WHAT DOES IT MEAN TO "UNDERSTAND" CONCAVITY AND INFLECTION POINTS?

Steven R. Jones
Brigham Young University
sjones@mathed.byu.edu

The calculus concepts of concavity and inflection points are often given meaning through the shape or curvature of a graph. However, there appear to be deeper core ideas for these two concepts, though the research literature has yet to give explicit attention to what these core ideas might be or what it might mean to "understand" them. In this paper, I propose a framework for the concavity and inflection point concepts, using the construct of covariation, wherein I propose conceptual (as opposed to mathematical) definitions that can be used for both research and instruction. I demonstrate that the proposed conceptual definitions in this framework contain important implications for the teaching and learning of these concepts, and that they provide more powerful insight into student difficulties than more traditional graphical interpretations.

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At the level of calculus, and beyond, the twin concepts of concavity and inflection points are essential components of a complete understanding of function behavior. As such, several researchers have begun to examine how students think about these two concepts (e.g., Baker, Cooley, & Trigueros, 2000; Gómez & Carulla, 2001; Tsamir & Ovodenko, 2013). The way these concepts have been studied, as well as how they are often portrayed in textbooks, is usually deeply connected to graphical interpretations and meanings of the concepts (e.g., Baker, Cooley, & Trigueros, 2000; Stewart, 2014). Yet, on occasion, calculus education researchers seemed to have acknowledged other important interpretations or meanings of these concepts. For example, Tsamir and Ovodenko (2013) discussed how students used symbolic representations, including $f'(x)$ and $f''(x)$, to define and reason about the two concepts, and Berry and Nyman (2003) described teaching activities that develop these two concepts through physical movement. Furthermore, Carlson, Jacobs, Coe, Larsen, and Hsu (2002) explained how covariational reasoning is deeply connected to making sense of function behavior, which is closely linked to concavity and inflection points. All of this research has given us important information on how students use concavity and inflection points in activities like graphing, or on general difficulties students may have with them, or on types of reasoning that are required to make sense of them. Yet, these studies have often taken implicit stances on what the concepts concavity and inflection points actually mean. Thus, the question is raised: What are the *core ideas* we might say are contained in the concepts concavity and inflection points, and, consequently, what does it mean for a student to *understand* these ideas? While it is possible to claim that these answers can, or should, reside purely in graphical terms, such as the shape or curvature of a graph, the research literature seems to suggest that there are ideas more fundamental than simply the "shape of a graph" for these concepts. Since research has often not made explicit what these core ideas are, beyond graphical interpretations, it is important that we, as a research community, debate what might make up the core ideas of concavity and inflection points and an "understanding" of them.

Like the research literature, calculus textbooks also tend to focus on graphical interpretations of concavity and inflection points (e.g., Hughes-Hallett et al., 2012; Stewart, 2014; Thomas, Weir, & Hass, 2009). However, despite the heavy graphical treatment of the concepts, they are frequently given definitions through non-graphical language. For example, Thomas et al. (2009) define concavity based on whether "f' is increasing [or decreasing] on I" (p. 203), and in Foerster (2010),

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concavity is defined by the sign of the second derivative (p. 373). Even though these books subsequently focus on graphical meanings in their treatment of the concepts, we see hints at other ideas for concavity and inflection points beyond just the shape of a graph. Again, we are left with the question as to what the core ideas are that we want to ascribe to concavity and inflection points and what an "understanding" of them might consequently look like.

In this paper I propose an argument for a particular conceptual framework that could be used as a way to *conceptually* define concavity and inflection points, which then provides one possible answer for what the core ideas are and what it might mean to "understand" them. I articulate my stance through covariational reasoning (Carlson et al., 2002), in that I see covariation as more than just "important" to an overall understanding of these concepts, but I have come to see it as the single core idea that makes up the *essence* of these two concepts. In laying out this conceptual framework, I discuss its connection to the common graphical approach to these concepts, and I discuss some pedagogical implications. I hasten to note that my stance is influenced by my examination of these concepts in real-world contexts (see Gundlach & Jones, 2015), and as such, I fully acknowledge that my view is certainly not the *only* point of departure for a conceptual discussion on concavity and inflection points. The fact that many studies and textbooks discuss concavity in other ways speaks to this. However, I propose my perspective as a way to launch a debate on what a shared idea of the *core ideas* of concavity and inflection points might be and what an *understanding* of them might look like for calculus education.

Arguments For and Against Having a Conceptual Framework

In one sense, one could argue that the various ideas about concavity and inflection points in the research literature and in the textbooks that I outlined in the introduction are all basically different ways to express the same idea, and that consequently there would be no need for a framework such as this. However, contained in this argument is exactly the issue I want to address: if they are all different ways of expressing the same idea, *what*, exactly, is that underlying idea? Similarly, one could argue that we just define an "understanding" of these concepts through the idea of making connections between representations, much as some have couched an understanding of *function* through representational connections. Yet, just as Carlson et al. (2002) have shown that there is a deeper level of function understanding in covariational reasoning, I believe concavity and inflection points also have deeper meanings beyond just connections in the ways we externally represent them to each other through graphs or symbols.

In another argument, one could dismiss the need for a framework by indicating that these different approaches in the literature and textbooks are all equivalent mathematically and that one approach or representation can be translated into another approach or representation. However, despite the *mathematical* consistency across the different approaches and representations, I argue that they are quite *conceptually* different from one another and would therefore each have separate learning implications (as opposed to mathematical implications). To develop a shared notion of what it means to *understand* concavity and inflection points, the conceptual nature, and not purely the mathematical nature, of these two concepts must be attended to. In other words, my argument is similar to—though not congruent to—Tall and Vinner's (1981) distinction between concept definition and concept image. I use the comparison to Tall and Vinner simply to illustrate the difference I see between mathematical consistency versus conceptual consistency. Despite the possible mathematical equivalence of the various definitions and uses of concavity and inflection point, it is important that we distinguish what are the core conceptual ideas that might make up the concepts.

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A Proposal of a Conceptual Framework for Concavity and Inflection Points

Covariational Reasoning

Since I am using covariational reasoning as the foundation of my proposed conceptual framework, in this section I briefly describe the covariation construct as laid out in the work of Carlson and colleagues, which is rooted in research on understanding function behavior (Carlson, 1998; Carlson et al., 2002; Oehrtman, Carlson, & Thompson, 2008). Covariational reasoning is defined to be "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354). In the framework, five levels of "mental actions" are described, which correspond to increasingly sophisticated cognitive activities. These begin with simply recognizing that the two variables depend on each other, moving to coordinating the "direction" and "amount" of change, followed by a coordination of how the rate of change changes. The five mental actions of the covariational framework are summarized in Table 1.

Table 1: Mental Actions of the Covariation Framework
(Carlson et al., 2002, p. 357; Oehrtman et al., 2008, p. 163)

Mental action	Description of mental actions
1	Coordinating the *dependence* of one variable on another variable
2	Coordinating the *direction* of change of one variable with changes in the other
3	Coordinating the *amount* of change of one variable with changes in the other
4	Coordinating the *average rate-of-change* of the function with uniform increments of change in the input variable
5	Coordinating the *instantaneous rate of change* of the function with continuous changes in the independent variable for the entire domain of the function

Conceptual Objects Produced by the Mental Actions

While Carlson and colleagues obviously bring up the concepts of concavity and inflection points in their work, the framework is focused on an understanding of *function*. Yet, since mental actions 4 and 5 deal quite explicitly with concavity, I propose to define the "concavity concept" in terms of the conceptual *objects* (in the spirit of Sfard, 1991; Sfard & Linchevski, 1994) potentially produced by covariational reasoning at mental actions 4 and 5. In other words, simply put, I define concavity conceptually as "the covariation between the rate of change and the independent variable." Inherent in this definition is the claim that concavity cannot be truly understood in the absence of a mastery of mental actions 4 and 5. I acknowledge that this proposed definition extends the covariation framework beyond its original intent, in that the framework was originally meant to capture cognitive *activities* involved in coordinating change (Carlson et al., 2002, p. 354). By contrast, in this paper I am proposing a definition for a conceptual *object*. Yet, I believe the cognitive activities performed in the mental actions 4 and 5 can produce a conceptual object, which is the very covariation that exists between the rate of change itself and the independent variable. Thus, the concept of covariation dealing explicitly with the rate of change is, then, exactly the concept of concavity I propose.

To define the "inflection point concept," I note that mental actions 4 and 5 can be seen as essentially recycling through mental actions 1–3, with the "dependent variable" quantity being replaced by the "rate of change" quantity. In other words, mental actions 4 and 5 recursively trace through the first three mental actions again, but with the more sophisticated layer of one of the variables being the rate of change (see Table 2). Note that mental action 2 coordinates the *direction* of change, which I interpret to mean *increasing* or *decreasing*. Within mental action 2, one can track places where a *switch* in the *increasing/decreasing* of the rate of change takes place. Thus, within this mental action, if a switch in increase/decrease is identified, a conceptual object can be produced,

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which is “change in the direction of covariation,” which has much in common with the concepts of maximum or minimum. I then define this “change in covariation” as the “inflection point concept” precisely when one of the covarying quantities is the rate of change itself. Note that in this definition, an inflection point is *not* a point on a graph, but rather a “switch in increase/decrease” of the rate of change as the independent variable changes. Of course, it can be represented graphically as a point, but it is only that—a visual representation of a deeper idea centered on covariation.

Table 2: Mental Actions 4 and 5 Recycling through Mental Actions 1, 2, and 3

Mental Actions: 4. Coordinating *average rate-of-change* 5. Coordinating *instantaneous rate of change*	1. Coordinating *dependence* of rate of change on independent variable 2. Coordinating *direction* (*i.e. increase/decrease*) of change in the rate of change with respect to the independent variable 3. Coordinating *amount* of change in the rate of change with changes in independent variable
Conceptual objects produced	→ Concavity concept = *covariation* between rate of change and variable → Inflection point concept = change in *direction* of covariation

I point out that I am careful to state definitions for the concavity *concept* and the inflection point *concept*, as I am not attempting to define them mathematically. A mathematical definition is a way to take a conceptual idea and create a formal statement for it (see Tall & Vinner, 1981; Zandieh & Rasmussen, 2010). By contrast, this framework is not intended to create a standardized *formal* definition, but rather to put forward an argument for what we, as a field, should consider to be the core conceptual ideas for these two concepts, which may then be represented in various possible ways in textbooks or other mathematical writings.

Comparing the Framework to the Common Graphical Approach

In this section, I address two issues: First, if the framework is simply a repeat of the ways that concavity and inflection points are already typically approached and used, then it is of little use. This framework should offer something beyond what one could already find in other presentations on these two concepts. Second, despite the need for the conceptual framework to offer something new, it must also resonate with the way in which that concept *is* commonly used, defined, and represented in the mathematical community, including in educational research literature and textbooks. Otherwise, it may be that the framework does not even capture what the community considers to be “concavity” and “inflection point.”

To address the first of these issues, I describe how concavity and inflection points are often approached and represented in the research literature and in textbooks. Many studies that deal with concavity and inflection points discuss them through the activity of graphing, or through graphical images (e.g., Asiala, Cottrill, Dubinksy, & Schwingendorf, 1997; Baker et al., 2000; Gómez & Carulla, 2001; Tsamir & Ovodenko, 2013). Similarly, many textbooks I have examined primarily define and discuss these concepts through the graphical register (e.g., Finney, Demana, Waits, & Kennedy, 2012; Hughes-Hallett et al., 2012; Smith & Minton, 2008; Stewart, 2014; Thomas et al., 2009; Zill & Wright, 2011). Many textbooks do provide a definition of concavity based on the first derivative increasing or decreasing, yet these books still seem to ascribe concavity as only a feature of a *graph*. For example, Finney, Demana, Waits, and Kennedy (2012) begin their definition with, “The *graph* of a differentiable function $y = f(x)$ is…” (p. 197, emphasis added), and Smith and Minton (2008) start their definition with, “The *graph* of f is…” (p. 238, emphasis added). The book

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by Briggs, Cochran, and Gillett (2015) is the only book I have examined that indicates that the *function* itself might be considered concave up or down. Consequently, my conceptual framework offers a new perspective that takes a stance against convention, in that I do not define the concavity and inflection point concepts through graphical terms at all, but as concepts rooted in the ideas of covariation.

Moving to the second issue, given that my stance goes contrary to convention, is my conceptual framework even consistent with what the community thinks of as "concavity" and "inflection points?" In many of the studies and textbooks listed in the preceding paragraph, the discussion regarding the relationship between an increase in the derivative, f', and the shape of the graph focuses on how the slopes of the tangent lines become more steep or less steep. However, one can ask, steeper according to what? Implicit in these kinds of statements is the idea of getting steeper as one moves *left to right*. If one moves *right to left*, the steepness trend would reverse. Thus, even these strictly graphical approaches have covariation inherently embedded in them: the slopes (quantity one) change with respect to the independent variable (quantity two).

Similarly, "inflection point" is often described as a change in one kind of graphical shape to another kind of graphical shape. Inherent in this description is the need to attend to whether the slopes' steepness increases or decreases (i.e. "direction"). These increases or decreases correspond to the mental action 2 with slope or rate of change as one of the quantities. A *switch* in that increase/decrease of slopes or rates of change aligns with the definition of the "inflection point concept" in this framework. Thus, this framework does speak to the same "concepts" discussed in the typical graphical approaches.

Implications for the Teaching and Learning of Concavity and Inflection Points

So far in this paper, I have (a) outlined potential *conceptual* (not mathematical) definitions of concavity and inflection points, (b) shown that this framework differs from conventional approaches, and (c) shown that despite the difference, it still speaks to the same concepts of concavity and inflection points used by the community. However, if this framework yielded no worthwhile implications for how concavity and inflection points are to be taught and learned, then it would still be little more than an academic exercise of small value. I believe that this conceptual framework for concavity and inflection points contains significant implications for the teaching and learning of these concepts. In this section, I first outline what it might look like for a student to "understand" these concepts according to my framework. I then re-examine examples of student difficulties described in the literature regarding the concavity concept and the inflection point concept to show how this framework can further illuminate the nature of some of these difficulties and provide ways to address them.

I was recently involved in a study in which we examined how students made sense of concavity and inflection points in real-world contexts (Gundlach & Jones, 2015) . While the students in the study exhibited a range of interpretations of concavity and inflection points, one student in particular showed an ability to think about, reason about, and make sense of concavity and inflection points in a range of contexts, from intuitive contexts (temperature) to more abstract contexts (the size of the universe). He also demonstrated a facility with the type of graphing problem used in Baker et al. (2000). I believe his proficiency with these concepts stemmed from the fact that he had mastery of the covariation mental actions and seemed to have codified these into some kind of conceptual objects—or at least he had begun to. For example, in one prompt, he was asked to describe how concavity related to a person's height over their lifetime. He first discussed the early period of a person's life, which he stated reflected "concave up."

Student: The way I thought about it is that, over time, it seemed like the height increases. There's a bigger increase of height as you get older, up to a certain point… So from here to here [indicates two points in time] it's less increase of height, whereas from here to there, about the same [time] length, it's a greater increase of height, and this would be concave up.

Here the student's comments reflected mental action 4, wherein average rates of change over equal-sized intervals were considered. What is important to note is that he realized that one of the covarying "quantities" in this case is the "increase of height," or the growth rate. Thus, the student has demonstrated a *dependence* of the growth rate on the time variable. As he moved to discuss a concave down period, he exhibited mental action 5 by dropping finite intervals of time from his explanations, and switched instead to continuously changing rates.

Student: They're starting to get close to their full height, whatever that height happens to be. It'll still be increasing, it'll be increasing at a decreasing rate, which means that—that would have to be an inflection point [in order] for that rate, at which their height is changing, to change. And at that point it would begin to be concave down.

Interviewer: …If the curve [i.e. the graph of the height function] doesn't go back down, is it not concave down? What would you say about that?

Student: I'd still say it's concave down… that rate at which they're growing slows down, it still is concave down.

The student used the idea of a continuously changing growth rate, and stated that an increasing growth rate is defined as concave up, while a decreasing growth rate is defined as concave down. Note that he also described the inflection point as being dependent on a "change" in the growth rate from a growth rate that is increasing to a growth rate that is decreasing. Thus, he seemed to have encapsulated the "change in direction" from mental action 2, with one "quantity" being the growth rate. These sophisticated mental actions seemed to have produced understanding of covariation and change that were associated with the terms "concavity" and "inflection point." As such, this student seemed to demonstrate an *understanding* of these concepts according to this framework.

In this illustrative example of understanding, I wish to highlight that this student's discussion of and meaning for concavity and inflection points were *not* dependent on the shape of a graph, even though a graph was drawn during the discussion. Rather, it was his coordination of a changing *rate of change* with the dependent variable that drove his thinking and seemingly produced an understanding of the conceptual objects defined in this paper.

I now switch to focus on examples from the literature regarding student difficulties. First, I discuss a student difficulty described in Baker et al. (2000), in which at least two of the students in the study (Carol on page 567 and Jack on page 568) held the idea that concave up meant the graph was increasing and concave down meant the graph was decreasing. This idea was a source of difficulty for the students when it conflicted with information about the first derivative that seemed to contradict it. The question then becomes, if these two students knew that the *first* derivative dealt with increase/decrease, why did they also apply the idea of increase/decrease to the *second* derivative? Baker et al. proposed the idea that the students were not at an "inter-property" level, meaning they could not coordinate information from two different properties simultaneously. However, it seemed like the students attempted to do so, which is what created the conflict in the first place. To provide an alternate reason, note that this difficulty is consistent with the framework's interpretation of mental actions 4 and 5 being a recycling of mental actions 1–3. That is, making sense of concavity goes through the same cognitive activities as making sense of slope or rate of change. The problem is that the activities required in making sense of concavity take on an additional layer of sophistication since the recycled mental actions 1–3 now deal with one of the quantities being the "rate of change." Since it is likely that current graph-based approaches provide little instructional support to developing these sophisticated, layered mental actions, then it is possible that the recycled layer of mental actions 1–3 collapsed to the original layer of mental actions 1–3. It may be that "concavity" was reduced in some sense to the same thing as "rate of change." Thus, the two distinct conceptual objects (concavity versus rate of change) became blurred into a single conceptual object, rate of change. To separate out these two objects, instruction would have to highlight the fact that concavity essentially retraces through the same concept as rate of change, but now *using* rate of change as one of the covarying quantities. I am in no way claiming that this would be an easy feat, but a necessary one if students are to fully understand these concepts.

In a second example, Tsamir and Ovodenko (2013) describe several students identifying inflection points as places where "the graph keeps increasing, but the slope changes dramatically," like a mountain trail changing from a gentle upward slope to a steep, difficult climb (p. 421). The reasons given by Tsamir and Ovodenko for this difficulty was that students look too holistically at graphs, or base their reasoning too much in real-world contexts (p. 421). However, this explanation does not provide much by way of how to address this problem, other than to be "less holistic" or to not use "real-world contexts." By contrast, I believe my framework provides a much deeper reason for this difficulty. In general, an "inflection point" is typically presented as a *change* in something (e.g., Hughes-Hallett et al., 2012; Stewart, 2014; Thomas et al., 2009), and the issue is in *what* changes exactly. There is a significant cognitive demand in recycling through mental actions 1–3 inside of mental actions 4 and 5, replacing one "variable" with the "rate of change" quantity. Specifically, an inflection point arises by noting a change in the "increase/decrease" inside mental action 2, if one quantity is understood to represent the rate of change. The students in Tsamir and Ovodenko's study may have been looking for something that met the usual criterion of a "change in something" when seeking to identify inflection points. Indeed, the idea of the graph going from a slower increase to a sudden, dramatic increase seemed to give the students some kind of "change" happening in the rate of change. Yet, despite the students' possible recognition of the need for a change in the rate of change, they did not have a fully-formed object associated with a switch in the *direction* from mental action 2. Only a switch of *increase-to-decrease* (or vice versa) in the rate of change is appropriate for identifying an inflection point. Thus, this framework would indicate that the students in that study were, in fact, making decisions intelligently, but without a fully formed conceptual object of *what* exactly should change about the rate of change. In other words, instead of looking at students' inability to attend to relevant features of a graph, or of particular shapes, it may be that we should help students see that a change from increasing rates of change to decreasing rates of change (or vice versa) is what constitutes an inflection point.

Conclusion

In this paper, I have shown that despite the heavy graphical emphasis on concavity and inflection points, these concepts may have more important core meanings than as the "shape of a graph." What the core ideas are has important ramifications for the teaching and learning of these concepts, and how we view, understand, and address student difficulties. I have proposed a framework that *conceptually* (not mathematically) defines concavity as the concept of *covariation* in which one quantity is the rate of change, and inflection points as a *change in direction* of this covariation. Thus, this framework provides one possible answer to the question, "What does it mean for a student to understand concavity and inflection points?"

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