A LEARNING TRAJECTORY FOR EARLY EQUATIONS AND EXPRESSIONS FOR THE COMMON CORE STANDARDS

Nicole Panorkou[1]
Montclair State University
panorkoun@mail.montclair.edu

Alan P. Maloney
North Carolina State University
alan_maloney@ncsu.edu

Jere Confrey
North Carolina State University
jconfrey@ncsu.edu

This paper describes the learning trajectory for Early Equations and Expressions (EELT), one of the 18 learning trajectories presented at www.TurnOnCCMath.net, developed to interpret the Common Core Standards for Mathematics based on research on student learning. The EELT is foundational for introducing concepts of early algebra and setting the foundations for the most advanced mathematical ideas in the later grades of schooling. The theoretical framework and research literature that the unpacking was based on are presented by giving examples from the descriptors of the standards.

Keywords: Expressions, Equations, Standards, Learning Trajectories

Learning Trajectories and the Common Core Standards for Mathematics: TurnOnCCMath.net

The Common Core State Standards for Mathematics (CCSS-M) (CCSSO, 2010) have been adopted by 45 states and the District of Columbia. Major goals include strengthening students' mathematical conceptual understanding, weaving eight key mathematical practices throughout mathematical instruction and learning, and bringing U.S. educational standards to par or better with education in countries whose preparation of students in mathematics is stronger than that of the U.S., based on international assessments (CCSSO, 2010). The CCSS-M represents a major effort to improve the coherence in learning expectations across states. The new standards include significant changes in depth, emphasis and timing of instruction in numerous topics. Some topics are introduced earlier, some later, than in previous standards from various states, and several with more intensive or expanded treatment (e.g. early number and operations; ratio and proportional reasoning; statistics and probability).

How then to translate the intent of the new standards into instructional practice, to interpret the CCSS-M in ways that strengthen students' conceptual development? Research on student learning by mathematics educators during the past 20 years or more has generated *learning trajectories* (LTs) as an organizing framework for student conceptual growth (Clements & Sarama, 2004; Confrey et al., 2009; Simon, 1995). Depicting student learning of major mathematical ideas over time, LTs describe student prior knowledge of particular ideas, and networks of instructional experiences that support students in transiting likely intermediate states of understanding on their way to robust domain goal understanding (Confrey et al., in press).

Learning trajectories can provide a framework for organizing the instructional core (Confrey & Maloney, 2011)—that spectrum of instruction, instructional practices, enacted curriculum, and classroom discourse and assessment that comprise the learning environment that children experience—to focus on the consistent development of student conceptual understanding over time.

[1] On October 22 of 2013, PME-NA committee was informed of a mistake in authorship. Therefore, the paper authorship herein differs from the version of this paper included in the PMENA 2013 Conference Proceedings that were posted on the conference website on Monday, October 21, 2013. The authors of this paper listed here are correct as of Thursday, October 24, 2013.

Martinez, M. & Castro Superfine, A (Eds.). (2013). *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Chicago, IL: University of Illinois at Chicago.

The CCSS-M calls for incorporation of "research-based learning progressions detailing what is known today about how students' mathematical knowledge, skill, and understanding develop over time" (CCSSO 2010, p. 4). Our research priorities are to bring to bear the research on student learning to support educators in interpreting the CCSS-M and support practical instructional implementation of the CCSS-M.

We have constructed 18 learning trajectories that cover all of the K-8 CCSS-M standards (http://www.turnonccmath.net) (Confrey et al., 2011; Confrey, 2012). A hexagon map of the CCSS-M was created as a visual model for navigating the connections among major topics and standards and illustrating learning trajectories within the standards. Each learning trajectory (LT) comprises descriptors that incorporate the Common Core Standards into text- and graphics-based descriptions of student movement from naïve to more sophisticated mathematical understanding. The descriptors identify (1) conceptual principles, (2) student strategies and representations or *inscriptions* along with misconceptions; (3) meaningful mathematical distinctions and multiple models; (4) coherent structure or schemes of reasoning as topics become more complex; finally, (5) bridging standards that are added to provide conceptual continuity for more fully articulated learning trajectories than could be provided in the compact CCSS-M. The iterative development of these learning trajectory descriptors, and their continual improvement and strengthening based on feedback and review, have been described previously (Confrey, 2012). In this paper, we describe in more detail the learning trajectory for Early Equations and Expressions (EELT).

Early Equations and Expressions Learning Trajectory

Algebra "encompasses the relationships among quantities, the use of symbols, the modeling of phenomena, and the mathematical study of change" (NCTM 2000, p. 37). By linking these topics to, and developing them simultaneously with, topics of early mathematics, the aim of early algebra in elementary schools is for students to establish the foundations for development of algebraic reasoning in middle and high school (Carraher et al. 2007). The main goal of algebra is for students to learn how to form generalizations by seeing the "general through the particular" and "the particular in the general" (Mason et al., 2005), to shift students' focus "from thinking about relations among particular numbers and measures toward thinking about relations among sets of numbers and measures, from computing numerical answers to describing and representing relations among variables." (Carraher et al., 2007, p. 266). For competency in algebraic reasoning, student learning must encompass and coordinate several different "big ideas" such as generalized arithmetic, functional thinking, the concept of equality, and the concept of a variable (Blanton & Kaput, 2011; Carraher et al., 2008; Carraher et al., 2007).

The goal of the EELT was to describe the development of students' reasoning about patterns and the early foundations of functional relationships (Section 1, "Exploring Patterns and Their Relationships"), equality and equations (Section 2, "Exploring and Solving Equations"), and expressions, including variables as quantities that vary (Section 3, "Working with Expressions"). These interrelated concepts span grades K through 7, concurrently during many of the grades, with standards from the Operations and Algebraic Thinking and the Expressions and Equations domains of the CCSS-M. Other topics important in the foundation of algebraic reasoning are covered in other TurnOnCommonCoreMath LTs: Inequalities and more advanced notions of functional thinking are emphasized in the Linear and Simultaneous Functions LT; operations including inverse operations, and properties are explored in detail in the Addition & Subtraction, and Division and Multiplication LTs, and ratio, rates, and proportional reasoning are developed in the Ratio and Proportion and Percents LT. Explicit connections among these and the EELT are made, where appropriate in each of these LTs. The EELT contains a total of 17 CCSS-M and 6 Bridging Standards, with their corresponding descriptors. The following sections describe the unpacking of the EELT as it was developed and presented in www.TurnOnCCMath.net.

Section 1: Exploring Patterns and Their Relationships

Section 1 describes the development of student reasoning in identifying repeating and growing patterns, and the characterization of relationships between two patterns. The CCSS-M introduces arithmetic patterns in Grade 3 (Standard 3.OA.9), but does not explicitly treat geometric patterns, which are also a type of growing pattern. Repeating patterns can act as the basis for introducing growing patterns, because they are foundational for expressing initial generalizations of rules, functional relationships, proportional thinking, and number theory (Threlfall, 1999; Warren, 2005; Zazkis & Liljedahl, 2002). The CCSS-M introduces shape patterns in Grade 4 (Standard 4.OA.5), however, research has demonstrated that non-numeric patterns such as pictorial, verbal and symbolic should occur in earlier grades to help students recognize generalization in relationships that are "independent of the numbers or objects being operated on" (Warren, 2005, p. 759). Building on such insights, the learning trajectory therefore incorporates introduction of students first to non-numeric patterns (both repeating and growing) and then to numeric patterns (both repeating and arithmetic/geometric); we added four bridging standards to support these (Bridging Standards K.EQX.A, 1.EQX.A, 1.EQX.B and 2.EQX.A).

We also discerned a systematic scheme for pattern exploration (Figure 1), an example of *coherent structure* (element 4, above). For each type of pattern, student understanding is strengthened through first (a) identifying the terms and the core unit, (b) extending the pattern to subsequent terms, (c) identifying missing terms, (d) debugging mistakes, (e) recognizing common structure among patterns (i.e. the pattern 2, 4, 6, 8, … has the same structure as the pattern 86, 88, 90,…) and finally (f) creating new patterns from a given core unit and initial element.

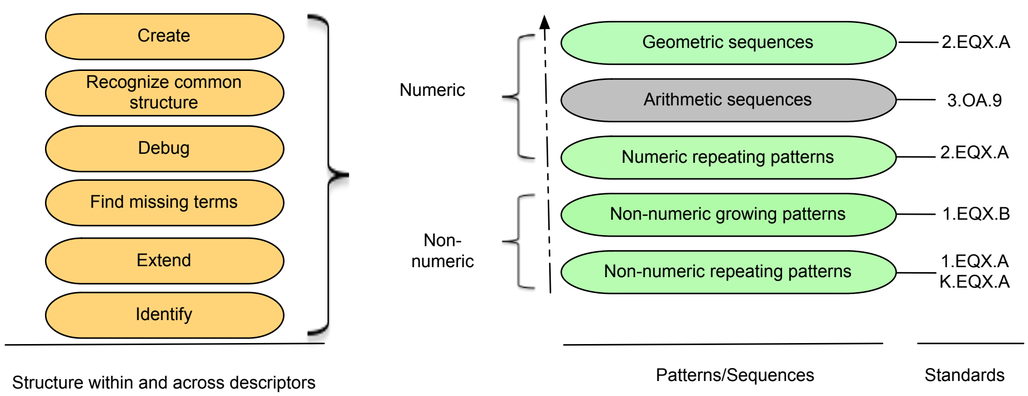

Figure 1: CCSS-M Standards in Grey and Bridging Standards in Green

A more sophisticated description of patterns, identifying a rule for a given pattern in standard 4.OA.5 (i.e. given the pattern 2, 7, 12, 17, … students identify the rule as "Add 5"), supports the second goal of this section, namely the identification and characterization of relationships between two patterns (Standard 5.OA.3). A correspondence description for two patterns expresses a rule that relates corresponding value pairs of two patterns. A foundation of the development of correspondence rules is a covariation relationship, which explains how the values of two patterns change simultaneously (Blanton & Kaput, 2011; Confrey & Smith, 1991, 1994, 1995). The distinction between covariation and correspondence relationships can be introduced through contextual problems, for example the following: "A roadrunner, being pursued by Wile E. Coyote, runs 3 meters every two seconds. Make a graph that describes the distance the roadrunner travels in relation to time. Predict the roadrunner's location after 5 seconds." The learning trajectories highlight multiple representations: e.g. for characterizing functional relationships, the use of diagrams, tables, graphs, and dynagraphs (Figure 2).

Time (*s*, seconds)	Distance *m* (meters)
0	0
2	3
4	6
6	9
8	12
10	15

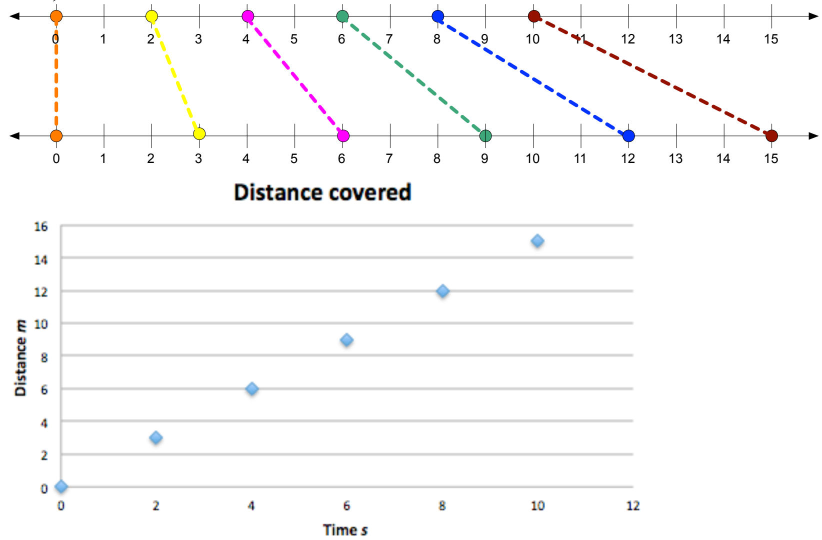

Figure 2: Table, Dynagraph and Graph as Representations

The two patterns in this example covary: values in the first column increase by 2 while values of the second column increase by 3. A correspondence description here delineates a rule that describes one of the sequences in terms of (dependent on) the values of the other: any value to distance covered by the roadrunner (*m*) is $^3/_2$ times the corresponding value of the time (*s*) in seconds. Students in 5th grade are not expected to write a symbolic equation for this type of relationship. However, they are expected to extend the patterns and identify the rule (correspondence). Studies have shown that students are capable of reasoning about covariation and correspondence relationships as early as first grade (Blanton, & Kaput, 2004; Martinez & Brizuela, 2006; Stephens et al., 2012). By 5th grade in the EELT, students have experience exploring both covariation and correspondence relationships and a variety of arithmetic and geometric sequences, all of which lay a strong foundation for the development of ratio and functions in middle school (Carraher & Schliemann, 2007; Zazkis & Liljedahl, 2002).

Section 2: Exploring and Solving Equations

The goals of Section 2 focus on a) understanding the concept of equality and solving equations and b) the concept of a variable as representing a single unknown. Equality is a pivotal concept with multiple meanings. It is key to development of a progressively sophisticated understanding of and flexibility with equations. It is developed from a way to make the equations "balance," to determining which equations are true or false, leading to the more sophisticated solving of equations (Carpenter, & Levi 2000; Carpenter et al., 2003; Van de Walle et al., 2013). The aim is for students to recognize that the equal sign signifies a claim that the quantities or expressions on either side of the equal sign represent the same value; this serves as a foundation for creating equivalent algebraic expressions at the higher levels of the learning trajectory.

The CCSS-M introduces the meaning of the equal sign in equations in Grade 1 (Standard 1.OA.7). Before formal introduction to the equal sign and writing equations, students learn to model the equality of (a) two additive parts (which can be joined) and their sum, and (b) equipartitioned parts (which can be reassembled) and their original collection or whole. To link students' prior understanding of joining and separating in counting (Samara & Clements, 2009; Nunes & Bryant, 1996; Fuson et al., 1983; Piaget, 1965) and equipartitioning and reassembly (Confrey et al., 2009) to the concept of equality, we added a preceding bridging standard (1.EQX.C) that contains multiple well-known ways to concretely model equations, such as unifix cubes, counters, and drawings. Balance scales also serve as visual models for equations: children use different combinations of quantities to represent and achieve balance (equality) and to show equivalence between an original collection and all of the corresponding fair shares reassembled (Baroody & Ginsburg, 1983; Van de Walle et al., 2013). Balance diagrams (Figure 3) anticipate the balancing of equations in algebra (Herscovics & Linchevski, 1994). Another visual model for equations is the "bar model" (Figures 4a/b), in which children use different combinations of horizontal bars to establish claims of equal lengths or to show equivalence between a collection of items and the reassembly of equipartitioned fair shares (Fong Ng & Lee, 2009; Hoven, 2007). For example, students may recognize that a 12-inch paper strip shared among three people can be represented with 4-inch long strips per person, which can then be reassembled lengthwise to form the original 12-inch paper strip (Figure 4a). They may also demonstrate that the three fair shares of 4-inch strips all have the same length (Figure 4b). We conjecture that as students coordinate such models with the writing of equations, they reveal misconceptions about the role of the equal sign, such as using it as an operator symbol (Carpenter & Levi, 2000; Carpenter et al., 2003; Kieran, 1981). Teachers can then, in a timely way, instructionally address such a misconception.

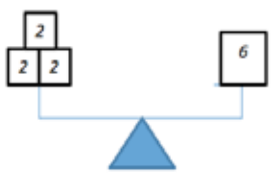

Figure 3: Balance Scale

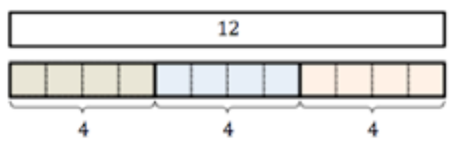

Figure 4a: Bar Model 1

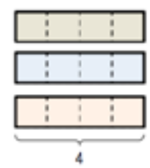

Figure 4b: Bar Model 2

Following the writing of equations, students begin to solve equations with single unknowns. They progress from addition and subtraction equations to division and multiplication equations relating three whole numbers (Standards 1.OA.8 and 3.OA.4 respectively), move to equations that involve four whole numbers (3.EQX.A, bridging standard added) and then to the more sophisticated solving of two-step word problems using all four operations in equations (Standard 3.OA.8). For the solving of all the types of equations above, the EELT includes four different strategies suggested by research (Carpenter, & Levi 2000; Carpenter et al., 1996): a) *substitution* of different values, including checking to see if the two sides of an equation have equal value; b) using *models* to illustrate equivalence, such as rearranging arrays (i.e. 6 x 2 is equal to 3 x □); c) applying inverse operation (as in "fact families"), such as rewriting 35 ÷ 7 = □ as 7 x □ = 35 to solve; and d) applying known properties (commutative property of addition and identity) to solve problems, such as recognizing that the equation 1 x □ = 35 ÷ 7 can be simply □ = 35 ÷ 7 through application of the identity property (1 x □ = □).

Section 3: Working with Expressions

Section 3 of the EELT has goal understandings of a) distinguishing between variable as representation of a single unknown value and variable as a quantity that varies, according to context, b) using variables in different contexts to represent quantities, and c) identifying and generating equivalent expressions at a more advanced level.

Students first encounter the *variable* as representing a single unknown value in equations (Section 2) but now broaden the concept to represent a quantity that varies, in exploring expressions (Standard 6.EE.2.a). Studies have shown that students of an early age can represent quantities that vary (Carraher et al., 2007; Marum et al., 2011). Again, multiple representations support strengthening student understanding: variables should be represented in many forms including multiple letters, shapes, and symbols, in part to avoid the development of misconceptions and prototypes such as believing that letters are general referents (i.e. *h* stands for height of multiple people) (MacGregor & Stacey, 1997) and that different symbols can have the same value (Carpenter et al., 2003). Contextual problems such as the following, from Brizuela & Earnest (2008), support distinctions: "Raymond has some money. His grandmother offers him two deals: *Deal 1*: She will double his money. *Deal 2*: She will triple his money and then take away 7. Raymond wants to choose the best deal. What should he do? How would you figure out and *show him* what is the best to do? Is one deal *always* better? Show this on a piece of paper." Generating expressions for these problems, students recognize that a) a variable can represent a parameter whose value determines the characteristics or behavior of other quantities [i.e. if *m* is the value money above, then Deal 1 is $2 \bullet m$ and Deal 2 is $(3 \bullet m)-7$], b) while the value of a parameter can vary, its value is always fixed in the context of a specific problem, and c) the same variable used in expressions related by context must represent the identical value, but different variables may represent the same value or different values (Blanton & Knuth, 2009-2013). By exploring which deal is better to take, students distinguish between an equation or inequality being true for *all* values of *x*, and for *some* (or only one) values of *x*.

While exploring equivalent expressions, students typically identify two expressions as equivalent if, for any value substituted for the variable, the values of the two expressions are equal (Standard 6.EE.4). This implies that a single counterexample (a value for the variable that yields unequal values of the expressions) demonstrates the expressions' non-equivalence (Carpenter & Levi, 2000). Students express numerous common misconceptions and errors when simplifying expressions, but by substituting values for the variables before and after simplifying, and looking for counterexamples, they are able to strengthen their understanding and improve their accuracy. This experience is critical for generating equivalent expressions, bringing many of students' skills with operations, in a variety of contexts, together with their work in algebra, to explore and demonstrate the power of algebra as a generalized solution method (Standards 7.EE.1 and 7.EE.2), thereby advancing their algebraic reasoning and problem-solving abilities.

Implications and Discussion

The EELT builds on previous research on how students' thinking progresses within patterns and sequences, equations, and expressions, and is foundational for the development of their algebraic thinking. Aiming to show the progression of "big ideas" within the CCSS-M, the LT maintains the grade level of each mathematical idea as it is presented in the CCSS-M (for instance, not introducing the variable as a quantity that varies until 6th grade, despite research showing that this can be done earlier) while providing bridging standards to fill gaps in the CCSS-M for instructional continuity and a coherent structure (e.g. introducing repeating patterns and geometric sequences). Blanton and Knuth (2009-2013) have suggested five curricular learning progressions for algebra education: a) equality, expressions, equations and inequalities, b) functional thinking, c) variables, d) generalized arithmetic, and e) proportional reasoning. Ongoing research such as theirs will provide additional fundamental insight for revising the EELT and other learning trajectories of TurnOnCCMath.net. Interpretation and implementation of the CCSS-M using learning trajectories can spur strategic critical areas of further research into student learning of mathematics, and contribute to constructive critique of the CCSS-M, both of which will lay the groundwork for systematic revision of the CCSS-M in the coming years.

Acknowledgements

Supported by National Science Foundation, Qualcomm, and Oak Foundation. We gratefully acknowledge K. Lee, T. Avineri, K. H. Nguyen, A. K. Corley, and J. Nickell (members of the EELT writing and reviewing team) and Pedro Larios, APS, Inc., (website engineering).

References

Baroody, A., & Ginsburg, H. (1983). The effects of instruction on children's understanding of the "equals" sign. *The Elementary School Journal 84*, 198-212.

Blanton, M., & Kaput, J. (2004). *Elementary grades students' capacity for functional thinking.* Paper presented at the 28th annual conference of the international group for the psychology of mathematics education Bergen, Norway.

Blanton, M., & Kaput, J. J. (2011). Functional Thinking as a Route Into Algebra in the Elementary Grades. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 5-13): Springer Berlin Heidelberg.

Blanton, M., & Knuth, E. (2009-2013). Developing Algebra-Ready Students for Middle School: Exploring the impact of early algebra. (Curricular learning progressions) Retrieved January 26th, 2013, from http://algebra.wceruw.org/progressions.html

Brizuela, B. M., & Earnest, D. (2008). Multiple notational systems and algebraic understandings: The case of the "best deal" problem. In D. Carraher, J. Kaput, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 273-301). Mahwah, NJ: Lawrence Erlbaum and Associates.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). Thinking mathematically: Integrating arithmetic and algebra in the elementary school. Portsmouth, NH: Heinemann

Carpenter, T. P., & Levi, L. (2000). Developing conceptions of algebraic reasoning in the primary grades. Research report. Madison, WI: Wisconsin University, Madison, National Center for Improving Student Learning and Achievement in Mathematics and Science.

Carpenter, T. P., Fennema, E., & Franke, M.L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal, 97*, 3-20.

Carraher, D. W., Martinez, M., & Schliemann, A. D. (2008). Early algebra and mathematical generalization. *The International Journal on Mathematics Education, 1*, 3-22.

Carraher, D. W., & Schliemann, A. D. (2007). Early Algebra and Algebraic Reasoning. In F. Lester (Ed.), Handbook of Research in Mathematics Education. Greenwich, CT: Information Age Publishing, pp. 669-705.

Carraher, D. W., Schliemann, A. D., & Schwartz, J. (2007). Early algebra is not the same as algebra early. In J. J. Kaput, D. Carraher & M. Blanton (Eds.), *Algebra in the Early Grades* (pp. 235-272). Mahwah, NJ: Erlbaum.

CCSSO. (2010). Common Core State Standards for Mathematics, from http://www.corestandards.org/assets/CCSSI_Math Standards.pdf

Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical thinking and learning, 6*, 81-89.

Confrey, J. (2012). *Articulating a learning sciences foundation for learning trajectories in the CCSS-M.* In Van Zoest, L. R., Lo, J.-J., and Kratky, J. L. (Eds.) Proceedings of the 34th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, pp. 2-22. Kalamazoo, MI: Western Michigan University.

Confrey, J., & Smith, E. (1991). *A framework for functions: Prototypes, multiple representations, and transformations.* Paper presented at the 13th annual meeting of the North American chapter of the international group for the psychology of mathematics education, Blacksburg, VA.

Confrey, J. & Maloney, A. (2011). Engineering [for] effectiveness in mathematics education: Intervention at the instructional core in an era of Common Core Standards. Paper prepared for the National Academies Board on Science Education and Board on Testing and Assessment for "Highly Successful STEM Schools of Programs for K-12 STEM Education: A Workshop."

Confrey, J., Maloney, A. P., & Nguyen, K. H. (in press). *Learning over time: Learning trajectories in mathematics education*: Information age.

Confrey, J., Maloney, A., Nguyen, K. H., Mojica, G., & Myers, M. (2009). *Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories.* Paper presented at the 33rd Conference of the International Group for the Psychology of Mathematics Education, Thessaloniki, Greece.

Confrey, J., Nguyen, K. H., Lee, K., Corley, A., Panorkou, N., & Maloney, A. (2011). Learning trajectories hexagon map and descriptors for the Common Core State Standards for Mathematics Retrieved January 25, 2013, from www.turnonccmath.net

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics, 26*, 135-164.

Confrey, J., & Smith, E. (1995). Splitting, Covariation, and their role in the development of Exponential Functions. *Journal for Research in Mathematics Education, 26*(1), 66-86.

Fong Ng, S., & Lee, K. (2009). The model method: Singapore children's tool for representing and solving algebraic word problems. *Journal for Research in Mathematics Education, 40*(3), 282-313.

Fuson, K. C., Secada, W. G., & Hall, J. W. (1983). Matching, counting, and conservation of numerical equivalence. *Child development, 54*, 91-97.

Herscovics, N. & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics 27*(1), 59-78.

Hoven, J. G., B. . (2007). Singapore math: Simple or complex? . *Educational Leadership, 65*(3), 28-31.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*, 317-326.

MacGregor, M., & Stacey, K. (1997). Students' understanding of algebraic notation: 11-15. Educational Studies in Mathematics, 33(1), 1-19.

Martinez, M., & Brizuela, B. M. (2006). A third grader's way of thinking about linear function tables. *Journal of Mathematical Behavior, 25*, 285-298.

Marum, T., Isler, I., Stephens, A., Gardiner, A., Blanton, M., & Knuth, E. (2011). *From specific value to variable: Developing students' abilities to represent unknowns.* Paper presented at the 33rd Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education, Reno, Nevada.

Mason, J., Graham, A. & Johnston-Wilder, S. . (2005). *Developing thinking in algebra*. London: Paul Chapman Publishing.

McCallum, B., Black, A., Umland, K., & Whitesides, E. (2010). Tools for the Common Core Standards Retrieved January 26th, 2013, from http://commoncoretools.me/tools/

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*: Reston: National Council of Teachers of Mathematics.

North Carolina Department of Public Instruction. (2011). NC Common Core Instructional Support Tools - Math unpacking standards Retrieved January 26th, 2013, from http://www.dpi.state.nc.us/acre/standards/commoncore-tools/

Nunes, T. & Bryant, P. (1996). *Children doing Mathematics*. Cambridge, MA: Blackwell Publishers Inc.

Piaget, J. (1965). *The child's conception of number*. New York: Norton.

Sarama, J. & Clements (2009). *Early Childhood Mathematics Education Research: Learning Trajectories for your children*. New York: Routledge.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Stephens, A., Isler, I., Marum, T., Blanton, M., Knuth, E., & Gardiner, A. (2012). *From recursive pattern to correspondence rule: Developing students' abilities to engage in functional thinking.* Paper presented at the 34th Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education, Kalamazoo, MI.

Threlfall, J. (1999). Repeating Patterns in the Early Years. In A. Orton (Ed.), *Pattern in the Teaching and Learning of Mathematics* (pp. 18-30). London and New York: Cassell.

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2013). *Elementary and middle school mathematics: Teaching developmentally; The professional development edition*. New Jersey: Pearson Education.

Warren, E. (2005). *Patterns supporting the development of early algebraic thinking.* Paper presented at the Building connections: Research, theory and practice. Proceedings of the 28th Annual Conference of the Mathematics Education Research Group of Australia Melbourne, Sydney.

Zazkis, R., & Liljedahl, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics, 49*, 379-402.