UNPACKING THE COMMON CORE STATE STANDARDS FOR MATHEMATICS: THE CASE OF LENGTH, AREA AND VOLUME

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Adoption of the Common Core State Standards present challenges to school districts, school administrators, and teachers. To assist in this endeavor, we present our work on unpacking the CCSS-M for the length, area, and volume Learning Trajectory (LT). The overarching theme of "genetic epistemology" and a five-characteristic framework guided our work of unpacking the Standards. As a result, we added "Bridging Standards" to mediate students' progression through the length, area, and volume LT to provide a coherent structure through this trajectory. The implications of our work were discussed.

Keywords: Standards; Learning Trajectories; Curriculum

Objective

The Common Core State Standards for Mathematics (CCSS-M) (CCSSI, 2010) are a major revamping of existing and past state standards. Adoption of the CCSS-M presents many challenges for school administrators and teachers. In particular, the learning trajectories that ostensibly undergird the Standards are not readily accessible to readers because they are *abridged* within the standards and do not contain a full treatment of the research base (Confrey, 2012). Hence, there are gaps between standards reducing their cohesiveness. Finally, the Standards authors state: "These Standards do not dictate curriculum or teaching methods" (CCSSI, 2010, p. 3) which is commendable, however, this implies that teachers need resources and support to understand the gradual evolution of the "big ideas" within the Standards.

Our research group has unpacked the grade K–8 standards for the CCSS-M (http://www.turnonccmath.com) (Confrey et al., 2011) by mapping each of the K–8 standards onto 17 LTs (Confrey, 2012). For each trajectory, we unpacked the Standards, or parts of a Standard if it had a few parts (e.g., 3.MD.7 had three parts: 3.MD.7.a, 3.MD.7.b, and 3.MD.7.c), into descriptors to include a careful discussion of the full learning trajectory. The descriptors include: (1) Conceptual principles; (2) Misconceptions, strategies, and representations; (3) Introduction of meaningful distinctions about mathematical concepts and multiple models of situations; (4) A coherent Structure of Development underlying the LT; and (5) Bridging Standards. Other groups who are unpacking the standards tend to elaborate on the mathematical content in each Standard (e.g., McCallum, Black, Umland, & Whitesides, 2010) or make comparisons between existing standards and the CCSS-M (e.g., North Carolina Department of Public Instruction, 2011). Though important, these approaches do not always give perspective on how students' mathematical ideas advanced under instruction. In this paper, we present our work on unpacking the K–5 CCSS-M Standards for the *length, area, and volume* LT. Drawing upon the literature, we created an initial draft to reveal a coherent structure for this LT.[1]

Literature Review

Learning Trajectories

The term *learning trajectories (LT)*, has different meanings among researchers in mathematics education. Simon (1995) first defined a *hypothetical learning trajectory* (HLT) to be, "The learning goals, the learning activities, and the thinking and learning in which students might engage" (p. 133). Our research group defines a learning trajectory to be,

> a researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e., activities, tasks, tools, and forms of interaction), in order

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to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time. (Confrey, Maloney, Nguyen, Mojica, & Myers, 2009)

We view LTs as expected probabilities of students' progresses in their development of mathematical knowledge in terms of sequence and likelihood. LTs permit one to specify at an appropriate and actionable level of detail what ideas students need to know during the development and evolution of a given concept over time. This definition allowed us to unpack and sequence of the CCSS-M Standards guided by the research literature on spatial measurement.

Learning Trajectories for Length, Area, and Volume

Synthesizing the literature in length and area measurement (Nguyen, 2010), we found three different viewpoints on measurement: (1) those who have built LTs for length and area using an external iterating unit (Barrett, Clements, Klanderman, Pennisi, & Polaki, 2006; Battista, 2007; Battista, Clements, Arnoff, Battista, & Borrow, 1998; Clements & Sarama, 2009; Outhred & Mitchelmore, 2000); (2) those who have investigated the use of common units as measure (Lehrer et al., 1998; Nguyen, 2010); and (3) those who have built an entire numeration system based on measurement (Dougherty & Venenciano, 2007). By approaching measurement as a "systematic process to compare two or more quantities" (Confrey, 2011), we expanded the meaning of measurement beyond association a number of units with a given quantity to include building number-unit relationships using units that are internal of and external to the object being measured. In our work, we treated students' learning of the concepts and skills of length, area and volume as progressions through a single LT instead of separate LT for the above reason.

Development of length trajectories. Sarama and Clements (2009) have proposed a LT for length measurement based on a mixed method analysis and synthesis from other studies (e.g., Hiebert, 1981; Lehrer, 2003; Piaget, Inhelder, & Szeminska, 1960; Stephan, Cobb, & Gravemeijer, 2003). Their LT identified five areas of how students build concept and skills through instructional experiences: (1) alignment of endpoints to compare lengths (Piaget et al., 1960); (2) comparing the lengths of two objects using a third object and transitive reasoning (Hiebert, 1981); (3) finding the lengths of an object by "tiling" or "iterating" smaller identical objects as length units and associating higher counts with longer objects (Hiebert, 1981; Lehrer, 2003; Stephan et al., 2003); (4) understanding that length measure requires equal-length units (Ellis, Siegler, & Van Voorhis, 2000); and (5) using rulers and length measures to investigate real-world phenomenon (Lehrer, 2003; Stephan et al., 2003).

The evolution of students' concepts and skills on length measurement is described in terms of students' developmental progressions and their action schemes (see Sarama & Clements, 2009, pp. 289–291 for details). Seven levels were identified in the LT: (1) Pre-length quantity recognizer; (2) Length quantity recognizer; (3) Length direct comparer; (4) Indirect length comparer; (5) End-to-end length measurer; (6) Length unit relater and repeater; and (7) Length measurer. Sarama & Clements' (2009) work and its supporting corpus of studies provided the research input needed to unpack the Length Standards (Sarama, Clements, Barrett, Van Dine, & McDonel, 2011).

Development of area trajectories. Researchers have documented that to have a deep understanding on area, students must first understand the idea of systematic coverage (no overlaps or gaps) by a square unit (Outhred & Mitchelmore, 2000). They learn to align the units into an array of rows and columns, relating rows and columns to the lengths of the sides, and finally to calculate area from the number of units of length and width (Battista et al., 1998). Other aspects of a more complete learning trajectory for area would include developing student understanding about measuring with a square unit versus with a ruler, linking to lattice point arrays, the impact of different sized units on the magnitude of the area, linking area and perimeter, and extending to triangles or circles. Finally, it would include student understanding of the calculation of fractional area with an anticipation that the product of two numbers produce an area that is smaller than an area of either one of the linear dimensions by 1 unit (e.g. ½ in. x $^3/_4$ in. = $^3/_8$ sq. in. is less than ½ in by 1 in. or $^3/_4$ in by 1 in.).

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Nguyen (2010) documented that students could construct common units to compare areas when asked to compare two or more areas without the provision of an external unit. Through equipartitioning (Confrey et al., 2009) of the two areas into smaller areas, students created a same-sized area unit embedded in the original areas to be used as the basis of comparison. He also demonstrated that students eventually generalized that if two areas are equal, they must be measured by the same-sized unit the same number of times. As a result, his students were able to correctly predict the effects of changing the unit size on the measure of an area. Others have investigated a number of these ideas (Simon & Blume, 1994), but work remains to synthesize these findings into a unified description linked to student behaviors.

Development of volume trajectories. Battista and Clements (1996) showed five levels of student behaviors when working volume tasks. At Level A, students only begin to conceptualize a set of cubes that forms a rectangular array. At Level B, students have conceptualized the cubes, but do not utilize the inherent layer structure of a 3-dimensional cube. At Level C, cube faces are used, however, either all of the face cubes are counted or outside the cubes. At level D, students use the volume formula and count a row of face cubes to calculate volume. Lastly, level E is reserved for outliers. Students who were not yet at Level A were generally unable to find out how many cubes there were in a 3-dimensional box, since seeing a mental array picture is only the beginning step to Level A understanding. To such students, the L × W × H formula means very little. Those who applied the formula tended to ignore the three-factor product that results from volume measurement. Multiplication was also not the only operation relied on to calculate volume. Addition, skip counting, and repeated addition were also used.

Battista (1999) followed with a teaching experiment to see if fifth graders could enumerate cubes. All six students in the study were able to structure and enumerate 3D cube arrays. However, their use of layering did not immediately lead to its use in subsequent predictions. Battista (2007) currently claims seven levels of sophistication in students' uses of cubic arrays to construct volume, ranging from organization or location of units in arrays, to introducing composite units, emergent array structures, and spatial structuring and enumeration.

Curry and Outhred's (2005) work distinguishes "packing volume" with cubes and "filling volume" with liquid or sand. While investigating students' understanding of the relationship between length, area, and volume, they discovered that student scores on packing volume tasks were highly correlated with scores on length. In these tasks, students were asked to pack an area with a unit box. They performed much better on tasks involving filling volume with water or sand. The authors conjectured that a filling procedure and length iteration were related processes. This literature informed our consideration of the contents to be included in the descriptors.

Unpacking the Length, Area, and Volume Trajectory

An overarching theme of our work is to consider the "genetic epistemology" (Piaget, et al., 1960) of how instruction refines students' informal mathematical idea successively and develop more complex ideas, as informed by research from a cognitive and instructional standpoint. The adoption of the genetic epistemology approach motivated a five-characteristic framework for unpacking the mathematical content of the Standards into the descriptors. First, the descriptors provide an explicit breakdown of complex mathematical ideas into its *conceptual principles*. For example, the descriptor for standard 1.MD.2 spells out the principles of using a length unit to measure. Second, the descriptors address the *misconceptions, strategies, and representations* that students may encounter as their informal ideas evolve into complex mathematical ideas. For example, the descriptor for standard 2.MD.1 addresses the misconception in using a ruler, where students may misinterpret the number of tick marks spanned by an object as its length. Third, the descriptors identify *meaningful distinctions* about a mathematical concept. These distinctions lead to *multiple models* of problems and support students' generalizations. For example, the descriptor for standard 3.MD.5.b makes three distinctions about the idea of "an area of n square units" as: (1) iterating an area unit $n - 1$ times, (2) "n times as big" as an area unit, and (3) a sweep of a line segment over a distance. Fourth, the organization of the descriptors of a LT reflected a genetically *coherent structure* of development through which students develop "big ideas." For example, the descriptors of this LT are organized to highlight the genetic sequence in which students develop length, area, and volume by: (1) Defining the attribute, (2) Direct comparison, (3) Indirect comparison, (4) Measuring using a unit with no gaps or overlaps, and (5) Compensatory and Additive principles. Fifth, we introduce “Bridging Standards,” additional mathematical knowledge that mediates students’ progression from prior concepts in earlier Standards to more sophisticated and formal concepts in later standards. These Bridging Standards and their descriptors provide a complete genetic epistemological account of a LT. For example, qualitative comparison of area and volume were added as Bridging Standards, since this mathematical knowledge was instrumental to the coherent structure underlying students’ development of measurement, but was not included in the CCSS-M.

We approach the task of unpacking the CCSS-M by describing students’ development in terms of the characteristics mentioned above. Our unpacking proceeded in the following manner. First, we sequenced the relevant Standards in a way that generally reflects research findings about how students progressively learn the ideas. A set of sequenced Standards can be regarded as an *abridged* LT. Second, based on the abridged LT, we built an unabridged version where we incorporated research findings to bridge the instructional gaps between and within the standards of a LT. For length, area, and volume, we synthesized different research findings in the domain of spatial measurement into a unified description of how students’ mathematical knowledge evolved as they encounter activities, tasks, tools, and forms of interaction. Third, we added Bridging Standards when we felt the research suggested mediating ideas that were necessary to be learned before progressing to the next standard in the LT.

We drafted the text of the unpacked LTs in the format of a two-column table, in which the left column showed the standards and its codes as sequenced in the LT and the right column showed the descriptor of the standard. We used Confrey’s (2010) hexagons map to represent how the LTs develop over time and to depict how they are relate to each other visually. The length, area, and volume LT was organized into six sections: (1) Attributes, Measuring Length and Capacity by Direct Comparison; (2) Length measurement using units and tools; (3) Area and Perimeter; (4) Volume Measurement; (5) Conversion; and (6) Area and Volume of Geometrical Shapes and Solids. The move to subdivide the entire LT into sections does not signify some disconnect between the contents of the descriptors but rather permit us to focus on unpacking the more intertwined connections among some Standards. In fact, cross-references between the Standards were often made when drafting the descriptors.

Report of the Unpacking of Length, Area and Volume Standards

We wrote 50 descriptors in the length, area, and volume LT (36 from CCSS-M and 14 Bridging Standards). Below we present a summary of the mathematical knowledge that we have unpacked, according to the five-characteristic framework. The most updated edition of the descriptors can be accessed online (http://www.turnonccmath.com).

Conceptual Principles of Length, Area, and Volume

In the descriptors, we unpacked a list of conceptual principles to be mastered by students across length, area, and volume. They are: the Conservation Principle, the Compensatory Principle, the Principle of Unit Placement, the Principle of Unit Conversion, and the Additive Principle. The Conservation Principle states that the length (or area or volume) of an object remains unchanged under any rigid transformation. The Compensatory Principle states that there is an inverse relationship between the size of the unit (length, area, or volume) used for measurement and the measure (count of the units). The Principle of Unit Placement states that the units used to measure the length (or area or volume) of an objects must be placed without gaps or overlaps and along a path aligned with the object's length (or arrays in the case of area and volume). The Principle of Unit Conversion states that smaller units can be composed to form larger units and that larger units can be regrouped into smaller units. The Additive Principle states that the joining of two lengths (areas or volumes) are sums of the lengths (areas or volumes). From the LT perspective, these principles are foundational to students' development across length, area, and volume. This does not imply that they are taught directly, but rather that the students’ understanding of them evolves gradually through the course of activities and tasks.

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Misconceptions, Strategies and Representations

We identified a number of misconceptions informed by NAEP results. These concerned students'' use of rulers and their understanding about area and perimeter. For example, when measuring the length of an object, many students do not check if the object aligns with the zero mark. They also tend to treat tick marks on the ruler as the length of the object instead of the interval between the tick marks. In area and perimeter, students tend to measure the perimeter of a rectangle using square tiles around the corner and believe that increasing the perimeter of a rectangle always increase its area.

We described length as being represented on a number line by equally spaced intervals from 0 as a useful representation of addition and subtraction. Addition of two numbers ($a + b$) could be thought of as combining a length of a units with another length of b units. Subtraction of two numbers, $a – b$ can be thought of as comparing the difference between two line segments or taking away b units from a line segment of a units. For strategies, we also highlighted various ways in which students can directly compare two lengths, two areas and two volumes. Because length, area, and volume have different spatial properties, the strategies of direct comparison varied. For example, straight lengths can always be directly compared, while some areas may overlap and need decomposition to compare. Likewise, the capacity of two containers can be directly compared if poured into cylinders with the same base, whereas volumes of solids will require a systematic means of decomposition.

Distinctions and Models

While the Standards did not introduce any distinctions between volume of a solid and the volume of a container, we use "capacity" to refer to the latter in the descriptors. We also make distinctions among concepts of area and volume which were not explicit in the Standards. For example, the area of a rectangle can be viewed as composed of smaller square units versus the sweeping of a length over a distance. Likewise, we distinguished between volume as the packing of space-filling units versus the sweeping of an rectangular area over a height.

We also distinguished the area formula of rectangles involving fractions from whole-number lengths and introduced four models of fractional multiplication of lengths based on equipartitioning of areas in the descriptors: (1) a whole number and a unit fraction; (2) two unit fractions; (3) two proper fractions but not unit fractions; and (4) one or two mixed numbers (or improper fractions). This is consistent with the sequence in the standards for fractions for multiplication, which is developed fully in the division and multiplication LT. Likewise, in the unpacking of the volume formula of a rectangular prism, we introduced different models of Volume = L × W × H related to the associative property. Coordinating across learning trajectories and providing multiple models supports future development in these topics.

Coherent Structure

As Smith and Gonulates (2011) reported, the Standard's treatment of length measurement is the most complete in alignment with the research literature as students are expected to distinguish length as a measureable attribute (K.MD.1), directly compare two objects based on length (K.MD.2), order three objects based on length (1.MD.1), iterate a length unit to express the length of an object as a whole number of those length units (1.MD.2), use tools to measure the length of objects (2.MD.1), and measure the length of an object using different length units (2.MD.2).

However, for area measurement, the Standards writers presented an abridged version of this sequence where students immediately iterate a unit square to cover a rectilinear area and call this measure *n* unit squares (3.MD.5.a and 3.MD.5.b), then learn to measure area by counting unit squares (3.MD.6), and finally find the area of a rectangle by multiplying the length by the width (3.MD.7.a). They then include a standard for students to understand that areas are additive (3.MD.7.d), a Standard that was missing in the Length content. Similarly, for volume measurement, the sequence first started with students' measurement, estimation of volume and one-step volume problems of involving any of the four operations (3.MD.2). Due to the abridged treatment, the structure underlying students' development in Area and Volume was incomplete.

To ameliorate these issues, we identified from the length, area, and volume contents a template of key ideas found in students' development of spatial measurements. We then applied this template across length, area, and volume Standards in our unpacking. As a result, a coherent structure of the LT descriptors emerged across length, area, and volume, which showed how students' concepts and skills of Length and Area and Volume become more sophisticated under instructions over time: (1) Describe and recognize the measureable attribute; (2) Direct comparison of two objects; (3) Indirect comparison of two objects; (4) Comparison of three or more objects; (5) Define what is meant by *n* units; (6) Express the attribute as a whole number of the units. (7) Measure the attribute twice using different units (compensatory principle); (8) Measure to determine how much bigger or smaller; and (9) Recognize the attribute as additive. Describing students' development of mathematical knowledge within such a coherent structure leveraged on the relevant research in providing teacher readers a sense of an overall developmental progression of students' knowledge as well as the interconnectedness between different Standards when unpacked.

Addition of "Bridging Standards"

As a result of our undertaking of "generic epistemology" account of students' learning, we introduced a total of 14 Bridging Standards unpacked with descriptors based on the coherent structure ands. Five were associated to the conceptual principles of length, including the missing additive principle; five were associated to the conceptual principles of area; three were associated to the volume concepts; and the last one connected the surface area with the volume of the cylinder. The last Bridging Standard was added based on a suggestion from a district curriculum coordinator who noted its absence. When read as parts of the trajectory, these descriptors filled in the knowledge gaps between some Standards and provided a coherent structure for students' development of length, area, and volume.

Discussion

The length, area, and volume Standards in the CCSS-M provide an example of why carefully unpacking the Standards is important. We detailed a trajectory of weaving the relevant Standards together in our unpacking in place of a piece-wise Standard-by-Standard elaboration. Next, we discuss the implication of our work for State Standards and Curricula.

Cross-walk between CCSS-M, State Standards and Curriculum

Comparing the CCSS-M and existing State Standards provides a quick and pragmatic way of evaluating the amount of re-alignment needed for curricular and assessment purposes. However, this approach is insufficient in itself to prepare teachers for implementation. For example, how should matched State Standards be re-ordered to maintain a coherent learning path? Do unmatched State Standards matter to students' learning? A minimalist approach might do more harm in this case. Unpacked using a LTs perspective, the descriptors provide educational practitioners access to a research basis in making educated decisions. For example, the coherent structure of moving from "Defining attributes" to "Comparison" in the length, area, and volume LT provide grounds for including addition of areas as a grade-level objective in the CCSS-M. Similarly, an LT analysis supports a means to conduct content analyses of proposed curricula and CCSS-M. The five characteristics of our unpacked descriptors provided teachers with curricular "landmarks" in anticipation of identifying and filling in instructional gaps in curricula.

Endnote

[1] We thank the participants of the 2011 Measurement Mini-Center Conference (Rich Lehrer, Doug Clements, Jeff Barrett, Jack Smith, Mike Battista and others) for reviewing an earlier draft of these descriptors. This process of peer-review enriched our work with the current views of the research community.

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