THE USE OF STUDENT-CREATED DYNAMIC MODELS TO EXPLORE CALCULUS CONCEPTS

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Multiple representations, multiple modalities, and technology can be helpful in the understanding of mathematical concepts when used in an appropriate manner (Shah & Freedman, 2003; Goldman, 2003), but this alone does not account for the student benefits of creating and using dynamic models over teacher generated graphs to construct connections between representations. By uncovering the dynamic nature of mathematics, calculus becomes more transparent as relationships and patterns emerge. The struggle to understand becomes worthwhile and rewarding for students as they create and observe the action of a dynamic mathematical object. This study shows an improvement in attitude, and academic achievement when students develop dynamic mathematical object to understand calculus and poses new questions to explore.

Keywords: Technology, Curriculum, High School Education, Learning Theory

Introduction

Mathematics students across all ages and subtopics are encouraged to explore multiple representations of mathematical relationships (e.g., graphical, symbolic, tabular) using a variety of appropriate tools (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010; National Council of Teachers of Mathematics [NCTM], 2000). The use of tools ranging from pencil, paper, and rulers to calculators, computer algebra systems, and dynamic geometry software are encouraged to aid in the solution and exploration of mathematical problems and concepts. Since the invention of the computer in the mid-twentieth century, computers have decreased considerably in size, increased tremendously in processing power, and have become inexpensive enough to permeate private and public sectors, including education. The increasing availability of technology in education has led to digital textbooks, advanced handheld calculators, and one-to-one technology initiatives. New technologies such as these lead to new questions and studies regarding the efficacy of technology in the classroom, and the results have varied greatly (Bebell & Kay, 2010; Donovan, Hartley, & Strudler, 2007; Maninger & Holden, 2009). The opportunity to explore the affordances of effective practices is expanding as the technologies evolve, and more studies must be done in order to identify these affordances to help maximize learning in a mathematics classroom.

Although computers grant access to the Internet, powerful dynamic software, and the ability to collaborate in new ways, it is necessary that teachers thoroughly explore the multitude of options and determine what methods are effective and beneficial to teaching and learning. Studies of effectiveness should parallel the teachers' and students' explorations to confirm whether particular methods are beneficial, or possibly detrimental, to learning objectives and also strive to identify the particular affordances of the experience. This process will refine educators' understanding of the usefulness of technology and may help to evolve current methods of instruction by identifying the affordances of useful procedures. In this paper, these matters of efficacy and affordances will be explored by examining the use of software to create dynamic representations of functions, referred to as dynamic models, in a secondary calculus classroom. Specifically, this paper will address the question: does student creation of, and interaction with, dynamic models using Wolfram *Mathematica* increase performance on calculus assessments composed of a variety of subtopics (i.e., derivatives, tangent lines, relative extrema)? And, does this experience have an effect on student

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attitudes toward mathematics? First, the Literature Review section will provide context for the various tools that were utilized during the experiment. Then, the Design and Methods section will describe the quasi-experimental design and provide details about the intervention that took place during the experiment. Finally, sections will explore certain affordances of the learning activity and make suggestions for further clarifying research.

Literature Review

Dynamic mathematic learning environments allow students and teachers to explore and uncover relationships by manipulating aspects of a particular concept. These dynamic learning environments can be applied in a variety of ways to a diverse assortment of topics. Students can use dynamic geometry software (e.g., Geometer's Sketchpad, GeoGebra) to construct geometric figures and explore properties of the figures by clicking and dragging components to investigate patterns, make conjectures, and verify relationships. Bu and Haciomeroglu (2010) explore the specific use of sliders (see figure 1) in dynamic learning environments. A slider acts as a single mathematical object that has two representations. Algebraically, a slider acts as a variable within a defined interval that can be conceptualized as a constant in certain settings. "Graphically, a slider appears as a segment which allows the user to adjust the value of the corresponding variable through dragging" (Bu & Haciomeroglu, 2010, p. 214). The presentation of both graphical and algebraic representations gives learners the ability to make abstractions more visible and the opportunity to make connections between representations (Martinovic & Karadag, 2012). When sliders are used to represent a constant or a constant variable that acts as a parameter of a function, learners can explore multiple cases of a function without having to change the function definition (Bu & Haciomeroglu, 2010). By comparison, graphing multiple of cases of a function using pencil and paper, or even a traditional graphing calculator (e.g., the TI-84+), would invariably take more time. Among the various studies that examine the benefits of dynamic and interactive mathematics environments, there still exists a need to explore the importance of having learners generate dynamic representations as opposed to having them generated for them (Goldman & Petrosino, 1999; Schwartz & Bransford, 1998).

Design and Methods

During the course of a two-year period, assessments were collected from two introductory level calculus classes consisting of high school seniors at a midsized, rural high school in Texas. As seniors, students in this course had taken an advanced algebra course and a precalculus course prior to entering the introductory level calculus course. At the beginning of the experiment, the teacher had about four years of teaching experience and had taught the course multiple times before. During the 2013-2014 school year, the course content was taught to a class of 29 high school seniors in a traditional manner. The first semester of the course was designed to review algebra concepts while the second semester focused on connecting precalculus concepts to topics of differential calculus. Multiple representations were explored using pencil and paper. Graphing calculators (e.g., TI-84+) were used on occasion to verify relationships and check the reasonableness of solutions. The teacher often used graphical representations during instruction. Although graphical representations were an emphasis of the course, symbolic manipulation and interpretation were largely the primary foci, and the graphs that were explored were not dynamic. During the 2014-2015 school year, the same teacher taught course content to 25 high school seniors. Periodically throughout the second half of the year, the teacher guided students to create dynamic models that utilized sliders using Wolfram *Mathematica,* after which the teacher would formatively assess the students' understandings of the concept under investigation through the lens of the dynamic applet. All other instruction, assessments, and assignments were unchanged.

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An Overview of the Tasks

The students were first introduced to *Mathematica* as a replacement of the graphing calculator in order to gain familiarity with the software. By making use of *Mathematica's* freeform input, which essentially allows the user to input commands using English phrases and basic math notations, students could easily plot functions and note how *Mathematica* reformats the input into formal code. With this feature and varying amounts of support and guidance from the teacher and the Wolfram website, the students programmed an applet that accounted for the effect of one parameter of a sinusoidal function (e.g., amplitude). Then, the students were challenged to explore and manipulate the families of trigonometric functions by creating an applet in which all parameters of the functions could be changed (see figure 1). Upon completion, the teacher checked for understanding by asking questions about the effects of changing each parameter and by challenging the students to create sinusoidal functions that had certain attributes (e.g., a period of π, an amplitude of 4).

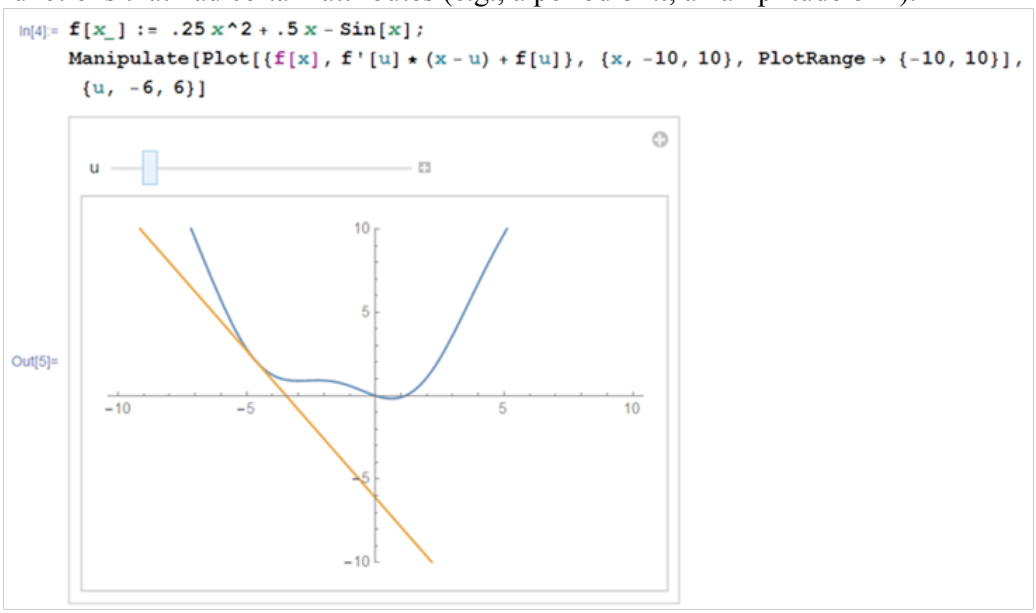

Figure 1. Screenshot of student created exploration of tangent lines.

Data, Analysis, and Results

All assessments were collected from the two senior classes during the 2013-2014 (Group1) and 2014-2015 (Group 2) school years. The formal assessment that occurred after the final use of *Mathematica* was chosen as the dependent measure to be compared in the quasi-experimental design. The assessment covered topics that aligned with the goals of the final applet-creation activity including the understanding of tangent lines and relative extrema. This assessment will be referred to as the post-test. The comprehensive semester exam (covering a review and extension of algebra topics) was chosen as a covariate to act as a control for the variability in math skills and experience between the two groups. The semester exam was graded on a traditional 100-point scale. The post test included an opportunity to score five bonus points for a maximum score of 105. As seen in table 1, the means and ranges of the scores on the semester exam vary between the groups, and it appears that the means of the post-test scores vary considerably. The Attitudes Toward Mathematics Inventory (ATMI) was given to each group, at the same time each year, after the post-test was given (see Tapia & Marsh, 2004). The ATMI is a brief survey consisting of 40 questions that is designed to measure high school and college students' attitudes toward mathematics.

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Table 1: Descriptive Statistics

		Minimum	Maximum	Mean	Std. Deviation
Group 1 *N* = 29	Semester Exam	49.00	101.00	75.93	12.23
	Post Test	46.00	99.00	69.48	16.40
Group 2 *N* = 25	Semester Exam	41.00	95.00	70.08	15.99
	Post Test	58.00	103.00	80.56	11.90

Results

A One-way ANCOVA revealed a statistically significant difference between the post-test scores of each group controlling for prior mathematical skill level and inconsistencies using the semester exam scores, $F(1, 51) = 14.075$, $p < 0.001$. It is important to note that the results were also significant when other prior assessments were used as a covariate. A partial eta-squared value of about 0.185 indicates that approximately 18.5% of the variance in the post test scores is attributable to the independent group variable (i.e., group 1 as the control group, group 2 as the post-intervention group).

A chi-square test of independence revealed a significant difference between the reported attitudes of group 1 and group 2, $\chi^2(2, N = 50) = 680.835$, $p < .001$. Students from group 2 reported a higher proportion of neutral and positive responses and reported fewer negative responses. Figure 3 shows the frequencies of responses for each group and level of response. The responses utilized a Likert scale from which the students chose the extent to which they agreed or disagreed with a statement involving mathematics. A response of rating one (1) corresponds to a choice that reflects the most negative attitude toward mathematics (e.g., Strongly Agree to the statement: "Mathematics makes me uncomfortable."; Strongly Disagree to the statement: "I think studying advanced mathematics is useful."). A response of rating three (3) corresponds to a selection of "neutral," and a response of rating five (5) corresponds to a choice that reflects the most positive attitude toward mathematics. Ratings of four (4) and two (2) correspond to intermediate responses, such as "Agree" or "Disagree."

Discussion and Conclusion

The significant difference in post test scores and reported attitudes suggests that the experience of creating and interacting with dynamic spaces did indeed aid the students in their quest to understand the nuances of calculus. This section will aim to explore some of the constructive aspects of this experience that may have implications for creating effective technological learning experiences in mathematics instruction. First and most obvious, the use of sliders allows for more time for productive discourse (Bu & Haciomeroglu, 2010). Once students learn how to operate the more advanced technology (i.e., *Mathematica*), students and teachers are enabled to spend more class time talking about the concept under investigation. As employed by the teacher in this study, discourse can evolve and move beyond addressing procedural features of a task (e.g., graphing multiple functions to identify a pattern) to attend to the qualities of the concept in a shorter amount of time while also granting more opportunities for feedback, revision, and reflection—a vital aspect of technology use in education (National Research Council [NRC], 2000). For example, during the task, questions (e.g., how can you use this model to find the where the lowest point on the function occurs?) used the dynamic aspect of the model to quickly make connections to concepts (e.g., relative extrema, concavity) while providing immediate feedback to each student. The efficiency of the dynamic models allows students and teachers to address broader conceptual qualities with less of an opportunity for students to lose interest or get distracted from the goal of the lesson.

The act of modeling and programming dynamic spaces can parallel the construction of underlying mathematics (Tall,1991). On the surface, the students are mirroring the common task of constructing tangent lines when they define a function and situate the correct values in position (e.g., $f''(u)$ as the slope, $f(u)$ as the y-intercept) (see figure 1). In this particular activity, several students initially programmed incorrectly by positioning functions (e.g., $f''\,x$ as the slope) rather than functions evaluated at a point (e.g., $f''\,u$ where u represents a constant). This mistake was quickly realized when the student's output returned graphs that clearly did not represent a tangent line which, in turn, led to conversations between the students and the teacher about the difference between a variable and a variable that represents a constant—an example of an opportunity for revision and reflection. While other software (e.g., GeoGebra, Geometer's Sketchpad) can construct similar dynamic spaces, programming using *Mathematica* introduce a practical application of a function. Specifically, by using the "Plot" and "Manipulate" functions of Mathematica, students gain experience with functions of multiple arguments (see figure **2**). At a deeper level, when students embed a function into another function (e.g., Manipulate [Plot[{f(x),…), they are reintroduced to the concept of function composition, a crucial topic in the understanding of the chain rule.

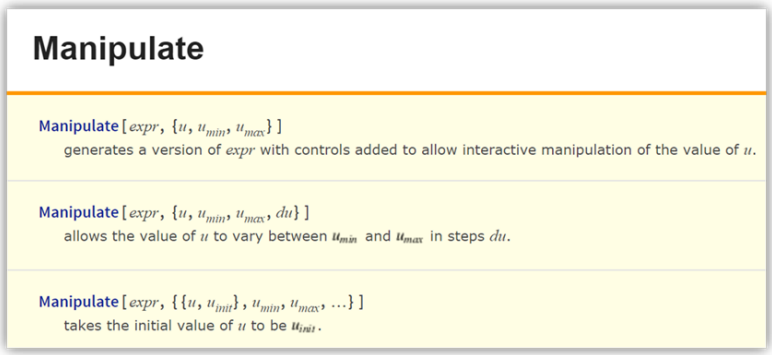

Figure 2. Screenshot showing multiple arguments of the Manipulate function.

The positive results are also consistent with NCTM Standards (2000) suggesting that the exploration of multiple representations is beneficial to the understanding of mathematical concepts. Since it is believed that "student difficulties in understanding calculus concepts result from an inadequate understanding of graphic and algebraic aspects of these concepts" (Haciomeroglu & Andreasen, 2013, p. 7), it is plausible that the experience with the dynamic graphical representations, the algebraic nature of coding *Mathematica,* and the use of a slider that acts as a medium between representations contributed to the easing of such difficulties and increased understanding evident in the results. For example, when using derivatives to determine relative extrema (i.e., a maximum or minimum value of a function) the analytic procedures employed align with a graphical interpretation. Specifically, to determine possible relative extrema, or critical points, of a single variable function f analytically, one may compute the derivative of the function f' and solve for the values at which the derivative evaluates to zero (i.e., solve for c such that $f'\,c = 0$). This process aligns with the graphical interpretation of finding the points along the function at which the line tangent to the point has a slope of zero (figure 3).

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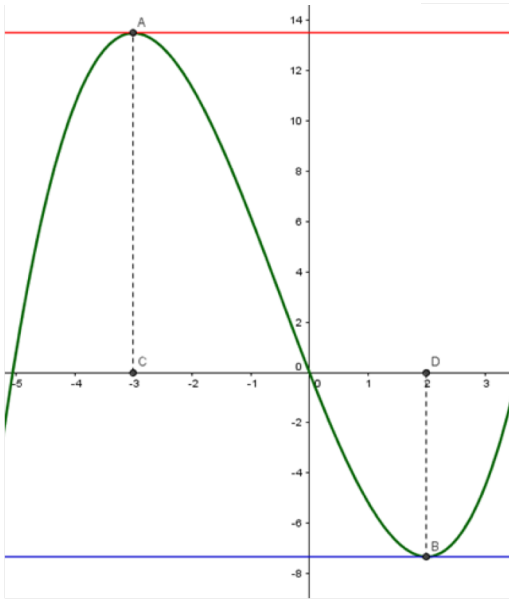
Graphical Process



Analytical Process

Given – –

$f(x) = 1/2\, x^2 + 1/4\, x^4 - 6x$,

Compute $f'(x)$,

$f'(x) = x4 + x - 6$

$= (x - 2)(x + 3)$

Let $f'(x) = 0$, solve for x,

$x4 + x - 6 = 0$

$(x - 2)(x + 3) = 0$

$x = 2, -3$

Conclude that f may have relative extrema at

$x = 2$ and/or $x = -3$.

Figure 3. Relationship between graphical interpretation and analytic.

The visualization of the graphical interpretation can act as a reference when selecting the appropriate analytic method to employ. In this case, by visualizing an example function, one can deduce that because the extrema occur where the tangent line has a slope of zero and the derivative of the function at a point can be interpreted as the slope of the tangent line, then the analytic process to find the extrema begins by setting the derivative equal to zero because that is the point where the slope is horizontal. When students fail to make this connection between the graphical and analytic procedures through visualization, many students haphazardly resort to analytic procedures without fully understanding what to use or why they are using them (Haciomeroglu & Andreasen, 2013).

Multiple representations, multiple modalities, and technology can be helpful in the understanding of mathematical concepts when used in an appropriate manner (Shah & Freedman, 2003; Goldman, 2003), but this alone does not account for the benefits of creating and using dynamic models over teacher generated graphs to construct connections between representations. One benefit of the dynamic feature of the models used is that they provide an external representation of tangent lines at various points on a function. Consequently, students do not have to maintain or mentally transform a mental representation of the model which may reduce the cognitive load required to comprehend the new concept being presented (Shah & Freedman, 2003). As Tall (1991) suggests,

> A computer can also give much-needed meaning to mathematical concepts that students may feel are not of the physical world but in the mind, or in some ideal world. It is generally agreed that ideas are easier to understand when they are made more "concrete" and less "abstract". When an abstract idea is implemented or represented in a computer, then it is concrete in the mind, at least in the sense that it exists (electro-magnetically, if not physically). Not only can the computer construct be used to perform processes represented by the abstract idea, but it can itself be manipulated, things can be done to it. (Tall, 1991, p. 235)

Tall's (1991) explanation may help shed light on how the dynamic model involving tangent lines aids in a concrete understanding of the abstract idea that a derivative of a function is, in many cases, a function itself. That is, the derivative of a function is more than just a slope at a single point, but

rather a sort of formula that contains enough information for the practicing mathematician to find a slope, or rate of change, at any point along a curve (granted the derivative is defined at each point on the curve). Also, Tall (1991) posits that it is often true that "whenever a person constructs something on a computer, a corresponding construction is made in the person's mind" (p. 235). Although this is a very bold remark, it is suggestive of constructivist learning theory in the sense that what is being learned is inextricably tied to how it is learned. Thus, by using animated (i.e., dynamic) visuals to decrease the cognitive load of the learner at the time of conceptualization (Shah & Freedman, 2003), students are enabled to make connections between graphical and analytic procedures through previously unrealized dynamic visualization and avoid haphazardly resorting to analytic procedures without a deep understanding of the reasoning behind them.

Finally, Shah and Freedman (2003) suggest that students' attention is drawn to electronic visual displays, that students are more apt to study electronically delivered content for longer periods of time, and that visualizations in electronic learning environments can be attractive and motivating. These ideas help to explain the increase in positive responses on the mathematical attitude survey.

The implications of this research align with the current trend and push to integrate technology in the mathematics classroom. Although it may be detrimental to assume that the use of technology automatically implies learning, it is evident that effective use of technology in the classroom by both teachers and students can have substantially beneficial impacts. The affordances involved with the effective use of technology should be identified, and this technology should continue enhancing learning opportunities in mathematics classrooms by taking "advantage of what technology can do efficiently and well - graphing, visualizing, and computing" (NCTM, 2000, p. 26).

Limitations

This study highlights the beneficial aspects of incorporating technology into the calculus classroom but has many limitations and leads to more questions about maximizing learning through the use of student-created explorative spaces. The study was primarily limited by the small sample sizes involved. The sample sizes could be expanded to include a more diverse population by including other schools and classes. This could also help eliminate the possibility that the positive results were due to teacher engagement or an excitement about using "new" technology, rather than the actual exploration of concepts and representations.

Opportunities for Future Research

Although the large effect size implies that the applet-creation and exploration was substantially valuable, from this study alone, it is unclear to what extent each aspect of the creative and investigative processes was beneficial to student learning. For this reason, among others, this study raises many more questions to be explored. Would the results have been different if the dynamic models were to be created and supplied by the teacher rather than created by the students? Would the results differ if another program were to be used? To distinguish between the benefits of investigating student-created dynamic spaces versus teacher supplied dynamic spaces, multiple groups could be included in a study. Particularly, the control group could be taught traditionally, a second group could explore concepts using teacher supplied dynamic spaces, while a third group could explore concepts using student-created dynamic spaces. This design need not be limited to concepts of calculus. Rather, it could be expanded to explore other statistical models and spaces used for real-world problem solving tasks. A similar three-group design could be used to distinguish between benefits of one program over another, given a particular topic, which might aid in determining if there is a strong benefit to the programming language employed by *Mathematica* over a more elementary software program. Although current popular programs used to explore mathematical concepts and create dynamic spaces (e.g., *Desmos, GeoGebra, Geometer's Sketchpad, Mathematic*a, *Matlab*) share some overlapping features of their functionality (e.g., the ability to

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create sliders) and user interfaces (e.g., programming language, inputting geometric figures by clicking and dragging), certain programs may be more appropriate for certain levels of development and age groups and may also vary among particular concepts and explorations. As Tall (1991) posits, it is important to note "the principal aim of the programming system of *Mathematica* is predominantly for doing mathematics, rather than learning mathematics" (p. 242). Since the technology that is used as an instructional tool develops over time due to updates in software, it could prove beneficial to identify the affordances of common features that aid in the development of mathematical skills in order to maximize the advantages of using technological tools in education. The future of technology in the classroom is promising; there are many questions left to explore.

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