YOUNG STUDENTS' EXPLORATIONS OF GROWING PATTERNS: DEVELOPING EARLY FUNCTIONAL THINKING AND AWARENESS OF STRUCTURE

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This study explored the development of early functional thinking in fifteen Kindergarten to Grade 2 students. As part of a teaching intervention that emphasized the co-variation between position numbers and number of items in each position of linear growing patterns, students were given a pre and post interview assessment to document their ability to predict the NEXT, NEAR and FAR terms of a pattern. Results of the interviews indicated that young students have the capability to recognize the relationship between two sets of values. Results also suggest that the recognition of visual and numeric structure in patterns influenced the extent to which students were able to reason recursively or explicitly. Finally, nine of the students developed the "perceptual agility" (Lee, 1996) to perceive the pattern in two ways. A "rows" perception was aligned with recursive reasoning, while a "columns" perception was aligned with explicit reasoning.

Keywords: Elementary School Education, Algebra and Algebraic Thinking

Early Years Algebra

In recent years there has been a shift in the research of mathematics education for young students from a focus on children's arithmetic thinking to the development of algebraic reasoning (Blanton & Kaput, 2002, 2003; Carraher & Schliemann, 2007). Researchers have begun to explore the kinds of learning experiences young students need in the elementary grades in order to develop algebraic thinking prior to formal algebraic instruction in high school. One area of research has been the study of elementary students' understanding of patterns and how this supports the development of functional thinking (Warren & Cooper, 2008).

Past research initially suggested that the connection between working with patterns to developing algebraic reasoning – that is, finding generalizations and expressing these as algebraic rules – is difficult even for students in middle and high school (English & Warren, 1998; Kieran, 1992; Lee & Wheeler, 1987; Orton et al., 1999; Stacey & MacGregor, 1999). One reason for this may be that current patterning curricula tend to emphasize the variation in only one set of data. For example, Figure 1 represents the kind of linear growing pattern found in elementary mathematics textbooks. Students from a young age are taught to focus on the systematic increase of tiles at each subsequent position of the pattern without thinking about the relationship between the tiles and their position in the pattern. Most students would describe and extend this pattern with the rule, "start with three red tiles and add two blue tiles each time". Asking for the 100th term highlights the problem of this method. Relying on this

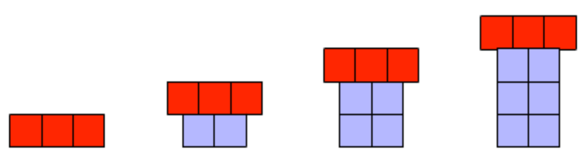

Figure 1: Growing pattern without position cards

recursive reasoning, adding two tiles to the previous term, makes it difficult to devise a general rule that would allow for the prediction of the number of tiles for *any* term of the pattern. Current instruction emphasizes this recursive reasoning by presenting growing patterns without term numbers, or with term numbers acting as labels for each term, but not as numeric quantities that can be used with a rule to determine the number of tiles.

More recently, researchers have been studying the potential of introducing specific pedagogical approaches to assist elementary students in developing algebraic reasoning through working with patterns (Rivera & Becker, 2008). Results of these studies have demonstrated the extent to which working with growing patterns can support students in developing algebraic thinking, particularly functional thinking, when the relationship between the term number and number of items at that term are emphasized during instruction (Beatty, 2007; Moss & McNab, 2011; Warren & Cooper, 2008). As an example, the pattern in Figure 2 represents the equation $y=2x+3$ with the coefficient represented by the blue tiles that increase by 2 at each term, and the constant represented by the red tiles that stay the same. The incorporation of term or position cards serves to help students identify the co-variation between one data set (i.e., the position number of each iteration) and another data set (i.e., the number of tiles at each position). The pattern rule, *number of tiles = term number x2+3* describes this relationship. Recognizing this relationship is referred to as explicit reasoning and allows for the prediction of elements far down the sequence and, ultimately, to identifying a general rule. Developing explicit reasoning is foundational to developing functional thinking, which is the perception of a generalization that relates two sets of objects.

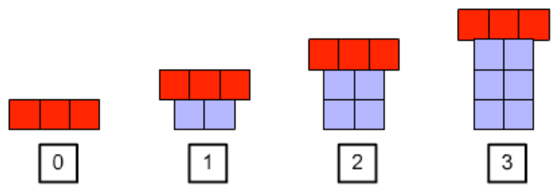

Figure 2: Growing pattern with position cards

Recent studies suggest that another important part of developing algebraic reasoning is the ability to identify mathematical structure (Mulligan et al., 2004; Mulligan & Mitchelmore, 2009). Linear growing patterns have both a numeric and geometric or visual structure. Students who focus on the numeric structure of the pattern in Figure 2 may attend to the fact that the elements at each position increase systematically (grow by 2) or that each element in the pattern is equal to twice the position number an additional three. Growing patterns also have a visual structure. Students who attend to the visual structure of the pattern may perceive it as composed of rows and/or columns of blue tiles with three red tiles on top. It is the development of an awareness of both numeric and visual structure, and identifying these as representations of a functional relationship, that supports algebraic reasoning.

There have been few studies that have looked at the potential to develop functional thinking in primary students (Moss & McNab, 2011; Warren, 2005; Warren & Cooper, 2008). The purpose of this exploratory study was to document how young students (Kindergarten – Grade 2) understand linear growing patterns. Specifically we were interested in the extent to which young students could accurately predict the NEXT (5^{th}), NEAR (10^{th}) and FAR (100^{th}) iteration of a linear growing pattern. Also of interest was whether these students would go beyond recursive reasoning to explicit reasoning by making the connection between the term or position number (the value of the position number) and the number of tiles at each position. We were also interested in how recognition of pattern structure (visual and/or numeric) influenced students' abilities to develop explicit reasoning.

Our research questions were:

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1. Can working with linear growing patterns support students in making the connection between the position number and the number of elements at that position (going beyond recursive to explicit reasoning)?
2. Can young students generalize (in this context, can they accurately predict NEAR and FAR iterations of a linear growing pattern)?
3. What role does recognizing numeric or visual structure play in young students' mathematical thinking?

Methods

Setting and Participants

This study took place in four classrooms in two elementary schools (one Kindergarten class, one Grade 1 class and two combined Grade 1 & 2 classes). Five teachers were involved in the study including one Kindergarten teacher, one Grade 1 teacher, two Grade 1 & 2 teachers and one resource teacher. A short 5-lesson teaching intervention was designed to support students in making explicit connections between position numbers and numbers of items in each position, and to generalize by making NEAR and FAR predictions. During the instruction, attention was also paid to the visual and the numeric structures of linear growing patterns.

Data Sources and Analysis

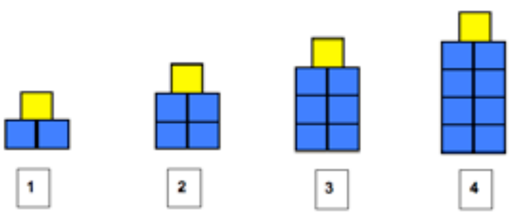

Figure 3: Interview Task Pattern

Individual task-based interviews were administered by the classroom teacher and recorded by the researcher. Fifteen students were interviewed (five Kindergarten, six Grade 1 and four Grade 2). The pre-interviews took place in December, and the post-interviews took place the following May. One of the interview tasks, the focus of this paper, asked children to describe and extend a linear growing pattern (Fig. 3) by building the 5th position. Students were asked to make NEXT (5th and 6th), NEAR (10th) and FAR (100th) predictions of the pattern. Transcriptions of the video recorded interviews were coded with respect to students' abilities to 1) identify predictable growth and constancy, 2) accurately extend the pattern, a NEXT prediction, 3) make accurate predictions down the sequence (NEAR 10th position or FAR 100th position). Transcriptions were also coded with respect to whether students recognized the numeric structure of the pattern, the visual structure of the pattern, and how recognition of structure supported an ability to make accurate generalizations.

Results

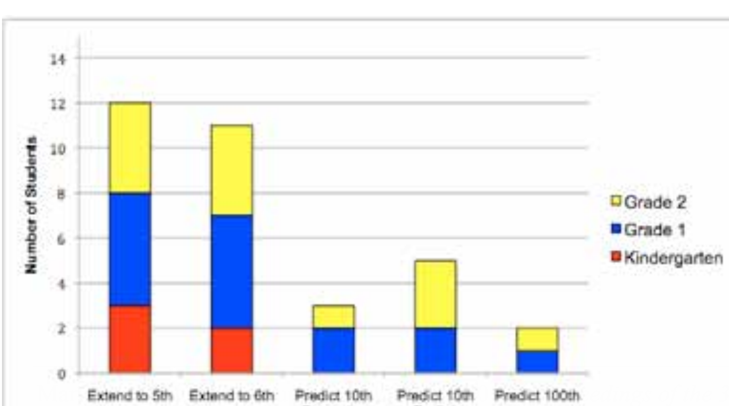

Figure 4: Pre-Interview Results

Pre-Interview Results. Results for the pre-interview are presented in Figure 4. As shown in the graph, most students were able to accurately extend the pattern to the 5th and 6th term. Seven of the students (including all of the Kindergarten students) were unable to make accurate numeric or geometric predictions about NEAR and FAR terms in the pattern, which suggested difficulty in recognizing the structure of the pattern either visually or numerically. However, eight of the fifteen students made accurate NEAR predictions using either recursive or explicit reasoning, and two students made accurate FAR predictions. We analyzed the thinking of these students to understand their reasoning.

NEXT predictions. Twelve of the fifteen students recognized that the pattern was increasing by two tiles each time, and extended the pattern by adding two more blue tiles, or by counting on from the 8 tiles at position 4. However, none of the Kindergarten and only three of the Grade 1 students could articulate how they knew how to build the 5th position. For example, Jesse (Grade 1) accurately built the 5th position, but stated that he "built a big pattern and all the tiles are blue. Blue, blue, blue, blue, blue, blue, blue, blue, blue, blue and one yellow." Jesse had only been exposed to repeated patterns, which are typically described by articulating the attributes of the pattern that repeat, for example, red, yellow, red, yellow. Although Jesse was accurate in his building, he was unable to quantify the growth of the pattern.

NEAR prediction using recursive reasoning. When predicting the 10th position two Grade 1 and one Grade 2 students employed recursive reasoning to determine the correct number of tiles. These students focused on the numeric structure, the fact that the pattern increased by two blue tiles each time, but when building the 5th term their visual patterns differed from the task pattern. For example, Isabelle (Grade 2) created an incomplete 3x4 array at position 5 using one yellow and ten blue tiles. When asked to predict the number of tiles for the 10th position, she coordinated two internal number lines to skip count by 2s (to represent adding 2 tiles each time) and simultaneously counted up by 1s (to keep track of the position number). She did this without using physical position cards or tiles.

Isabelle: [Pointing to the 5th position card – which had no tiles] 5…ten, 6 would be twelve [still pointing at the 5th position card because there are no other position cards on the table] [shifts her finger to the right of the 5th position card] 7 would be fourteen [shifts her hand to the right again] 8 would be sixteen, um 9 would be eighteen, and 10 would be twenty! Twenty blue and one yellow!

Isabelle incorrectly predicted that the 100th term would have 102 tiles.

NEAR and FAR prediction using explicit reasoning. Two Grade 1 and three Grade 2 students perceived the numeric structure of the pattern and recognized that the value of the tiles was double the value of the position number. They were able to predict the number of tiles at the 10th position using this reasoning, for example "because 10 plus 10 equals 20 blue and then one yellow." Three of these students were unable to make a FAR prediction and we observed that they were unable to accurately build the 5th and 6th position of the pattern (they used the correct number of tiles, but placed them randomly at each of the positions). This suggests that, although they understood the numeric structure of the pattern, these three students did not attend to the visual structure.

Two of these students (one Grade 1 and one Grade 2) built 5th positions that matched the other iterations of the task pattern. These students made accurate FAR predictions. For example, Chloe (Grade 1) explained the relationship she saw between the position card and the number of tiles when predicting positions of the pattern.

Teacher: How did you know there were going to be ten blue tiles at the 5th position?

Chloe: Because the ones on the top [touching the tiles] it's what they equaled. Like 1 plus 1 equals 2 [pointing to position cards and tiles], and 2 plus 2 equals 4, 3 plus 3 equals 6, and 4 plus 4 equals 8. So this one [pointing to the 5th card] would be 10 blue and one yellow. Because you look at the [position] card and that's what it equals.

Teacher: How about at the 100th position?
Chloe: 200 blue and one yellow.
Teacher: What about at the thousandth position?
Chloe: Two thousand blue and one yellow.

Post Interview Results

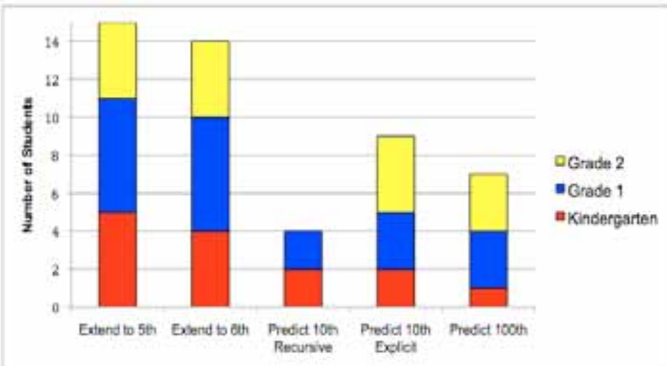

Figure 5: Post Interview Results

Results of the post interviews are shown in Figure 5. As shown in the graph, more students were able to make accurate NEXT, NEAR and FAR predictions during the post interview, and more students used explicit reasoning when making their predictions.

NEXT and NEAR predictions using recursive reasoning. When making NEXT predictions, six of the students demonstrated recursive reasoning that combined an understanding of the numeric structure (increasing by two) with a perception of the visual pattern structure as increasing rows of two tiles. Four of these students continued to use recursive reasoning to make NEAR predictions.

NEXT, NEAR and FAR predictions using recursive *and* explicit reasoning. When making NEXT and NEAR predictions, nine students used both recursive and explicit reasoning based on a shifting perception of the visual structure of the pattern (Figure 6). One perception, the rows perception, demonstrated an understanding that the pattern was increasing by successive increments of 2. This was evident when students justified their predictions either by stating the pattern "goes up by 2 each time" or by skip counting by 2s. Simultaneously these students saw the pattern structure as two columns of tiles with each column composed of a number of tiles equal to the position number. This perception was aligned with explicit reasoning as students justified their predictions by making connections between the position number, and the number of tiles in each column of the pattern. During the interviews, these students went back and forth between these two perceptions.

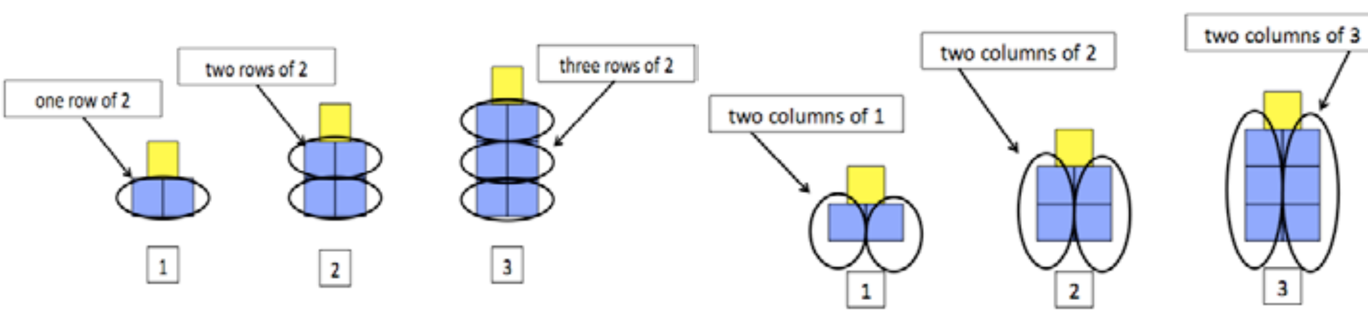

Figure 6: Two Pattern Perceptions

For example, Jesse (Grade 1) demonstrated how he used both the perception of the pattern as growing by one row of 2 tiles at each term, and of the pattern as two columns of tiles (with each column equal to the value of the position number) to make his predictions and justify his reasoning.

Teacher: [Built the first three positions]. How many am I going to need in the fourth position?
Jesse: Eight blues and one yellow! Because I know that for this one [pointing to the third position] you were going to put six because three plus three equals six [pointing to the two columns of three tiles]. And four plus four [pointing to the position card] is eight [pointing to the two columns of 4 tiles].
Teacher: What would you build in position 5?
Jesse: [Mouthing five plus five] ten!
Jesse then built the 5^{th} position.
Teacher: How did you know how many tiles went in the 5^{th} position?
Jesse: [Jessie uses two fingers to count by 2s from the bottom pair of tiles to the top pair.) Because you go 2,4,6,8,….then 10! Going up by twos!
Teacher: How many blue tiles would you need for the 10^{th} position?
Jesse:[Points to position card 10] Ten plus ten is 20!

Six of the students also used the columns perception when making FAR predictions. For example, Ava (Kindergarten) took a column approach when the teacher asked her to predict how many blue tiles there would be at the 100^{th} position and indicated that "there would be 100 up one side and then another row of 100, so two hundred blue and a yellow."

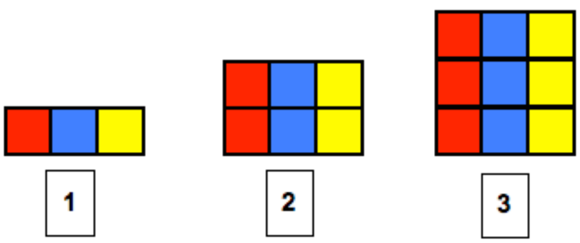

Figure 7: Ava's Pattern

To further probe her thinking, Ava was asked to build a pattern. She constructed a growing pattern with one column of red tiles and a column of blue tiles (Figure 7). The number of tiles in each column was equal to the number on the position card. The teacher then added a third column of yellow tiles and built the pattern to position 3. She asked Ava to predict how many tiles would be needed for the 10^{th} position. Immediately Ava said, "Thirty!"

Teacher: How did you know that?
Ava: Counting by 10s! Ten, twenty, thirty!
Teacher: So if you had four colours, how many tiles would you have at the 10^{th} position?
Ava: [laughs] Forty!
Teacher: What else that can you tell me?
Ava: If there's five colours, then fifty!

Discussion

The results of the pre-interview indicated to us that young students have the potential to work with growing patterns in a meaningful way, and can successfully focus attention on the relationship between two sets of data such as position cards and tiles. The initial analysis of the pre-task results indicated that eight of the fifteen students accurately made NEAR predictions based on their recognition of the numeric and/or visual structure of the pattern. After working through a series of five lessons that specifically focused on identifying the relationship between position number and number of items at the position, and the numeric and physical structure of linear growing patterns, we saw an increase in the number of students who demonstrated an ability to recognize and describe predictable growth, extend a linear growing pattern accurately, and make NEAR and FAR predictions. During the post-interview thirteen of the fifteen students were able to make NEXT and NEAR predictions, and seven were able to make FAR predictions. This indicates that the initial difficulties some students experienced may have stemmed from a lack of experience, and not from a lack of capability. Their earlier instruction has focused almost exclusively on repeating patterns, which influenced their initial work with growing patterns. This was evidenced by such practices as describing the growing pattern as a repetition of one variable (for example, Jesse's description of the fifth position by articulating the word "blue" ten times).

Although this was an exploratory research project, and the teaching intervention was relatively brief, our results suggest that even very young students may be capable of functional thinking – an idea first posited by Blanton and Kaput (2004). There were some critical components of the teaching sequence that may have supported this capability. During the lessons an emphasis was placed on the relationship between the position number and number of tiles (or other elements) at each position. Students were also asked to make predictions from the NEXT term to NEAR and FAR terms in order to focus on the relationship between the two. Another important component was the use of arrays to represent different growing patterns, and the connection between the position number and number of "groups" in the array. Unlike typical pattern instruction for this age level, the instruction emphasized multiplicative rather than additive thinking, which is necessary for developing an understanding of functional relationships.

Of interest was the "perceptual agility" (Lee, 1996) demonstrated by nine of the fifteen students when making their NEXT, NEAR and FAR predictions. Perceptual agility is the ability to perceive a pattern in multiple ways, and utilize one or more perception as the basis for making generalizations. Because we had incorporated the use of arrays into our instructional design, students developed two ways to consider the growing pattern used during the assessment interview. As the teacher built the pattern for the interview task, she emphasized the series of rows of two tiles. However, the students were able to perceive it as either a pattern of increasing rows of two, or as a pattern of two columns of tiles equal to the position number. The "rows" perception seemed to be connected to recursive reasoning, adding a row of two more tiles each time. Students with this perception considered the variation in one data set and identified the recursive relationship within the pattern. The "columns" perception, on the other hand, seemed to allow students to make the leap to explicit reasoning by considering the pattern as two columns with each column composed of a number of tiles equal to the position number. Students identified the relationship between the position number and the number of elements at each position as evidenced by their ability to make NEAR and FAR predictions without having to rely on "adding two more tiles each time." This was demonstrated by Ava's recognition that at the 10^{th} position of the pattern that she built, if she had 5 colours of tiles there would be 50 tiles. Rather than thinking of ten groups of 5 (rows), she was clearly thinking of 5 groups of 10 (columns) and used that perception in her prediction.

Also of interest is the development of an awareness of visual and numeric structure. This was evident as we analyzed the change in students who had trouble making NEAR and FAR predictions during the pre-interview. Some students could extend the pattern in a way that was visually accurate with two columns of blue tiles and one yellow tile on top, but was not accurate numerically. Other students could make numeric NEAR predictions, but did not represent the visual structure of the pattern accurately, which hampered their ability to make FAR predictions (most of these students predicted 102 tiles would be needed for the 100^{th} term). During the post-interviews, all of the students who made accurate NEAR and FAR predictions had developed an

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ability to recognize both the visual and numeric structure of the pattern. This is important because previous research suggests that the ability to recognize both numeric patterns and visual pattern structure are key elements in developing algebraic reasoning (Arcavi, 2003; Berch, 2005; Mulligan & Mitchelmore, 2009; Papic et al., 2011).

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