INVESTIGATING INTEGER ADDITION AND SUBTRACTION: A TASK ANALYSIS

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Fifteen elementary and secondary teacher candidates solved sixteen integer addition and subtraction problems during think-aloud interviews. Investigators further probed participants' solution strategies as well as what they noticed first when starting a new problem. Task analyses of participants' solutions led to the creation of two distinct maps detailing the procedural relations among and conceptual elements underlying the six problem types. The task analysis results indicate that participants relied on procedural knowledge and had weak strategic knowledge with little focus on number values in their four main solution strategies. These findings highlight possible areas for instruction to support conceptual understanding.

Keywords: Number Concepts and Operations, Cognition

Purposes of the Study

Students' transition from working with whole numbers to integers in middle school causes cognitive dissonance. They have to reinterpret the meaning of the minus sign (Bofferding, 2010; Vlassis, 2004) and the meanings of addition and subtraction (Bofferding, 2010; Bruno and Martinon, 1999), both of which play a role in how they solve integer problems (Bofferding, 2012). Researchers have explored the relative difficulty of integer problems based on percent correct data and detailed some of students' difficulties in solving the problems (e.g., Murray, 1985); however, their descriptions of students' solution methods are inconsistent, incomplete, focus on a few problem types, and/or pertain to specific contexts. Moreover, the existing studies provide little explanation for how students correctly solved problems or whether their solutions were mathematically rich. Lack of detail regarding students' solutions is troublesome because students can get correct answers to integer problems using incorrect methods (Tatsuoka, 1983).

Developing a classification scheme of integer problems and typical solution paths that students take to solve these problems could unite research efforts in this area. Certain solution paths may draw on procedural thinking while others tap into more complex types of knowledge (e.g., schematic or strategic); such information has important instructional implications. Further, a detailed framework could help teachers develop models of their students' thinking processes and better equip them to adjust their instruction to build on students' knowledge, as was found with Cognitively Guided Instruction (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989).

Research Questions

This paper begins to address these issues by exploring the processes and knowledge involved in solving integer arithmetic problems. Specifically, we explore the following questions:

1) What are the processes through which teacher candidates (who may later teach this topic) solve integer addition and subtraction problems?
2) What knowledge (declarative, procedural, schematic, strategic) do students use to solve the integer problems, and how does this relate to the thinking processes students use?

Theoretical Framework

Negative Number Instruction

One area of insight into how students might solve integer problems comes from the instruction they receive. Two typical methods for teaching addition and subtraction with negative numbers are movements on the number line and cancellation. Common rules for moving on the number line include the following: when you add, move forward; when you subtract, move backward; if you are adding or subtracting a positive number, face the positive numbers; if you are adding or subtracting a negative number, face the negative numbers (Liebeck, 1990). Given these rules, a student would likely start at whatever number is first in the problem, pretend to face the direction of the second number's sign, and then move up (forward) or down (backward) on the number line or counting sequence.

A common way to think about cancellation is through the use of chips, where 1 positive chip cancels out 1 negative chip. People who use this method might focus on the amounts of the two numbers in the problems. Further, they will need to have rules for dealing with situations when they have to take away more positives or negatives than are originally given (e.g., -3 - -5) (Liebeck, 1990). Other forms of cancellation instruction use different contexts for exploring the canceling operations, such as colored beads on a double abacus (Linchevski & Williams, 1999), cutting and gluing input and output trains (Schwarz, Kohn, & Resnick, 1993), and adding and removing balloons and weights to a hot air balloon (Janvier, 1985); however, the rules remain the same.

Results from instructional and interview studies provide a sample of strategies students use to solve integer problems. For example, to solve -4 + -3, a student might add 4+3=7 and then add a negative sign to get -7 (Bofferding, 2010). Students might also count down to solve problems like 3-5 (Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011). Tatsuoka (1983) created a system to classify students' response patterns (particularly in terms of their erroneous rules) to a series of integer problems based on possible rules students might use to manipulate the absolute values and signs of the numbers. Although they might not directly match students' thinking, results such as these can be used as a starting place for developing a map of how students solve integer problems.

Task Analyses

Another way researchers have attempted to characterize students' solution processes for solving mathematics problems is through the use of task analysis, "a decomposition of a complex task into a set of constituent subtasks" (Gardner, 1985, p. 157). Recent task analyses – such as the work of Piaget and others – focus on analyzing students' mental processes and conceptions as they engage in tasks (see Resnick, 1976, for a more detailed description of the history). Thus, task analysis is a useful tool in cognitive research (Siegler, 2003) for mapping out the procedures students use and highlighting the conceptual knowledge needed to solve problems (Greeno, 1976).

There are several reasons why task analyses are necessary (especially if modeled after students' solution processes). First, they identify conceptual prerequisites or structures that underlie instructional outcomes (Anderson & Schunn, 2000; Greeno, 1976). For example, Griffin, Case, and Capodilupo (1995) identified that knowledge of number order (such as on a number line) serves as an essential support for other number concepts; they found that when given explicit instruction on the number line, low-performing students caught up to their high-performing peers in mathematics. Task analyses can also be used to determine where students are likely to deviate from correct procedures, which can help teachers target instruction (Anderson & Schunn, 2000). Additionally, from a series of related arithmetic problems, researchers can use task analysis to determine which problems elicit more potential methods and which strategies are most efficient. This information can help instructors identify problems that might engender better classroom discussions about strategies.

Mathematical Proficiency and Knowledge Types

Not only are the processes through which people solve mathematics problems important but so are the types of knowledge they use in relation to the strands of mathematical proficiency, which is our goal for students in mathematics classes (National Research Council, 2001). The first strand, conceptual understanding, relates to understanding why a procedure works and can help students avoid errors. Further, students with conceptual understanding "understand why a mathematical idea is important and the kinds of contexts in which it is useful" (National Research Council, p. 118). Students with procedural fluency, the second strand, know when and how to use procedures to solve a wide range of problems efficiently and accurately; students who use procedures without conceptual understanding are likely to make errors and lack procedural fluency (National Research Council). Strategic competence, the third strand, involves understanding problems and knowing which strategy is best to use. Further, students with strategic competence understand how problems are related. Adaptive reasoning is the fourth strand, which refers to students' ability to justify their answers, and the final strand, productive disposition, includes "see[ing] sense in mathematics" (National Research Council, p. 131).

The four types of knowledge underlying these interwoven strands require different amounts of cognitive demand. Declarative knowledge is knowledge of facts, which are memorized and recalled. Procedural knowledge requires more cognitive effort and involves knowing the steps for solving a problem. On the other hand, schematic knowledge involves knowing why certain processes or strategies work. Finally, strategic knowledge, or knowing about knowing, includes monitoring one's own knowledge and knowing when a particular strategy is the optimal one to use (Li, Ruiz-Primo, & Shavelson, 2006).

Methods

Participants and Site

Participants in the study included fifteen teacher candidates (8 secondary-focused, 7 elementary-focused) taking courses in a teacher education program at a Midwestern state university. All had taken some mathematics content courses at the undergraduate level. Two of the secondary student participants were male, while the remaining thirteen people were female.

Materials and Data Collection

In order to access the teacher candidates' solution methods as they solved the integer problems, we conducted individual think aloud interviews. Participants were asked to "say everything that you might say to yourself silently or think in your head" as they solved 16 naked number addition and subtraction problems involving integers (see Table 1). Although the use of think-alouds slows processing time when solving problems, they do not change their thinking (Ericsson & Simon, 1993; Ericsson, 2006). Participants started with a practice question to get used to talking out loud while solving the problems. One or two researchers conducted each interview and took notes about participants' solutions. Follow-up questions asked after they solved all of these problems included clarifying questions about their strategies as well as the following: "Which questions were the easiest? Why?" and "When you looked at a new problem, what is the first part you noticed? Which part is most important? Why?" Their work for each problem was collected, and interviews were audio and videotaped, then transcribed.

Data Analysis

To analyze the data, we first developed an initial task analysis of the steps we thought were needed to solve the problems based on accounts of students' strategies in the research and the common ways of teaching integer addition and subtraction. The task analysis was adjusted throughout the coding process, based on participant responses. Rather than making separate task analysis maps for each procedure students might use, we combined them into a nondeterministic map; therefore, a student could take one of many paths through the map (Greeno, 1976).

During the coding process, we found that determining how people were "starting" the problem, or what they noticed first, was too open to interpretation to be used as the first step in the task analysis. Therefore, we reorganized the analysis based on identifying the problem type as the first step. As we read the transcripts of each participant's solution steps, we coded each step in their processes based on the final task analysis maps. We double-coded about 20% of the naked number problems with >85% agreement, then discussed differences, and coded another 14 problems with >90% agreement.

Table 1. Integer Problems Solved and Organized by General Problem Type

Addition: Two Negatives	Addition: Negative, Positive	Subtraction: Negative, Positive	Subtraction: Two Negatives	Subtraction: Two Positives
-7+-1	-2+7, -4+6	-3-5, -5-9	-4 - -7, -6 - -9	3-9, 6-8
	-9+2	4 - -5	-8 - -5	
	5+-2	5 - -3, 9 - -2	-8 - -8	

Results

Task Analysis

The final task analysis consists of two maps, showing the six general problem types (those shown in Table 1, as well as Addition: Two Positives), the relations among them, and branches showing the sequential steps participants made when solving problems of these types. Map 1 consists of addition of two positive numbers, subtraction of a positive and negative number, and addition of two negative numbers. These problems are related because problems of one type can be changed into one of the other types using sign rules (e.g., 5+3 is equivalent to 5 - -3, and -2-7 is equivalent to -2+-7). Map 2 consists of subtraction of two positive numbers, addition of a positive and negative number, and subtraction of two negatives. The problems are also related (e.g., 3-9 is equivalent to 3+-9, which is equivalent to -9 - -3). If a person solved -2+7 by identifying the first number as negative, switching the order of the numbers (7+-2), and changing the operation using sign rules (7-2), the map would show the following connections: Addition, Negative and Positive → 1st Number Negative → Commute → Change to Subtracting a Positive → Subtraction, Two Positives → Recall.

Solution Processes by Problem Type

Based on coding from the task analysis, certain problems stand out as related based on the similar ways in which the participants solved the problems. Across participants, the most consistently solved problem type was subtracting a negative number from a positive one (4 - -5, 5 - -2, and 9 - -2). For all three of these problems (except in 1 instance), they identified the second number as negative, changed the problem into adding a positive, and recalled the answer.

Results for subtracting a positive number from a negative one were less consistent. J02 nicely explained how she solved -3-5: "I'll make the five a negative, instead, and so then all I have to do is add the actual numbers together and then just make sure that the sign after that is negative." Most participants used this reasoning – solving the problem as they would if adding two negatives – for both -3-5 and -5-9. However, two people for -3-5 and five people for -5-9 focused on number order, starting at the negative number and counting down as they would for a normal subtraction problem. One of them thought of this as a hole that was getting deeper.

Subtracting a larger number from a smaller one (3-9 and 6-8) also had fairly consistent responses from participants. The most popular response was to indicate that the larger number was second, subtract the smaller from the larger, and make the answer negative. On 6-8, A05 used zero as a landmark number: “I'll go back 6, then I'll be at 0. Then I'll have 2 more to go, so that's -2.” One person for 3-9 and two for 6-8 changed the problems into adding a negative (3+-9 and 6+-8) then either recalled the answer, started at the negative number and counted toward zero, or canceled equal numbers of positives and negatives to get the resulting negative answers.

Although -2+7, -4+6, -9+2, and 5+-2 appear similar because they involve adding a positive and negative number, participants treated -9+2 very differently. For the other three problems, participants typically changed the problems into subtraction by using the commutative property to switch the order if needed (7+-2 and 6+-4, although only a few verbalized this step), changing the problem into subtracting a positive (7-2, 6-4, 5-2), and recalling the answer. Similarly, when solving -4 - -7 and -6 - -9, participants first changed these into adding positive numbers (-4+7 and -6+9) and then followed the same steps.

Changing -9+2 into a subtraction problem was rare because the number with larger absolute value was negative, and in general, the participants did not choose to subtract larger numbers from smaller ones. Instead, on this problem, they either started at -9 and counted up 2 or subtracted 2 from 9 and used the sign of the number with the larger absolute value, reporting the answer -7. Likewise, when solving -8 - -5, students first changed the problem into adding a positive number (-8+5) and followed the same steps as -9+2.

These results present a much more complicated picture of the interrelationships among the problems (see Figure 1) than shown in Table 1. Solid lines indicate problems that are part of the same category from Table 1. Solid arrows indicate problem types that can be changed into each other, and dotted arrows indicate problem types that can be solved using similar processes.

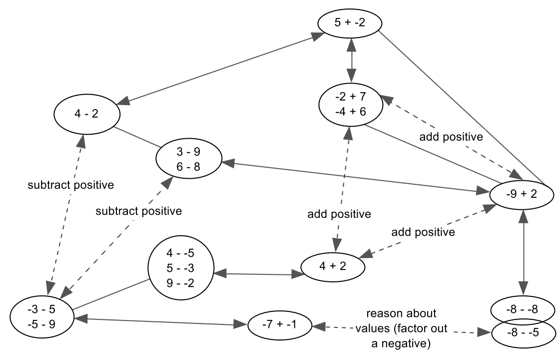

Figure 1: Relations Among Problems

Participants’ Solution Process Consistency

Although solutions for the problems were fairly consistent, participants approached the set of problems in four main ways. Five of the participants (A01, A03, A06, O01, R02) solved problems in a way that suggests they preferred to solve problems with *all positive or all negative numbers*, unless it made the problem more complicated. For example, when solving -2+7, they changed the problem into 7-2. In some cases they did not explicitly change -9+2 because they knew the answer or because they knew they could solve 9-2 and add a negative sign. For -3-5 and -5-9 they solved them as addition (-3+-5), adding the absolute value of the numbers and making the answer negative. Also, although problems like -4 - -7 contain two negative numbers, participants still changed them to the form 7-4.

A second group of participants (A02, A05, A04, K01, K02, R01) solved or changed problems so that they could *add or subtract positive numbers*. Therefore, although all of them changed 5+-2 into 5-2, none of them changed -9+2. Further, none of them changed -3-5 or -5-9; instead, they talked about starting at the negative number and going back the positive number value. The two participants who talked about solving -3-5 and -5-9 by adding the absolute value of numbers and making the answer negative also changed -7+-1 into -7-1 before solving it.

Participant K03 had a different approach than the others; she changed problems so that she could always *add positive and/or negative numbers* because she mainly used a cancellation strategy. She kept -2+7, -4+6, -9+2, and 5+-2 as is and reasoned about the relative number of positives versus negatives to determine what would be left over if each positive cancelled out a negative. Further, she changed 3-9 and 6-8 into 3+-9 and 6+-8 and changed problems like -4 - -7 into -4+7 using sign rules in order to use the same cancellation strategy. For the other problems, she either added two positive numbers (e.g., 4 - -5 became 4+5) or added two negative numbers (e.g., -3-5 became -3+-5, and -7+-1 stayed the same).

Unlike participant K03 and the others, participant J02 primarily focused on starting with the number of larger absolute value. When the number of smaller absolute value was first, she changed the format of the problem. For example, on 6-8, she changed it to 6+-8 and then further changed it to -8+6. However, when solving -2+7, she flipped the numbers to 7+-2 and also changed it to 7-2. This suggests she preferred adding positive numbers, which is confirmed by her statement, “I think adding is easier than subtracting, trying to subtract negative numbers.”

Finally, O02 and J01 did not use a consistent way to solving the problems. For example, they both solved -4+6 by starting at -4, adding the opposite (4) to get to 0, and determining the answer was 2. However, on a similar problem, -2+7, they changed it into 7-2.

Knowledge Types

Overall, participants drew heavily on procedural knowledge when solving the problems; however, they sometimes lacked procedural fluency due to insufficient schematic knowledge. For example, several participants changed problems such as 5 - -3 into addition problems by drawing vertical lines through the minus and negative signs (5 + + 3). When asked how they knew they could do this, they referred to declarative rules rather than conceptual explanations: “When you have two minus signs you can put them together to make a plus sign” (A02) or “You can slash it but then you dash it…cause the minus minus makes a plus” (A06). When asked, “That’s just a rule that you know?” most agreed and restated the rule or indicated that they were told this rule as a child, and one replied, “The theory behind that, I don’t [know].” The one participant who was more successful at explaining how subtracting a negative was similar to adding a positive drew on a description of negative amounts as holes underground, explaining, “Subtracting a negative number, so that's like subtracting part of a hole, which is like adding” (A04).

In limited cases, participants’ lack of schematic or conceptual knowledge led them to change the problems incorrectly (solving -9+2 as 9-2) or use sign rules incorrectly. For example, one participant changed -2+7 into 7-2=5 and answered negative five, claiming, “Because I’m taking away from a negative number,” suggesting she may have thought 7 was negative originally or that because -2 was the first number in the original problem, the answer should be negative.

One of the most striking patterns in participants’ responses was their lack of focus on the values of the numbers, resulting in inefficient methods and suggesting weak strategic knowledge and possibly a lack of conceptual understanding of integers. For -8 - -8, only one person stated that “When you subtract a number from itself you get zero” (A02). All others changed the problem to -8 + 8, and rather than use the additive inverse property, two people further changed the problem to 8-8 before answering 0. Further, although participants solved -7+-1 by solving 7+1 and adding a negative, they never used the same reasoning to solve -8 - -5 as -(8-5).

Discussion and Implications

The results of the task analysis and participant responses illuminate ideas that teachers need to keep in mind when teaching about integers and suggest concepts that students need to know as they learn integers. A prominent way that the participants solved the problems was to change the problems into a different form. Some of them used the phrase "slash and dash" to remember when and how to change the problems, but they had difficulty explaining the meaning behind it. The National Research Council (2001) warns against this type of operating:

> Mnemonic techniques learned by rote may provide connections among ideas that make it easier to perform mathematical operations, but they also may not lead to understanding. These are not the kinds of connects that best promote the acquisition of mathematical proficiency (p. 119).

Rather, if we want students to develop conceptual understanding of integer operations, teachers need to know how to help students meaningfully explain the relation between the operations.

One potential way teachers can help students make these connections is through the use of contexts or the number line. Stephan and Akyuz (2012) successfully used the idea of net worth to help students make sense of losing debt as gaining overall net worth. Further, participants in the study were able to use movements on the number line to add or subtract positive numbers from negative ones. However, they did not use it to add or subtract negative numbers, suggesting that number line instruction needs to address these situations more effectively. By using the number line, the holes context, and cancellation reasoning, some participants broke up numbers in strategic ways, making zero and then adding or subtracting the rest. This method promotes numerical reasoning and works for many problems – especially when changed into the form of adding a positive and negative number – and highlights the importance of zero.

There were a few concepts in particular that participants used frequently and that would likely be important for students to know as they begin to work with integers: relative number values (including number order) versus absolute values and the commutative property. When solving problems, such as -9+2, some participants decided whether the answer would be closer to or farther from zero based on the relative values of the two numbers. On the other hand, when adding a positive and negative number, participants often compared the absolute values of the two numbers to determine if there was enough of the positive number to cancel out the negative number or to move from the negative number through zero. Knowledge of opposites also helped participants solve -8 - -8 as -8+8; however, a stronger focus on number values could have helped them solve the problem more efficiently.

Participants used the commutative property when they changed problems like -4+6 into 6-4; however, they rarely explained this intermediary step (6+-4). In fact, they never explained why they could solve -9+2 as -(9-2) either. When discussing these processes with students, teachers would need to be more explicit about the reasoning behind these steps or students might develop misconceptions about the operations and sign, such as believing -9+2 could be solved as 9-2.

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