THE IMPACT OF ONLINE ACTIVITIES ON STUDENTS' GENERALIZING STRATEGIES AND JUSTIFICATIONS FOR LINEAR GROWING PATTERNS

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This study explores the impact of working with online learning activities of linear growing patterns (CLIPS) on students transitioning into Grade 9. Fifty students were interviewed about their understanding of linear growing patterns. Twenty-five students had participated in a study involving an experimental instructional approach that emphasized exploration of multiple representations of linear relationships when they were in Grade 8. They were then assessed five months later, and their reasoning compared to twenty-five students who did not take part in the study. Results indicate that students who worked with CLIPS were able to find explicit, generalized rules for patterns and offered higher levels of justifications than their counterparts. These students were also more likely to refine their thinking.

Keywords: Algebra and Algebraic Thinking; Instructional Activities and Practices; Technology

Context

Studies have shown that the transition from primarily arithmetic thinking in elementary school, to algebraic thinking in high school, is difficult for most students (Kieran, 1992). This transition entails moving from a focus on mathematical operations (addition, subtraction, multiplication, division) to thinking about relationships between sets of numbers, and identifying generalized mathematical structures with or without specific numeric values. Traditional algebra is often initially presented in high school as a pre-determined syntax of rules and symbolic language to be memorized by students. Students are expected to master the skills of symbolic manipulation before learning about the purpose and the use of these symbols. In other words, algebra is presented to students with no opportunity for exploration or for meaning making (Kaput, 2000).

In response, a series of online learning objects was designed as an alternative way to introduce the concepts of algebraic relations, specifically linear relations, to Grade 8 students prior to formal algebraic instruction in Grade 9. The activities are based on an approach that emphasizes the observation of relationships among quantities, and among multiple representations, which allows for the construction of understanding rather than rote memorization of procedures. As part of a larger long-term study, I have been investigating the affordances of this instructional approach that prioritizes visual representations of linear relationships – specifically, the building of linear growing patterns and the construction of graphs (e.g., Beatty 2010). Previous research on the lesson sequence has shown that it supports students' progression from working with linear growing patterns as an anchoring representation to considering graphical representations of linear relationships. Students also make connections among different representations – pattern rules, patterns and graphs (Figure 1).

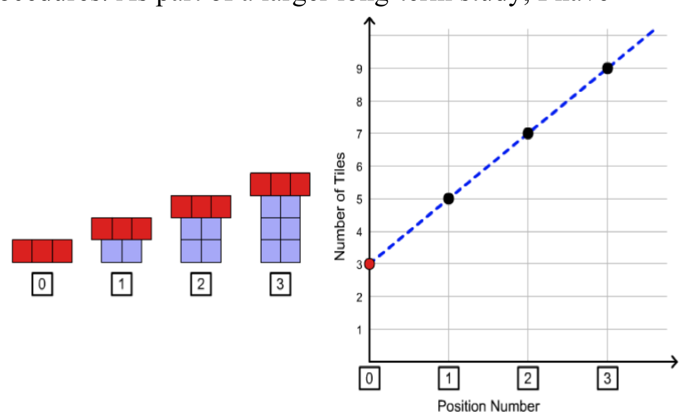

Figure 1: Representations of the rule *y*=2*x*+3

The online activities, called CLIPS LGP (Critical Learning Instructional Paths Supports – Linear Growing Patterns) were designed using Flash technology and offered the possibility of combining a proven instructional sequence with unique properties of digital technology. The online activities were integrated into the instruction in five classes of Grade 8 students. The students accessed the online

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activities for 2 months in order to develop an understanding of linear relationships via linear growing patterns. As part of the instruction, students were supported to develop sophisticated generalizing strategies by considering the explicit relationship between the term number of a pattern and the number of tiles in the pattern, and to express this relationship using pattern rules such as "the number of tiles is equal to the term number x2+3" or "tiles = term number x2+3." Students also engaged in classroom discussions based on the online activities, and developed a disposition for providing justifications for their pattern rules.

In this study we wanted to assess how much content material was retained by these Grade 8 students as they transitioned into Grade 9. We also wanted to compare the problem-solving processes of Grade 9 students who had been part of the CLIPS study in Grade 8 with those who had not in order to determine whether there was a difference in students' generalizing strategies and justifications.

Developing Generalizing Strategies

A main component of algebraic reasoning is the ability to generalize. In the domain of linear relations, particularly when thinking about linear growing patterns, a generalization can be thought of as the articulation of a pattern rule that applies across all cases in the situation (for example figure numbers and number of toothpicks in a linear growing pattern.) Studies have shown that students have difficult moving from particular examples (for instance, focusing on particular iterations of a pattern) towards creating generalizations (a generalized pattern rule that holds for infinite iterations of the pattern). Numerous researchers have reported that the route from working with liner growing patterns to finding generalized rules (and later, algebraic expressions for those rules) is difficult (Kieran, 1992; Orton, Orton & Roper, 1999; Noss et al., 1997). However, we have found in our previous studies that the instructional approach that underpins the CLIPS activities has facilitated students' abilities to find and articulate general rules for linear growing patterns (Beatty & Bruce 2012).

Researchers have identified many generalizing strategies that students adopt when working with problems involving linear relations, including problems based on linear growing patterns (Lannin, 2005; Mason, 1996; Lee, 1996). Below we identify three of these strategies from the least to the most sophisticated. They are presented with reference to a well-known generalizing problem (one that we used in our study), the Toothpick Trees problem (Figure 2). In this problem, students are shown a series of Toothpick Trees and asked to predict how many toothpicks would be needed to build the 10th figure, and how many would be needed to build the 100th figure.

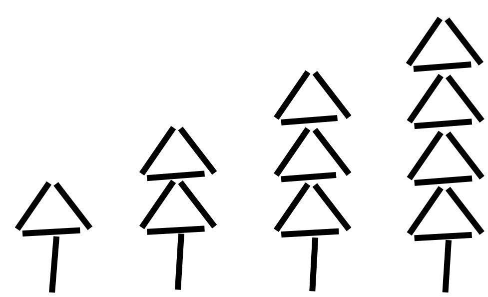

Figure 2: The toothpick trees problem

Counting strategy. Students draw a picture or constructing a model to represent the situation in order to count the desired attributes. For example, students draw the 10th figure and count the number of toothpicks required. The limitation of this strategy is evident to students when they are asked to predict the number of toothpicks for the 100th figure.

Recursive reasoning strategy. Students build on the previous term in the sequence to determine subsequent terms. In our example, students would state that the rule for the pattern is "add 3 each time." To find the 10th figure they add three to the 4th figure, then three more to the 5th figure etc. This strategy generally results in the correct answer to predict the number of toothpicks for "near" terms of a pattern (for example, figure 10) but is problematic for finding the 100th term. It also does not allow for the articulation of the rule, which would allow for the prediction of the number of toothpicks for *any* figure.

Explicit reasoning strategy. Students construct the explicit rule that expresses the co-variation of two sets of data, based on information provided in the situation. An explicit rule can allow for the prediction of any term number in the pattern. An example of an explicit rule would be, "the number of toothpicks is equal to the figure number x3+1."

Research suggests that when working linear relationships and linear growing patterns it is rare for students to go beyond limited kinds of mathematical generalizations – namely counting or recursive reasoning – primarily because these are the strategies that are supported by traditional approaches to teaching patterning and algebra (Noss et al., 1997). However, the instructional approach in CLIPS prioritizes explicit reasoning in order for students to determine and articulate the mathematical structure of linear growing patterns.

Importance of Justification

When students justify their solutions strategies they are able to provide reasoning and evidence to validate their generalization. This has been found to be challenging for most students (Lannin, 2005). However, providing a justification for a generalized rule helps students to see the generalized relationships that exist in the problem context. Just as there is a framework for generalization strategies, there is also a five-level framework for justification strategies (Table 1) (Simon and Blume, 1996). Higher levels of justification have been shown to support higher levels of generalization (Lannin, 2005).

Table 1: Levels of Justification

Level	Descriptions
0	No justification.
1	Appeal to external authority. Reference is made to the fact that a solution is correct because it is stated by some other individual (teacher or a peer who is regarded as more successful) or some other reference material.
2	Empirical evidence. A justification is provided through the correctness of particular example but with no indication of an understanding of <u>why</u> the rule is correct. For instance, "The rule is 'add 3' because for the first figure there are four toothpicks, then you add 3 more for figure 2."
3	Generic example. Deductive justification is expressed for a particular instance, a generic example, which the students uses as a proxy for "any" instance. For example, "I know the rule is "toothpicks = figure number x3+1" because for, say, the fifth figure, there are five triangles, and five times three is fifteen. And then there is one more, so plus one is sixteen."
4	Deductive justification. Validity is given through a deductive argument that is independent of particular instances. For example, "At any figure number, the number of triangles equals the figure number, so that means multiplying the figure number by three, and then there's always an additional one for the trunk."

When working with CLIPS, students engaged in classroom discussions during which they were encouraged to justify their solutions by making connections between their solutions and the context of the problem, with a focus on deductive reasoning.

Methodology

Participants

Fifty Grade 9 students participated. Of these, 25 students had been part of the CLIPS study and 25 had not. The students were drawn from 8 different classrooms in two different school boards with equal number of CLIPS and non-CLIPS students selected from each classroom. Students were interviewed individually for approximately 30 – 35 minutes.

Data Sources and Analysis

In order to track the content knowledge and algebraic reasoning of students, we chose to conduct task-based clinical interviews during which students were asked to describe what they were thinking while solving ten linear relationship problems. This form of interview opens a window into the participants' content knowledge, problem-solving behaviours and reasoning (Koichu & Harel, 2007; Schoenfeld, 2002). In this study, the clinical interviews were semi-structured, which allowed for prompting or questioning students in order to clarify our understanding of the students' reasoning. Validity of the subjects' verbal report corresponds to the extent to which the subjects' talk represents the actual sequence of thoughts mediating solving an interview task (Clement, 2000; Ericsson & Simon, 1993). Therefore, all interviews were digitally video recorded so that verbal report and non-verbal gestures were captured in order to develop a comprehensive analysis of student thinking.

Overall students answered five items that were taken directly from the CLIPS activities, which we termed "near transfer" items because they test the retention of understanding of items that are similar to items students experienced while working with CLIPS. The other five items came from sources such as TIMSS (Third International Math and Science Survey) and NAEP (National Assessment of Educational Progress). We termed these "far transfer" items because they are dissimilar to the CLIPS content, and so assess understanding of underlying conceptual concepts. For this report we will focus on students' responses to one "far transfer" item – the Toothpick Trees problem described above. The students were asked to predict how many toothpicks would be needed for the 10^{th} and 100^{th} figure, and to explain their thinking. However, we did not explicitly ask for a pattern rule in order to determine whether students would use the information presented in the patterns to formulate a general rule that would give the number of toothpicks required for any figure of the pattern.

The scoring guide for these items, based on the generalization framework, is given below.

Table 2: Scoring Guide for Generalization Strategies

Score	Description
0	Incorrect answer
1	Counting strategy. The student drew out the figure(s) and then counted the number of matchsticks/toothpicks (drew out the 10^{th} figure, drew out the 4^{th} to the 10^{th} figure).
2	Recursive strategy. The student has articulated the rule as "add three more each time" or created an ordered table of values that increased by three each time.
3	Explicit strategy. The student has articulated the explicit rule as "matchsticks = figure number x3 + 3" and "toothpicks = figure number x3+1"

Video recordings of task-based interviews were transcribed and coded. Codes were based on the generalization and justification frameworks outlined above.

Results

Generalizing Strategies

Table three shows the level of generalizing strategy demonstrated by students who had experienced CLIPS and those who had not.

Table 3: Generalization Strategies Used by CLIPS and Non-CLIPS Students

	Score 0	Score 1	Score 2	Score 3
CLIPS	0	0	7	18
Non CLIPS	11	8	4	0

Most CLIPS students used an explicit generalizing strategy to find a general rule using the context of the problem in order to find the correct solution. In contrast, many non-CLIPS students did not find a viable solution, and those that did used a counting strategy or recursive reasoning strategy, rather than finding an explicit pattern rule.

Counting strategy. Eight of the 25 non-CLIPS students used a counting strategy for this problem, meaning that they drew the 10th figure and then counted the number of toothpicks. A striking finding was that all eight of the students who used a counting strategy could not make the connection between counting by threes, and multiplying by three. For example, in the transcript below the student is not able to transition to multiplicative thinking in order to predict the 100th term.

Malinda: Well I drew it out and counted and found out that it just kept adding three. So I drew it to the tenth and then counted them to find the right number of toothpicks.
Interviewer: How are you counting? Can you count out loud?
Malinda: Three, six, nine, twelve, like that?
Interviewer: Yeah.
Malinda: (pointing to each triangle as she counts) Three, six, nine, twelve, fifteen, eighteen, twenty-one, twenty-four, twenty-seven, thirty plus one is thirty-one. I was counting by three's.
Interviewer: Could that help you think about how many toothpicks you'd need for any figure number? Like the 100^{th} figure?
Malinda: Um…well…I know that I would draw out three more for each figure. So I would just keep drawing three for each next figure, and then count by threes.

It should be noted that Malinda, like most of the students we interviewed, was considered to be, and considered herself to be, capable of engaging in mathematical operations like multiplication. However, the majority of students who were not part of the CLIPS study could not make the connection between "counting by threes" and multiplying the term number by three.

Differences in recursive thinking. The seven CLIPS students who used recursive thinking created a generalized rule "add three each time plus the one for the trunk" that took into account both the multiplier and the constant of the rule with reference to the figural context of the problem.

> I added three toothpicks every time, like one triangle every time up to the tenth one and got thirty toothpicks and then you have to add the stump part to it. So it's always going up by three each time, but with the little stump so you add one for that. So for 100 you'd add 3 100 times and then add 1.

The two non-CLIPS students who used recursive thinking articulated their pattern rule as "start with four and add three." This is a common way that students are taught to articulate linear growing rules. Students were able to find the number of toothpicks for the 10th tree, but then simply guessed for the 100th term.

> Ok so I started at the first tree with four toothpicks. Then as it added we still had the four and I added it to each new tree every time. So then every time I'd get an answer with three more. I'd added 4 and three and three and three and keep count of where I was until I hit the tenth figure. Then that was my answer. For 100 it would be…maybe 101? I don't know.

Explicit thinking. Eighteen CLIPS students found a generalized rule for this pattern. Most of the students used language and concepts that are part of the CLIPS LGP instruction.

> I looked for the rule. So I could see the triangles were growing, so that meant the multiplier would be times 3. And then the one that stays the same, that's the constant. So for 10 it would be 10x3+1, which is 31. And for 100 it would be 100 x 3 is 300 plus 1 is 301.

Justification Strategies

Transcripts were coded for the level of justification offered by students as they articulated their thinking during the task-based clinical interview. Justifications were scored from Level 0 to Level 4, based on the framework outlined above. Table 4 below shows the level of justification provided by students who had experienced CLIPS and those who had not.

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Table 4: Levels of Justification Given by CLIPS and Non-CLIPS Students

	Level 0	Level 1	Level 2	Level 3	Level 4
CLIPS	0	0	2	7	16
Non CLIPS	11	2	8	0	0

Seven CLIPS students offered a Level 3 justification for their rules. An example of a Level 3 justification is as follows: “The tenth tree would have three triangles, which is 3 times 10 or 30. And then you add the one, so it’s 31.”In this example the student refers to a particular figure number to describe how each component of her rule relates to how her rule determines the number of toothpicks. She notes the ten groups of 3 for the ten triangles in the tenth figure and explains the need to add the one extra for the trunk. In this case a particular example is used to communicate generality across all cases.

Sixteen CLIPS students offered a level 4 justification: “I know my rule is correct because you just multiply the figure number by the group of three for the triangles because the figure number tells how many triangles there are, and triangles are always going to be 3 toothpicks. And then the little trunk means you always add one more.” Students clearly explain why the rule applies to all cases of the situation by relating it to the context of the pattern. The students describe the “groups of three” they see in each of the patterns, so the multiplier x3 represents the number of toothpicks in these groups. The extra 1 toothpick (for the trunk) is added to the rule to express the total number of toothpicks needed for any figure number. Unlike a generic example, this justification does not describe a particular instance. Instead, it describes a general relationship that would apply to any case.

In contrast, the majority of non-CLIPS students did not offer a justification, and when asked why their rule worked replied, “I’m not sure” or “because it just does.” Two students offered justification that were scored as Level 2, empirical reasoning based on the correctness of one specific example, but with no demonstrated understanding of why this was correct, or how their solution was related to the context of the problem. “From the first to the second tree you add three more so the rule is plus three.”

Students’ Refining Their Own Thinking

One of the most striking results revealed by an analysis of the interview transcripts is the extent to which students who had been part of the CLIPS project refined their thinking during the course of the interview as a result of explaining their solution process. Overall, there were 28 such episodes coded in the interview transcripts for the 25 CLIPS students. However, there were no such episodes coded for the transcripts of the non-CLIPS students. Non-CLIPS students did not revise their thinking, and when they discovered an error between their rule and the problem context (their rule would lead to an incorrect number of toothpicks) they either gave up or were not aware of the disconnect between their rule and the values given in the problem.

The CLIPS students were more likely to try to find an alternative solution strategy, or to refine their answer based on new evidence. In this example, Deepak had briefly looked at the first figure of the pattern and written “x4.” He was then asked to explain his rule. During his explanation, Deepak realized that he had misperceived the structure of the pattern, and that his perception did not coincide with the numerical value of the pattern for each figure number. Rather than dismiss this discrepancy, Deepak went to work to try to discern a rule that would work with all iterations of the pattern.

Deepak: Well I just looked at figure 1, and found that the tree is made up of 4 toothpicks, so the rule is figure number times 4 So, if you look at figure 2, it would be 2 times 4 which is 8, and there are…wait…oh that’s not right.

[Deepak spent 54 seconds working on a new rule.]

Deepak: Ok I see what I did wrong. I didn’t see it right, I thought the whole tree was made of 4 not 3. But if you check the numbers, it’s growing by 3, so the three that are growing are these three that make up the triangles so it’s times 3. And then the trunk is made of one, so it’s plus 1. And that works with all the figures. So figure 3 is 3 times 3, 9, and then plus 1, 10. So the 10^{th} figure would have 31.

Interviewer: So how many toothpicks would be needed for the 100th figure?
Deepak: Easy! I can just use the rule! 301!

Conclusions

One focus of this study was to assess the enduring understandings students developed while working with the CLIPS activities, and how much of this understanding was retained during the five months from the end of the instructional intervention (June, 2010) to the time of the interviews. The interviews were held near the end of the first semester of school, during November and December 2010. The students had not yet had any formal instruction in linear relations. Given that our intervention was relatively short, these results indicate that students retained a great deal of understanding both of content material, and of the importance of providing justifications for their answers.

Another focus of the study was to compare the thinking of Grade 9 students who had been part of the CLIPS study with those who had not. There were two main areas of algebraic thinking that we assessed – the level of generalizing strategy used by students when solving linear problems, and the level of justification offered for their solutions. We found differences in the kinds of generalization strategies used by CLIPS and non-CLIPS students. In this study, students who had not been part of CLIPS, but who had experienced traditional approaches of instruction, had great difficulty in finding generalized rules for patterns. For those who did find a correct solution for the tenth figure of a linear growing pattern, their solutions were based on counting or, less frequently, recursive reasoning. These limited solutions strategies allowed students to find the number of toothpicks for the tenth figure (a near generalization), but did not aid them in finding the number for the one hundredth figure (a far generalization).

In addition, few students who had not been part of the CLIPS study offered any kind of justification for their solutions. In contrast, students who had been part of CLIPS tended to offer level 3 and 4 justifications. They explained their solutions using the context of the problem, and could articulate why their general rule would work for any case.

Finally, there was the unexpected, yet striking result, of the extent to which CLIPS students revised their thinking. This happened numerous times with the CLIPS students, who, during the course of explaining their solutions, caught and corrected their own mistakes. Past research suggests that students typically do not attempt to revise their rules (Bednarz, Kieran, and Lee, 1996; Mason, 1996; Stacey, 1989). In fact, Cooper and Sakane (1986) suggest that once students select a rule for a pattern, they tend to persist in their claims even when finding a counter example to their hypothesis. Students would rather refute the data presented than modify their original rule. This was the behaviour we observed with the non-CLIPS students, who had no interest in revising their rules.

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