ANALYSIS OF STUDENTS' PROPORTIONAL REASONING STRATEGIES

Michele Carney
Boise State University
michelecarney@boisestate.edu

Ev Smith
University of Illinois – Chicago
evsphd@gmail.com

Gwyneth Hughes
Boise State University
gwynethhughes@boisestate.edu

Jonathan Brendefur
Boise State University
jbrendef@boisestate.edu

Angela Crawford
Boise State University
angelacrawford1@u.boisestate.edu

Tatia Totorica
Boise State University
tatiatotorica@u.boisestate.edu

Proportional reasoning is key to students' acquisition and application of complex mathematics and science topics. Research is needed regarding how students' progress towards and come to demonstrate key developmental understandings within proportional reasoning. To this end we created and administered assessment items to 297 middle grades students. We categorized student solution processes qualitatively, followed by Rasch analysis to examine item difficulty and strategy use in relation to an anticipated trajectory. Our findings indicate that different strategies manifest themselves in a hierarchical manner, providing initial confirmation of categories based on strategy efficiency and emphasizing the importance of teacher (and researcher) analysis of classroom assessments from a student cognition perspective.

Keywords: Learning Trajectories; Measurement; Number Concepts and Operations

Purpose

Proportional reasoning is a lynchpin for future success in mathematics and science (Lesh, Post, & Behr, 1988). Based on a substantial body of proportional reasoning research (e.g., Lamon, 2005; Lobato, Ellis, & Charles, 2010; Tourniaire & Pulos, 1985), there have been several calls for shifting instruction from the typical focus on the cross-multiplication algorithm to students' meaningful understanding and application of ratio related concepts (e.g., National Governors Association & Chief Council of State School Officers, 2011). However, implementing this shift in instruction is difficult. Schools and teachers need resources to support this change and more information is needed on how students' proportional reasoning develops from less efficient to more efficient strategies.

The overall purpose of our research is to develop measures to assess students' flexibility and efficiency in proportional reasoning situations. Our work revolves around: (a) measuring and identifying qualitatively different categories or aspects of student reasoning, and (b) determining whether these categories manifest themselves along a hierarchical progression. The qualitative and quantitative confirmation of different categories or aspects of students' proportional reasoning along a continuum would contribute to a better awareness of how students' progress towards important understandings and assist in designing classroom instruction, curriculum, assessment, and teacher professional development.

More specifically, the present research uses Simon's (2006) KDUs as a theoretical framework for examining student work samples to identify qualitatively different categories or aspects of proportional reasoning in simple missing value contexts. From there we use the structure of Steinthorsdottir's (2009) hypothetical trajectory to create assessment items which measure students' proportional reasoning and enable us to analyze and categorize the resulting student thinking. Lastly, we use Rasch modeling to determine whether our identified categories for student thinking manifest themselves hierarchically, indicating the potential usefulness of the assessment items and qualitative rubric for teachers and researchers in their analysis of students' thinking.

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Theoretical Framework

Simon's (2006) articulation of key developmental understandings (KDUs) provides a framework for analyzing students' flexibility and efficiency in proportional reasoning situations. Simon (2006) describes key developmental understandings by stating,

> …I am not claiming that these understandings exist in the student; rather, specifying understandings is a way that observers (researchers, teachers) can impose a coherent and potentially useful organization on their experience of students' actions (including verbalizations) and make distinctions among students' abilities to engage with particular mathematics (p. 360).

We see KDUs as a potential framework for identifying important categories of students' reasoning when analyzing work samples. The following section articulates a KDU important to students' initial development of proportional reasoning.

Research indicates that students' demonstration of flexible and efficient use of the scalar and functional perspectives in proportional reasoning situations may be a KDU (Lobato, Ellis, and Charles 2010; Lamon 2005). A scalar perspective entails recognizing a ratio as a composed unit that can be scaled up or down by multiplying each quantity in the ratio by a constant factor. For example, given the problem "Callie bought 7 cookies for $3. How many cookies can Callie buy for $12?" a student recognizes the original 7 cookies to $3 ratio can be scaled up by multiplying each quantity in the ratio by 4 to generate the 28 cookies for $12 ratio (see figure 1a). A functional perspective entails recognizing and using the constant multiplicative relationship between the two quantities within the ratio and applying this relationship to create equivalent ratios. For example, given the similar context "Callie bought 6 cookies for $2. How many cookies can Callie buy for $13?" a student recognizes the number of cookies to be purchased is three (6 ÷ 2) times the number of dollars paid. This understanding allows the student to quickly realize Callie can purchase 3 x 13 or 39 cookies (see figure 1b).

Figure 1a: Callie bought 7 cookies for $3. How many cookies can Callie buy for $12?

Scalar Perspective

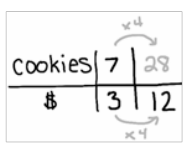

Figure 1b: Callie bought 6 cookies for $2. How many cookies can Callie buy for $13?

Functional Perspective

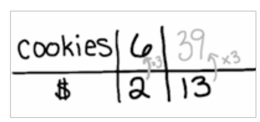

Figure 1. Scalar and functional assessment items and solution perspectives.

In simple missing value problems, students demonstrate attainment of this initial proportional reasoning KDU by flexibly and efficiently demonstrating knowledge of either the scalar or functional strategies based on the situation or number relationship presented (Steinthorsdottir & Sriraman, 2009). For example, given the situation illustrated in Figure 1a, an efficient and flexible strategy is to scale up by a factor of four. A student who applies the functional multiplier of 2.33 is likely performing a standard procedure without reasoning through the proportional relationships, indicating a possible lack of flexibility in their proportional reasoning.

In addition to examining a students' work for application of the scalar or functional perspective, one must also examine scalar situations for the level of efficiency used in the scaling process (Authors, 2015). For example, in missing value situations with an integral scalar relationship strategies can often be differentiated as additive or multiplicative (see figure 2).

Additive strategies may indicate initial understanding of the scalar relationship but a multiplicative understanding is needed for eventual generalization of the scalar perspective to non-integral

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relationships. Therefore, student work must also be examined for use of an additive versus multiplicative approach in addition to the scalar and functional perspectives.

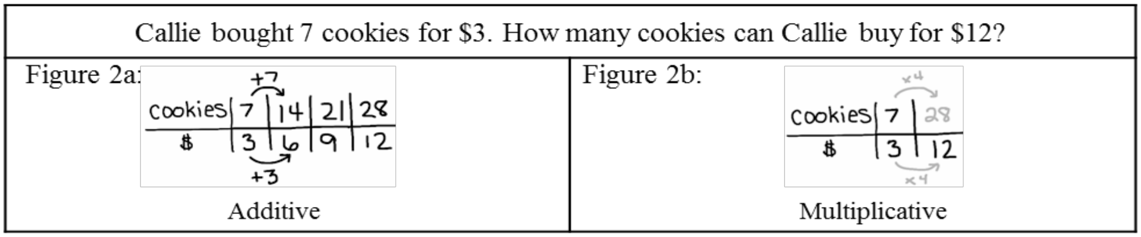


Figure 2. Additive and multiplicative solution strategies for a scalar problem.

In sum, we observe students' flexible and efficient application of the scalar or functional perspective in simple missing value situations as an initial KDU in a proportional reasoning learning trajectory. In students' application of the scalar perspective, student solution strategies must be examined for an additive versus multiplicative approach to ensure students are able to eventually generalize their strategy to non-integral situations. The next section details a potential developmental trajectory for these ideas, followed by a description of the assessment items designed to capture students' understanding of these concepts.

Steinthorsdottir and Sriraman (2009) articulated a potential progression for students' proportional reasoning. They identified four levels of increasingly sophisticated strategies students used to solve missing value problems. In level one, students incorrectly focus on the difference in quantities either within or between the ratios. In level two, students focus on either additively iterating or multiplicatively scaling the given ratio as a composed unit to reach the missing value in the equivalent ratio (scale-up). Level three involves scaling down a given ratio, and includes two sub-levels, the ability to partition the given ratio as a composed unit to reach the missing value in the equivalent ratio (scale-down) and the ability to combine iteration and partitioning to reach the missing value (scale up and down). Level four involves the flexible use of either the scalar or functional relationship depending upon the ease of calculation with the numbers in the problem.

Based in part on the progression outlined by Steinthorsdottir and Sriraman (2009) and the identified KDUs for scalar and functional perspectives, we developed an assessment with a focus on manipulation of number relationships to examine students' types of reasoning and the level of efficiency in their solution process. For the sake of brevity we focus on presenting three exemplar problems from the assessment and their solution strategy analysis (Table 1).

Table 1: Assessment framework and anticipated solution strategy analysis

Item Construction		Solution Strategy Analysis	
Type of Reasoning	*Integral Number Relationship*	*Less Efficient*	*More Efficient*
1. Scalar	Scale Up	Additive	Multiplicative
2. Scalar	Scale Down	Additive	Multiplicative
3. Functional	Functional	Scale Up and Down	Functional

Methods

Research Questions

1. Can we identify qualitatively different categories of student thinking related to use of an additive or multiplicative solution strategy and/or fluent and flexible use of the scalar or functional relationship depending on the number relationship presented?

2. Do the items manifest themselves as anticipated in relation to a progression from easiest-scale up, moderate-scale down, to functional-hardest?
3. Do the strategy labels and associated scoring codes based on strategy efficiency manifest themselves as anticipated along a continuum? In other words, are less efficient strategies used by less able students and more efficient strategies associated with more able students?

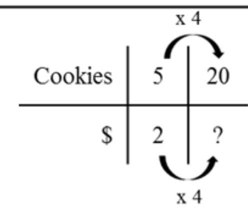
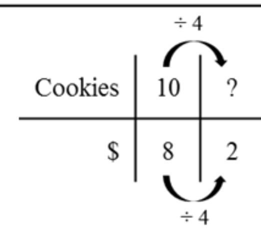
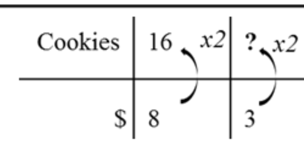
Participants

The respondents represent a convenience sample of 297 students from fourth to ninth grade with the majority of the students coming from 6th and 7th grade (n=198).

Measure

As described in Table 1, we constructed a measure based on the Steinthorsdottir progression. We focus on presenting data from three of the assessment items: scale up, scale down, and functional. Table 1 provides the 3 items and the anticipated 'efficient' strategy based on the number relationship.

Table 2: Assessment items included in analysis.

Strategy Focus	Context	Anticipated Strategy
Scale Up	Callie bought 5 cookies for $2. How much will it cost to buy 20 cookies?	
Scale Down	Thomas found a cookie deal with 10 large cookies for $8. How many cookies can he buy for $2?	
Functional	Jason found a cookie deal with 16 cookies for $8. How many cookies can he buy for $3?	

Timeline and Setting

In order to focus on initial cognitive understanding rather than procedural knowledge, assessment items were administered in the fall prior to formal proportional reasoning instruction. Older students in our sample should have received instruction around proportional reasoning and we would expect more efficient strategies from these students. However, contact with teachers in our study indicated instruction was based primarily on implementation of cross-multiplication, with little or no emphasis on scalar or functional perspectives.

Results

Qualitative analysis of the outcome space

We analyzed the student strategies for each assessment item. We coded each for an additive or multiplicative solution strategy and/or fluent and flexible use of the scalar or functional relationship. Table 3 provides an overview of the coding framework with example student work for each efficiency level. In addition to the qualitative coding of strategy name, description and example, we

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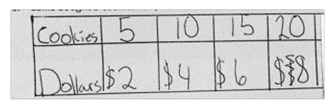
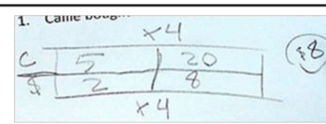
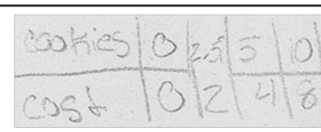
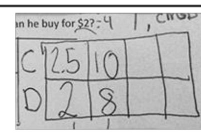
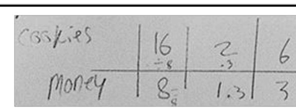
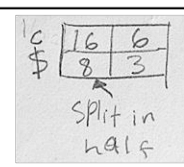
identified the following scoring categories with respect to strategy efficiency: 0 = incorrect, 1 = correct but inefficient strategy, and 2 = correct and efficient strategy.

Rasch Analysis

We selected Rasch modeling for our quantitative analysis due to its usefulness in (a) identifying the difficulty level of an item in relation to other items, and (b) evaluating the strategy thresholds of our efficiency-based scoring model (Van Wyke & Andrich, 2006). Assessments created to fit the Rasch model consists of items designed to assess a single (unidimensional) construct. Rasch analysis situates test takers' understanding (person ability) and item difficulty along a common equal interval scale, often with a score range between -4 to 4 with 0 as the mean. Therefore, person ability and item difficulty scores can be interpreted in relation to one another through probabilistic language. In situations involving dichotomous scoring (0=incorrect, 1=correct), when person ability and item difficulty are the same, this indicates a 50% probability that the individual would respond correctly (or incorrectly). When a person ability score is higher (e.g., 1) than the item difficulty (e.g., -1) the person is more likely to solve the problem correctly and vice versa. Figure 3 provides an example of a Rasch

Table 3: Strategy coding framework for assessment items with example student work.

Number Relationship	Number Relationships	Rubric Points	Strategy **Name** and Description	Strategy Example
Scale Up	Cookies: 5, 20 $: 2, ?	1	**Scale Up Additive:** Iterate initial ratio additively or double and continue to double resulting ratios until desired number of given component is reached	
		2	**Scale Up Multiplicative:** Determine and use scale factor that scales initial ratio to desired number of given component in one step	
Scale Down	Cookies: 10, ? $: 8, 2	1	**Scale Down Additive:** Halve initial ratio and continue to halve resulting ratios until desired number of given component is reached	
		2	**Scale Down Multiplicative:** Determine scale factor that scales initial ratio down to desired number of given component	
Functional	Cookies: 16, ? $: 8, 3	1	**Scale Up and Down:** Use a combination of scaling up/down to reach desired number of given component	
		2	**Functional:** Determine constant of proportionality and apply to the number of given component	

item scale map based on our analysis. Typically Rasch item maps display person ability on the left side of the scale and item ability on the right. However, for ease of interpretation, we focus on item difficulty and are not displaying person ability.

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Analysis 1

In the context of our present work, we first used Rasch analysis to examine item difficulty in relation to the number relationships being manipulated across problems. Our initial analysis examined whether the item difficulties manifested themselves as anticipated in relation to our hypothetical progression; easiest-scale up, moderate-scale down, and functional-hardest?

Figure 3a presents the results of Rasch analysis using the Winsteps program with dichotomous item scoring (0=incorrect, 1=correct). The difficulty scores in the box represent the item difficulty (and standard error) and determine an item's placement on the scale. For example in figure 3a, for the scale up item, -2.99 is the point on the continuum where students with an estimated ability below -2.99 are more likely to get the problem incorrect. Students with an estimated ability score above -2.99 are more likely to get the problem correct. If a student had an estimated ability score of 0, it is likely they would correctly solve the functional and scale up item but incorrectly solve the scale down item.

Findings from Analysis 1

The item order from easiest to most difficult was (1) scale up, (2) functional, and (3) scale down. Our empirical data indicated more students were likely to correctly solve the functional item than the scale down item. This was different than we anticipated. However, we recognized potential issues with this analysis. First, there was the issue with the scale down item not resulting in an integer answer (i.e., 2.5 cookies) and it is highly possible this non-integer result influenced the level of item difficulty.

However, perhaps more importantly, there was also an issue with examining the data from a dichotomous or correct/incorrect perspective instead of investigating students' strategy approach given our qualitative scoring rubric. We knew students used different strategies to correctly solve an item. Our dichotomous scoring model did not allow us to take student strategies into account, nor did it allow us to examine whether students had selected a flexible and efficient approach based on the number relationships presented in the problem. For example, on the scale up problem we wanted to determine whether students who were more likely to use an additive strategy (less efficient) versus a multiplicative strategy (more efficient) differed in ability score.

Analysis 2

Analysis 2 evaluated the qualitative rubric and associated scoring model of less and more efficient strategies. Our specific question was, did our strategy labels and associated scoring codes related to strategy efficiency manifest themselves as anticipated along the interval scale? In other words, are less efficient strategies associated with less difficult threshold scores and more efficient strategies associated with more difficult threshold scores. Thresholds are the point on the continuum where adjacent categories (or strategies) are equally probable. For example, the transition point between correct but inefficient (1) and correct and efficient (2) for the scale- up problem is .48 (see table 4). A student with an ability score of 1 would be more likely to use an efficient strategy, while a student with an ability score of 0 would be more likely to use a correct but inefficient strategy. To conduct this analysis we used a partial credit Rasch model in the Winsteps program. The strategy categories were scored as more efficient (2 pts), less efficient (1 pt), and incorrect (0 pts).

Findings for Analysis 2

Table 4 provides the Rasch item statistics for the three items and the associated category thresholds (other assessment items administered on the test are not included for ease of interpretation). The fit statistics indicate the items are 'fitting' the model. Figure 3b presents findings from analysis 1 and 2 in conjunction with each other. For example, the less efficient scale up additive strategy for the scale up item had a difficulty threshold of -3.85, and the more efficient scale up strategy for the same item had a difficultly threshold of 0.48. This indicates that a student with an ability level below -3.85 would likely get the item incorrect. Those students with ability levels between -3.85 and .48 would likely get the item correct, but use an additive strategy. Those students with an ability level above .48 would be more likely to use a multiplicative strategy.

Table 4: Rasch item statistics for proportional reasoning items

	Strategy Category Threshold	Threshold Difficulty	SE	Infit MNSQ	Outfit MNSQ
Scale Up	0-1	-3.85	0.22	1.01	1.00
	1-2	0.48	0.21	1.06	1.12
Scale Down	0-1	-0.38	0.18	0.89	0.80
	1-2	1.54	0.37	1.20	1.07
Functional	0-1	-0.97	0.18	0.98	0.92
	1-2	1.57	0.37	1.07	1.00

The threshold levels for additive versus multiplicative strategies also held true for the scale down item. While still preliminary, these findings support analysis of students' use of additive or multiplicative strategies in classroom and assessment practices to determine their depth of understanding of the scalar perspectives. This will support later work on related topics, such as geometric scaling where students must be flexible and efficient in applying a scalar multiplicative strategy. In addition, threshold levels for the functional item strategy categories supported the progression articulated by Steinthorsdottir and Sriraman (2009).

In conclusion, we recognize a continuum of ordered strategies related to students' ability is not equivalent to a progression of how students' develop understanding of a KDU. However, Rasch analysis has the potential to support our qualitative findings in ways that would assist in identification of hierarchical relationships between strategy approaches. This can provide important information to inform future research. In addition, the fact that the observed strategy thresholds match the scoring rubric indicate the usefulness of the qualitative rubric in analyzing student work through an efficiency perspective.

Lastly, while not a focus on this investigation, comparison of the two scoring models highlights the importance of analyzing students' strategies in the evaluation of students' understanding of a topic. We cannot assume students have demonstrated knowledge of a key development understanding simply because they provide the correct answer. A student could (and did) correctly solve all three assessment items using a scalar additive or scale up and down strategy. However, these strategies do not demonstrate understanding of the scalar and functional perspectives and will not continue to work as number relationships increase in difficulty.

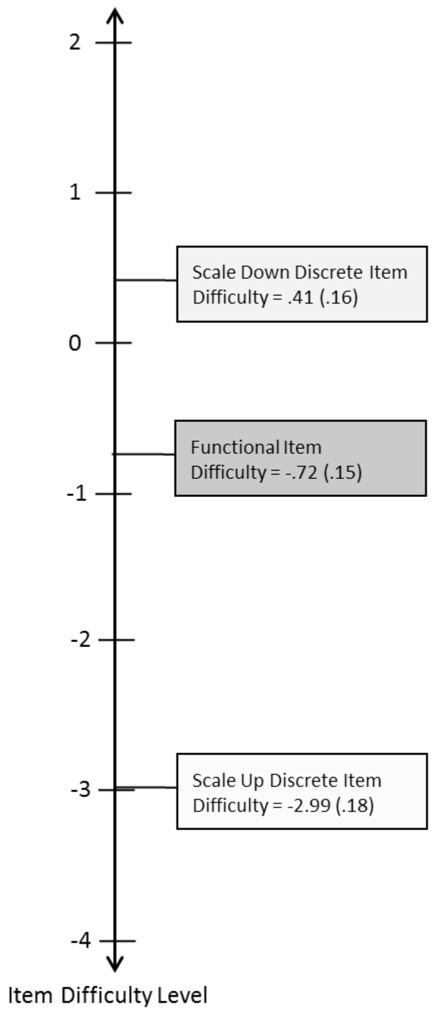

Figure 3a: Analysis 1 with dichotomous scoring

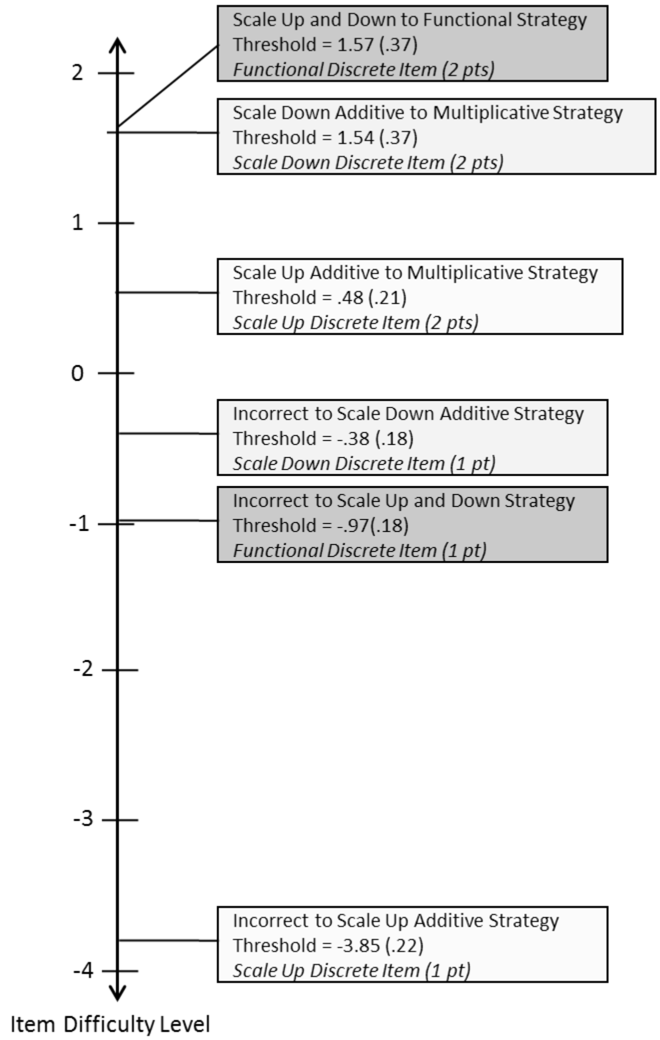

Figure 3b: Analysis 2 with partial credit scoring by strategy

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