FRACTION OPERATION ALGORITHMIC THINKING: LEVERAGING STUDENTS' USE OF EQUIVALENCE AS A TOOL

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The data used for the qualitative analysis reported here were generated as part of a larger study to understand and characterize teacher practice related to engaging students in algorithmic thinking associated with the fraction operations of addition, subtraction, multiplication and division. This paper presents ways in which teachers used students' emergent ideas to leverage the use of equivalence as a tool, rather than a procedure, to support students as they work to develop algorithms for operating with fractions.

Keywords: Rational Numbers; Instructional Activities and Practices; Middle School Education

Purpose

Prior work on teacher practice acknowledges the complexity of instruction when teachers aim to engage students in authentic mathematical activity where the instructional path is not specified and teachers themselves engage in sense-making as they make instructional decisions (Ball & Bass, 2003; Kazemi & Stipek, 2001; Stein, Smith, Henningsen, & Silver, 2000). In their review of the collective literature on teaching and classroom practice, Franke, Kazemi and Battey (2007) offer that effective teaching involves more than having a rich task or eliciting students' thinking. They argue that the field would benefit if the complexity of teacher practice were examined using a domain-specific approach leading to the identification of routines of practice, or core activities, that should occur regularly within particular mathematical domains.

From an instructional perspective, fraction operations are especially complex (Lamon, 2005; Ma, 1999; Borko et al., 1992). The literature (e.g., Kamii & Warrington, 1999) has documented that students can invent, or reinvent, procedures for operating with fractions. However, there has been little consideration of the role that a teacher might play in supporting students to construct such strategies and procedures. In this paper we draw from our work with four experienced and "skillful" teachers whose approach to teaching fraction operations involves positioning student to invent, or reinvent, their own procedures for operating with fractions. It is argued that the ways in which the teachers leveraged student reasoning to draw out perspectives on equivalence is an important aspect of teacher practice associated with instruction that emphasizes a *guided-reinvention* approach to fraction-based algorithm development (Gravemeijer & van Galen, 2003).

Theoretical Framework

In their discussion of a *guided-reinvention* approach to algorithm development, Gravemeijer and van Galen (2003) emphasize that instead of concretizing mathematical algorithms for students, teachers can use an instructional approach where students develop or reinvent algorithms for themselves. Given the opportunity to reinvent mathematics in somewhat the manner that it played out historically, students can experience mathematical knowledge as a product of their own activity. "The core idea is that students develop mathematical concepts, notations, and procedures as organizing tools when solving problems" (Gravemeijer & van Galen, 2003, p. 117). Related to guided-reinvention is the notion of emergent-modeling (Gravemeijer, 2004). When instruction is designed to support emergent-modeling, instead of trying to concretize mathematical knowledge, the objective is to help students model their own informal mathematical ideas. From this informal modeling, more formal ways of reasoning can emerge. The teacher plays a role in supporting this development. This work characterizes practice where teachers supported students' mathematical activity related to fraction operations, and the role of equivalence, without taking over the guided-reinvention process or reducing the cognitive demands of the work.

Equivalence concepts are fundamental if students are going to be able to operate meaningfully with fractional quantities. The flexibility to understand and view fractional quantities as having many names all representing the same number, the ability to generate equivalent fractions meaningfully, and the ability to perceive the relationship between equivalent fraction representations, are important features in algorithm development (Lamon, 2005). The students in this study explored equivalence as a conceptual idea and as a skill in an instructional unit that preceded the unit where data was collected for the study reported here. In the data we focus on ways in which teachers drew from students' informal reasoning in order to support the notion of equivalence as a tool when operating with fractions. It was not suggested to students in advance that they needed to have or use equivalent fractions. It emerged from their mathematical activity. It was present in their informal work when making sense of and solving problems that would lead to adding, subtracting, multiplying and dividing fractions.

Methodology

The settings for this study were the classrooms of four sixth-grade teachers and their students. Each of the teachers used the Connected Mathematics Project (CMP) II instructional unit *Bits and Pieces II: Using Fraction Operations* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006a) as their primary curriculum source. This unit uses a guided-reinvention approach to developing meaning for fraction operations. It allows algorithms to arise through student engagement with both contextual and number-based situations. In this setting, assumptions can be made about the tasks used and about the fraction-related concepts that were developed prior to, and during the unit on fraction operations. In the timeline for the sixth graders who are part of this study, students came to the fraction operation unit with previous experiences that supported their understanding and ability to use equivalent fractions. Prior to implementing the *Bits and Pieces II* unit, the *Bits and Pieces I: Understanding Fractions, Decimals and Percents* (Lappan, Fey, Fitzgerald, Friel & Phillips, 2006b) unit was also implemented.

This study used a qualitative design. During the teaching of the *Bits and Pieces II* unit, classroom lessons were videotaped each day during the 5–6 weeks it took to cover the unit. In addition, the teachers wore an audio recorder during each lesson. The audio recorder was used to record the small group conversations teachers had with students. When a teacher completed a lesson, they also audio recorded a short 5-minute reflection on the lesson. When visiting, the researchers engaged in participant observation. This included observing, taking field notes, interacting with students during small group work time, and meeting with the teacher after the lesson to seek their perspectives on the lesson. During the summer the researchers and teachers came together for three days to discuss their teaching. The three days of summer work were also videotaped for data analysis. Ways teachers purposefully leveraged the use of equivalence as a mathematical reasoning tool was one of the topics discussed.

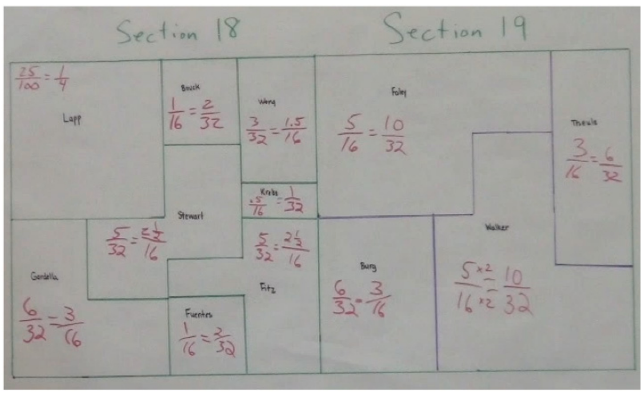
Data analysis was guided by Erickson's (1986) interpretive methods and participant observational fieldwork, which addresses the need to understand the social actions that take place in a setting. The multiple data sources allowed for triangulation. The school-year data was transcribed and analyzed for emerging themes. The analysis led to characterizations for leveraging equivalence as a tool. It focused on themes related to what teacher elicited from students during whole class discussions when they were sharing strategies for solving problems, and how these elicitations positioned students to move from informal to formal mathematical reasoning.

Findings

In order to capture how a teacher might leverage students' informal reasoning with equivalence in support of helping them articulate strategies for operating with fractions, characterizations of practice are provided for each addition/subtraction, multiplication, and division. Specific project teachers are not identified in the dialogue. These findings are presented as a collective view of what was observed in the classroom data and what emerged in the collaborative work that took place during the summer workshop.

Addition/Subtraction

The work on addition and subtraction began with a task where students drew on past work that involved partitioning and naming fractional quantities. The problem, referred to here as The Land Problem (see Lappan et al., 2006a, pp. 17–19 for full problem), used an area model where square sections of land where divided into smaller sections for farming. Initially, the task asked students to determine what fraction of a section of land each farmer owned. Depending upon how students partitioned the land, various equivalent fractional names emerge for different farmers. As part of their arguments, students visually partitioned their map into equal-size parts and showed, for example, that Bouck owned 1/16 of a section or Foley owned 5/16 of a section. Some students would cut out a farmer's section, for example a Kreb's piece, and then show 32 Krebs-size pieces filled Section 18 and that Bouck's land could also be called two 32nds of the section. Figure 1a shows a map where a teacher recorded the fractional values that emerged from students' work. An important idea that emerged from this part of the problem was that collectively students offered more than one possible fractional name for each section.



(1) 5/16 + 3/16 = 8/16 = ½

(2) 10/32 + 6/32 = 16/32

(3) 10/32 + 6/32 = 16/32 = ½

(4) 5/16 + 6/32 = 16/32

(5) 5/16 + 3/16 = 8/16

Figure 1a. Land map solutions

Figure 1b. Number sentences

The next part of the Land Problem asked students to combine various sections of land and write a number sentence for their solution. One problem posed was: *Lapp and Bouck combine their land. What fractions of a section do they now own together?* These number sentences were offered by students during discussion: 4/16 + 1/16 = 5/16 and 8/32 + 2/32 = 10/32. Here, as is typical of students who solve this problem, they used fractions with common denominators to write their number sentences. This emerges intuitively. Students did not do this because they were prompted to. When presenting their number sentences, students were asked to show on the map, how they knew their number sentences were true. The teacher then extended students' ideas to draw upon equivalence as a tool by asking them to consider ideas like the following:

- When we put Lapp plus Bouck together some of you said the answer was 10/32 and some of you said the answer is 5/16. Are those amounts the same? Or are they different?
- I am going to throw up another example. I have some kids who look at Lapp and say that Lapp is ¼ of Section 18. Is Lapp ¼ of that section? [class says "yes"] And we are supposed to add Bouck to it. So for example, I could say that Bouck is 1/16. Is Bouck 1/16? [class says "yes"] So I am going to write the number sentence ¼ + 1/16 = 5/16. Is that a true statement?

In response to the later scenario, some said "yes" and others said "no." The teacher asked students to talk to their groups and prove if it was true. The class conversation then went:

T: Tara I heard you say something. Would you share it?
Tara: It is right but if you wanted to make it an easier addition problem to do, you could change the ¼ into 4/16. Then you would have the same denominators.
T: Would that make it easier?
Class: Yes.
T: How many of you agree with that?
Class: [Most students raise hand.]
T: Is this sentence right here [1/4 + 1/16 = 5/16] a true sentence?
Class: Yes.
T: Can someone say what it is about this sentence [1/4 + 1/16 = 5/16] that makes it hard to say if that is right or wrong?
Sam: Because the denominators are different?
T: What does that tell us about the size of the pieces.
Liam: They are different.
T: We are talking about a unit here [points to Lapp on map] that is fourths and then a unit here [points to Bouck on map] that is sixteenths. And it is kind of hard to put that together and say what it is.

There was a similar discussion when the teacher posed the following: *Foley and Burg combine their land*. *What fraction of a section will they now own together?* Figure 1b contains a string of number sentences that emerged during this discussion. Again, students were asked to use the Land Problem map to argue that their solutions were sensible. A student offered number sentence 1 in Figure 1b. Another student then offered number sentence 2 in Figure 1b.

T: You didn't get the same fraction that the other group had…Can someone talk to us about that? One is 8/16 and one is 16/32. Who is right?
Drew: They are both equal.
T: How do you know they are both equal?
Kayla: Because 8 times 2 is 16 and 16 times 2 is 32.
T: So if you have 16/32 of the whole section, how much to you have?
Kayla: ½. [Writes number sentence 3 in Figure 1b.]
T: So, Isabel, what does that say about both answers?
Isabel: They are both equal.
T: Let me ask another question. I had a kid last year that did it a different way. He said Foley was 5/16. Then he looked at Burg [on map] and wrote 6/32. Then for the answer, he wrote 16/32. [See number sentence 4 in Figure 1b.] Would that number sentence work?
Class: [There were both yes and no responses.]
T: Talk with your table. [Students talk.] Daniel.
Daniel: The answer works but not the sentence. But the answer is the right answer.
T: Oh. So the answer works but not the sentence.
Leah: If it gives the right answer then it is true.
Others: No.
Tara: Really, you can change 6/32 to 3/16 and 16/32 is 8/16…[this gets recapped and number sentence 5 in Figure 1b is recorded.]
T: Let me ask again, is this sentence [5/16 + 6/32 = 16/32] a true sentence?
Class: [some said yes and some said no]
T: Yes. This is a true sentence. Because we know that Foley really is 5/16. We know Burg really is 5/32. But, what helps us think about it? Daniel, I heard you say you don't like this sentence. What is it about that sentence that made it hard for you?
Daniel: There are different denominators

T: Yeah. The pieces are not the same size. So what were you guys doing to make these easier for you? So they weren't confusing.
Lacey: Changing the denominators.
T: Yeah. You were changing the denominators and then adding the amounts.

Using equivalence as a tool, the teacher leveraged students' reasoning to draw out several ideas. One idea was to make explicit why students were choosing fractions with common denominators. They were doing this intuitively. By leveraging equivalence as a tool, students were able to explain why this was helpful. This is a key component of the algorithm they are working toward. A second idea involved using equivalence to compare three different solutions (i.e., 8/16, 16/32, and ½) in order to verify they were all correct. This supported students' ability to read and work with mathematical symbolism. When students move on to problems that do not have a context, they will need to use equivalence as a tool to show others how they are working with and manipulating quantities.

Multiplication

There were opportunities to leverage equivalence as a tool when working with fraction multiplication. In one conversation students were finding fractional sections of fractional parts (parts of parts) in a scenario that involved brownie pans (see Lappan et al., 2006a, pp. 32–33 for full problem). For example, *What fraction of a pan will I have if I buy ¾ of a pan that is ½ full?* In these scenarios, the problems were presented as "part of part" problems. At this point in the work, an algorithm was not established nor pushed for explicitly. When modeling ¼ of 2/3, two different diagrammatic approaches were used leading to two different number sentences: ¼ of 2/3 = 2/12 and ¼ of 2/3 = 1/6. Students were asked to consider whether these were both true and how they knew. Students used their diagrams and equivalence as a tool to argue that both 2/12 of a pan and 1/6 of a pan were the same amount.

After working through numerous brownie pan problems, a student offered that when she wrote her number sentences she noticed that it looked like you could just multiply the numerators across and the denominators across and it would work too. Many students were still trying to understand what ¼ of 2/3 meant and so the teacher suggested that this student continue to draw her brownie pan models and test her idea to see if worked across numerous problems.

On the second day of the unit, students were introduced to multiplication symbolism where ¾ of ½ is formally written as ¾ × ½. They were also asked to use estimation and number sense to consider whether the following problems would lead to products greater than or less than one whole: 5/6 × ½, 5/6 × 1, 5/6 × 2, and 3/7 × 2. The student who had been contemplating how to operate symbolically started the following discussion.

Libby: When there is a whole number, not so much when estimating, but remember how I told you before [referring to her idea to multiply numerators and multiply denominators]. For 5/6 × 2, couldn't you turn the two into 12/6 and do it my way and I could figure it out?"
T: [Rewrites 5/6 × 2 as 5/6 × 12/6 on board.] Change this into 12/6?
Libby: Yeah.
T: I don't know why not. It is another name for 2. Right?
Libby: Then I could do 5 times 12 and 6 times 6.
T: If that way works. It seems like every time you tried it, it has matched your model. I don't know if I would want to draw all those things but you could. For 2 you would have to draw two whole pans.
Ginny: I agree with Libby on her way but I think you could do it in a simpler way. You could turn two into two halves.
T: Would that be one? [writes 2/2 = 1].
Ginny: No.
T: So that is not equal. That seems like I would be finding 3/7 of 1 instead of 3/7 of 2 if I made it 2 halves. This is an excellent discussion and it is exactly what I want everyone to be doing…You are thinking and I love it. Keep thinking.

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Here the teacher prompted students to draw upon equivalence as a tool. We also saw the student, Libby, inquiring about the use of equivalence as a tool. In her understanding of fractions, she recognized that a fraction can have many names. She was also starting to realize that by choosing a specific equivalent name, she could use her theory about how to multiply fractions symbolically. This is an idea the entire class would eventually explore together.

Division

With division the long-term goal was to support development of the common denominator algorithm for fraction division. Most of the work used quotative division problem contexts. For example: *You have 7/8 of a pound of hamburger. If you make patties that are each 3/8 of a pound, how many patties can you make?* The initial problems involved simple fractions with common denominators, then simple fractions with unlike denominators, and finally mixed numbers with both common and unlike denominators. In the initial days of the work on division, the focus was on creating a picture or visual representation for the problem and writing a corresponding number sentence. Students were using drawings, rate tables, and number sentences to talk about why 7/8 ÷ 3/8 was like finding how many 3 eighths are in 7 eighths or how many groups of 3 are in 7.

In this scenario, students had moved to working with mixed numbers. The problem being worked on was *You have 2 2/3 pounds of hamburger and you are making 2/3 pound patties. How many patties can you make?* A student presented a picture where three wholes were partitioned into thirds. Two and two-thirds was marked. The student then marked and counted out how many two-thirds were in 2 2/3. The number sentence they wrote was 2 2/3 ÷ 2/3 = 4.

T: Did anyone have a different number sentence then what she had written there? She had 2 2/3 ÷ 2/3 = 4 which is correct. But I think there is another number sentence that could help make the answer stand out even better.
Cody: 8/3 divided by 2/3 equals 4.
T: Can you write that number sentence up there? [pause] Look at that number sentence. We are purposefully putting these up there so you can look at those and start seeing if there is a faster way to do this then drawing a picture or making these [rate] tables that we are making. 8/3 divided by 2/3 is 4. I can see that really easily, but I had a hard time seeing it with 2 2/3 ÷ 2/3. So keep thinking about that.

Next, students worked work on the problem *You have 2 1/4 pounds of hamburger and you are making 3/8 pound patties. How many patties can you make?* In his diagram, a student partitioned each pound of hamburger into eighths and marked groups of 3/8. Along with a drawing to support an answer of 6, the student presenting his work wrote the number sentence 9/4 ÷ 3/8 = 6.

T: I am looking at his number sentence. I am having a hard time seeing that the answer is 6. Does any one have a way that we could write that number sentence that could help us see the answer better. I saw some other number sentences on peoples' work.
Chris: You could write 18/8 ÷ 3/8 equals 6. [This is recorded on the work being displayed.]
T: Why is the first sentence [9/4 ÷ 3/8 = 6] so hard to deal with?
Ali: We don't have common denominators.

The class continued to discuss why having common denominators were helpful. In these examples the teacher was drawing out the basis behind using the common denominator algorithm. The students' number sentences did not capture how they were using common-size parts in their drawings. Leveraging equivalence as a tool was one way to draw out a connection between students' diagrams, their symbolism and a potential algorithmic approach.

Discussion and Significance

The contribution of this work is an articulation of specific ways that teachers might leverage equivalence for a particular fraction-based operation without reducing the cognitive complexity of the students' work. In the bigger picture of supporting algorithm development, there were connections made between symbolism, visual models, and equivalence. The leveraging of equivalence as a tool supported students to make their implicit or informal ideas, found in their various representations, explicit for public discussion. While the data presented did not share the actual emergence and articulation of specific algorithms for each operation, it highlighted ways a teacher might use equivalence as a tool to support students to invent (or reinvent) for themselves algorithmic procedures for operating with fractions based on their informal work. This focus on leveraging equivalence as a tool is in contrast to presenting equivalence as a rote procedural step as is common when instruction presents algorithmic procedures as ready-made.

While it was not the direct focus of this paper, an important part of the work students were doing involved developing visual representations or models for scenarios that enacted the four fraction operations. Students then attached symbolism in the form of number sentences to their visual representations. The data shared revealed ways in which the visual representations and the symbolic representations were important in the algorithm-development process. While there was not enough space here to display full development from informal to formal, equivalence is presented as one important tool that teachers might leverage to help students engage in mathematical reasoning that supported the emergence of meaningful procedures and algorithms for fraction operations.

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