is much less room for repetition of content at each grade.

We found learning trajectories useful in supporting implementation, because they focus attention on gradual and systematic student learning over time, a form of "genetic epistemology" (Piaget, 1970). The idea behind explicitly mapping learning trajectories onto the CCSS-M is to help teachers and students build consistently stronger understandings of big ideas by revising and modifying prior views in light of new conditions and challenges. Rather than emphasize a standard-by-standard view of implementation of new or revised content, learning trajectories support "vertical teaming" by teachers. This allows an exciting chance for teachers to discuss and plan their instruction based on how student learning progresses. An added strength of a learning trajectories approach is that it emphasizes why each teacher, at each grade level along the way, has a critical role to play in each student's mathematical development.

Our effort to build a website that synthesizes the relevant research and to lay out a manageable number of learning trajectories for the CCSS-M began as a result of a meeting of the *Measurement Mini-Center*.[3] Many of the group's participants had conducted pioneering work on learning trajectories, and each has his or her preferences about how to characterize, emphasize or order underlying proficiencies and concepts. Concerned that the interpretation of the CCSS-M should be better and more publicly informed by "learning sciences research," my research team drafted a synthetic trajectory built around the CCSS-M, drawing from these scholars' work, and brought it to the meeting for discussion. The Mini-Center's response to the effort was positive and constructively critical—the group reviewed the proposed trajectory, offered valuable suggestions and distinctions, and labored until an acceptable synthesis was negotiated. This specific trajectory as finalized is represented on the turnonccmath.com site (Confrey et al., 2011) and is described in more detail in a 2012 PME-NA paper (Lee, Nguyen, & Confrey, 2012).

Buoyed by this experience and stimulated by requests from the field, our NCSU team decided to undertake a full learning trajectories analysis of the K–8 Standards. Using a hexagon map of the CCSS-M (designed by Jere Confrey and ©Wireless Generation) to display the Standards and learning trajectories visually, I dissected the CCSS-M into 18 learning trajectories. Over a concentrated period of six months, the research team undertook writing, revising, and interlinking *descriptors*, which are text-based descriptions of standards in terms of students' movement from more naïve to more sophisticated ideas for each of the trajectories. Our working assumptions were that the web-based environment would: (1) provide the opportunity for continuous incremental improvements in the descriptors that would serve the needs of the field for rapid access to the associated learning trajectories for the Standards, and (2) permit us to gradually strengthen the site based on feedback and review. In the next sections, the hexagon map is introduced along with an explanation of the framework used to analyze the trajectories and unpack them into descriptors.

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Turnonccmath: by Grade

The website http://www.turnonccmath.com displays a "hexagon map" of the CCSS-M. In designing this map, decisions to use a predictable and consistent method to assign standards to hexagons were largely pragmatic. Standards in the CCSS-M are of many different grain-sizes, which added considerable challenge to the effort in mapping them to hexagons. Standards were assigned to individual hexagons using the following scheme: (1) If a Standard has no subparts, the hexagon represents the entire standard. However, multipart Standards were too dense to be summarized in a single hexagon. Therefore, (2) for any Standard with subparts (e.g., a, b, c, etc.), each subpart was assigned its own hexagon. The map can be displayed in three views: by grade levels, by LT with the LTs labeled, and by LTs without labels. The topics within the standards generally proceed from less complex (lower left) to more complex (upper right).

The hexagons for the different grade levels occur in bands that are more or less orthogonal to the progression of the topics. In the grade level display, the lower left ends of any relevant learning trajectory contain the earliest grade-level standards, beginning (if applicable) with kindergarten standards, followed by first through eighth grade Standards built on top and to the right, and coded such that a hexagon's background color represents its grade level. The text color in each hexagon represents the content strand; for example in K–8, blue text corresponds to Number and Operations; red text corresponds to Measurement and Data, and black text corresponds to Geometry. In terms of the relative positions of different main content strands and learning trajectories, I chose to put Number and Operation-related standards on the bottom with Measurement-related standards on top of those, diagonally, and then Geometry-related standards above measurement. At the very top is a peninsula where the very thin learning trajectory for Elementary Data (Statistics) and Modeling is placed. This trajectory comprises K–5 standards in the Measurement and Data cluster that address how to build and interpret data representations. Having opposed the writers' decision to reduce the treatment of statistical reasoning in the CCSS-M at the elementary level, I left space to expand these standards in future revisions.

From the grade-level display, one can discern certain patterns. For instance, one can see that third grade is almost entirely comprised of standards on number and measurement, with only one standard in geometry. In contrast, one can see that in sixth grade, there are three distinct clusters of topics: (1) statistics, (2) ratio and proportion, and (3) equations and expressions.

The Relationship Between the Learning Trajectories and the CCSS-M

The purpose of a learning trajectory is to describe and synthesize what is known about how students reason over time. The term *Learning Trajectory* (LT) has varied meanings in mathematics education. Simon (1995) first defined the term hypothetical learning trajectory (HLT) to be "The learning goals, the learning activities, and the thinking and learning in which students might engage" (p. 133). We define it as, "a researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e., activities, tasks, tools, and forms of interaction), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time" (Confrey, 2008; Confrey, Maloney, Nguyen, Mojica, & Myers, 2009, p. 2-346). We view a learning trajectory as a path through a conceptual corridor in which there are predictable obstacles and landmarks and thus a student's particular path is an issue of expected probabilities and likelihoods: LTs permit one to specify at an appropriate and actionable level of detail what ideas students need to know during the development and evolution of a given concept over time.

Learning trajectories provide a way to create coherence within the CCSS-M by drawing attention to how knowledge develops over time. If teachers try to implement the CCSS-M standard-by-standard, they will be unlikely to leverage the underlying structure of the standards and support gradual transformations in student reasoning. When we have worked with teachers in unpacking our learning trajectories, they have commented on the value of creating a "story" which illustrates how the ideas are likely to evolve in the minds of students when they are provided appropriate curriculum tasks, instruction, and opportunities for discourse. Therefore, our goal is to provide this type of support to teachers by providing them efficient and coordinated access to related research. In the end, the success of the CCSS-M rests on its potential to

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support alignment, including curriculum, assessment (formative *and* summative), and professional development, at a level not previously possible. But to achieve the deep and lasting change envisioned by the Common Core State Standards Initiative and the mathematics education community, the knowledge of learning trajectories must be made clear, accessible, compact, and well-integrated within the CCSS-M.

The relationship between the learning trajectories and the Standards is complex. To a degree, the CCSS-M were built on the foundation of learning trajectories. But it would not be accurate to say that there is an isomorphic relationship between the CCSS-M and the learning trajectories. In fact, acknowledging this, the Standards' writers call the progressions in the standards, "standards progressions" (Common Core Writing Team, 2011). The reasons include:

1. Different researchers have differing views of learning trajectories, even within strands;
2. Not all topic areas have been studied as learning trajectories; and
3. The writers took suggestions from mathematicians who conflated learning trajectories with logical progressions created by "thought experiments," independent of empirical verification.

This outcome is to be expected in a document resulting from negotiations and differences of opinion among disciplinary scholars, researchers and practitioners; moreover, it creates the possibility now to systematically test, compare, and refine those trajectories in light of students' work. Also, in order to construct "fewer" and "clearer" standards, the learning trajectories in the CCSS-M are of necessity *abridged*; that is, they do not and could not contain a full treatment of all the big ideas contained in the research literature. To address this in our analysis, we added "bridging standards" as needed. These statements are similar in structure to the CCSS-M standards, but represent topics that would be required in a more fully articulated (i.e., unabridged) learning trajectory. Because of the dual nature of standards as both assessment targets and targets of understanding, bridging standards can permit one to describe standards that need to be addressed in preparation for a later standard but which will not be assessed directly at that specific time. Finally, even after debate and review, there are a few standards that were poorly constructed, inconsistent, or unadvisable, based on mathematics education or learning sciences literature; a bridging standard may be added to improve the coherence of the trajectory overall.

Standards, by themselves, can serve as a skeleton for learning trajectories, but they need to be interpreted and made unabridged to serve this purpose. Moreover, the interpretation must make explicit the connections to the research base and provide a more complete articulation of how the ideas in a trajectory evolve in light of students' documented behaviors, emergent relations and properties, and generalizations (Confrey, Maloney, Wilson, & Nguyen, 2010). To this end, and so that there would not be too many LTs to manage, we decided to create a mapping such that every standard would belong to exactly one LT, each targeting a key "big idea" or set of related big ideas. The CCSS-M document itself does not suggest an instructional sequence or rigid ordering of the Standards beyond specifying grade level, as the authors have stated: "These Standards do not dictate curriculum or teaching methods" (CCSSO, 2010, p. 3). Therefore, we reorganized standards within a trajectory if this would show the student learning development more clearly (while keeping the grade level position of standards and topics). Thus, sequencing within grade was malleable; we adjusted it to fit the learning trajectories structure (hence the numbering of the standards can be "out of order" within a grade). We also assisted readers in seeing the internal structure of and the relations among the learning trajectories by (a) creating sections to reveal underlying development, (b) providing structural overviews, and (c) cross-referencing and referencing forward and backward within a LT.

Turnonccmath: by Learning Trajectories

The hexagon map of the CCSS-M, with learning trajectories labeled, is shown at www.turnonccmath.com (Figure 1). The two-dimensional structure of the map lends itself to parallel structures among some learning trajectories, in some cases, to represent close relationships between various big ideas. One of these is the fundamental role played by (1) *counting*, (2) *equipartitioning*, (3) *addition and subtraction*, and (4) *place value and decimals* in developing an early sense of number and operations. These four learning trajectories are situated at the lower left portion of the map. *Counting* is

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directly tied into *addition and subtraction* and develops in tandem with *place value and decimals*. *Equipartitioning* leads directly to supporting the development of (5) *division and multiplication*, and subsequent rational number reasoning, with contributions from *addition and subtraction*. (6) *Fractions* are most closely related to equipartitioning and division and multiplication, with (7) *ratio and proportion and percents* being most closely tied to division and multiplication and fractions in topic and grade-level development within the CCSS-M. (8) *Rational and irrational numbers* link to *ratio and proportion and percents*.

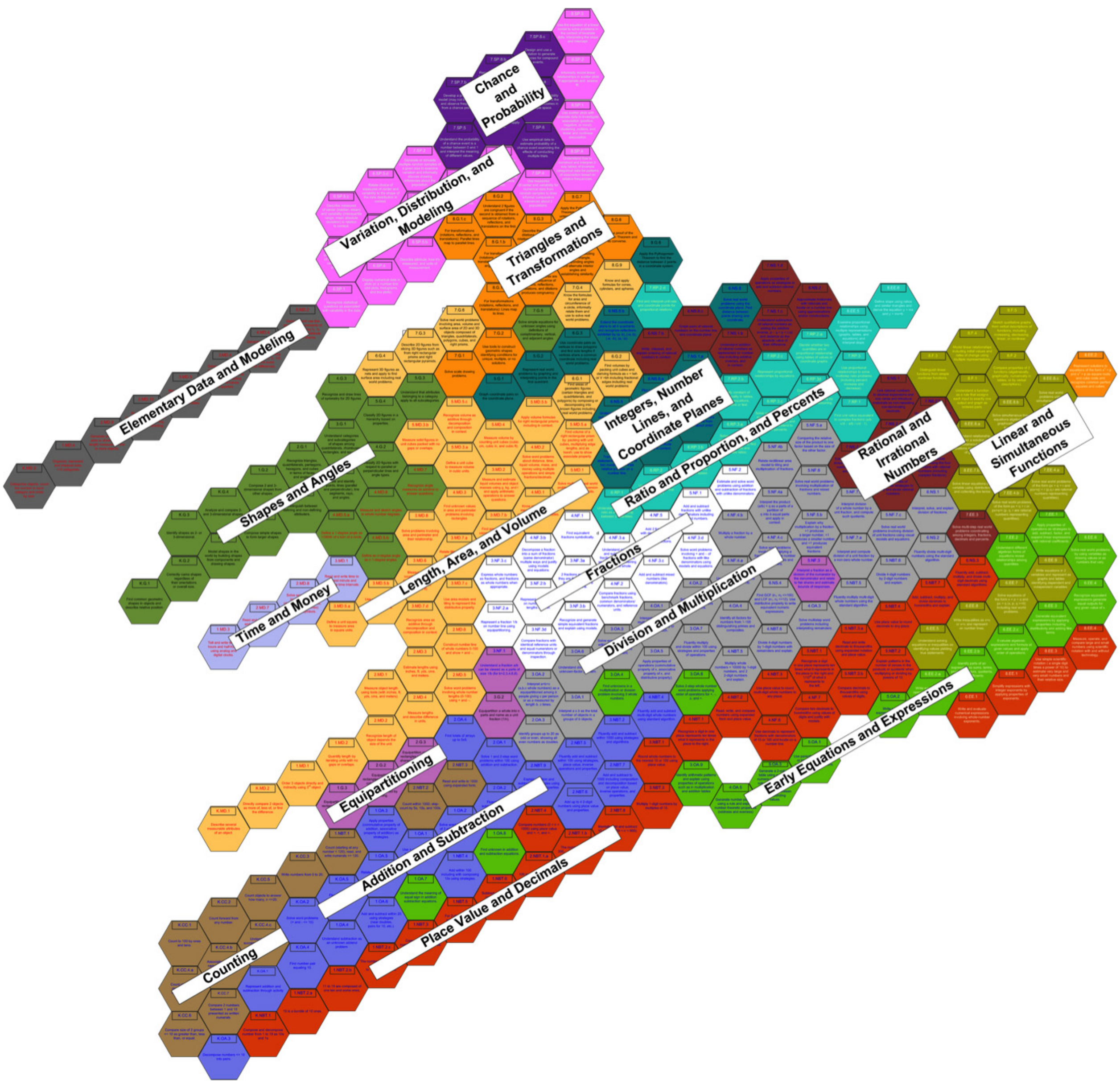

Figure 1. Hexagon Map of K–8 Common Core State Standards for Mathematics with individual learning trajectories color-coded and labeled

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The learning trajectory for (9) *length, area and volume* is situated next to equipartitioning in recognition of their close relationship in early reasoning about shapes and measurement, and because they cover a considerable amount of conceptual development in spatial, measurement, and geometrical reasoning. This forms a large anchor LT. (10) *Time and money*, a small early-grades set of topics, is tucked into the left side of the map. The measurement cluster has close links to (11) *shapes and angles*, which carries into (12) *triangles and transformation* as students progress into the middle grades. *Integers, number lines and coordinate planes* (13), a mostly 6th-grade set of topics, are placed close by to support further development of other middle-grades trajectories linking geometry and number systems. The cluster of learning trajectories that comprise data, statistics, and probability—(14) *elementary data and modeling,* (15) *variation, distribution and modeling,* and (16) *chance and probability*—are located along the top of the map, as they were most closely related to each other. The limitations of the two-dimensional space on which the map was constructed prevented us from linking them more closely to measurement and ratio reasoning.

Further upwards and to the right are the more complex topics of (17) *early equations and expressions,* which are built on the four operations and which link to (18) *linear and simultaneous functions* to create a foundation for algebra in the 9–12 Standards.

A Framework for Unpacking Learning Trajectories

When one hovers the cursor over a hexagon on the hexagon map of www.turnonccmath.com, the full Standard is presented verbatim in a box in the bottom left corner. If one clicks on a hexagon or learning trajectory, a new window with the descriptors for the selected learning trajectory appears. The descriptors are organized as follows: A "Structural Overview" is presented at the beginning of each LT, identifying the sections of the LT and showing its development across the relevant grades. Sections are then used to create a sub-organization of the learning trajectory. In addition, a framework of five elements was created to systematize the unpacking of the each trajectory:

1. *Conceptual principles*: These are a list of underlying cognitive principles, identified by researchers, which support the overall development of the ideas.
2. *Strategies, representations, and misconceptions:* When students encounter new tasks that are presented as a cognitive challenge, they invent strategies and representations as they solve them, demonstrating their ways of thinking and, often, revealing related misconceptions that need to be addressed instructionally. Because misconceptions typically have a kernel of "right thinking" (Confrey, 1990), these thoughts must be elicited and then refined into alternative conceptions or valid intermediate steps on paths to more sophisticated thinking.
3. *Meaningful distinctions and multiple models:* All educators recognize the value of prior knowledge and the importance of identifying clear targets for learning. A major challenge, however, lies in identifying and evaluating intermediate states of proficiency and understanding their role in moving students forward in their thinking. To describe these intermediate states, teacher and researchers must recognize or invent meaningful distinctions; vocabulary terms for these tend to exhibit properties that are both cognitive and mathematical, such as partitive vs. quotative division, which later simply collapse to "division." We refer to these as "meaningful distinctions." In addition, for "big ideas"—also described as a learning trajectory's "domain goal of understanding"—there are often multiple earlier models that correspond to the different schemes that govern recognition of situations in the real world. These big ideas are typically captured as a "generalization" that, while "encapsulating" their meanings in the minds of experts, hides or loses the details of the distinctions and models, so students should be afforded sufficient opportunity to explore the distinctions and models *before* they move to the generalization, in order to understand its many referents and applications.
4. *Coherent structure:* In a learning trajectory, a pattern often emerges in how a topic is developed; commonly, that pattern is repeated as the students expand it at later grades and apply it to increasingly complex cases, representations, tools, choices of numeric values, or spatial

dimensions. For example, students' understanding of area is expanded as the lengths of the sides take on fractional values. Understanding such structure, and considering which parts of it remain invariant and which change under these expansions, is a characteristic of mathematical reasoning.

5. *Bridging standards*: Moving from "abridged" learning trajectories represented in the CCSS-M to more fully-articulated, "unabridged" standards requires the addition of "bridging standards" that might not have represented major intellectual targets within the CCSS-M but which may nonetheless be necessary to support a successful progression of learning for students. Based on our structural analysis, we sometimes found gaps or inconsistencies in the Standards. In these cases we also added bridging standards. The bridging standards are identified by their use of a capital letter (A, B, C, …) at the end of the standard number, and the use of brown font. Each bridging standard includes an explanation for its addition to the descriptors document.

A question can be raised about the relationship of the eight mathematical practices to our learning trajectories analysis of the CCSS-M. We do not address the practices directly in the analysis, although the practices are critical elements of the curricular instantiations of the CCSS-M. First of all, we emphasize that a learning trajectories analysis is not a curricular analysis, although one can conduct analysis of curricula using the learning trajectory construct (Nguyen & Confrey, in press) by considering the learning trajectory as a boundary object (Confrey & Maloney, in press; Star & Griesemer, 1989). Furthermore, as students progress along a learning trajectory, they will employ the various mathematical practices, such as applying repeated reasoning, and using precision, articulating arguments, or building or critiquing new modeling.

An Example: The Division and Multiplication LT

Data on large-scale assessment show weakness in U.S. student knowledge and understanding of division and multiplication (NAEP, 2009). Furthermore, division and multiplication are topics around which there is considerable research. Fischbein et al. (1985) introduced the idea of primitive schemes for division and multiplication, claiming two for division (partitive and quotative) and only one for multiplication. Partitive division was linked to schemes based on dealing (usually to obtain the size of a share or group) while quotative division, later commonly referred to as "measurement division" (Simon, 1993), was linked to repeated subtraction or addition, in an iterative manner.

Elaborating further on how children learn multiplication, many researchers (Kamii, 1985; Steffe & Cobb, 1998) describe a process of accumulating equal-sized groups by describing how children learn to coordinate the process of differentiating the roles of numbering the groups and naming the group size. In doing so, they derive multiplicative structures from additive ones. They describe a gradual process of skip counting, double counting, and eventual description as a product, *ab,* comprised of a number of groups, *a,* of a particular size *b*. Because multiplication then is comprised of two elements, group size and number of groups, these researchers tend to follow Fischbein et al. (1985), in recognizing the two types of division, one focused on finding the size of the group (partitive) and the other the number of groups (quotative).

Other researchers categorize word problem types in multiplication or division (e.g., equal groups, rates, comparison, Cartesian products, scaling, etc. undertaken by scholars such as Nesher (1980, 1988, 1992), and Carpenter, Fennema, and Romberg (1993). These scholars have a tendency to associate multiplication with a certain set of problems and each type of division with other sets of problems. For example, equal groups problems are associated with multiplication, fair sharing problems are associated with partitive division, and measurement problems (e.g., How many 3 inch ribbons are there in a ribbon that is 36 inches long?) with quotative division. It is preferable, in our opinion, to distinguish among the questions asked (e.g., the size of a group or fair share and the number of groups or the number of shares) and to associate these questions, and not problem categorizations, with the processes students use to solve a problem. One advantage is that this leaves open the possibility of students using other approaches (e.g., co-splitting (Corley, Confrey, & Nguyen, 2012), or the use of arrays or area models models (Battista, Clements, Arnoff, Battista, & Borrow, 1998; Outhred & Mitchelmore, 2000). Researchers who rely on categorization schemes (CGI, others) tend to focus on these as applications of operations rather than to go further to use them to define the underlying cognitive schemes (Carpenter & Fennema, 1992).

A contrasting trend in research was introduced by Vergnaud in his work on multiplicative conceptual fields (MCF) (Vergnaud, 1983, 1988), when he articulated the relations among ratio and proportion and multiplication and division. The MCF, he argued, consisted "of all situations that can be analyzed as simple or multiple proportion problems and for which one usually needs to multiply or divide" (Vergnaud, 1988, p. 141). He connected the many parts of the MCF to a four part relationship (visually, a two-by-two arrangement) among quantities in which movement horizontally was described as a functional, demonstrating a direct variation relationship between two quantities (i.e., $f(x) = ax$) and vertical movement was referred to as an "*isomorphism of measures*."

In a related vein, in 1988, I articulated my splitting conjecture (Confrey, 1988), arguing that multiplication and division could be linked to ratio and proportion as derived from an early application of an operation I labeled *splitting*, and subsequently also labeled *equipartitioning*. In a three-year teaching experiment of children in 3rd–5th grade, I demonstrated the advantages to student learning of co-defining multiplication, division, and ratio (Confrey & Scarano, 1995) and showed the effects of teaching fractions as expressing a particular subset of ratio relations.

Data suggest that, contrary to most textbook sequencing, equipartitioning and partitive division are understood at an early age (Bell, Fischbein, & Greer, 1984; Confrey et al., 2009; Confrey & Scarano, 1995). Moreover, approaching division and multiplication through early experience with ratio has been supported by research on protoratio (Noelting, 1980a; Noelting, 1980b; Resnick & Singer, 1993), on splitting (Confrey, 1988; Confrey & Scarano, 1995), and on distribution (Streefland, 1984, 1991).

Schwartz (1988) distinguished between referent-transforming and referent-preserving operations, suggesting that additive structures are referent-preserving (preserves the referent unit, e g., 4 apples plus 3 apples equals 7 apples) while multiplicative ones are referent-transforming (does not preserve the referent unit, e.g., 20 coins shared among (divided) 5 people results in 4 coins per person). He also introduced the distinction between extensive quantities (magnitude) and intensive quantities (indirectly measured as composed from other quantities). However, I argue that multiplication can also be referent-preserving when only the particular unit changes (e.g., in the case of measurement conversion, the use of groups, or scaling).

This second set of approaches deemphasize the role of addition and subtraction in the construction of division and multiplication. Instead I view division and multiplication as related operations describing the same situations in reverse. The two operations are interlocked in a four-part relationship that can be described by ratio relations. For example, in the "division problem" 20 coins shared among 5 people results in 4 coins per person, the ratio relationship is 20 coins : 5 people :: 4 coins : 1 person. Multiplication can be used to describe the movement from 4 coins to 20 coins and 1 person to 5 people and division can be used to describe the reverse movement. Because they rely on ratios, this treatment of division and multiplication is necessarily related to the use of two distinct quantities: the case of referent-preserving division and multiplication is cast as the reduced case where groups, unit-changes, or a scalar are introduced. These approaches also tend to support the extension of the operations to non-whole numbers, and more intuitively anticipate the operator construct of rational numbers (Behr, Harel, Post, & Lesh, 1994), which I locate in this trajectory.

Both generalized approaches recognize the use of division and multiplication in area measurement and find ways to incorporate it. In the first approach through counting and additive structures, arrays can be viewed as a transitional tool. If the groups are lined up in columns and placed side by side, then the resulting array can be viewed as representing both the number of groups (rows) and the size of the groups (columns). Proceeding from the discrete case to the continuous case can still support a definition of the multiplication operation in terms of the number of groups and their sizes. The integrated approach also uses area problems but does so through the application of scaling operations from the single unit on the lengths of the sides of a rectangle, and subsequently on the area of the resulting rectangular figure.

In deciding how to approach the learning trajectory, I sought ways to:

1) combine the strengths of both models, while emphasizing importance of multiplicative structures;
2) build from what the children already knew from the related learning trajectories of equipartitioning, length, area and volume, and addition and subtraction;
3) ensure the approaches were sensitive to the variety of situations connected to division and multiplication; and
4) anticipate how sufficient the models would be as the numeric values in the problems changed from whole numbers to non-whole rational numbers.

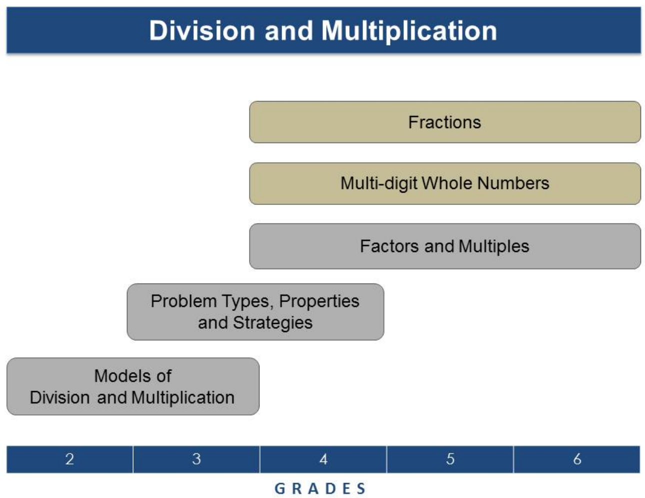

Figure 2. Structural Overview diagram for Division and Multiplication learning trajectory

Framework for Learning Trajectories, Applied to the Division and Multiplication LT

The Structural Overview of the learning trajectory is shown above (Figure 2) whereby one can see that the LT stretches from second through sixth grade. Students develop three models and then apply them to a variety of problem types. As they become fluent in the number facts, they learn about factors and multiples and then extend their knowledge to more complex cases. In the following sections, a window into the structure of the division and multiplication learning trajectory (DMLT) is provided using the five-element framework described previously.

The Target of the Learning Trajectory for Division and Multiplication

Learning trajectories always incorporate assumptions about what students have experienced and know, and what the target of that learning should be at the upper end of the trajectory. The primary target of the DMLT is for students to understand the relationships captured in the equation: $ac/bd \div a/b = c/d$. As explained below, these relationships can be understood either as they reside in a ratio box or in relation to two-dimensional area relations (which can later be extended to higher dimensions).

Ratio boxes relate two quantities such that the relationship is preserved across multiplicative changes to both quantities. All but elementary uses of the ratio box for fair sharing explicitly show the preservation of the ratio across multiplicative changes by using two pairs of "arrows," one which shows the multiplicative or divisional operation that relates the two sets of numbers vertically and showing the other relationship horizontally (Confrey, 1995). Noelting refers to these as, respectively, "between" and "within ratio relations (Noelting, 1980a, 1980b). Characteristic of a ratio box is that the pairs of opposite arrows are identical.

The DMLT can be summarized as an evolving sequence of types of ratio boxes and area models. Those ratio boxes start with a "fair sharing box," and proceed to a division/multiplication box (D/M box) to complete the DMLT. In the ratio and proportion and percents LT, the boxes evolve into a fully developed ratio box. Figure 3, below, illustrates the fully developed ratio box. Given any three values students find a fourth unknown value of the proportion, and describe the relationships represented by the operator arrows, either as shown here as multiplication, or its inverse, division (not shown).

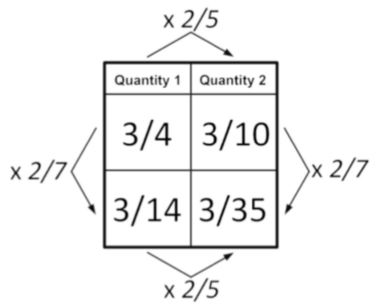

Figure 3: A ratio box solution, with multiplication shown

The DMLT begins from a "reduced ratio box" known as a fair-sharing box in the equipartitioning LT (EQLT). Second-graders can fill in the column headers and the two rows when sharing, for example when fair-sharing 12 coins among 3 people, they fill in 12 and 3 in the top row, and 4 and 1 in the bottom row (Figure 4a). Also based on the EQLT, they express the sizes of upper row numbers relative to lower row numbers as "*b times as many*." At this young age and lacking any formal introduction to multiplication or division, children are not expected to use the arrow notation. For the EQLT, the final target goal can be expressed in a ratio box (Figure 4b) corresponding to Standard 5.NF.3 ("Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem").

Figure 4a: Fair share box for equipartitioning a collection of 12 coins

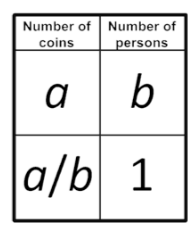

Figure 4b: Generalized fair share box for equipartitioning collections

Building from the fair sharing box, the first target for the DMLT is a slightly more sophisticated reduced ratio box called a "division/multiplication box" (D/M box). The D/M box (Figure 5a) also has a 1 in the lower right corner because in the four-part relations for MCF, for division and multiplication, one cell is equal to 1. For example, in the problem "at a tire shop, six cars are getting their 4 tires changed. How many tires are needed?," the final D/M box would have two columns—one for the number of tires and one for the number of cars—and show 24 tires associated with six cars and 4 tires with *one* car. The number facts, $6 \times 4 = 24$, $24 \div 6 = 4$, and $24 \div 4 = 6$, do not show the *one*. At first, the use of the D/M box can be constrained to whole numbers only. The D/M box differs in two respects from the fair-sharing box. Firstly, it is not restricted to fair-share situations, and secondly, as students learn to work with division and multiplication operations symbolically, they add arrows to define the relationships (operators) explicitly. The associated area model, can also initially use whole numbers (Figure 5b).[4]

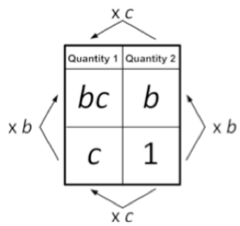

Figure 5a: D/M box adapted for whole-number multiplication

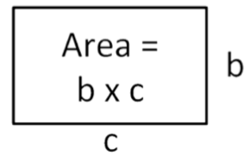

Figure 5b: Whole-number multiplication model for the area of a rectangle

In order to understand the D/M box and the rectangle area model, students describe and work with all three related equations of $a \times b = ab$, $ab \div$ b = a and $ab \div$ a = b. These intermediate goals are presented here in symbolic form for brevity, for the benefit of experts; students, however, are expected to understand where they come from, explain and represent them, relate them to prior and related knowledge with justifications, and apply them to solve a rich variety of problems. In addition to correctly producing their answers, students are expected to be able to move about flexibly and fluently in multiplicative space using factors, including primes and multiples, and recognize, discover, and use the relevant properties and practices.

The final target for the DMLT is a D/M box showing division, multiplication, and a rectangular area model (Figures 6a, b, c) where the non-one values in the cells can be any rational numbers. The DMLT can be understood now as poised between (a) equipartitioning, and (b) ratio and percent. As will also be shown, it draws on elements of other LTs on the length, area and volume, addition and subtraction, and place value and decimals.

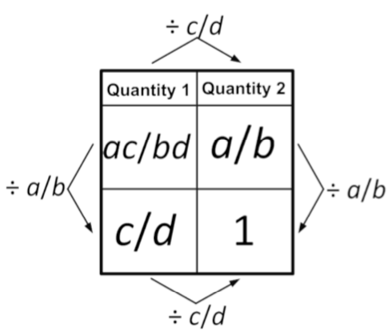

Figure 6a. D/M box (division)

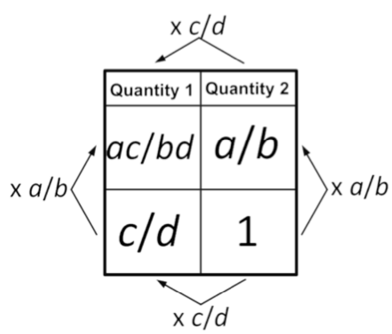

Figure 6b. D/M box (multiplication)

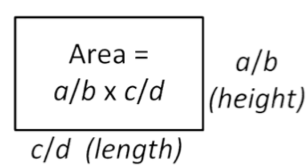

Figure 6c. Area model

Also, later in the length, area, and volume LT, the product can include more than two dimensions (essential for the associative property), so that one can explain volume as $v = l \times w \times h$, or as v = area $\times$ h, and one can increase dimensionality as required for modeling multiplication in higher dimensions that lack obvious spatial analogues. This set of related learning trajectories: equipartitioning, division and multiplication, ratio and proportion and percents, and length, area and volume, together with similarity (within the triangles and transformations LT), comprise the majority of the content that resides in the multiplicative structures.

It is important to understand as fully as possible the target or domain goal understanding for a learning trajectory, because while it often cannot be directly taught, it must be reached as the product of a careful series of transformations based on empirical study of student learning. By delineating it carefully, one can

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distinguish intermediate states that are productive from ones which will limit students' chances of obtaining a full and nuanced perspective.

Distinctions and Models

A synthesis of the literature yields three fundamental *models* for the joint operations of division/multiplication, each of which generate both division and multiplication contexts. These are (a) referent-transforming, (b) referent-preserving, and (c) referent-composing models. These three models are necessary to sufficiently link division and multiplication to its related trajectories, from equipartitioning and addition/subtraction to ratio and proportions and percent, fractions, chance and probability, and length, area, and volume, and to support mathematical modeling. The three models are described below:

a) *Referent-Transforming.* Division/multiplication in these models involves changes in the attributes or referents connected with the quantities, or action on a quantity of one attribute or referent by a quantity of another attribute or referent. For instance, in fair sharing, coins are shared among people to produce coins per person (Figure 7). Rate problems also fit in this category. In relation to the D/M Box, the student sees $6 \times 3 = 18$ as shifting from 6 people to 18 coins by means of a multiplication by 3 coins per person, which transforms the referent using an intensive quantity as an operator. There are two associated division problems for fair sharing $18 \div 6 = 3$ and $18 \div 3 = 6$, each of which is referent-transforming. Students are likely to solve the first one partitively and the second quotatively.

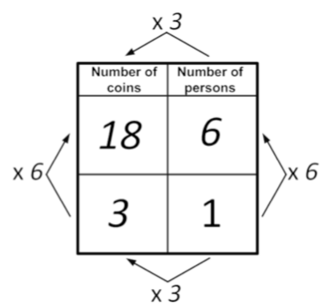

Figure 7. D/M box used to model referent-transforming multiplication

b) *Referent-Preserving*. Division/multiplication in these models involves a multiplicative comparison of two amounts of a single quantity. This can be accomplished using a new unit, a composite unit such as a group or a scale, or by using one amount to measure another while the referent or attribute is maintained. For example, if one is told that the distance from New York to Kansas City is six times the distance from New York to Baltimore (approximately 200 miles), the D/M box would look like Figure 8a:

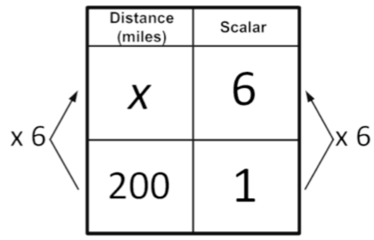

Figure 8a: Referent-preserving multiplication problem modeled with a D/M box

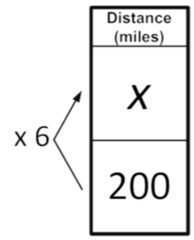

Figure 8b: The same D/M box with arrow indicating the scalar

The scale in the right-hand column is, by most accounts, unit-less, but the right column is used to establish the vertical arrow, or the "within" or referent-preserving relation, "multiply by 6." Thus to solve this problem, one maps miles to miles, multiplying by the dimensionless scalar 6, to get 1200 miles. Because the left-hand column with the scalar multiplication is sufficient to solve the problem, a two-by-one display of this relationship is sufficient as shown in Figure 8b. Likewise we suggest that problems involving groups and measurement conversions can and probably should be treated as referent-preserving because only the unit and not the referent changes.

We note that because the D/M box always has a 1 in one cell, collapsing it to a 2×1 box or a 1×2 box is always possible because the operator arrows will "carry" the information from the non-one cell as illustrated in figure 8b. These collapsed views permit one to assert a single model for division/multiplication; a drawback of this curtailment, if done too early, conceals some of the richness of the relational reasoning.

c) *Referent-Composing*. Division/multiplication in these problems involves the creation of a new referent or attribute not previously associated with the other quantities. For example, the division/multiplication associated with area produces square inches from side lengths in inches. In Cartesian products, a number of shirts and a number of pants produce a number of outfits, and so on. Volume as a product of three length measures or as a product of length and area, and higher dimensions also fit in this category. Arrays can form a transitional representation linking referent-preserving and referent-creating, such that the product can be computed by multiplication of the number of dots in each of the two sides, but the product remains a number of dots so no new referent is composed. The row and column structure, while geometrically extending in two dimensions (length and width) still produces a product that is a total number of dots.

These three models of division and multiplication can be summarized as shown in Table 1 along with examples of problem contexts associated with each model.

Table 1. Three Models of Division/Multiplication, Along with Common Contexts for Each

Model 1: Referent-Transforming	**Model 2: Referent-Preserving**	**Model 3: Referent-Composing**
Fair Sharing	Unit Conversion	Arrays
Rate	Scaling	Area
Equal-sized Groups		Cartesian Product

Note the placement of the equal-sized groups context, in which one reasons with the number of groups, the size of the group; the resulting product is placed in both models 1 and 2. A problem such as "a bookshelf has four shelves with six books on each, how many books are there?" can be viewed as referent-transforming (number of books per shelf × number of shelves = number of books) or as referent-preserving (4 groups of 6 books).

As a result of this analysis, the team recognized that the transition to division and multiplication needed to be broadened and strengthened. We analyzed the experiences of children that would support these varied models, especially in the earlier trajectories of equipartitioning and addition/subtraction. The expectation for the DMLT was that students would encounter the models as simple whole-number cases until they built up their repertoire, became fluent and flexible in their knowledge of the associated facts, and explored the properties. As the numbers became larger, the algorithms would be developed. Though not fully developed in this paper, students' introduction to non-whole quantities in the LT division and multiplication involves reconceptualizing meanings based on their understanding of relational naming (describing 12 shared among 4 as 1/4 of the collection) and reassembly from EQLT. Over time, students generalize across the various number types, models and applications as division/multiplication more abstractly. However, by avoiding overgeneralizing and simplifying to one single model, students should remain flexible in selecting appropriate models for division and multiplication in modeling activities.

Bridging Standards

From EQLT, children enter third grade with experience in fair sharing, relational naming, and composition of splits, all of which can support their movement to division/multiplication. Composition of splits refers to children splitting a split (such as a rectangle into two parts vertically and three parts horizontally) and learning to predict six (2 × 3) instead of five (2 + 3) resulting parts. The addition and subtraction LT also links to DMLT through a standard on the array structure and repeated addition. The length, area, and volume LT also contributes to students' conceptions of division and multiplication, and the relevant commutative and distributive properties with such activities as finding a common unit for area measurement and composing and decomposing rectangular areas. Nonetheless, a set of bridging standards were needed—first, to make the necessary connections to these earlier learning trajectories, and secondly, to interpret the meaning of the standards in light of our targets and distinctions.

There are four Standards in CCSS-M that specifically carry the weight of introducing division and multiplication:

- 3.OA.1: "Interpret products of whole numbers";
- 3.OA.2: "Interpret whole-number quotients of whole numbers";
- 3.OA.3: "Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem"; and
- 3.OA.6: "Understand division as an unknown-factor problem."

(Note: 3.OA.4 is placed in elementary algebra because it involves solving for an unknown in any position in $a \times b = c$; 3.OA.5 (concerned with properties) and 3.OA.7 (concerned with fluency) are placed in the next section of the DMLT.)

While these four standards are sufficient to support the distinctions offered above, they are awkward to interpret standard by standard: three of them are required to introduce and link multiplication and division (3.OA.1, 2,and 6), and the examples mentioned along with the first two in the CCSS-M document seem to imply that a problem type is linked to an operation (groups to multiplication and fair sharing to division). Furthermore, 3.OA.3 seems to imply that the problem situations are used to apply the operations rather than that the operations are developed to model the situations. This bias seems to be pervasive in the K–8 Standards.

However, what appears to be awkwardness in the Standards can be addressed because the examples therein are not intended to limit the cases but only to illustrate them. Therefore in our interpretations, we explain the three cases of multiplication (referent-transforming, referent-preserving and referent-composing), then treat division similarly, using Standard 3.OA.6 to link the operations. While the model remains referent-transforming, the observed processes for the division problems may appear as partitive or quotative.

Standard 3.OA.3 provides an opportunity to summarize the entire framework with descriptions of the overall D/M box for whole numbers and the area model. In preparation for Standard 3.OA.3, three bridging standards were required for the model for referent-composing D/M. The bridging standard 3.OA.F ("Students reason with arrays using multiplicative relationships") was added to provide students opportunities to work multiplicatively with arrays. This was necessary because the standard authors had restricted the approach to arrays in second grade to repeated addition (2.OA.4: "Use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns; write an equation to express the total as a sum of equal addends"). This constraint ruled out other approaches such as by decomposing and composing arrays into other equivalent arrangements (for instance, rearranging a 6 × 4 array as a 12 × 2 or a 24 × 1), or using skip counting.

Building on a bridging standard from the EQLT (2.G.C: "Equipartition a rectangle using vertical and horizontal cuts and predict the resulting number of parts."), another bridging standard, 3.OA.D ("Students learn to code composition of splits as multiplication and can state the associated division problem"), supports students in coding compositions of splits as multiplication and division. From the length, area and volume LT, the standard 3.MD.7.b ("Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning"), links to the emerging DMLT. To complete the idea of referent composition then for both area and for pairing of attributes to create Cartesian products, bridging standard 3.OA.B was added, stating "Relate multiplication and division problems to rectangular area (e.g., 3 inches × 4 inches = 12 square inches) and Cartesian products (e.g., 3 pants × 2 shirts = 6 possible outfits)."

With this set of three bridging standards carefully linked to the four CCSS-M Standards, third grade students who accomplish the related content should be able to apply all three models to situations to produce both division and multiplication problems and solve for unknowns in all of the three positions of the problem in standard 3.OA.3. Well-prepared with three models, students can be carefully introduced to the cases in which non-whole numbers are involved, topics that are discussed more fully on the website. As argued previously, this approach is also powerful because it builds explicitly from prior learning trajectories and anticipates later ones.

Strategies, Representations, and Misconceptions

The previous section on distinctions and models supports students in creating a rich variety of representations for multiplication and division (groups, tree diagrams, measures, scaled drawings, and Cartesian products shown as two dimensional cross products). A second important area of development involves how children learn their "multiplication and division facts." Confrey and Scarano (1995) had demonstrated that children are not given adequate support to "move in multiplicative space." Most

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teachers assume that multiplication should be introduced separately from division and that learning number facts should proceed in the same order as addition facts, from small to large numbers. Instead, the LT research shows how many forms of interrelationships among and between multiplication facts can be fostered by teaching children rich strategies that build on early understanding of numbers. For example, instead of teaching multiplication facts in the order of the counting numbers (i.e., ×1, ×2, ×3, etc.), Confrey showed that a sequence of double (×2), double-double (×4), double-double-double (×8), then multiplying by 10 and then by 5 (×10 ÷2), then tripling (×3), multiplying by 6, (triple-double, or ×3 ×2), and by 9 (triple-triple), and then, finally, by 7, is more readily understood by students, and makes more sense to them. (The related division facts are practiced simultaneously with multiplication facts in this sequence.) Instead of viewing multiplication facts as simply a list of things to be memorized, students begin to get a foundation of the multiplicative relationships among numbers—what I have previously called "moving around in multiplicative space" (Confrey, 1995).

Two misconceptions are addressed in the DMLT. An early standard in the LT regards the idea of "evenness" (as contrasted with "oddness"), and the descriptors carefully articulate two approaches, (1) fair sharing by two, and (2) pairing up. In addition, the descriptors warn that students use the term "even" to describe when a collection can be fairly or evenly shared, for example, in the sentence, "It came out even." The descriptors discuss how the term "even" therefore can be used simultaneously by students in two conflicting ways, (1) to describe when a factor divides evenly—then the result is even (so that six shared among two is three which is "even" or fair), and (2) to describe that when a number is "even," i.e., is divisible by two. The two meanings must be distinguished by students, so they avoid or resolve a "misconception." This is a prime example in which we wrote into the descriptors an important distinction that we believe many teachers would not readily recognize and discuss with their students.

The second, more widely recognized, misconception is "multiplication makes bigger and division makes smaller" (MMBDMS) (Greer, 1992). The CCSS-M address this misconception directly in 5.NF.5.b ("Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1").

In the DMLT, the misconception is addressed in relation to each of the three models. In the unit transforming model, the descriptors illustrate that any two numbers can be related in an equation, such as rate × time = distance, so that 30 mph can be multiplied by a half hour to produce 15 (i.e., fewer) miles. Students also learn to interpret division of two quantities, in the form a/b and c/d, as a ratio of fractions or ratios ($3/4 \div 1/2 = 3/2$). This example demonstrates that division can result in a larger quantity than the quantity one begins with. In referent-preserving situations, division by n is shown to be equivalent to multiplication by $1/n$, with students learning to predict the effects of multiplication by a/b as a composition of multiplication and division, just as was done originally in Dienes's work on operators (e.g., stretchers and shrinkers) (Dienes, 1967). Finally, for contexts using the area model, students learn that area measured in square units can be of a smaller magnitude than the magnitudes of either of the sides.

Conceptual Principles

The development of conceptual principles in the DMLT can revolve first around the ideas of factors and multiples. Overreliance on multiplication as exclusively derived from repeated addition leaves students insensitive to the distinctions between additive and multiplicative reasoning. As noted above most students are not given enough experience moving in multiplicative space. In the descriptors, we also offer the view that students should be challenged to find multiple ways using only multiplication and division to move among numbers, such as between 15 and 24 (dividing by 5 and multiplying by 8). I called these types of problems "daisy chains" in earlier work (Confrey & Scarano, 1995). This encourages students to work with common factors. In addition, it helps students to develop knowledge of the principles of multiplication by 1 (identity), multiplication by zero, the commutative property of multiplication, the associative property of multiplication, and, later, multiplicative inverses. It can also lead to students recognizing rational number multiplication and division. In the DMLT, we also treat distributivity very carefully and explicitly, as it is the means by which the additive structures are linked to the multiplicative structures.

Coherent Structure

The coherence of the DMLT's structure can now be summarized. The LT builds from the prior LTs of (a) equipartitioning; (b) length, area, and volume; and (c) addition and subtraction to establish the three models applied to whole numbers. The interrelationships among the ideas of factors and the patterns in the multiplicative table are used to support the evolution of the properties and draw connections to multiplicative vs. additive comparison. Then at the upper end of the LT, two types of extensions occur: the application of the problems to multidigit algorithms using the distributive property, and the inclusion of fractions and ratios as operators. These extensions are carefully constructed in the context of the three underlying models. The extensions to fractional operators are also connected to the learning trajectory on length, area and volume where the MMBDMS misconception can be most readily remediated.

Overall the LT is designed to set up the movement to ratio reasoning through connections to the two Standards on tables of values, 4.MD.1 on conversions and 5.OA.3 on tables of values. Finally, students are prepared for the culmination of equipartitioning in the fifth grade standard (5.NF.3: "Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem"). The target goal of the LT is reached in a set of Standards that include 6.NS.1 ("Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem"), and 7.RP.1 ("Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units").

With this example of how an LT is related to the standards, one can see that the process of linking an LT to standards requires careful and synthetic applications of empirical research literature. The overall framework for multiplication and division is thin in the early grades and tends to overemphasize a relationship to additive structures, resulting in an underdeveloped framework for multiplicative structures. We have attempted to articulate a stronger framework for a stronger multiplicative structures approach by adding a few key bridging standards within the learning trajectory which link to equipartitioning and help to explain how multiple models of division and multiplication can be supported in classroom instruction. The authors of the CCSS-M left room for such interpretations by avoiding the mistake of *defining* multiplication as repeated addition (which had been included in early drafts of the CCSS-M). The learning trajectory also makes the case for both strong distinction among the strategies, and strong relationships among the models, strategies, and associated properties.

Implications for Researchers and Professional Developers

The www.turnonccmath.com website was visited more than 7000 times between its release in April 2012 and late May 2012. The primary visitors have been state and district personnel and teachers looking for a means to make sense of and make instructional interpretations from the CCSS-M. Some found the website on their own while others have found it as a result of presentations and mailings. We are currently in the process of improving the site in several ways. We are adding in the relevant references to research that we were unable to do in the first round due to the pressures of time and the focus on creating coherence and consistency in the descriptors; as one can imagine, this has been hard work. We are also preparing to undertake an expert review process, similar to the process we conducted for vetting the LT on length, area and volume with the researchers from the Measurement Mini-Center.

We are also committed to working with districts and states using the LTs and their descriptors as a basis for professional development. These efforts include both pre-service and in-service teachers. We have worked with Colorado, West Virginia, North Carolina, and Washington, and have received requests from other states. In this work, it becomes clear that the foundation of knowledge in the unpacking is not on its own sufficient to support professional development; the examples in this paper make it clear that the written descriptors by themselves can serve as an important part of efforts to help teachers understand the mathematical knowledge embedded in the trajectories and to translate them into robust learning trajectory-based classroom practice.

There are numerous opportunities for college and university faculty and state and district mathematics coordinators to use these materials to support professional development. We have engaged in creating digital presentations to show, in a more visual and story-based way, how the LTs are linked to the standards. One could imagine building webinars and course materials to provide hands-on experiences for teachers with these ideas as well, assuming sufficient available resources. Some of the teams developing the original LTs have already created related professional development materials that can be used in creating a nationwide application of this work.

Perhaps even more relevant to the PME-NA audience is the potential professional value of the website to the research community. To some degree, the influence of learning trajectories/progressions on the CCSS-M was mitigated by ambiguity, dispute, or lack of synthesis by the research community. While this is not surprising in a field as young as ours, its maturation depends on our willingness to undertake synthesis, and suggests it would be wise to engage in more of this kind of activity. While researchers may wish to “do their own thing” or await some other body to interpret and synthesize the development of the Standards, it would improve our professional reputation as a field if we were to take up this challenge ourselves.

It is often reported that in medicine, prior to the famous Flexner report (Flexner, 1910), physicians received education in general basic science and then apprenticed to a working physician until they were ready to establish their own practice. If that mentor was a strong and knowledgeable role model, the apprentice was likely to emerge as a well-qualified and very competent physician as well. If not, another “quack” might be added to the rolls. After the Flexner report, the medical field stepped up to create a practitioner-informed practice-oriented knowledge base for “clinical training” of physicians and to standardize medical education. In some ways, we are in a similar predicament in mathematics education research. Someone studying in a strong program, or apprenticing with a strong faculty member, tends to move into teacher education well prepared. Study in a less rigorous program and navigating the literature without any guidance leaves one tasked with “inventing” a deep understanding of the literature: the job is highly inefficient, at best, and likely to leave a student poorly prepared to take up highly informed work or to make insightful contributions. Synthesis work is challenging, sometimes grueling, and yet remarkably satisfying. The www.turnonccmath.com website is meant to serve as one contribution to increasing the accessibility, completeness, and consistency of the interpretation of the significant portion of the research base in mathematics education on student learning.

Our research group has been the beneficiary of one of the REESE synthesis grants to bring together a literature on rational number reasoning that consists of some 600 articles. This experience has led us to this synthesis of the LTs work with the CCSS-M. It may be the case that the idea of LTs will fade, just as so many movements in mathematics education do (e.g., metacognition, problem solving, differentiated instruction, active mathematics teaching, and individualized instruction; the list is, sadly, quite long). Many valuable lessons resided in those movements, and for the field to become robust for guiding the conduct of practice, it must create a means for its empirical work to accrue progressively and be refined over time. Such a means would help reduce the frequency with which we see the same studies conducted (e.g., students mistaking the visual path of a function’s representation for the behavior of the function has been studied too many times to count), and help to define a cutting edge field where scholars can aim to make progress. All of these suggestions fulfill the vision of the conference organizers for this PME-NA annual meeting to discuss transitions. The bulk of this paper addressed how to create supports for teachers as they transition to the CCSS-M, but the discussions herein also address transitions for professional developers and researchers in the everyday conduct and sharing of our practices.

Endnotes

[1] The author was a member of the National Validation Committee for the Common Core State Standards.

[2] This meeting was jointly hosted by the DELTA research group (directors Confrey and Maloney) and the Consortium for Policy Research in Education (co-sponsors F. Mosher, P. Daro, and T. Corcoran).

[3] The Mini-Center comprises faculty and senior researchers (J. Smith, organizer, J. Confrey, J. Barrett, R. Lehrer, M. Battista, D. Clements, B. Dougherty, D. Heck) and associated postdoctoral researchers and graduate students.

[4] One can also use the D/M box (Figure 5a) to apply to area, if one starts with a unit square and views b as stretching b into a strip of b units, for example, as a strip along the top of Figure 5b. Then if c represents a c × 1 strip vertically along the left edge, then stretching it by b produces bc; and the ratios are preserved. This model seems too abstract and so we prefer to introduce the area model separately.

Acknowledgments

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