STAGES IDENTIFIED IN UNIVERSITY STUDENTS' BEHAVIOR USING MATHEMATICAL DEFINITIONS

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This paper reports and describes some of the observations and conclusions drawn from a study developed to find information on undergraduate students' spontaneous actions and reactions to mathematical definitions that are new to them. There were 23 participants from a transition-to-proof course. They were interviewed individually on a particular mathematical definition. The analysis of the interviews was iterative and consisted mainly of three phases. During the first two phases students' behaviors were observed and identified in order to categorize commonalities of their responses and actions. In the third phase, a search-back through the data was done for additional supporting evidence of the commonalities previously observed. This third phase allowed making a more general classification of the various behaviors observed. Participants were classified into five different stages in using mathematical definitions according to their behaviors when working with a particular definition.

Keywords: Reasoning and Proof, Advanced Mathematical Thinking, Mathematical Knowledge for Teaching

Introduction and Research Questions

The role of definitions in mathematics is fundamental. Professors apparently expect students to be able to grasp definitions and then proceed to do something with them (Alcock, 2010). Nevertheless, "Many students do not categorize mathematical definitions the way mathematicians do; many students do not use definitions the way mathematicians do, even when the students can correctly state and explain the definitions; many students do not use definitions the way mathematicians do, even in the apparent absence of any other course of action." (Edwards & Ward, 2004). This study aims to shed light on how undergraduate students proceed when they are presented with new (to them) mathematical definitions by addressing the following questions: How do students make use of mathematical definitions new to them? What are their spontaneous reactions? What contributes to their difficulties with "unpacking" and using abstract mathematical definitions? How do they use a definition in three different settings: examples, proofs, and true/false questions? This research falls within the scope of a framework developed by Selden and Selden (1995, 2008). I particularly focus on the part of the framework referring to operable interpretations of statements and formal and informal forms of a mathematical statement.

Literature Review

Research has revealed that students have a variety of difficulties understanding and using definitions, many of which could be attributed to the "structure of mathematics as conceived by mathematicians and the cognitive processes involved in concept acquisition" (Vinner, 1991). According to Tall (1980), "concept image is regarded as the cognitive structure consisting of the mental picture and the properties and processes associated with the concept…Quite distinct from the complex structure of the concept image is the concept definition which is the form of words used to describe the concept." A mismatch between the concept image developed by an individual and the actual implications of the concept definition often leads to obstacles in learning. The work of several researchers has confirmed this. Parameswaran (2010) has addressed how mathematicians approach new definitions; her research shows that examples and non-examples play a very important role in the process of learning a new definition. However, students are not frequently asked to generate

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examples, most of the time they are provided with a worked-out example or an illustration (Watson & Mason, 2002). Although there are some studies addressing students' and mathematicians' use of definitions in the construction of proofs, there seems to be a lot more investigate in respect to students' perceptions of mathematical definitions.

Methodology

I conducted a series of semi-structured task-based interviews with voluntary participants taking a transition-to-proof course. There were five definitions: function, continuity, semigroup ideal, isomorphism, and group, spanning most of the course. I interviewed 23 volunteer students individually. For each definition, 4-5 students were interviewed approximately two weeks before that particular definition came up in the course, to assure they had not seen it before (in class). The interviews were audio-recorded, and the students used LiveScribe pen in order that their real-time responses could be analyzed. I wanted the interviews to address the following four main points. First, I wanted the students to be presented with a definition for the first time, that is, they were interviewed about a definition that they had not yet seen in class. Second, I also wanted to test their ability to interpret the definition, so I asked the participants if they were able to come up with some examples that could illustrate the definition. Third, I looked for information on their ability to make use of the definition in the construction of a proof that required no more than the definition itself. And fourth, I was interested in the way they could reason about true/false statements involving the definition. These four points were addressed by the design of five handouts, presented one after the other to each student individually in a 60 to 90 minute interview. This design was partly inspired by the work of Dahlberg and Housman (1997) and the work of Housman and Porter (2003). The first phase of the analysis was done considering each of the five definitions separately. I concentrated on only one definition at a time, analyzing all the data on the five handouts for the four to five respective students considering that definition. The second analysis was done by handout. I looked at the general performance across all handouts, considering one handout at time, across all participants.

Results

From a detailed observation and analysis of students' actions working with definitions newly introduced to them, a general perspective on students' behaviors has been developed. There seem to be various stages that the participants were in during their attempts to use mathematical definitions in the three different settings of this study; evaluating examples, constructing proofs, and answering true/false statements presented in five handouts. Students were initially provided with a handout containing only a definition, and then other four handouts with the different tasks were given. As a result of the analysis five stages (0-4), listed and described in detail in the following sections, were identified. These stages are not intended to be a definitive set of steps through which a student must pass in order to use definitions appropriately, but they describe and categorize the different behaviors observed amongst the 23 participants of this study. Each one of the participants worked on only one of the five definitions (function, continuity, group, ideal, and isomorphism) and was classified in one of the five stages according to their performance during the 60-90 minutes of the semi-structured interview. Further explanation and details about the design of the whole study (such as the mathematical definitions and the handouts) can be found in Holguin (2015). This paper exhibits only one of the main results.

Description and evidences of each of the stages

Stage 0: Unawareness. Students at Stage 0 see and treat mathematical definitions as everyday words. A student is in Stage 0 if he/she does not see mathematical definitions as having meanings separate from the everyday linguistic meaning of the words used in them. Such students relate the words in a mathematical definition to words used in everyday language or to everyday situations.

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They make connections to their previous knowledge, but those connections are not necessarily mathematical. Mathematical symbols are often hard to interpret perhaps because in the everyday linguistic context definitions rarely have symbols involved as mathematical definitions often do. This behavior was observed in two of the 23 students, Fay and Gaby. Fay was working with the definition of function. She wants to become a secondary mathematics teacher. When she was asked to explain the definition in her own words she said:

Fay: …a function is like a machine; you put something in to get something out. Like a machine that makes copies of a newspaper in English and Spanish…

Then the interviewer asked her if she could write down her thoughts. Figure 1 shows what Fay wrote.

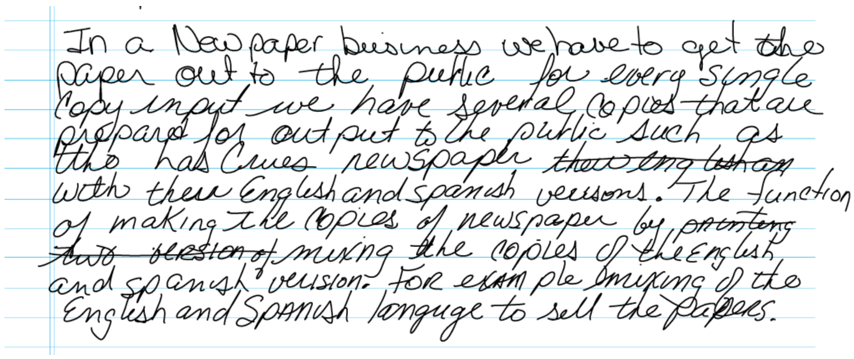
In a Newspaper business we have to get the
paper out to the public for every single
copy input we have several copies that are
prepared for output to the public such as
the Las Crues newspaper ~~that english~~
with their English and spanish versions. The function
of making the copies of newspaper by ~~printing~~
~~two version of~~ mixing the copies of the English
and spanish version. For example mixing of the
English and SPANISH languge to sell the papers.

Figure 1. Fay's thoughts about function.

Fay was classified in Stage 0 because the example she came up with exhibits a lack of awareness of the distinction between mathematical and everyday definitions. At best, she understood function as a process (machine) that gives a final product when some inputs are provided. But she gave an example of a newspaper copy machine that produces both English and Spanish versions. One can interpret this as indicating that she might be trying to express the mathematical property of a function of having one and only one output for every input, but has expressed it the wrong way around. She was assigning several copies to a single input, which violates the second condition of the given definition of function. In addition, she was blending her somewhat fuzzy mathematical ideas with an inappropriate real life situation.

Gaby was an engineering major; she was also asked to explain her understanding of the given definition. She read the definition of a group and she replied the following.

Gaby: A group is not a group if there is not different components or… you know, several elements. So if it were only one element you wouldn't call it a group. I think that's why [it] is telling you that if there is a g element of G and then there is a g' which is an element also of G, and those things comb... you know, make a group. To be a group I think you need to have more than one element."

One can see that Gaby is using the everyday meaning of the word "group". Later in the interview Gaby was asked to provide an example:

Interviewer: Can you think of a particular example? Can you tell me the properties that an object would need to have in order to be a group?

Gaby: OK, so you have to have an element, or actually each… you have to have a semigroup, each element has to have a subset... for example, I don't know, this is what I can picture, like the school. The school is a [with emphasis] school, as a whole, but it has different colleges, the university has colleges, there is colleges that belong to the university and each college has

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departments, like for example like a g', so departments, colleges, you know, are composed for and there... you know, the university, which equal university. That's the way I see it.

Gaby's example, as Fay's, is a blending of a real life situation with pieces of mathematical knowledge (some ideas from set theory). In Gaby's case, I find it harder to speculate which mathematical properties might be involved in her explanation. The only thing I see is that she might be thinking that the prefix "semi" before the word "group" implies that a semigroup is something smaller than the group but contained in it. While explaining the definition in her own words, she also drew the diagram in Figure 2.

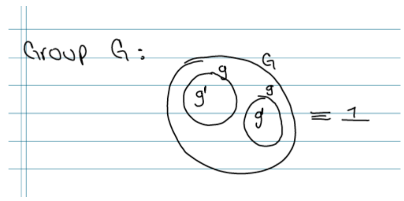

Figure 2. Gaby's diagram about group.

From this diagram and the conversation we had about it, I see that she might have been thinking of g (by itself) as a subset of G instead of as an element. And perhaps she was trying to illustrate the property of having an identity element, that is to say, the part of the definition of group stating that for each g in G there is a g' in G such that $gg'=g'g=1$. For the aforementioned reasons, Fay and Gaby were classified in Stage 0.

Stage 1: Awareness. Some students seemed able to recognize the differences between mathematical definitions and everyday definitions. They seemed to be aware of the importance of mathematical definitions, but they seemed to find them too complicated to make use of them or to think of examples of them. They seemed to make strong connections to definitions previously seen in other mathematics classes or to any mathematical concepts that seemed familiar to them, or that they thought were related to the particular presented definition. Students in Stage 1 did not demonstrate a recognition of how, why, what, or when to use mathematical definitions; there five out of the 23. The following excerpts illustrate these students' behaviors.

Carlos was majoring in engineering technology and was interviewed on continuity. He was told at the beginning of the interview, as every participant had been, that he was being given a definition. He read it and immediately afterwards he started talking. The following is a piece of what he said.

Carlos: This is like a weird... a weird problem
Interviewer: Have you seen this definition before?
Carlos: I have seen some of the symbols... in this class... in Algebra, Precalculus, Technical Calculus. I haven't seen absolute value symbols in this class yet, or continuous.
Interviewer: How about other classes?
Carlos: Continuous? Oh yeah! yeah! in calculus! Like natural log e be continuous… or they already gave a problem like saying here is a chemical that grows continuously at so and so, yeah that's the kind of problems I had.
Interviewer: And what was the meaning of continuity in those cases?
Carlos: The symbol e, then is e like decaying continuously or growing continuously, so yeah that's where. But for this problem, I don't know, is kind of like... I never... I don't how to approach it.
Interviewer: Well I am not giving a problem; it is just a definition, all I am asking is to read it and I'm just trying to see what you can see in it. So do you get some meaning from it? Do you get what it is telling you?

Carlos: Yeah, just slightly... well you are telling there is a function from R to R, so something from R has to be in R. And the a is an element of R which is in f function continuous and... I don't really know about continuous 'cause we never did continuous so... or either greater than... I'm like what is that? Even delta... I know what is going from here, [pointing to the function] but I get knocked out here...[pointing to the inequalities] I'm like what is going on? I don't even know.
Interviewer: Well, but you said you have seen or heard the word "continuous" before, can you come up with some examples of something that has to do with continuous functions?
Carlos: Yeah, OK. I can just make it up, right?
Interviewer: Yes, yes, definitely.
Carlos: Mmmm... should I use like a chemical? or just say a dead body decays...

[Carlos continues to write what is shown in Figure 3]

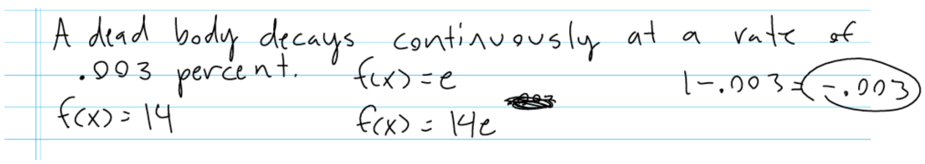

Figure 3. Carlos' continuous function example.

It seems interesting that Carlos' first reaction was to think there was a problem to solve. His first words after reading the definition were "this is like a weird problem", when there was no problem at all. Every participant of the present study was provided with the same explanation and instructions about the procedure of the interview. It is not clear to me why Carlos thought of the definition as a problem. It might be due to the fact that, as stated by him, he did not understand much about what was being stated.

Another behavior worth noticing is the connection with previous knowledge from other mathematics classes about a function that grows or decays continuously. This once more demonstrates that students use their concept images whether or not they match the concept definition. These behaviors identify Carlos in Stage 1; he used mathematical terms to communicate his thoughts but he didn't get much from the information in the definition, he basically ignored it.

Stage 2: Contextualization. By contextualize, I mean being able to identify when and where to use a certain mathematical definition. Some students seemed to understand that a given definition makes sense within a particular situation of a particular area of mathematics, and that it is in such context that the definition should be used in the way it is stated. They were aware of their previous mathematical knowledge, but they seemed to understand that their previous knowledge might not always be helpful. There were 3 participants presenting this behavior.

Fred was majoring in mathematics. He worked with the definition of function.

Fred: I have never seen this definition before ... I have heard the word function from previous math classes, Calc I, II, III, trig and Precalc… It helps with graphing, but we are not going into graphing here.

One could see that, from looking and reading the definition, Fred had noticed a difference. He said that he hadn't seen this definition before, but he had definitely heard the word "function" before. He distinguished, without being told to, that this was a different context compared to others where he had used the definition previously.

Frank was an engineering major. He considered the definition of function. The following excerpt shows what he said after reading it.

Frank: So… what do I do?

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Interviewer: Well… there is nothing to do now…It is only a definition, for now it's just reading it… Have you seen this definition before?
Frank: It does sound like a derivative, because of the prime. [Clarification provided.]
Interviewer: Now, have you heard the word "function" before?
Frank: Yes, in Calculus, 191, 192, Differential Equations, I don't think I heard of function when I did Statistics, but yeah, in almost every math class … A function is like machine. And so a function is basically like a, this is the way some people do it, like here is a machine [drawing] and then so you have *x*, which is the independent variable and it goes through this machine.

Frank was classified in Stage 2 because although he was not strictly following the given definition, he seemed to know that some mathematical concepts are common to different mathematics courses. He was also using his knowledge from previous mathematics classes.

Stage 3: Implementation attempts. In Stage 3, a student is able to recall, look for, and attempt to use/follow mathematical definitions, not necessarily with success. Some participants appeared to understand the context in which the given definition was relevant; they tried to stick to it as much as possible. This does not necessarily imply that the student had acquired/developed the ability to use the definition correctly, but at least he/she appeared able to understand the importance of considering the explicit and/or implicit details of the definition. Students in this stage seemed to know there is something embedded in each part of the definition that needs to be unpacked. But still they didn't fully unpack the definition and use it on the tasks included in the handouts. There were 9 students identified in this stage.

Candy worked with the definition of continuity at a point. She was a graduate student in the Department of Curriculum and Instruction. This excerpt from the interview shows her explanation of how she was trying to prove a statement that required only the use of the definition, namely, that f:R→R defined by $f(x)=2x+3$ is continuous.

Interviewer: So what is your approach? What are to trying to prove?
Candy: OK, so I'm trying to prove this function is continuous. So you just let out your specifications, let *R* be the real numbers, let it be a function defined by the $2x+3$...and so... there is going to be... I want to use this definition... the definition for continuous and just sort of work that, so I didn't know if I should... suppose... that there is... but...[whispers reading the definition] I don't know if I need to prove that there is an 'a' also in the reals or... or not, or should that... yeah, I guess I'm not really quite sure... we have the reals over here and they are following this function $2x+3$, and there is an element in here [R] that is mapping to this element in here [R] and... so that... I want to see...oh! I guess that would be, I guess I should use this... OK... so to me, maybe I wanna assume that there is an "*a*" in *R* such that this and that is true but... yeah... I... I'm not quite sure… I don't know I'm going around in circles…

Above we can notice that Candy was trying to follow the definition, she even stated that. Candy was the participant that best represented Stage 3. In her interview she seemed to be trying very hard to use and understand what the definition stated.

Stage 4: Accomplishment. This last stage can be described as the stage of successful manipulation of the definition. Students at this stage were able to use the definition appropriately and according to the particular mathematical setting. These students had passed, perhaps implicitly, through all previous stages and they were, somehow, able to determine whether or not a mathematical object satisfied the given definition. Please notice that I am not claiming that students at this stage had achieved conceptual understanding of the definition, however they demonstrated the ability to stick to the definition and used it appropriately during the interview. Conceptual understanding might have occurred but I'm not accounting for it in this study. The following excerpts

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show some of the students' work that I considered a successful usage of the given mathematical definition. There were 4 students in this stage.

Scott worked with the definition of isomorphism. Scott was a mathematics major. In the following portion of his work (see Figure 4) one can see that he was able to unpack and use the definition appropriately, in particular, in attempting a proof.

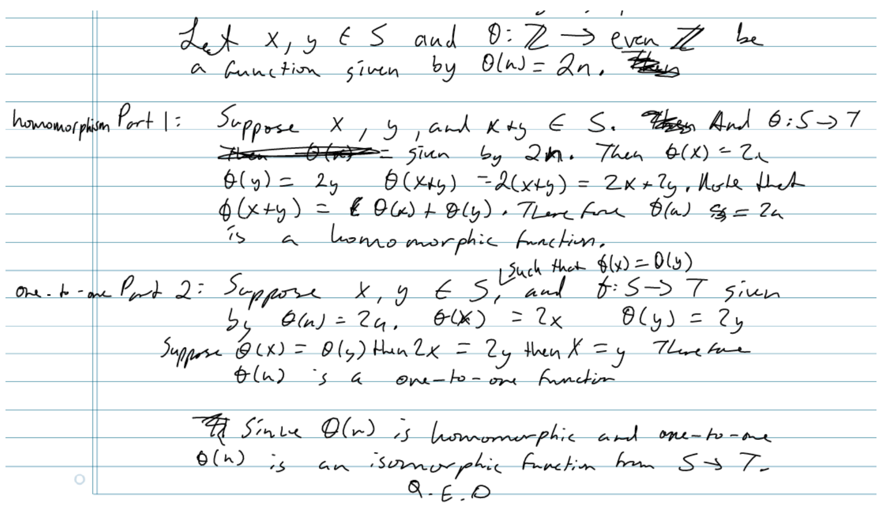

Figure 4. Scott's proof attempt.

Scott was using the proof framework suggested in class. This proof could have been written better, but Scott's attempt let me see that he was able to use and unpack a definition that had been newly introduced. For this reason, he was classified in Stage 4.

Summary and Conclusions

After identifying each participant as in one of the five stages, I noticed that these stages appear to be nested. Thus it is a conjecture that the process of getting to use a mathematical definition appropriately is teachable. One can attempt to learn how to be competent at it. Many opportunities for student success might be hidden behind the quality of communication between the teacher and the learner. However, to test this conjecture further research would be needed. The present study provides evidence that using mathematical definitions can be a difficult task for students, often expected to do it straightforwardly. I find it important to remark that conceptual understanding of the mathematical definition is not necessarily involved in these stages. These stages describe solely students' behavior reading and using mathematical definitions. Although to account for conceptual understanding was not the aim of this study, it is plausible that knowing in which of these stages a student is situated could be a fundamental early step towards the development of conceptual understanding. Being aware of the stage in which a student is, can make us more sensitive to their behaviors and the obstacles they face in attempting to consolidate their formal mathematical skills.

References

Alcock, L. (2010). Mathematicians' perspectives on the teaching and learning of proof. *Research in Collegiate Mathematics Education VII*, 63-91.

Dahlberg, R. P., & Housman, D. L. (1997). Facilitating learning events through example generation. *Educational Studies in Mathematics*, *33*(3), 283-299.

Edwards, B. S., & Ward, M. B. (2004). Surprises from mathematics education research: Student (mis) use of mathematical definitions. *The American Mathematical Monthly*, *111*(5), 411-424.

Holguin, A.V. (2015). *Undergraduate students' use of mathematical definitions new to them: evaluating examples,*

constructing proofs, and responding to true or false statements (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses.
Housman, D., & Porter, M. (2003). Proof schemes and learning strategies of above-average mathematics students. *Educational Studies in Mathematics*, *53*(2), 139-158.
Parameswaran, R. (2010). Expert mathematicians' approach to understanding definitions. *The Mathematics Educator*, *20*(1), 43-51.
Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, *29*(2), 123-151.
Selden, J. & Selden, A. (2008) Consciousness in enacting procedural knowledge. In Proceedings of the Conference on Research in Undergraduate Mathematics Education (2008). Available at http://cresmet.asu.edu/crume2008/Proceedings/Proceedings.html.
Tall, D. O. (1980). *Mathematical intuition with special reference to limiting processes*. Paper presented at the meeting of the Fourth International Conference for the Psychology of Mathematical Education, Berkeley, CA.
Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced Mathematical* Thinking (pp. 65-81). Dordrecht, The Netherlands: Kluwer Academic Publishers.
Watson, A., & Mason, J. (2002). Student-generated examples in the learning of mathematics. *Canadian Journal of Science, Mathematics and Technology Education*, *2*(2), 237-249.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.