HIGH SCHOOL STUDENTS' FORMING 3D OBJECTS USING TECHNOLOGICAL AND NON-TECHNOLOGICAL TOOLS

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We analyzed the ways in which two high school students formed 3D objects from the rotation of 2D figures. The students participated in a task-based interview using paper-and-pencil, manipulatives, and Cabri 3D. The results indicated that they had difficulty using paper-and-pencil to rotate 2D figures to form 3D objects. Their difficulty stemmed from thinking of the rotation in a 2D context. Although the use of manipulatives helped them reason about 3D problems, they still had difficulty representing 3D objects correctly. However, the students were able to relate the rotations applied to the 2D figure to the resultant 3D object using Cabri 3D.

Keywords: Geometry and Geometrical and Spatial Thinking, Technology

Introduction

Forming 3D objects from the rotations of 2D figures (e.g., forming a cone from the rotation of a right triangle about one of its legs) is important for the study of calculus (Baartmans & Sorby, 1996) and for developing students' visualization skills. Nevertheless, there is little research about students' forming 3D objects by rotating 2D figures and how different tools might influence those processes. Although *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA & CCSSO], 2010) states that students should be able to "identify the shapes of two-dimensional cross-sections of three-dimensional objects and identify three-dimensional objects generated by rotations of two-dimensional objects" (HSG.GMD.B.4) there is little guidance about how teachers can assist students in learning this important skill and the tools that may be useful. Research is needed to understand how students reason about 3D objects generated by rotations of 2D objects using different tools. The purpose of this research study is to understand how high school students' form 3D objects using two different types of tools: non-technological tools that consist of paper-and-pencil and physical manipulatives, and the technological tool Cabri 3D. Our research question is "How do high school students form 3D objects using different learning tools?"

Theoretical Framework

According to Duval (2006), changing one semiotic system into another system (e.g., statement to equation; figure to statement) using denotations, and coordinating these systems are essential for acquiring mathematical knowledge. Duval refers to *semiotic registers* that allows for representing mathematical objects, making transformations within the same semiotic system (e.g., figure to figure), and creating an equivalent representation in another semiotic system. Two types of transformations of semiotic representations are characterized: *treatment* and *conversion*. Treatments are transformations within the same register (e.g., rotating a triangle 360 degrees). Conversions involve transformations from one register to another one (e.g., finding an algebraic equation for a given graph). Treatments and conversions are given through *discursive* and *non-discursive* representations/apprehensions. Discursive representations are given through speech (articulation of thoughts) using mathematical properties, symbols, definitions, etc. (Duval, 1995).

Students who recognize shapes utilizing *perceptual* apprehension perceive geometric shapes by "figural organization laws, and pictorial clues" (Duval, 1995, p.145). Therefore, students are likely to be misled by perceptual characteristics of geometric shapes. Students who utilize *operative* apprehension make physical or mental manipulations of a geometrical shape such as dividing it into

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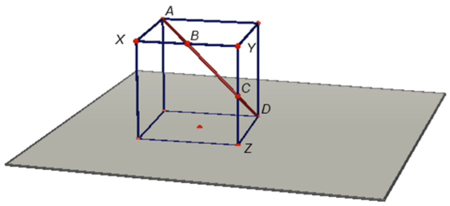
sub-figures, changing the orientation of it, shrinking-enlarging it, etc. (Duval, 1995). For example, in Figure 1, a student may state that the points are collinear if he or she uses perceptual apprehension. On the other hand, the student who utilizes operative apprehension notices the partial line segments are on different surfaces of the cube and/or changes his or her orientation toward the cube.



On the cube, point B is located on $\overline{XY}$, point *C* is located on $\overline{YZ}$. Are points *A*, *B*, *C*, *D* collinear? Why/why not?

Figure 1. A task related to apprehension of 3D objects.

Gorgorió's (1998) research indicated that students with low spatial abilities had difficulty describing the movements of rotations mentally and providing enough spatial information about 3D objects. Students may not think that they are to spin 2D figures to form 3D objects when they are given tasks in a paper and pencil environment. Especially students who use perceptual apprehension; they may consider this rotation in a 2D context. Under this circumstance, students may have difficulty making a conversion from one semiotic system to another one. There may be certain features of a task that prompt the student for particular actions. For example, the teacher may provide additional cues on the paper-based task such as including a spiral arrow in the question to suggest a rotation about a particular axis. Using manipulatives, students may be better able to see the different locations of the 2D object as it rotates that are not possible to see and difficult to draw in 2D. However, in both cases the learner is still responsible to bring the pieces together, and imagine the final 3D object because "most physical actions on physical manipulatives do not leave a trace sufficiently complete to reconstitute the actions that produced them" (Kaput, 1995, p.167). With the availability of dynamic geometry software (DGS), students can drag points, objects etc., observe the path of motion by activating the trace tool of DGS without being required to remember all of its locations and consider what is formed (Schumann & Green, 1997). With the help of the trace tool, technology allows students to observe and interpret the outcomes of fixed properties under different circumstances (Hollebrands & Dove, 2011).

In the current research study, we examine high school students' formation of 3D objects from the rotation of 2D objects. This study will investigate how students use manipulatives, and technology to form 3D objects.

Methods

The participants were selected among 29 students (18 females, 11 males) from a rural high school enrolled in a second year mathematics class. Students were given a spatial ability test. We selected six students (two above average, two average, two below average) to participate in one 90-minute clinical interview during 2013-2014 spring academic semester. In this report, results from two students, Andrea (female) and Pete (male), will be reported. After consulting their mathematics teacher and considering their test scores, we identified Pete as having low spatial abilities and Andrea as having high spatial abilities. The mathematics teacher of the participants reported that it had been over a year since they looked at 3D geometry, and all they would have done would be finding volume of different shapes. At the beginning of 2013-2014 spring academic semester they studied 2D transformations. None of the participants used Cabri 3D or any other DGS.

The interviewees were given five tasks. The tasks were sequenced beginning with paper and pencil. Afterwards, the participants solved the same tasks using manipulatives and then Cabri 3D. Square-centimeter grid paper was used for presenting the paper and pencil tasks, and students had

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access to a ruler and compass. In this report, results from the first two tasks are presented. In the first task, students rotated a rectangle about a line that passed through one of the longer sides. In the second task (Figure 2), students were asked to rotate a rectangle 2 cm away from the axis of rotation. In these tasks, the student needs to match the statement (e.g., intersects the plane perpendicularly, rotate the rectangle 360 degrees about line *KL*) with the figure given in the task (Figure 2) utilizing operative apprehension. In other words, a conversion from statement to figure (or vice versa) is needed. Also, a treatment from figure to figure that involves spinning the rectangle 360 degrees in a continuous motion is required.

> Task 2: Suppose the line in the figure (Figure 2) intersects the plane perpendicularly. In the figure, rectangle *ABCD* is 2 cm away from line *KL*. Draw the three-dimensional object that is formed if you rotate the rectangle 360 degrees about line *KL*.

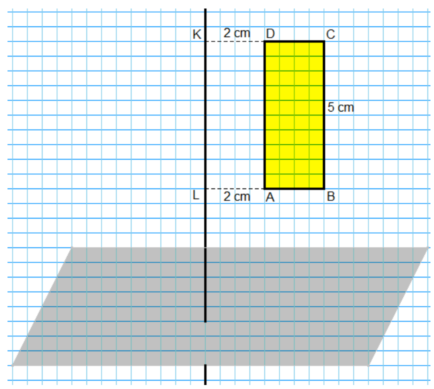

Figure 2. The figure given in Task 2.

A video camera, audio recorder, and a program that recorded the computer screen only when students used Cabri 3D were used to capture the participants' work. Because the participants were not familiar with Cabri 3D, the interviewer spent the first ten minutes teaching them the basic features of the program using some preliminary activities. The interviewees learned how to rotate objects and examine them from different perspectives, mark, label and trace points using Cabri 3D menu tools.

After the interviews, the researchers constructed verbatim transcripts. We identified the interviewees' apprehensions and production of registers within a semiotic system, by focusing on how the participants rotated 2D figures, what types of 3D objects they formed, and if they related the given lengths of 2D figures with the 3D objects (height, radius, etc.). We illustrated the participants' interpretation and constructions within and across the learning tools they used by comparing and contrasting the participants' representations and reasoning with regard to each tool they used.

Results

Results from the Participants' Uses of Paper and Pencil

In the first section of the interviews, the interviewees used paper and pencil to work on the tasks. Andrea and Pete stated that after rotating the rectangle 360 degrees it would arrive at its original location. Andrea summarized her thinking by saying: "basically, you take this figure and you rotate it 360 degrees by the line *AD*. So, that would just be the same figure cause you're rotating it 360 degrees just puts it back where it is." The interviewer asked her to describe how she rotated the rectangle 360 degrees. Andrea denoted the rotation step-by-step by rotating the rectangle 90 degrees each time separately.

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Andrea described the rotation as flipping although her rotation method involved rotating the rectangle about its centroid. After rotating the rectangle, she produced a semiotic representation in 2D. Andrea stated that it would be impossible to get a 3D object. She explained her thinking as follows: "if you start with a 2-dimensional object, I don't really think that you can turn into a 3-dimensional object." In the rest of the tasks, Andrea was consisted with her thinking and refused to produce a semiotic representation in 3D geometry, and said: "no matter what you do to it, it's still going to be the same flat shape it is. It's gonna stay 2-dimensional."

Pete conducted the 360-degree rotation by rotating each corner of the rectangle (Figure 3a) and $\overline{AD}$ (one of the sides of the rectangle) to exemplify how he rotated the rectangle. Similar to Andrea, Pete rotated the line segments 90 degrees each time separately, and stated that the rectangle would be back to its original location. Afterwards, he showed the rotation using an eraser as shown in Figure 3(b), he rotated the rectangle about its centroid. Then, he drew a rectangular prism and said: "most rectangles that one puts in a 3D perspective, it's gonna be a rectangular prism." After Pete had denoted the edge lengths, the researcher pointed at the height of the rectangular prism, and asked how he identified the length of it (Figure 3c). Pete said that he was looking at the object from the top view, and added:

> It states that 2 cm is right here (points at one of the short sides of the rectangle), 5 (cm) is here (points at one of the long sides of the rectangle). So, it never tells you the height but it does tell you the 5 cm on this line segment. So, I kinda took that as if possibly it could be saying that how tall it is. It is on a grid, it's not 3D, it could have that (inaudible) this is how tall it is…

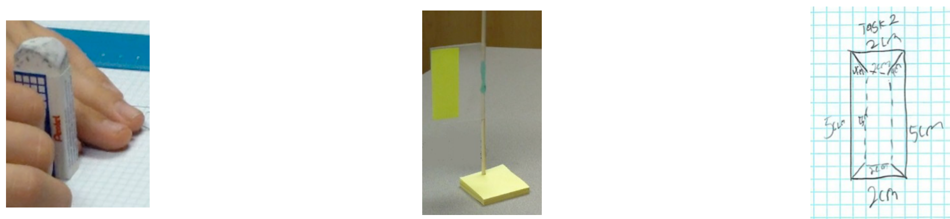
(a) **(b)** **(c)**

Figure 3. (a) Pete's illustration of the rotation, (b) Pete's illustration of the rotation using an eraser, and (c) Pete's illustration drawing for the first task.

In the second task, Pete rotated the rectangle similar to the first task (90-degree rotation each time separately) and associated the lengths of the rectangle with the edges of the rectangular prism he drew by saying: "if it is telling you to rotate it at, the rectangle 360 degrees about line *KL*, so you'd probably- rotate it towards *KL* that 2 cm could be telling us the height of the object like I said the height on the side." In his semiotic representations, Pete rotated the rectangles 360 degrees through their centroids and put them back in their original positions. He produced a semiotic representation in 3D geometry converting 2D figures into polyhedrons by matching one of the lengths he observed in the tasks.

Results from the Participants' Uses of Manipulatives

In the second session of the interviews, the interviewees solved the same tasks using manipulatives. The use of manipulatives helped Andrea and Pete utilize operative apprehension to reason about the problems since they had rotated the rectangles through the centroid of them in a 2D system using paper and pencil. They made sense of the crucial statements given in the tasks. Namely, the rotation line was perpendicular to the plane and rotating the rectangles about the line implied spinning objects. They noticed that the statement meant to spin the rectangles. However, Pete said that the same object would be formed although he was aware of the blind spots of the rotation. He demonstrated the rotation using the eraser (Figure 4(a)) and said: "when I am doing it this way, it's rotating, I am rotating it as towards me so it'd look like straight line (when the rectangle is rotated 90 degrees, it looked like a straight line) kind of but a 3-dimensional object."

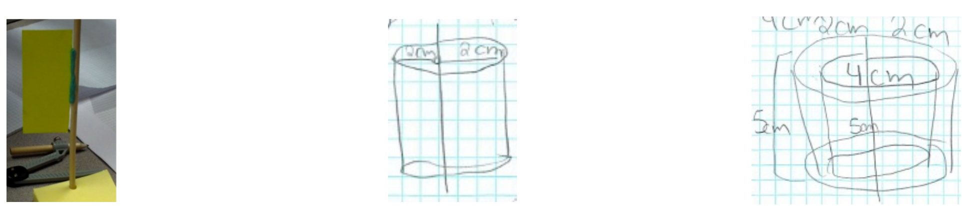


(a) **(b)** **(c)**

Figure 4. (a) Pete's illustration of spinning using an eraser, (b) Pete's use of manipulatives for the second task, and (c) Pete's modified drawing for the second task.

When Pete was reasoning about the second task using manipulatives (Figure 4(b)), he changed the height of the rectangular prism from 2 cm to 4 cm by adding *KD* and *CD* together by saying: "the 2 cm is here and here but also we added to the 2 cm that it is away. And, if you add them together so that it maybe 4 cm as the height of it from the ground now that I am looking at like this" (Figure 4(c)). Although Pete was doing a right treatment demonstrating how the rotation took place, he had difficulty coordinating the representations in the treatment. He focused on the outcome of the rotation and made little connection between his 3D drawings (e.g., Figure 4(c)) and how the rectangles were rotated.

On the other hand, Andrea held the same belief that one cannot form a 3D object by rotating a plane figure by saying:" it just goes around, stays the same. No matter what you do, if you flip 360 this way and go back to where it was...But no matter what you do if you spin it this way, this way, flip it upside down, it stays the same." She had difficulty thinking of the rotation in a continuous motion. However, using manipulatives, Andrea was able to notice the circular motion of the rotation. She described the previous rotation as flipping, and the new rotation using manipulatives as spinning. While Andrea was solving the third task in which she rotated a semi-circle, she highlighted that it would look like a sphere if one spins it fast enough by saying: "I assume the figure, if you're looking at it when you're spinning fast enough it will look as if it a whole sphere but it still won't be a sphere."



(a) **(b)** **(c)**

Figure 5. (a) Manipulatives designed for the first task, (b) Andrea's drawing for the first task, and (c) Andrea's drawing for the second task.

Afterwards, the researcher asked what she would form by rotating a rectangle fast enough. Andrea thought it would form a rectangular prism. However, soon after she rotated the rectangle using manipulatives (Figure 5(a)), she noticed that it would form a cylinder by saying:

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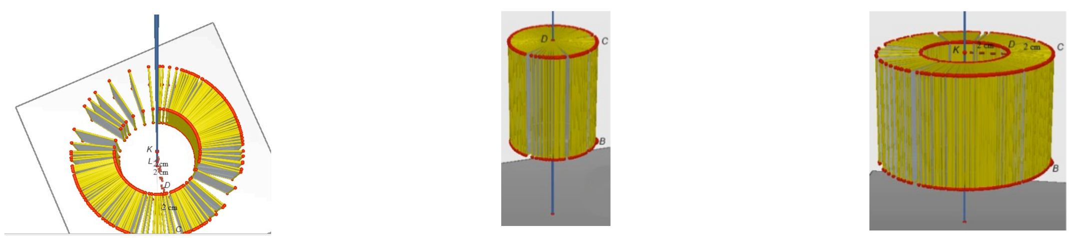
> If you spin it fast enough it would look like all the edges are coming up. Oh, actually, probably look more circular instead of actually rectangular. It would look like a flat, like a cylinder instead of a rectangular prism if you're spinning it so it could be a cylinder instead of a square thing (rectangular prism). Cause if you're spinning it'd look circular on the edges but look flat on the top.

Although Andrea figured out the correctly formed 3D objects after rotations, she mislabeled the radii of circles in her both drawings. Also, she misrepresented the location of line *AD* by drawing it at the left edge of the cylinders instead of in the center of them. The third task in which a semi-circle was rotated 360 degrees helped Andrea revise the radii lengths of cylinders and the location of line *AD*. While identifying the object after she rotated a semi-circle, Andrea highlighted that a sphere would be formed by saying: "yeah, like that makes a sphere. Because half of the line is kind of going down the middle of it. This is a semi-circle on this side, and if you're spinning fast enough, you can kinda see a sphere form out of it." Afterwards, the interviewer showed that she drew the line in the middle when she formed a sphere, while the line was at the edge of the cylinders in the previous tasks. With the prompt of the interviewer, Andrea modified her drawing as shown in Figure 5(b) and Figure 5(c).

Results from the Participants' Uses of Cabri 3D

In the third session of the interviews, the interviewees used Cabri 3D and the *Trajectory* tool. Because Andrea reasoned about the tasks correctly using manipulatives, she did not modify her drawings; instead she verified her drawings and restated her thinking. For example, in the second task, she looked at the object from the top view to indicate the inner cylinder as shown in Figure 6(a) by saying:

> *Andrea*: You can see there is a cylinder in the middle and a cylinder on the outside (Figure 6(a)). So, you see both of the shapes and stuff.
> *Interviewer:* So, why did we get a hole in it?
> *Andrea*: Because the lines are going outwards, they make it, right there, they made it so where it would have a space between this cylinder and this line and the outward cylinder.



(a) **(b)** **(c)**

Figure 6. (a) The rectangle in the second task traced out by Andrea, (b) The rectangle in the first task traced out by Pete, and (c) The rectangle in the second task traced out by Pete.

Andrea appreciated Cabri 3D by saying: "this one (Cabri 3D) helped a lot more because you can actually show everything that was going on..." On the other hand, Pete changed his reasoning by observing the objects formed in Cabri 3D. After he rotated the rectangle (Figure 6(b)), he said "I observe that it made a cylinder" and added "because you are dragging around 360-degree motion that it's tracing the whole time you go around. It's tracing the object during it went." The interviewer asked him to compare and contrast what he was thinking before and what he observed. He explained his previous thinking by saying "I was thinking before that it wanted to know the 3-dimensional shape of the object after you rotated it. So, I was thinking that the object that it wanted to know was

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what the shape would be after you rotated it, not the rotation would look like - the shape that rotation made." Using Cabri 3D, he made a connection between how the rectangles were rotated and the resulting 3D objects. Similarly, in the second task, Pete interpreted what he observed (Figure 6(c)) as follows:

> A cylinder but it's hollow in the middle… Because of the fact that it's 2cm away, and instead of like I was thinking, what I was thinking telling you the height of it, of the object. Instead of that, it's telling you that's how far away the rotation going around the object was.

While using Cabri 3D, he observed the rotations from different perspectives to articulate his thinking. The interviewer asked Pete to explain how he considered the tasks using three different learning tools. Pete explained his thinking as follows:

> I was thinking on this one (manipulatives) just like I was on this one (paper and pencil) that it was wanting to know what the 3-dimensional object was, and not what the actual rotation of the object would make if you were to trace it all the way around.

Discussion and Implications

In this research study, we analyzed Andrea and Pete's formation of 3D objects by rotating 2D figures using paper and pencil, manipulatives and Cabri 3D. They were reasoning about the problems primarily affected by perceptual apprehension because they made treatments rotating objects in a 2D semiotic system. They had difficulty understanding how they were asked to rotate the rectangles about the axes of rotations. In other words, they had difficulty making a conversion from the figures to the statements given in the tasks (or vice versa). As a consequence, Andrea was unable to produce a representation in a 3D semiotic system. Pete produced representations matching one of the lengths he observed in the tasks to denote the height of 3D objects. In Pete's drawings, there was little connection between the rotations and the resulting 3D objects. He had difficulty associating the rotation of 2D figures with the resultant 3D objects. Similarly, Gorgorió's (1998) research indicated students with low spatial abilities had difficulty describing the movements of rotations. Pete viewed prisms as a 3D form of 2D figures (e.g., rectangle $\rightarrow$ rectangular prism). These findings suggest a need to conduct research to examine how students can make connections between extruding (e.g. forming a cylinder translating a circle in a linear and continuous motion) and spinning 2D figures (e.g., forming a cylinder spinning a rectangle about one of its sides).

When Andrea and Pete were given manipulatives to solve the tasks, they observed the circular motion of the rotation utilizing operative apprehension. Pete focused on the outcome of the rotations, and provided a similar semiotic representation drawing rectangular prisms. On the other hand, Andrea, at first, held the same belief that one cannot form a 3D object by rotating a 2D figure until she rotated a semi-circle. Then, she identified the correct formed objects although she mislabeled the measures of the cylinders. After using manipulatives, the description of the rotation that Andrea used shifted from flipping to spinning. However, she needed to go beyond to identify the resultant objects because as Kaput (1995) emphasized manipulatives do not always provide sufficient information. She managed to form the objects associating the statements given in the tasks with the semiotic representations she produced. However, Pete who had low spatial abilities could not identify the objects. Schumann and Green (1997) emphasized that imagining a point's continuous motion in mind and conceive its path to look for mathematical relationships is closely related to students' visual imagination capabilities.

When Pete used Cabri 3D, he was aware of the continuous curricular motion of the 2D figures. Based on his observation of the objects formed in Cabri 3D, he attended to the properties of the formed objects. He compared and contrasted the representations in Cabri 3D with his previous drawings. With the feedback of Cabri 3D, Pete considered the rotation as a continuous process different from his previous methods that focused on outcomes of the rotations. As Hollebrands and Dove (2011) state, technology allows students to observe and interpret the outcomes of fixed properties under different circumstances with the help of the trace tool. On the other hand, Andrea verified her reasoning based on the feedback DGS provided, and supported her previous thinking.

Using manipulatives, the interviewees observed different locations of 2D figures as they helped them to think about the problems in a 3D context. However, the interviewees had difficulty imagining the resulting 3D objects. Manipulatives that give a concrete evidence about forming 3D objects as shown in Figure 7 can be given to students. Future research can examine short- or long-term influences of using different learning tools. Also, we had students use Cabri 3D for comparing and contrasting their previous thinking. New research is needed to indicate how students form 3D objects by the rotations of 2D figures.

Figure 7. Some hands-on materials related to spinning figures.

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