PRE-SERVICE TEACHERS' MATHEMATICAL HORIZON: THE CASE OF AN IRREGULAR HEXAGON

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This paper analyses pre-service secondary mathematics teachers' responses to a hypothetical situation concerning a student's struggle to determine the area of an irregular hexagon. The analysis focuses on individuals' Knowledge at the Mathematical Horizon, as it relates to philosophical notions of inner and outer horizons. Specifically, the paper attends to common features in participants' understanding of the mathematical task, and explores the interplay between their personal solving strategies and professed preferences when advising a student. Implications for teacher education and further avenues of research are suggested.

Keywords: Mathematical Knowledge for Teaching, Teacher Education-Preservice, Geometry

This research contributes to on-going conversations about teacher knowledge and the interplay between knowledge of mathematics and pedagogical decisions. It focuses on pre-service teachers' responses to a learning situation regarding a hypothetical student's struggle to determine the area of an irregular hexagon. The aim of the study was to investigate the connection between participants' personal strategies and preferences for solving a novel (for them) problem and their expectations and recommendations for student learning. Specifically, the paper considers the question: *What are pre-service secondary mathematics teachers' preferences when considering recommendations for how to determine the area of an irregular hexagon, and what are the bases for these preferences?*

Participant responses were analysed using the construct of Knowledge at the Mathematical Horizon (KMH), which was introduced by Ball and colleagues (e.g. 2008, 2009), and extended by Zazkis and Mamolo (2011). The analysis intends to shed new light on how different facets of KMH may manifest in a pre-service teachers' address of a teaching situation, and what consequences this might have for student learning. The paper concludes with pedagogical implications for pre-service education, as well as suggestions for future avenues of research.

Survey of Literature on Teacher Knowledge

Knowledge required for teaching mathematics has been widely discussed from a variety of perspectives. Attention has focused on what knowledge is required in teaching, for teaching, and of teachers (e.g. Ball, Thames, & Phelps, 2008; Davis & Simmt, 2006). Shulman's (1986) classic categorization of Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK) made an early distinction between teachers' knowledge of mathematics and knowledge of student learning, while more recent studies have sought to both refine these categories and connect them. For instance, Ball, et al. (2008) elaborate that SMK is knowledge required by teachers for the tasks of teaching, "which require a host of other mathematical knowledge and skills - knowledge and skills not typically taught to teachers in the course of their formal mathematics preparation" (p.402). They refined Shulman's categories by distinguishing between (for example) knowledge of subject matter specific to teaching (such as how to explain rules and procedures) and knowledge of subject matter that is common to other uses of mathematics. They

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also identify as necessary aspects of teachers' PCK: a familiarity with students' ways of thinking, as well as knowledge of effective examples or teaching sequences.

While this research was on distinguishing facets of teacher knowledge, Ball et al. (2008) acknowledge there are interconnected relationships. They suggested that "Teachers who do not themselves know a subject well are not likely to have the [pedagogical content] knowledge they need to help students learn this content" (2008, p.404). Similarly, Potari et al. (2007) observed that robust subject matter knowledge allowed teachers to interpret and develop student ideas with greater ease and effectiveness. They also suggested that teachers' ability to connect different mathematical areas and their awareness of the relevance of these connections were part and parcel to their ability to effectively integrate SMK and PCK to create a rich mathematical learning environment. Ball, Lubienski & Mewborn (2001) highlighted that: "It is not only what mathematics teachers know but also how they know it, and what they are able to mobilize mathematically in the course of teaching" (p.451).

Extending on these studies, this research considers pre-service teachers' address of a student's struggle to solve a non-routine problem, which can be addressed in a variety of ways with connections to core concepts in secondary school curricula. Participants' SMK is analysed via the sub-category Knowledge at the Mathematical Horizon, which is discussed in the following section. The intent is to offer a refined look at how this specific aspect of individuals' SMK can manifest in, and influence, a teaching situation.

Theoretical Framework

One of the facets of teachers' Subject Matter Knowledge, as introduced by Ball and Bass (2009) is Knowledge at the Mathematical Horizon (KMH), which is described as a structural, connected, and robust understanding of mathematics that goes beyond what is taught in school curricula. Ball and Bass present KMH as a teacher's knowledge of students' mathematical horizon, while other perspectives, such as that of Zazkis and Mamolo (2011), look to teachers' horizon. This study also attends to teachers' horizon and what lies "in and out of focus" as they consider the mathematics in a hypothetical teaching situation.

In line with the description of horizon knowledge as connected, robust, and beyond school curricula, Zazkis and Mamolo (2011) extend the construct of KMH to focus on teachers' horizon by connecting it to Husserl's philosophical notion of a (conceptual) object's horizon. Husserl's description of horizon relates to an individual's focus of attention – in particular, when an individual attends to an object (conceptual or physical), the focus of attention centers on the object itself, while the 'rest of the world' lies in the periphery (Follesdal, 2003). With this perspective, a teacher's horizon knowledge is dependent on the specific mathematical object under consideration – how that object is understood, what aspects of the object lie in focus or in the periphery, and what connections the teacher is able to make between the in-focus and peripheral facets. What lies in the periphery is in Husserl's perspective the object's horizon. Thus, Zazkis and Mamolo (2011) interpret teachers' horizon knowledge with respect to a specific mathematical object as knowledge of that object's horizon, which according to Husserl can be partitioned into an *inner* and an *outer* horizon.

Husserl's partition allows an analysis of different features in an object's periphery. An object's inner horizon refers to specific attributes of the object itself which are not (at that moment in time) in focus for the individual. With respect to a mathematical object or entity (say, a polygon), these attributes might relate to the polygon's symmetry or colour or area, if what is in focus for the individual is the polygon's number of edges and vertices, for example. Thus there is a reflexive relationship between what lies in the inner horizon of an object and what lies in focus for an individual. In contrast, an object's outer horizon refers to the "greater world" in which the object exists, and as such is not dependent on the individual's focus of attention. The outer horizon includes features which embed the object in a greater structure, and it consists of generalities which are exemplified by the particular object. For example, the fact that algebraic equations may be used to express or determine measurements of the polygon (such as its perimeter or size of interior angles) would lie in the outer horizon, exemplifying structural connections between strands (e.g. algebra and geometry) and between concepts (e.g. ratios of lengths and angles).

Teachers' KMH is thus interpreted as their knowledge of, and ability to access, elements of a mathematical object's inner and outer horizons. Zazkis and Mamolo (2011) use this construct to explore several examples of teachers accessing their KMH to inform their decisions in teaching situations. Extending on this work, this present study explores what mathematical knowledge is accessed or connected to the concepts "in focus" for pre-service teachers as they addressed a non-routine problem concerning the area of an irregular shape. The analysis attends to specific instances of inner and outer horizon knowledge being mobilized by participants and how this knowledge shaped and influenced their preferred recommendations for a hypothetical student.

Methodology

The participants in this study were 20 pre-service secondary school mathematics teachers enrolled in a teacher education program. Each participant had up-wards of three university level mathematics courses and professed a high level of confidence with secondary school material. Participation was voluntary and included responding to two written questionnaires, administered one week apart, and taking approximately 30 minutes to complete. Participants were informed of the scope of the questionnaires, which sought to explore their mathematical and pedagogical knowledge given a hypothetical teaching situation, but they were not made aware of the specific content in advance, aside from being told that the second questionnaire would follow up on ideas raised in the first session. Both questionnaires were administered with the instructions to answer honestly and reflectively, and that there was no "right answer".

> Imagine you are a teacher in the following situation: Delia, a high school student with good grades, is working on an extra-curricular math problem and approaches you for help. Here is the problem:
>
> You are given a hexagon ABCDEF, where the lengths of the sides are equal to AB = CD = EF = 1 and BC = DE = FA = $\sqrt{3}$, and AB is parallel to DE, BC parallel to EF, and CD parallel to FA.
>
> 1. What is the measure of each interior angle?
> 2. What is the area of the hexagon?
>
> Delia has found that all of the interior angles are of equal measure, but is unsure how to find the area. *How do you recommend Delia go about finding the area?*

Figure 1: The First Questionnaire

The first questionnaire, depicted in Figure 1, was designed to uncover participants' personal strategies and approaches when advising a student, Delia, on how to compute the area of an irregular hexagon. Delia was presented as a "good student" with the intent to encourage participants to reason freely with the problem, without a perception that they were being "tested" on their knowledge of specific curricular content (e.g. if Delia were in one class or another) or of special needs or math-anxious students. As mentioned above, this hexagon was chosen because of the many applicable concepts and solving strategies, which would (i) allow participants access to solving by various means, and (ii) provide information on what relevant (and irrelevant) mathematical concepts were evoked and remained "in focus" for participants. During this stage in the data collection, a diagram of Delia's hexagon was deliberately omitted since how (and whether) participants constructed their own diagrams would provide further insight into how they were interpreting the problem and its key features.

Since the second questionnaire followed up on the emergent themes in participants' recommendations to Delia, it is helpful to quickly summarize the observed trends and initial analysis before presenting the task, as they motivate its scope and intentions.

Of the 20 participants, 18 drew diagrams and all 20 described "deconstructing" the hexagon into smaller "easier" shapes. Of these 18 participants, 15 drew regular hexagons, with the most common diagrams depicted in Figure 2 below. The two most prominent trends in participants' recommendations for Delia were: (i) based on broad ideas, such as "put in lines to break up the hexagon into shapes which we have established rules and laws to work with" (Sophia); and (ii) giving step-by-step procedures of how to solve, such as "she can solve the area of the two triangles and the rectangle in the middle using formulas for the areas. Once this is calculated, she can just add the area of the rectangle and the two triangles" (Abigail).

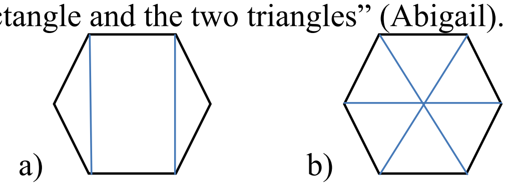

Figure 2: Decomposing Delia's Hexagon into a) Two Triangles and a Rectangle, and b) Six Equilateral Triangles Meeting at the Center Point

The initial analysis suggested that participants relied too heavily on the regularity of their depicted hexagons, using strategies that were either inappropriate (e.g. Figure 2a) or incomplete (e.g. Figure 2b) to generalize to the irregular case. As such, the follow up questionnaire (Figure 3, below) included: (i) a recommendation with a diagram of a regular hexagon, and one with an irregular hexagon, (ii) recommendations that could apply to both regular and irregular hexagons without introducing any additional mathematics, and (iii) recommendations that reflected and contrasted participants' inclination to decompose the hexagon.

In the following section, participants' responses to the second questionnaire are analysed in depth. For the purposes of this proposal, the focus is on identifying specific instances of participants' KMH via the refined lens of inner and outer horizons. Participants' preferences and recommendations are connected to their abilities to access these different components of horizon knowledge, with the intent to shed new light on what, and how, mathematical knowledge may be mobilized in a teaching situation.

Results and Analysis

Due to space limitations, the analysis in this section focuses on participant responses to the second questionnaire which exemplified the trends and themes observed more generally in the data. Before turning to an in-depth analysis, a summary of these trends is presented.

Briefly, of the 20 participants, 11 preferred recommendation A and 9 preferred recommendation B. A common feature in participants' justification of their preferences for both recommendations was the ease and clarity of the approach, yet differences emerged in why the recommendations were considered "easy." Specifically, participants who preferred A attended to structural features and consequences of the provided diagram, while those who preferred B attended to surface features of the solving proves and their prior personal experiences.

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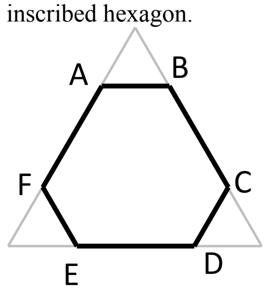
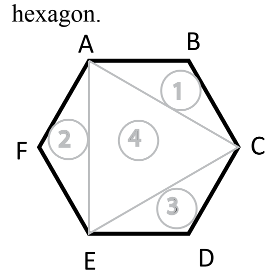
Recall Delia's hexagon ABCDEF, with sides lengths AB = CD = EF = 1 and BC = DE = FA = $\sqrt{3}$.

To determine the area, Delia was given a variety of different recommendations. Here are two of them:

Recommendation A:
Extend the hexagon into an equilateral triangle as in the figure below. Then use the areas of the large triangle, and small outer triangles, to determine the area of the inscribed hexagon.



Recommendation B:
Decompose the hexagon into three triangles (1, 2, 3, which are all equal), and an equilateral triangle 4, as in the figure below. Then sum the areas of the inscribed triangles to determine the area of the hexagon.



Which approach do you prefer, and why?

Figure 3: The Second Questionnaire

Considering first the respondents who preferred rec. A, their focus on structural rather than surface features has a direct connection to Knowledge at the Mathematical Horizon, via Husserl's notions of inner and outer horizons. For instance, Sarah explained:

> "I prefer rec. A because the image makes the idea very clear, where rec. B visually seems more complicated. Also with A, you're only using equilateral triangles… I know the smaller triangles are equilateral because the larger triangle is equilateral, making its interior angles 60 degrees by definition, which means the remaining two angles [of the small triangles] must be 60 degrees because they are all similar and the sum of all angles is 180 degrees."

Accompanying her explanation, Sarah drew the following diagram (Figure 4), noting lines of symmetry and illustrating her conclusion that the small and large triangles were similar:

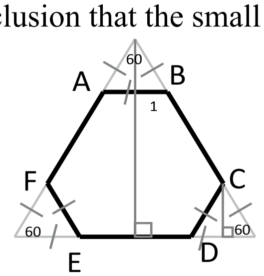


Figure 4: Sarah's Diagrammatic Reasoning

Sarah's attention focused on equilateral triangles and included properties of the formal definition and structural features which allowed her to assess the level of difficulty of this recommendation as opposed to the other. The markings on her diagram suggest Sarah's attention was at first focused on the encompassing triangle and its symmetries and angles. Her deductions regarding the smaller triangles can be interpreted as stemming from her knowledge of an equilateral triangle's outer horizon – e.g. the congruency of angles for all equilateral triangles, and the general property that all interior angles of any triangle must sum to 180 degrees, are part of a greater structure in which these specific triangles are embedded. Sarah further identified the area of the hexagon as the difference between the areas of the large triangle and three small ones, which again may be viewed as an instance of KMH at the outer horizon – that a shape's measurements may be determined indirectly through knowledge of other shapes' measurements. In addition, her attention toward the encompassing triangle suggests that the embedded hexagon shifted to Sarah's periphery, and as such became a part of the triangle's inner horizon (since Delia's hexagon was specific to this triangle). Sarah's continued focus on triangles, rather than the hexagon, suggests a certain comfort and flexibility in her horizon knowledge – she seemed to know that she need not explicitly address the hexagon to determine its properties.

In addition, Sarah also accessed these aspects of her KMH to critique the alternate recommendation. She contrasted the "easier" approach in A with the approach in B, which she identified as "unclear" since "I don't think I could find the side lengths [of the triangles] in B. It seems as though there is a lot more work to finding the areas of the triangles in B." Sarah's judgement that "there is a lot more work" may be seen as awareness that some important structural features of rec. A were not present in B. Thus, it seems as though what was in focus for Sarah – equilateral triangles – and the associated KMH she was able to access as she reasoned with rec. A, also influenced what was in focus for her as she considered rec. B.

Similar themes emerged from Miles, who focused on the specifics of the diagrams provided and the related reasoning. In explaining his preference for rec. A, he noted:

> "Negative space thinking is something that is rarely cultivated and can prove very useful in fields extending beyond pure mathematics. While method B is applicable for most polygons, method A offers a different approach that will get the student to begin thinking of alternative methods to exact the same end."

In this excerpt, Miles seems to be accessing aspects of his outer horizon knowledge, both in terms of situating "negative space thinking" in a world "beyond pure mathematics," as well as in acknowledging that "method B" is broadly applicable. There is also evidence of Miles accessing knowledge at the inner horizon when he attends to the particulars of the two diagrams:

> "It must also be noted that both figures shown are only one possible configuration. In fact, figure B is further from an accurate scale representation than figure A (the sides are not accurately proportioned)."

Miles' focus of attention seemed to be on the specific structure and shapes of the diagrams – in fact, Miles was one of only four participants who alluded to the inaccuracy of the hexagon in rec. B. His observation that there are other possible configurations suggests that while these alternate configurations were not in focus, they were certainly in his periphery. This led him to note that even "if the internal angles aren't equal, figure A's approach can still be used. The triangle form though, may not be equilateral, but it will be isosceles." In contrast to prior research which observed that teachers' images of hexagons tended to be restricted to regular prototypes (e.g. Ward, 2004), Miles was considering hexagons more broadly. His response suggests that he was able to reason with these shapes without having them directly in view. His consideration of how the encompassing triangle would differ depending on the specific hexagon and of how "A's approach can still be used" more generally, instantiate both inner and outer horizons, respectively. Miles also noted that "both methods should be shown and Delia should be encouraged to question both approaches.... Finding these different approaches will allow a student to choose a method that works well for him, and will deepen their understanding of the concept being taught / explored."

Miles' reflection on the value of both approaches is noteworthy as it contrasted with common responses that preferred rec. B as more familiar, comfortable, and more closely connected to strategies participants used when they were students. For these participants, personal preference and comfort seemed to take precedence, and also seemed to influence what participants "allowed in view" and what was kept to the periphery. For example, Abigail claimed that "rec. B is the approach I would take because of the way I learned geometry. The hexagon divided into triangles is the approach I learned in school" and because the "subtraction method [is] confusing to me, but adding small shapes to make a big shape is easy." Abigail's desire to use the approach she learned in school indicates a reluctance to access KMH – that is, knowledge that goes beyond school curricula. This reluctance also seemed to focus her attention toward superficial features of the recommendations (such as adding versus subtracting), which were not considered in relation to the specific entities that would be added or subtracted. It is unclear from Abigail's response whether she lacked appropriate KMH, or whether she viewed such knowledge as inapplicable. What can be said is that with a robust KMH, and a willingness to access it, then Abigail could have analysed the level of difficulty of the arithmetic with respect to the specific features of Delia's hexagon (inner horizon), and provided justification for her preference that spoke to the general validity or applicability of the proposed recommendation (outer horizon). The idea that adding is easier or more straightforward was echoed by the majority of participants who chose rec. B.

Comfort level was also important for Lexi, who chose B because "automatically I can recall the Pythagorem Theory [*sic*] to solve for the individual triangles 1, 2, 3, which are all equal" and "I prefer formulas in mathematics, and checking that my work is correct, as opposed to guessing what number relates to what side, which is probably how I would go about solving approach A." It should be noted that the Pythagorean Theorem is not applicable to the triangles in rec. B, thought it can be useful in determining the area of the equilateral triangle from rec. A. Lexi's strong preference for formulas and Abigail's preference for the approach she learned in school suggest that a desire to stay within a familiar comfort zone limited what mathematics they were willing or able to consider and mobilize in analysing the recommendations. For Lexi this led to an inappropriate approach, and for Abigail it led to a superficial consideration of what would be "easy" or "confusing." While personal comfort and familiarity are certainly valid reasons for choosing one approach over another, teachers must be able (and willing) to reason beyond their comfort zones in order to adequately meet the diverse needs of their students. In the data from this study, individuals who were reluctant to go beyond their comfort zones drew inappropriate conclusions about the solving methods and their levels of difficulty, and showed a limited access to knowledge beyond what was emphasized in their prior school experiences. In contrast, participants who looked beyond school curricula and made connections between and across ideas (KMH) were able to appreciate aspects of the recommendations on a deeper level, and also considered how these ideas related to a student's broader learning experience.

Concluding Remarks

What influences pre-service teachers' preferences and recommendations when advising a student on how to determine the area of an irregular hexagon? There were several factors. In accord with research done with children and elementary teachers (e.g. Ward, 2004; Walcott et al., 2009), participants with strong mathematics backgrounds also relied on prototypes of regular hexagons (first questionnaire). This resulted in recommendations that were inappropriate or incomplete for the problem at hand. Personal comforts with formulas or previously learnt school material were also key features in participants' preferences for how to advise a student (e.g. Abigail, Lexi, second questionnaire). While Delia's hexagon was new for all participants, some were more willing to 'dig in' and explore the problem on a deeper level, making connections beyond what was previously learnt in school to attend to structural features of the approaches (e.g. Sarah) or potential benefits to students' broader learning (e.g. Miles). These connections were interpreted as instances of Knowledge at the Mathematical Horizon, as it related to inner and outer horizons of the mathematical entity in focus. The analysis suggests that comfort and flexibility in what horizon knowledge is accessed and mobilized are important features in assessing and recommending appropriate solving strategies for learners, and it makes a case for developing such flexibility in teacher education programs.

Ball and Bass (2009) note that "knowledge of the horizon does not create an imperative to act in any particular mathematical direction" (p.10). This research suggests that while KMH may not provide an imperative, it can provide an invitation to act in ways that consider mathematics in a more structural and connected way, even if this may diverge from prior school expectations. Also, participants who accessed their KMH provided an analysis of the two recommendations that included more depth and accuracy than those who did not. This suggests that teachers with a more robust KMH (or one which is more readily accessed) are in a better position to guide and interpret student thinking. Participants in this study who were so able seemed more flexible in how they viewed the different recommendations, and were more willing to set aside their personal and initial preferences for decomposing the shape when interpreting the benefits and draw backs of each. This is particularly significant when considering the potential effects on a teacher's response to a student's unconventional or unexpected solutions or approaches.

Finally, this research opens the door to exploring a possible connection between a teacher's KMH and her Knowledge of Content and Students (KCS). Specifically, Miles' KMH seemed to influence what he believed would be important for student learning (e.g. "negative space thinking"), and this in turn influenced what horizon knowledge he was willing to mobilize when considering the alternate recommendation. Additional research regarding the relationship between KMH and KCS, how one might influence the other, and how one might develop in conjunction with the other may provide further insight into the experiences necessary for preparing future teachers of mathematics.

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